CS 109A/STAT 121A/AC 209A/CSCI E-109A: Homework 4

Regularization, High Dimensionality, PCA

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INSTRUCTIONS

- To submit your assignment follow the instructions given in canvas.
- Restart the kernel and run the whole notebook again before you submit.
- Do not include your name(s) in the notebook even if you are submitting as a group.
- If you submit individually and you have worked with someone, please include the name of your [one] partner below.

Your partner's name (if you submit separately):

Enrollment Status (109A, 121A, 209A, or E109A): AC209

Import libraries:

In [1]:
```
import numpy as np
import pandas as pd
import matplotlib
import matplotlib.pyplot as plt
from sklearn.metrics import r2_score
import statsmodels.api as sm
from statsmodels.api import OLS
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import Ridge
from sklearn.linear_model import Lasso
from sklearn.linear_model import RidgeCV
from sklearn.linear_model import LassoCV
import seaborn as sns
%matplotlib inline
```

```
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\statsmodels\compat\pandas.py:56: FutureWarning: The pandas.co
re.datetools module is deprecated and will be removed in a future version. Please use the pandas.tseries module in
stead.
  from pandas.core import datetools
```

Continuing Bike Sharing Usage Data

In this homework, we will focus on multiple linear regression, regularization, dealing with high dimensionality, and PCA. We will continue to build regression models for the Capital Bikeshare program in Washington D.C. See Homework 3 for more information about the data.

*Note: please make sure you use all the processed data from HW 3 Part (a)...you make want to save the data set on your computer and reread the csv/json file here.

In [2]:
```
dtype_dict = {"season":int, "month":int, "holiday":int, "day_of_week": int, "workingday":int, "weather":int, \
             "temp":float, "atemp":float, "humidity":float, "windspeed": float, "count":int}
```

In [3]:
```
# Load data from HW3
bike_train_df = pd.read_csv('data/bike_train.csv', index_col=0, low_memory=False)
bike_test_df = pd.read_csv('data/bike_test.csv', index_col=0, low_memory=False)
bike_train_df.head()
```

Out[3]:

	holiday	workingday	temp	atemp	humidity	windspeed	count	season_1	season_2	season_3	...	month_11	day_of_\
0	0	1	0.624743	0.651090	0.922058	-0.930164	6073	0	1	0	...	0	0
1	0	1	-0.180583	-0.054841	0.697907	-0.213825	6606	0	0	0	...	0	0
2	0	1	0.803704	0.852785	-0.449062	0.805143	7363	0	1	0	...	0	0
3	0	0	-1.522794	-1.567551	-0.332616	-0.269507	2431	0	0	0	...	0	0
4	0	1	0.535262	0.348548	1.978781	-1.200843	1996	0	0	1	...	0	0

5 rows × 30 columns

Part (f): Regularization/Penalization Methods

As an alternative to selecting a subset of predictors and fitting a regression model on the subset, one can fit a linear regression model on all predictors, but shrink or regularize the coefficient estimates to make sure that the model does not "overfit" the training set.

Use the following regularization techniques to fit linear models to the training set:

- Ridge regression
- Lasso regression

You may choose the shrikage parameter λ from the set $\{10^{-5}, 10^{-4}, \ldots, 10^{4}, 10^{5}\}$ using cross-validation. In each case,

- How do the estimated coefficients compare to or differ from the coefficients estimated by a plain linear regression (without shrikage penalty) in Part (b) fropm HW 3? Is there a difference between coefficients estimated by the two shrinkage methods? If so, give an explantion for the difference.
- List the predictors that are assigned a coefficient value close to 0 (say < 1e-10) by the two methods. How closely do these predictors match the redundant predictors (if any) identified in Part (c) from HW 3?
- Is there a difference in the way Ridge and Lasso regression assign coefficients to the predictors `temp` and `atemp`? If so, explain the reason for the difference.

We next analyze the performance of the two shrinkage methods for different training sample sizes:

- Generate random samples of sizes 100, 150, ..., 400 from the training set. You may use the following code to draw a random sample of a specified size from the training set:

In [4]:
```
X_train = bike_train_df.drop('count', axis=1).copy()
Y_train = bike_train_df['count']

X_test = bike_test_df.drop('count', axis=1).copy()
Y_test = bike_test_df['count']
```

In [5]:
```
MLR_result = OLS(Y_train, sm.add_constant(X_train)).fit()
coef_ind = MLR_result.params.index.tolist()
```

In [6]:
```
shrikage = 10.**np.arange(-5, 6)
```

In [7]:
```
coef_dict = dict()
for s in shrikage:

    ridge_cv_model = RidgeCV(alphas=[s])
    ridge_cv_result = ridge_cv_model.fit(X_train, Y_train)
    ridge_params = np.hstack((ridge_cv_result.intercept_, ridge_cv_result.coef_))
    ridge_series = pd.Series(data=ridge_params, index=coef_ind)

    lasso_cv_model = LassoCV(alphas=[s], max_iter=1e5)
    lasso_cv_result = lasso_cv_model.fit(X_train, Y_train)
    lasso_params = np.hstack((lasso_cv_result.intercept_, lasso_cv_result.coef_))
    lasso_series = pd.Series(data=lasso_params, index=coef_ind)

    all_params_df = pd.concat({'All':MLR_result.params, 'Ridge':ridge_series, 'Lasso':lasso_series}, axis=1)
    coef_dict[str(s)] = all_params_df
```

```
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\sklearn\linear_model\coordinate_descent.py:484: ConvergenceWa
rning: Objective did not converge. You might want to increase the number of iterations. Fitting data with very sma
ll alpha may cause precision problems.
  ConvergenceWarning)
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\sklearn\linear_model\coordinate_descent.py:484: ConvergenceWa
rning: Objective did not converge. You might want to increase the number of iterations. Fitting data with very sma
ll alpha may cause precision problems.
  ConvergenceWarning)
```

In [8]:
```
coef_dict.keys()
```

Out[8]:
```
dict_keys(['1e-05', '0.0001', '0.001', '0.01', '0.1', '1.0', '10.0', '100.0', '1000.0', '10000.0', '100000.0'])
```

In [9]:
```
coef_dict['1.0']
```

Out[9]:

	All	Lasso	Ridge
const	3192.209225	4747.892483	4204.708320
holiday	-284.356275	-185.908268	-244.125575
workingday	308.153142	367.097850	296.816742
temp	924.334403	892.793640	809.380072
atemp	311.961760	320.899770	381.424635
humidity	-547.663783	-545.623270	-544.331033
windspeed	-254.736916	-252.605100	-253.325313
season_1	-1226.186543	-1117.607718	-1059.725202
season_2	-327.357503	-231.522796	-228.539407
season_3	-193.304968	-166.721222	-221.729595
month_1	118.835819	0.000000	-23.756080
month_2	207.775911	89.033153	87.763838
month_3	358.016717	251.858672	266.633863
month_4	452.184905	363.970248	416.380937
month_5	53.023319	-0.000000	67.809242
month_6	-673.427080	-684.921111	-570.845620
month_7	-1161.151188	-1117.544788	-944.411268
month_8	-657.639671	-614.590506	-452.577697
month_9	523.980385	536.791951	645.226625
month_10	605.086722	618.304629	668.880786
month_11	231.517464	225.768431	268.391564
day_of_week_1	-123.751476	-193.015018	-133.416808
day_of_week_2	-195.285933	-244.056441	-175.661560
day_of_week_3	170.511347	99.470067	167.800956
day_of_week_4	61.256030	0.000000	70.861404
day_of_week_5	111.066899	44.222639	123.107175
day_of_week_6	465.145010	449.197520	438.207932
weather_1	1596.918031	18.028718	496.786237
weather_2	1580.351447	0.000000	473.455308
weather_3	14.939747	-1523.106354	-970.241545

In [10]:
```
for s in shrikage:
    zero_coef = coef_dict[str(s)]['Lasso'].index[np.abs(coef_dict[str(s)]['Lasso']) < 1e-10].tolist()
    print("With alpah = %s: The coefficients of the following predictors goes to 0:\n\t%s\n" % (s, zero_coef))
```

```
With alpah = 1e-05: The coefficients of the following predictors goes to 0:
        ['day_of_week_4']

With alpah = 0.0001: The coefficients of the following predictors goes to 0:
        ['day_of_week_4']

With alpah = 0.001: The coefficients of the following predictors goes to 0:
        ['day_of_week_4']

With alpah = 0.01: The coefficients of the following predictors goes to 0:
        ['day_of_week_4']

With alpah = 0.1: The coefficients of the following predictors goes to 0:
        ['day_of_week_4', 'weather_2']

With alpah = 1.0: The coefficients of the following predictors goes to 0:
        ['month_1', 'month_5', 'day_of_week_4', 'weather_2']

With alpah = 10.0: The coefficients of the following predictors goes to 0:
        ['holiday', 'season_2', 'month_2', 'month_5', 'day_of_week_4', 'weather_2']

With alpah = 100.0: The coefficients of the following predictors goes to 0:
        ['holiday', 'workingday', 'season_2', 'season_3', 'month_1', 'month_2', 'month_3', 'month_4', 'month_5',
 'month_6', 'month_7', 'month_8', 'month_9', 'month_10', 'month_11', 'day_of_week_1', 'day_of_week_2', 'day_of_wee
k_3', 'day_of_week_4', 'day_of_week_5', 'day_of_week_6', 'weather_1', 'weather_2', 'weather_3']

With alpah = 1000.0: The coefficients of the following predictors goes to 0:
        ['holiday', 'workingday', 'temp', 'humidity', 'windspeed', 'season_1', 'season_2', 'season_3', 'month_1',
 'month_2', 'month_3', 'month_4', 'month_5', 'month_6', 'month_7', 'month_8', 'month_9', 'month_10', 'month_11',
 'day_of_week_1', 'day_of_week_2', 'day_of_week_3', 'day_of_week_4', 'day_of_week_5', 'day_of_week_6', 'weather_
1', 'weather_2', 'weather_3']

With alpah = 10000.0: The coefficients of the following predictors goes to 0:
        ['holiday', 'workingday', 'temp', 'atemp', 'humidity', 'windspeed', 'season_1', 'season_2', 'season_3', 'm
onth_1', 'month_2', 'month_3', 'month_4', 'month_5', 'month_6', 'month_7', 'month_8', 'month_9', 'month_10', 'mont
h_11', 'day_of_week_1', 'day_of_week_2', 'day_of_week_3', 'day_of_week_4', 'day_of_week_5', 'day_of_week_6', 'weat
her_1', 'weather_2', 'weather_3']

With alpah = 100000.0: The coefficients of the following predictors goes to 0:
        ['holiday', 'workingday', 'temp', 'atemp', 'humidity', 'windspeed', 'season_1', 'season_2', 'season_3', 'm
onth_1', 'month_2', 'month_3', 'month_4', 'month_5', 'month_6', 'month_7', 'month_8', 'month_9', 'month_10', 'mont
h_11', 'day_of_week_1', 'day_of_week_2', 'day_of_week_3', 'day_of_week_4', 'day_of_week_5', 'day_of_week_6', 'weat
her_1', 'weather_2', 'weather_3']
```

For shrikage = 1e-5 to 1e-1

> Both Lasso and Ridge give very similar coefficients compared to the plain linear regression method without penalty, except for some terms. The main difference is in the intercept, holiday, day_of_week_3 and 4 and the weathers. In particular, most of Ridge's coefficients are almost identical. This makes sense since alpha, which can be thought of as the "amplification" of the penalizing term, is small. Thus not much penalty is added.

For shrikage = 1 to 1e5

> Both Lasso and Ridge diverge from the original model. As the shrikage parameter alpha keey increasing, more coefficients in the LASSO regularization model go to zero. At the extreme case alpha = 1e5, all coefficients go to 0 as the penalty term dominate the linear regression term. However, the coefficients in Ridge model never go to 0 but shrink together

At shrikage = 10, only the coefficients for "temp", "atemp", "humidity", "windspeed" and "season_1" has non-zero coefficients, which are included in the Forward and Backward selection (redundancy removal) performed in HW3

For smaller shrikage alpha (< 1), the coefficients for "temp" and "atemp" predicted from Lasso and Ridge are similar to each other and to the original method. For large alpha, Lasso and Ridge predictions give a decreasing coefficients for "temp" and an increasing one for "atemp", where Lasso has a more drastic change.

In [11]:

```
#--------  sample
# A function to select a random sample of size k from the training set
# Input:
#      x (n x d Dataframe)
#      y (n x 1 Dataframe)
#      k (size of sample)
# Return:
#      chosen sample of predictors and responses

def sample(x, y, k):
    n = x.shape[0] # No. of training points

    # Choose random indices of size 'k'
    subset_ind = np.random.choice(np.arange(n), k)

    # Get predictors and reponses with the indices
    x_subset = x.loc[subset_ind, :]
    y_subset = y.loc[subset_ind]

    return (x_subset, y_subset)
```

- Fit linear, Ridge and Lasso regression models to each of the generated sample. In each case, compute the R^2 score for the model on the training sample on which it was fitted, and on the test set.
- Repeat the above experiment for 10 random trials/splits, and compute the average train and test R^2 across the trials for each training sample size. Also, compute the standard deviation (SD) in each case.
- Make a plot of the mean training R^2 scores for the linear, Ridge and Lasso regression methods as a function of the training sample size. Also, show a confidence interval for the mean scores extending from **mean - SD** to **mean + SD**. Make a similar plot for the test R^2 scores.

How do the training and test R^2 scores compare for the three methods? Give an explanation for your observations. How do the confidence intervals for the estimated R^2 change with training sample size? Based on the plots, which of the three methods would you recommend when one needs to fit a regression model using a small training sample?

Hint: You may use `sklearn`'s `RidgeCV` and `LassoCV` classes to implement Ridge and Lasso regression. These classes automatically perform cross-validation to tune the parameter λ from a given range of values. You may use the `plt.errorbar` function to plot confidence bars for the average R^2 scores.

In [12]:

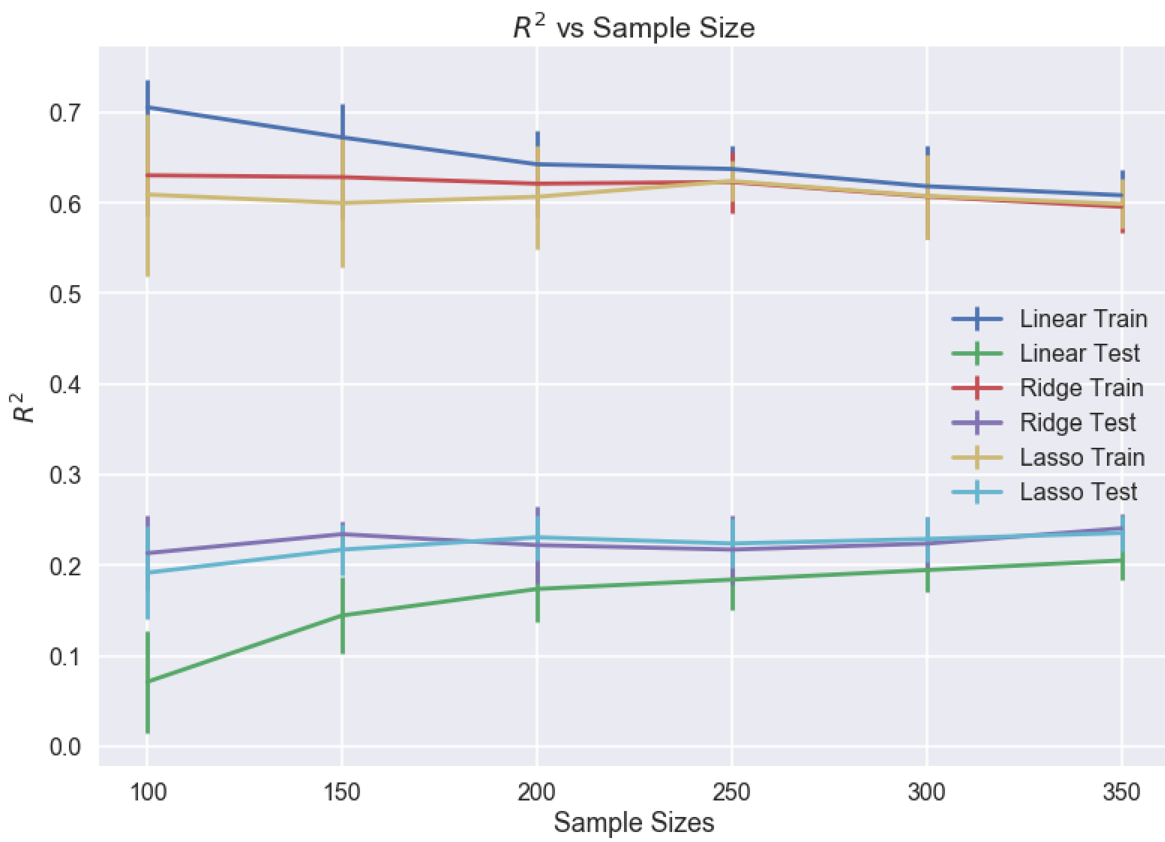
```
sample_sizes = np.arange(100, 400, 50)
sample_sizes
```

Out[12]:

```
array([100, 150, 200, 250, 300, 350])
```

In [13]:
```
linear_R2_train = []
linear_R2_test = []
ridge_R2_train = []
ridge_R2_test = []
lasso_R2_train = []
lasso_R2_test = []
trials = 10

for s in sample_sizes:
    for i in range(trials):
        X_train_sub, Y_train_sub = sample(X_train, Y_train, s)

        MLR_result = OLS(Y_train_sub, X_train_sub).fit()
        linear_R2_train.append(r2_score(Y_train_sub, MLR_result.predict(X_train_sub)))
        linear_R2_test.append(r2_score(Y_test, MLR_result.predict(X_test)))

        ridge_cv_model = RidgeCV(alphas=shrikage)
        ridge_cv_result = ridge_cv_model.fit(X_train_sub, Y_train_sub)
        ridge_R2_train.append(r2_score(Y_train_sub, ridge_cv_result.predict(X_train_sub)))
        ridge_R2_test.append(r2_score(Y_test, ridge_cv_result.predict(X_test)))

        lasso_cv_model = LassoCV(alphas=shrikage, max_iter=1e5)
        lasso_cv_result = lasso_cv_model.fit(X_train_sub, Y_train_sub)
        lasso_R2_train.append(r2_score(Y_train_sub, lasso_cv_result.predict(X_train_sub)))
        lasso_R2_test.append(r2_score(Y_test, lasso_cv_result.predict(X_test)))

linear_R2_train = np.array(linear_R2_train).reshape((len(sample_sizes), trials))
linear_R2_test = np.array(linear_R2_test).reshape((len(sample_sizes), trials))
ridge_R2_train = np.array(ridge_R2_train).reshape((len(sample_sizes), trials))
ridge_R2_test = np.array(ridge_R2_test).reshape((len(sample_sizes), trials))
lasso_R2_train = np.array(lasso_R2_train).reshape((len(sample_sizes), trials))
lasso_R2_test = np.array(lasso_R2_test).reshape((len(sample_sizes), trials))
```

In [14]:
```
sns.set_context('poster')
plt.errorbar(x=sample_sizes, y=np.mean(linear_R2_train, axis=1), yerr=np.std(linear_R2_train, axis=1), label='Linea
r Train')
plt.errorbar(x=sample_sizes, y=np.mean(linear_R2_test, axis=1), yerr=np.std(linear_R2_test, axis=1), label='Linear
 Test')
plt.errorbar(x=sample_sizes, y=np.mean(ridge_R2_train, axis=1), yerr=np.std(ridge_R2_train, axis=1), label='Ridge T
rain')
plt.errorbar(x=sample_sizes, y=np.mean(ridge_R2_test, axis=1), yerr=np.std(ridge_R2_test, axis=1), label='Ridge Tes
t')
plt.errorbar(x=sample_sizes, y=np.mean(lasso_R2_train, axis=1), yerr=np.std(lasso_R2_train, axis=1), label='Lasso T
rain')
plt.errorbar(x=sample_sizes, y=np.mean(lasso_R2_test, axis=1), yerr=np.std(lasso_R2_test, axis=1), label='Lasso Tes
t')
plt.legend(loc='best')
plt.title('$R^2$ vs Sample Size')
plt.xlabel('Sample Sizes')
plt.ylabel('$R^2$')
plt.show()
```



With smaller sample sizes, plain linear regression gives the best R^2 score in the training set but the worst in the testing set. This can be explained by over-fitting of the simple linear regression model. Although Ridge model has better training R^2 score than Lasso for smaller sample sizes, their performance for testing set are really close to each other, except that Ridge has smaller STD (confidence interval). Hence, for small sample size, I would recommend Ridge model for fitting.

Part (g): Polynomial & Interaction Terms

Moving beyond linear models, we will now try to improve the performance of the regression model in Part (b) from HW 3 by including higher-order polynomial and interaction terms.

- For each continuous predictor X_j, include additional polynomial terms X_j^2, X_j^3, and X_j^4, and fit a multiple regression model to the expanded training set. How does the R^2 of this model on the test set compare with that of the linear model fitted in Part (b) from HW 3? Using a t-test, find out which of estimated coefficients for the polynomial terms are statistically significant at a significance level of 5%.
- Fit a multiple linear regression model with additional interaction terms $\mathbb{I}_{month=12} \times temp$ and $\mathbb{I}_{workingday=1} \times \mathbb{I}_{weathersit=1}$ and report the test R^2 for the fitted model. How does this compare with the R^2 obtained using linear model in Part (b) from HW 3? Are the estimated coefficients for the interaction terms statistically significant at a significance level of 5%?

Polynomial Terms

In [15]:
```
# Expand the selected column with their polynomials
# X: Dataframe
# col: List of column names
# order: the orders of polynomials

def expand_poly(X, col, order):
    X_temp = X.copy()
    for c in col:
        for o in range(2, order+1):
            X_temp[c+'_'+str(o)] = X_temp[c]**o

    return X_temp
```

In [16]:
```
X_train_poly = expand_poly(X_train, ['temp', 'atemp', 'humidity', 'windspeed'], 4)
```

In [17]:
```
X_train_poly.head()
```

Out[17]:

	holiday	workingday	temp	atemp	humidity	windspeed	season_1	season_2	season_3	month_1	...	temp_4	atemp
0	0	1	0.624743	0.651090	0.922058	-0.930164	0	1	0	0	...	0.152337	0.4239
1	0	1	-0.180583	-0.054841	0.697907	-0.213825	0	0	0	0	...	0.001063	0.0030
2	0	1	0.803704	0.852785	-0.449062	0.805143	0	1	0	0	...	0.417239	0.7272
3	0	0	-1.522794	-1.567551	-0.332616	-0.269507	0	0	0	0	...	5.377301	2.4572
4	0	1	0.535262	0.348548	1.978781	-1.200843	0	0	1	0	...	0.082085	0.1214

5 rows × 41 columns

In [18]:
```
MLR_result = OLS(Y_train, sm.add_constant(X_train)).fit()
MLR_result_poly = OLS(Y_train, sm.add_constant(X_train_poly)).fit()
```

In [19]:
```
MLR_result_poly.summary()
```

Out[19]:

OLS Regression Results

Dep. Variable:	count	**R-squared:**	0.670
Model:	OLS	**Adj. R-squared:**	0.625
Method:	Least Squares	**F-statistic:**	15.13
Date:	Wed, 11 Oct 2017	**Prob (F-statistic):**	7.98e-50
Time:	21:14:37	**Log-Likelihood:**	-2790.9
No. Observations:	331	**AIC:**	5662.
Df Residuals:	291	**BIC:**	5814.
Df Model:	39		
Covariance Type:	nonrobust		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
const	4255.8774	360.286	11.812	0.000	3546.780	4964.975
holiday	-189.7675	365.157	-0.520	0.604	-908.451	528.916
workingday	351.2739	150.615	2.332	0.020	54.841	647.707
temp	770.3204	758.967	1.015	0.311	-723.441	2264.082
atemp	895.9192	712.094	1.258	0.209	-505.588	2297.426
humidity	-667.9033	157.118	-4.251	0.000	-977.136	-358.671
windspeed	-445.8335	148.704	-2.998	0.003	-738.505	-153.162
season_1	-1523.2288	467.580	-3.258	0.001	-2443.496	-602.961
season_2	-756.7981	536.808	-1.410	0.160	-1813.316	299.720
season_3	55.5255	419.974	0.132	0.895	-771.046	882.097
month_1	555.6676	479.543	1.159	0.248	-388.146	1499.481
month_2	230.5990	478.558	0.482	0.630	-711.275	1172.473
month_3	250.8185	472.431	0.531	0.596	-678.997	1180.634
month_4	137.6431	605.236	0.227	0.820	-1053.552	1328.838
month_5	-481.5367	647.272	-0.744	0.458	-1755.466	792.392
month_6	-900.5181	648.671	-1.388	0.166	-2177.201	376.164
month_7	-861.3206	653.672	-1.318	0.189	-2147.844	425.203
month_8	-1160.2713	644.832	-1.799	0.073	-2429.397	108.854
month_9	-517.7332	527.018	-0.982	0.327	-1554.984	519.518
month_10	-370.2035	420.771	-0.880	0.380	-1198.343	457.936
month_11	-269.8653	380.406	-0.709	0.479	-1018.560	478.830
day_of_week_1	-93.3265	156.015	-0.598	0.550	-400.387	213.734
day_of_week_2	-133.4279	184.734	-0.722	0.471	-497.012	230.156
day_of_week_3	147.7313	195.071	0.757	0.449	-236.197	531.660
day_of_week_4	30.5924	187.547	0.163	0.871	-338.528	399.713
day_of_week_5	209.9371	182.024	1.153	0.250	-148.313	568.187
day_of_week_6	471.0834	246.557	1.911	0.057	-14.178	956.345
weather_1	1746.9551	190.401	9.175	0.000	1372.217	2121.693
weather_2	1805.9670	177.651	10.166	0.000	1456.324	2155.610
weather_3	702.9554	413.144	1.701	0.090	-110.173	1516.084
temp_2	-1805.5466	814.442	-2.217	0.027	-3408.491	-202.602
temp_3	8.5688	274.482	0.031	0.975	-531.653	548.790
temp_4	-44.9184	170.385	-0.264	0.792	-380.261	290.425

atemp_2	1171.9491	786.481	1.490	0.137	-375.962	2719.861
atemp_3	-302.5595	244.983	-1.235	0.218	-784.722	179.603
atemp_4	-20.6433	146.714	-0.141	0.888	-309.398	268.112
humidity_2	-53.5087	154.914	-0.345	0.730	-358.402	251.384
humidity_3	-15.9849	44.689	-0.358	0.721	-103.940	71.970
humidity_4	-24.6869	31.291	-0.789	0.431	-86.272	36.898
windspeed_2	-34.0621	126.569	-0.269	0.788	-283.168	215.044
windspeed_3	44.6309	65.163	0.685	0.494	-83.619	172.881
windspeed_4	-20.0552	30.144	-0.665	0.506	-79.382	39.272

Omnibus:	29.995	**Durbin-Watson:**	1.959
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	10.202
Skew:	-0.094	**Prob(JB):**	0.00609
Kurtosis:	2.161	**Cond. No.**	1.36e+16

In [20]:
```
X_test_poly = expand_poly(X_test, ['temp', 'atemp', 'humidity', 'windspeed'], 4)
```

In [21]:
```
r2_MLR = r2_score(Y_test, MLR_result.predict(sm.add_constant(X_test)))
r2_MLR_poly = r2_score(Y_test, MLR_result_poly.predict(sm.add_constant(X_test_poly)))

print("Linear R^2: %.5f\nPolynomial R^2: %.5f" % (r2_MLR, r2_MLR_poly))
```

```
Linear R^2: 0.24934
Polynomial R^2: 0.27724
```

In [22]:
```
MLR_result_poly.pvalues[MLR_result_poly.pvalues < 0.05]
```

Out[22]:
```
const         1.443714e-26
workingday    2.036885e-02
humidity      2.870343e-05
windspeed     2.950904e-03
season_1      1.256241e-03
weather_1     8.573877e-18
weather_2     5.660778e-21
temp_2        2.740200e-02
dtype: float64
```

R^2 for the polynomial fitting is slightly better than that of the plain linear regression model. According to the p values information, two more terms' coefficients (workingday and $temp^2$) become significant in the polynomial fitting model.

Interaction Terms

In [23]:
```
# Expand the selected two columns with their interaction term
# X: Dataframe
# cols: List of column pairs e.g. cols = [(col1, col2), (col3, col4)]

def expand_inter(X, cols):
    X_temp = X.copy()
    for c in cols:
        col1 = c[0]
        col2 = c[1]
        X_temp[col1+'_X_'+col2] = X_temp[col1]*X_temp[col2]
    return X_temp
```

In [24]:
```
X_train_inter = X_train.copy()
X_test_inter = X_test.copy()

X_train_inter['month_12_X_temp'] = (1 - X_train_inter.loc[:, 'month_1':'month_11'].sum(axis=1)) * X_train_inter['te
mp']
X_test_inter['month_12_X_temp'] = (1 - X_test_inter.loc[:, 'month_1':'month_11'].sum(axis=1)) *
X_test_inter['temp']
```

In [25]:
```
X_train_inter = expand_inter(X_train_inter, [('workingday', 'weather_1')])
X_test_inter = expand_inter(X_test_inter, [('workingday', 'weather_1')])
```

In [26]:
```
X_train_inter.head()
```

Out[26]:

	holiday	workingday	temp	atemp	humidity	windspeed	season_1	season_2	season_3	month_1	...	day_of_week_2
0	0	1	0.624743	0.651090	0.922058	-0.930164	0	1	0	0	...	1
1	0	1	-0.180583	-0.054841	0.697907	-0.213825	0	0	0	0	...	1
2	0	1	0.803704	0.852785	-0.449062	0.805143	0	1	0	0	...	0
3	0	0	-1.522794	-1.567551	-0.332616	-0.269507	0	0	0	0	...	0
4	0	1	0.535262	0.348548	1.978781	-1.200843	0	0	1	0	...	0

5 rows × 31 columns

In [27]:
```
MLR_result_inter = OLS(Y_train, sm.add_constant(X_train_inter)).fit()
```

In [28]:
```
MLR_result_inter.summary()
```

Out[28]: OLS Regression Results

Dep. Variable:	count	**R-squared:**	0.580
Model:	OLS	**Adj. R-squared:**	0.540
Method:	Least Squares	**F-statistic:**	14.36
Date:	Wed, 11 Oct 2017	**Prob (F-statistic):**	2.43e-41
Time:	21:14:37	**Log-Likelihood:**	-2830.5
No. Observations:	331	**AIC:**	5721.
Df Residuals:	301	**BIC:**	5835.
Df Model:	29		
Covariance Type:	nonrobust		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
const	3958.9074	587.527	6.738	0.000	2802.727	5115.088
holiday	-268.7464	400.237	-0.671	0.502	-1056.364	518.871
workingday	102.9274	266.927	0.386	0.700	-422.353	628.207
temp	905.5879	474.450	1.909	0.057	-28.072	1839.247
atemp	274.3553	429.137	0.639	0.523	-570.134	1118.844
humidity	-573.2500	113.820	-5.036	0.000	-797.234	-349.266
windspeed	-275.3288	81.613	-3.374	0.001	-435.933	-114.724
season_1	-884.3187	550.763	-1.606	0.109	-1968.152	199.515
season_2	-110.9431	587.150	-0.189	0.850	-1266.381	1044.495
season_3	-75.5976	453.682	-0.167	0.868	-968.388	817.193
month_1	-1211.6733	1040.400	-1.165	0.245	-3259.052	835.706
month_2	-1095.9921	1020.067	-1.074	0.283	-3103.357	911.373
month_3	-896.4352	975.450	-0.919	0.359	-2816.000	1023.129
month_4	-705.5649	997.870	-0.707	0.480	-2669.249	1258.119
month_5	-1011.9024	1004.051	-1.008	0.314	-2987.750	963.945
month_6	-1737.2977	980.173	-1.772	0.077	-3666.157	191.562
month_7	-2140.0440	941.775	-2.272	0.024	-3993.341	-286.747
month_8	-1640.2677	934.999	-1.754	0.080	-3480.230	199.695
month_9	-421.5095	838.850	-0.502	0.616	-2072.263	1229.244
month_10	-290.2825	766.708	-0.379	0.705	-1799.069	1218.504
month_11	-681.9367	777.020	-0.878	0.381	-2211.016	847.142
day_of_week_1	-151.6213	173.066	-0.876	0.382	-492.193	188.950
day_of_week_2	-236.3003	205.676	-1.149	0.252	-641.045	168.444
day_of_week_3	113.1613	217.128	0.521	0.603	-314.119	540.442
day_of_week_4	31.0892	209.658	0.148	0.882	-381.491	443.670
day_of_week_5	77.8521	202.159	0.385	0.700	-319.972	475.676
day_of_week_6	423.1840	269.763	1.569	0.118	-107.676	954.044
weather_1	1691.8385	296.339	5.709	0.000	1108.679	2274.998
weather_2	1915.1451	264.936	7.229	0.000	1393.784	2436.507
weather_3	351.9238	431.980	0.815	0.416	-498.160	1202.008
month_12_X_temp	1043.5538	728.600	1.432	0.153	-390.241	2477.349
workingday_X_weather_1	313.2194	352.789	0.888	0.375	-381.025	1007.464

Omnibus:	33.048	**Durbin-Watson:**	1.921
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	10.362
Skew:	0.037	**Prob(JB):**	0.00562
Kurtosis:	2.136	**Cond. No.**	1.33e+16

In [29]:
```
r2_MLR_inter = r2_score(Y_test, MLR_result_inter.predict(sm.add_constant(X_test_inter)))

print("Linear R^2: %.5f\nInteraction R^2: %.5f" % (r2_MLR, r2_MLR_inter))
```

```
Linear R^2: 0.24934
Interaction R^2: 0.26852
```

In [30]:
```
MLR_result_inter.pvalues[MLR_result_inter.pvalues < 0.05]
```

Out[30]:
```
const        8.156785e-11
humidity     8.186394e-07
windspeed    8.390803e-04
month_7      2.377039e-02
weather_1    2.721882e-08
weather_2    4.041032e-12
dtype: float64
```

R^2 for the interaction fitting is slightly better than that of the plain linear regression model. According to the p values information, the significant coefficients are exactly the same as the plain linear regression model.

Part (h): PCA to deal with high dimensionality

We would like to fit a model to include all main effects, polynomial terms up to the 4^{th} order, and all interactions between all possible predictors and polynomial terms (not including the interactions between X_j^1, X_j^2, X_j^3, and X_j^4 as they would just create higher order polynomial terms).

- Create an expanded training set including all the desired terms mentioned above. What are the dimensions of this 'design matrix' of all the predictor variables? What are the issues with attempting to fit a regression model using all of these predictors?
- Instead of using the usual approaches for model selection, let's instead use principal components analysis (PCA) to fit the model. First, create the principal component vectors in python (consider: should you normalize first?). Then fit 5 different regression models: (1) using just the first PCA vector, (2) using the first two PCA vectors, (3) using the first three PCA vectors, etc... Briefly summarize how these models compare in the training set.
- Use the test set to decide which of the 5 models above is best to predict out of sample. How does this model compare to the previous models you've fit? What are the interpretations of this model's coefficients?

In [31]:
```
X_train_PCA = X_train_poly.copy()
X_test_PCA = X_test_poly.copy()
```

In [32]:
```
col_all = X_train_PCA.columns.tolist()
col_pairs = []
for i in range(len(col_all)-1):
    for j in range(i+1, len(col_all)):
        # Check the two columns are not correlated
        #     Situation 1: Polynomial terms
        #     Situation 2: Same category terms (will be all 0s in interactive term)
        if col_all[i].split('_')[0] != col_all[j].split('_')[0]:
            col_pairs.append((col_all[i], col_all[j]))
```

In [33]:
```
X_train_PCA = expand_inter(X_train_PCA, col_pairs)
X_test_PCA = expand_inter(X_test_PCA, col_pairs)

X_train_PCA.shape
```

Out[33]:
```
(331, 761)
```

In [34]:
```
MLR_PCA_all_result = OLS(Y_train, sm.add_constant(X_train_PCA)).fit()
```

In [35]:
```python
MLR_PCA_all_result.pvalues[MLR_PCA_all_result.pvalues < 0.05]
```

```
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\statsmodels\regression\linear_model.py:1353: RuntimeWarning:
 divide by zero encountered in double_scalars
  return np.dot(wresid, wresid) / self.df_resid
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\statsmodels\base\model.py:1118: RuntimeWarning: invalid value
encountered in multiply
  cov_p = self.normalized_cov_params * scale
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\scipy\stats\_distn_infrastructure.py:879: RuntimeWarning: inv
alid value encountered in greater
  return (self.a < x) & (x < self.b)
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\scipy\stats\_distn_infrastructure.py:879: RuntimeWarning: inv
alid value encountered in less
  return (self.a < x) & (x < self.b)
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\scipy\stats\_distn_infrastructure.py:1818: RuntimeWarning: in
valid value encountered in less_equal
  cond2 = cond0 & (x <= self.a)
```

Out[35]:
```
Series([], dtype: float64)
```

With all predictors' interaction/polynomial terms, there are 761 predictors in total, which are more than twice of the sample size. Trying to fit a regression model using all of these predictors will generate non-unique parameters (since there are more coefficients than samples, the solution is not unique)

In [36]:
```python
# Normalized the continous data before applying PCA
# The test set should be normalized by the training set's parameters
for col in X_train_PCA.columns.tolist():
    if X_train_PCA[col].dtype == np.float64:
        mean_val =  X_train_PCA[col].mean()
        std_val = X_train_PCA[col].std()

        X_train_PCA[col] = (X_train_PCA[col] - mean_val)/std_val
        X_test_PCA[col] = (X_test_PCA[col] - mean_val)/std_val
```

In [37]:
```python
X_train_PCA.head()
```

Out[37]:

	holiday	workingday	temp	atemp	humidity	windspeed	season_1	season_2	season_3	month_1	...	atemp_4_X_wind
0	0	1	0.623798	0.650106	0.920664	-0.928758	0	1	0	0	...	-0.096003
1	0	1	-0.180310	-0.054759	0.696852	-0.213502	0	0	0	0	...	-0.097008
2	0	1	0.802489	0.851495	-0.448383	0.803926	0	1	0	0	...	-0.095348
3	0	0	-1.520492	-1.565182	-0.332113	-0.269099	0	0	0	0	...	-0.096770
4	0	1	0.534453	0.348021	1.975789	-1.199027	0	0	1	0	...	-0.096779

5 rows × 761 columns

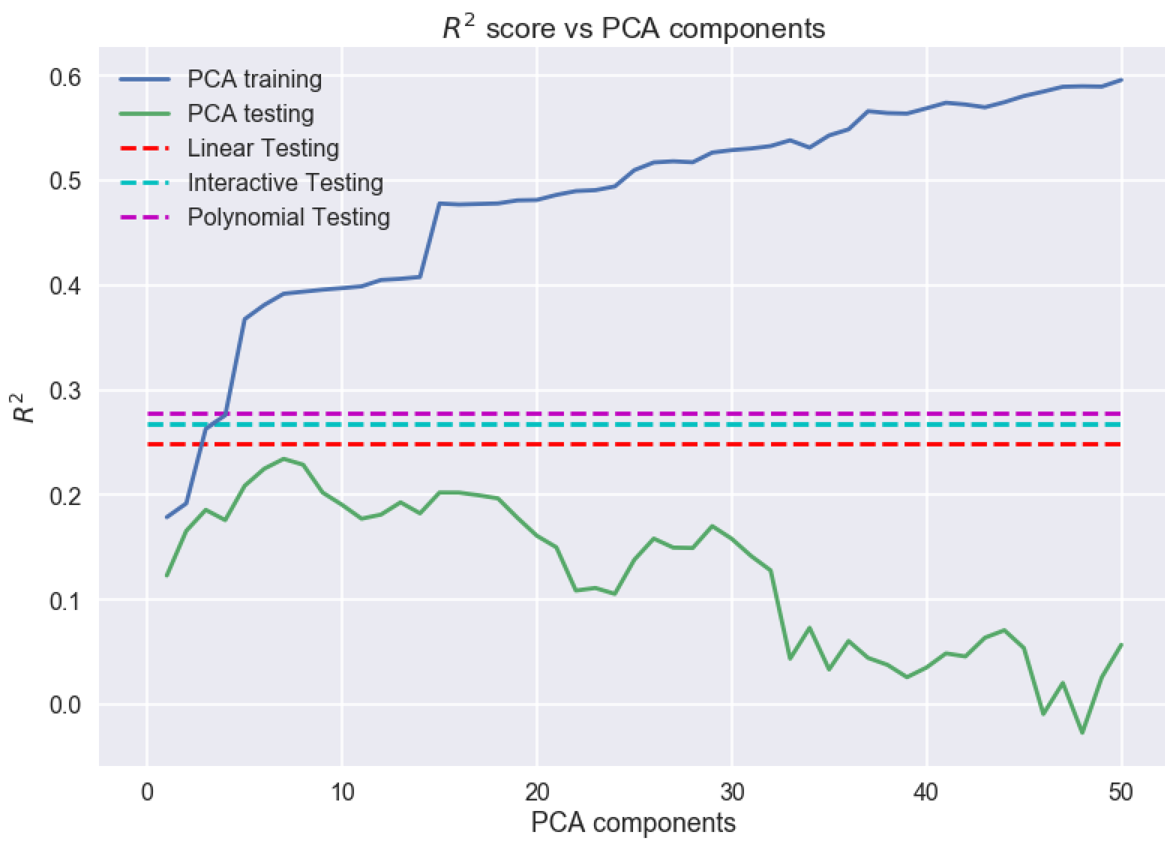
In [38]:
```python
from sklearn.decomposition import PCA
```

In [39]:
```python
R2_train_PCA = []
R2_test_PCA = []
max_PCA_comp = 51
for n in range(1, max_PCA_comp):

    pca = PCA(n_components=n)
    Z_train_PCA = pca.fit_transform(X_train_PCA)
    MLR_result_PCA = OLS(Y_train, sm.add_constant(Z_train_PCA)).fit()

    Z_test_PCA = pca.transform(X_test_PCA)

    R2_train_PCA.append(r2_score(Y_train, MLR_result_PCA.predict(sm.add_constant(Z_train_PCA))))
    R2_test_PCA.append(r2_score(Y_test, MLR_result_PCA.predict(sm.add_constant(Z_test_PCA))))
```

In [40]:
```python
np.max(R2_test_PCA), np.argmax(R2_test_PCA)
```

Out[40]:
```
(0.23463200887601854, 6)
```

In [41]:
```
sns.set_context('poster')
plt.plot(np.array(range(1, max_PCA_comp)), np.array(R2_train_PCA), label='PCA training')
plt.plot(np.array(range(1, max_PCA_comp)), np.array(R2_test_PCA), label='PCA testing')
plt.hlines(y=r2_MLR, xmin=0, xmax=max_PCA_comp-1, color='r', linestyles='dashed', label='Linear Testing')
plt.hlines(y=r2_MLR_inter, xmin=0, xmax=max_PCA_comp-1, color='c', linestyles='dashed', label='Interactive
Testing')
plt.hlines(y=r2_MLR_poly, xmin=0, xmax=max_PCA_comp-1, color='m', linestyles='dashed', label='Polynomial Testing')
plt.legend(loc='best')
plt.xlabel('PCA components')
plt.ylabel('$R^2$')
plt.title('$R^2$ score vs PCA components')
plt.show()
```



The plot above shows the R^2 score for the PCA with number of components from 1 to 50. As expected, the R^2 score in the training set increases as the component number increase. However, surprisingly, the R^2 score of the PCA testing set never reach the level of the Interactive or polynomial testing for n < 50, and reachs its maximum value of 0.235 at n = 6

Part (i): Beyond Squared Error

We have seen in class that the multiple linear regression method optimizes the Mean Squared Error (MSE) on the training set. Consider the following alternate evaluation metric, referred to as the Root Mean Squared Logarthmic Error (RMSLE):

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(log(y_i+1)-log(\hat{y}_i+1))^2}.$$

The *lower* the RMSLE the *better* is the performance of a model. The RMSLE penalizes errors on smaller responses more heavily than errors on larger responses. For example, the RMSLE penalizes a prediction of $\hat{y}=15$ for a true response of $y=10$ more heavily than a prediction of $\hat{y}=105$ for a true response of 100, though the difference in predicted and true responses are the same in both cases.

This is a natural evaluation metric for bike share demand prediction, as in this application, it is more important that the prediction model is accurate on days where the demand is low (so that the few customers who arrive are served satisfactorily), compared to days on which the demand is high (when it is less damaging to lose out on some customers).

The following code computes the RMSLE for you:

In [42]:
```
#--------  rmsle
# A function for evaluating Root Mean Squared Logarithmic Error (RMSLE)
# of the linear regression model on a data set
# Input:
#      y_test (n x 1 array of response variable vals in testing data)
#      y_pred (n x 1 array of response variable vals in testing data)
# Return:
#      RMSLE (float)

def rmsle(y, y_pred):
    # Evaluate sqaured error, against target labels
    # rmsle = \sqrt(1/n \sum_i (log (y[i]+1) - log (y_pred[i]+1))^2)
    rmsle_ = np.sqrt(np.mean(np.square(np.log(y+1) - np.log(y_pred+1))))

    return rmsle_
```

Use the above code to compute the training and test RMSLE for the polynomial regression model you fit in Part (g).

You are required to develop a strategy to fit a regression model by optimizing the RMSLE on the training set. Give a justification for your proposed approach. Does the model fitted using your approach yield lower train RMSLE than the model in Part (g)? How about the test RMSLE of the new model?

Note: We do not require you to implement a new regression solver for RMSLE. Instead, we ask you to think about ways to use existing built-in functions to fit a model that performs well on RMSLE. Your regression model may use the same polynomial terms used in Part (g).

In [43]:
```
rmsle_train_poly = rmsle(Y_train, MLR_result_poly.predict(sm.add_constant(X_train_poly)))
rmsle_test_poly = rmsle(Y_test, MLR_result_poly.predict(sm.add_constant(X_test_poly)))

print("RMSLE Training: %.5f\nRMSLE Testing: %.5f" % (rmsle_train_poly, rmsle_test_poly))
```

```
RMSLE Training: 0.31064
RMSLE Testing: 0.52427
```

```
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\ipykernel_launcher.py:13: RuntimeWarning: invalid value encou
ntered in log
  del sys.path[0]
```

To impletment the RMSLE cost function with existing built-in function, we can use the ordinary linear regression model with a new response:

$$Y_{new} = ln(Y + 1)$$

Then the linear regression cost function on Y_{new} automatically become the RMSLE on Y

In [44]:
```
# Use OLS to fit log(y+1) with X
Y_train_log = np.log(Y_train + 1)
Y_test_log = np.log(Y_test + 1)

MLR_result_log = OLS(Y_train_log, sm.add_constant(X_train_poly)).fit()

Y_train_log_pred = np.exp(MLR_result_log.predict(sm.add_constant(X_train_poly))) - 1
Y_test_log_pred = np.exp(MLR_result_log.predict(sm.add_constant(X_test_poly))) - 1

rmsle_train_poly = rmsle(Y_train, Y_train_log_pred)
rmsle_test_poly = rmsle(Y_test, Y_test_log_pred)

print("RMSLE Training: %.5f\nRMSLE Testing: %.5f" % (rmsle_train_poly, rmsle_test_poly))
```

```
RMSLE Training: 0.29117
RMSLE Testing: 0.52355
```

In [45]:
```
MLR_result_log.pvalues[MLR_result_log.pvalues < 0.05]
```

Out[45]:
```
const         1.025111e-180
workingday     1.266310e-03
humidity       2.374314e-05
windspeed      1.067498e-03
season_1       3.455829e-05
month_8        3.387338e-02
weather_1     8.926599e-136
weather_2     1.869153e-145
weather_3      1.548309e-43
dtype: float64
```

Comparing the RMSLE score on the ordinary linear regression and the "logarithmic" regression, the latter has a lower RMSLE score (0.02 difference) on the training set, while testing set is very similar (0.0007 difference).

Part (j): Dealing with Erroneous Labels

Due to occasional system crashes, some of the bike counts reported in the data set have been recorded manually. These counts are not very unreliable and are prone to errors. It is known that roughly 5% of the labels in the training set are erroneous (i.e. can be arbitrarily different from the true counts), while all the labels in the test set were confirmed to be accurate. Unfortunately, the identities of the erroneous records in the training set are not available. Can this information about presence of 5% errors in the training set labels (without details about the specific identities of the erroneous rows) be used to improve the performance of the model in Part (g)? Note that we are interested in improving the R^2 performance of the model on the test set (not the training R^2 score).

As a final task, we require you to come up with a strategy to fit a regression model, taking into account the errors in the training set labels. Explain the intuition behind your approach (we do not expect a detailed mathematical justification). Use your approach to fit a regression model on the training set, and compare its test R^2 with the model in Part (g).

Note: Again, we do not require you to implement a new regression solver for handling erroneous labels. It is sufficient that you to come up with an approach that uses existing built-in functions. Your regression model may use the same polynomial terms used in Part (g).

Thoughts:

In linear regression modeling, we assume the error from the predictive value and the true value has normal distribution with 0 mean and a non-zero STD. If we assume this model is correct here, then the further a error is from 0, the less likely this data point belongs to the dataset (and thus more likely to be the erroneous term/outlier). However, not all erroneuous terms are outliers. Hence, we can first find the 5% outliers datapoints. Take the outliers out one by one and fit a regression model using cross-validation in every step and store their scores. The model (with removal of certain outliers) with the best scores should be the one best decribe the data.

In [46]:
```
from sklearn.model_selection import KFold
```

In [47]:
```
Y_diff = Y_train - MLR_result_poly.predict(sm.add_constant(X_train_poly))
```

In [48]:
```
count_min = len(X_train_poly)
s_min = 0
for s in np.arange(1, 3, 0.1):
    count = np.abs(np.sum(np.abs(Y_diff) > s*Y_diff.std()) - len(X_train_poly) * 0.05)
    if count < count_min:
        s_min = s
        count_min = count
s_min
```

Out[48]:
```
1.7000000000000006
```

In [49]:
```
ind_outlier = list(X_train_poly[np.abs(Y_diff) > s_min*Y_diff.std()].index)
```

In [50]:
```
error_indexes = [([], 0)]
# Use forward selection
for k in range(1, len(ind_outlier)+1):
    best_k_minus_1 = error_indexes[-1][0]
    new_index = list(set(ind_outlier) - set(best_k_minus_1))
    score = []

    r2_k = []
    for ind in new_index:
        k_ind = best_k_minus_1 + [ind]

        k_X_train = X_train_poly.drop(k_ind)

        r2_valid = []
        # Cross-Validation
        for train_ind, valid_ind in KFold(10, shuffle=True).split(k_X_train):

            k_X_train_cv = sm.add_constant(k_X_train.iloc[train_ind, :])
            k_X_valid_cv = sm.add_constant(k_X_train.iloc[valid_ind, :])

            Y_train_cv = Y_train.drop(k_ind).iloc[train_ind]
            k_MLR_result = OLS(Y_train_cv, k_X_train_cv).fit()
            Y_valid_pred = k_MLR_result.predict(k_X_valid_cv)

            Y_valid_cv = Y_train.drop(k_ind).iloc[valid_ind]
            r2_valid.append(r2_score(Y_valid_cv, Y_valid_pred))

        r2_k.append(np.mean(np.array(r2_valid)))

    best_k = best_k_minus_1 + [new_index[np.argmax(r2_k)]]
    error_indexes.append((best_k, np.max(r2_k)))
```

In [51]:
```
error_indexes
```

Out[51]:
```
[([], 0),
 ([204], 0.51957419687774453),
 ([204, 251], 0.5468436322616379),
 ([204, 251, 284], 0.53742333878433723),
 ([204, 251, 284, 224], 0.53304902624649908),
 ([204, 251, 284, 224, 267], 0.54649554797090594),
 ([204, 251, 284, 224, 267, 145], 0.54511410773908642),
 ([204, 251, 284, 224, 267, 145, 154], 0.55468937253352912),
 ([204, 251, 284, 224, 267, 145, 154, 60], 0.57481381013297528),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287], 0.54534430569381753),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305], 0.56730719402511509),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62], 0.58074357516164998),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62, 83],
  0.56161286897462037),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62, 83, 249],
  0.56469308814668107),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62, 83, 249, 172],
  0.57654799250849265),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62, 83, 249, 172, 29],
  0.57195604691503166),
 ([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62, 83, 249, 172, 29, 135],
  0.57113150879227481)]
```

In [52]:
```
best_indexes_forward = sorted(error_indexes, key=lambda i: i[1])[-1]
best_indexes_forward
```

Out[52]:
```
([204, 251, 284, 224, 267, 145, 154, 60, 287, 305, 62], 0.58074357516164998)
```

In [53]:
```
X_train_filter = X_train_poly.drop(best_indexes_forward[0])
Y_train_filter = Y_train.drop(best_indexes_forward[0])

MLR_result_filter = OLS(Y_train_filter, sm.add_constant(X_train_filter)).fit()

r2_train_filter = r2_score(Y_train_filter, MLR_result_filter.predict(sm.add_constant(X_train_filter)))
r2_test_filter = r2_score(Y_test, MLR_result_filter.predict(sm.add_constant(X_test_poly)))

print("Training R^2: %.5f\nTesting R^2: %.5f" % (r2_train_filter, r2_test_filter))
```

```
Training R^2: 0.71288
Testing R^2: 0.27245
```

In [54]:
```
print('R^2 score before removal of errneous terms: %.5f\nR^2 score after removal of errneous terms: %.5f' %
     (r2_MLR_poly, r2_test_filter))
```

```
R^2 score before removal of errneous terms: 0.27724
R^2 score after removal of errneous terms: 0.27245
```

APCOMP209a - Homework Question

Question 1: Student's t MLE

Use Maximum Likelihood Estimation to generate a linear regression model on the data provided in `beerdata.csv` considering two statistical models for noise: a) iid Normal and b) iid Student's t-distribution with $\nu = 5$ and scale factor σ=0.5.

Compare the two models performances and comment why it is perhaps appropriate to use the Student's t-distribution instead of the Normal?

HINTS:

1. Use the probability density function for the Student's t distribution with location μ and scale factor σ.
2. If the MLE regressions coefficients can not be derived analytically consider numerical methods.
3. You can use sklearn or statsmodel for the Normal case

In [55]:
```
beerdata = pd.read_csv('data/beerdata.csv', index_col=0)
x = beerdata['x'].values.reshape(-1, 1)
y = beerdata['y'].values.reshape(-1, 1)
```

Normal Model

In [56]:
```
from sklearn.linear_model import LinearRegression
```

In [57]:
```
normal_model = LinearRegression()
normal_model.fit(x, y)
```

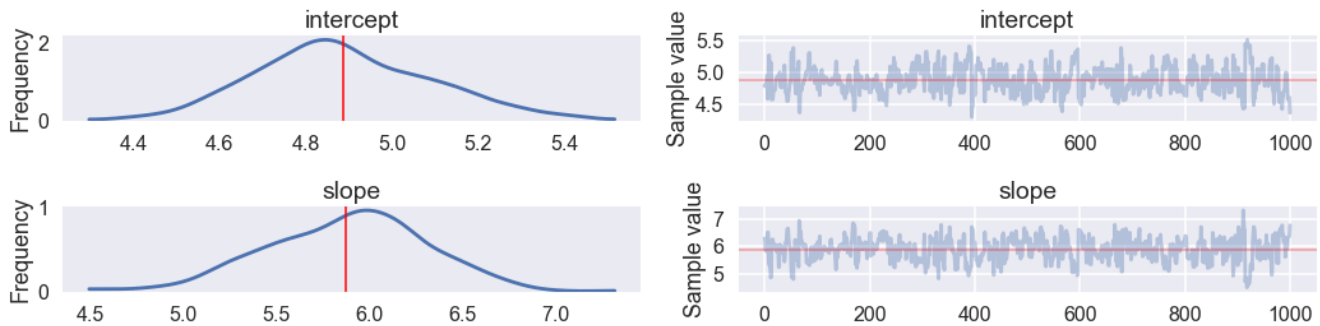
Out[57]:
```
LinearRegression(copy_X=True, fit_intercept=True, n_jobs=1, normalize=False)
```

t-distribution model

In [58]:
```
import pymc3 as pm

with pm.Model() as model:

    # Priors for unknown model parameters
    intercept = pm.Normal('intercept', mu=0, sd=10)
    slope = pm.Normal('slope', mu=0, sd=10)

    # Likelihood (sampling distribution) of observations
    y_pred = pm.StudentT('y_pred', nu=5, mu=intercept+slope*x, sd=0.5, observed=y)
```

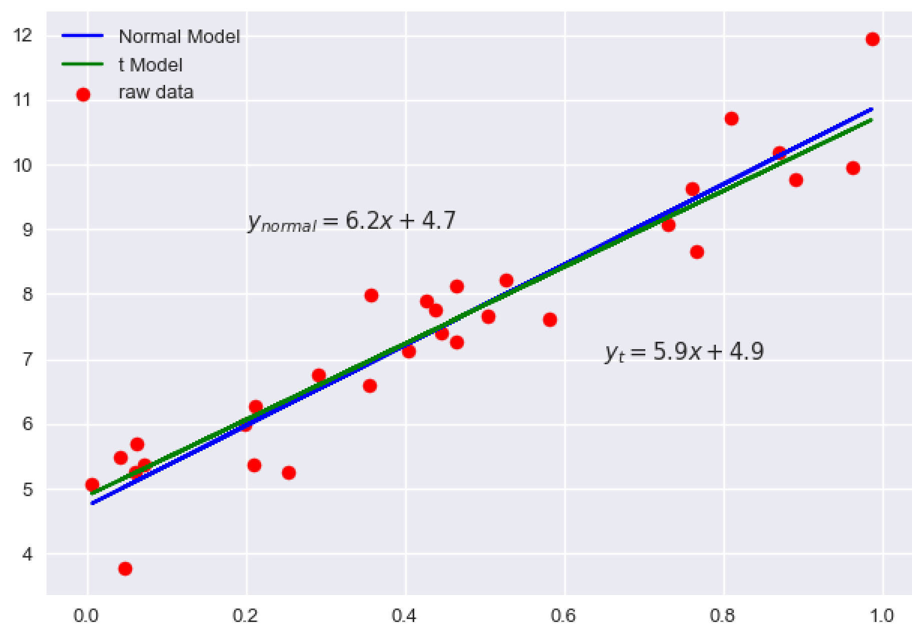
In [59]:
```
with model:
    trace_studentt = pm.sample(1000)

_ = pm.traceplot(trace_studentt[-1000:], figsize=(15,len(trace_studentt.varnames)*2),
                 lines={k: v['mean'] for k, v in pm.df_summary(trace_studentt[-1000:]).iterrows()})
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using advi...
Average ELBO = -35.433: 100%|██████████| 200000/200000 [00:07<00:00, 27156.28it/s]
Finished [100%]: Average ELBO = -35.424
100%|██████████| 1000/1000 [00:01<00:00, 825.21it/s]
```



In [60]:
```
intercept = trace_studentt["intercept"].mean()
slope = trace_studentt["slope"].mean()
y_pred_t = x*slope+intercept
```

In [61]:
```
sns.set_context('talk')
plt.scatter(x, y, c='r', label='raw data')
plt.plot(x, normal_model.predict(x), c='b', label='Normal Model')
plt.plot(x, y_pred_t, c='g', label='t Model')
plt.text(x=0.2, y=9, fontsize=15,
         s='$y_{normal} = %.1fx + %.1f$' % (normal_model.coef_[0], normal_model.intercept_[0]))
plt.text(x=0.65, y=7, fontsize=15,
         s='$y_{t} = %.1fx + %.1f$' % (slope, intercept))
plt.legend(loc='best')
plt.show()
```



In [62]:
```
r2_normal = r2_score(y, normal_model.predict(x))
r2_t = r2_score(y, y_pred_t)
print("Normal Model R^2: %.5f\nt model R^2: %.5f" % (r2_normal, r2_t))
```

```
Normal Model R^2: 0.90792
t model R^2: 0.90539
```

t-distribution is more appropriate when the sample size is small and the true population STD is unknown. The sample STD is weighted less in fitting the linear regression model, thus the outliers in the sample are less significant.

Question 2 (continued from HW2)

Read sections 1 and 2 of this paper (https://www.researchgate.net/profile/Roberto_Togneri/publication/45094554_Linear_Regression_for_Face_Recognition/links/09e4150d243bd8b987000000/Line Regression-for-Face-Recognition.pdf).

Briefly, the model leverages the concept that "patterns from a single-object class lie on a linear subspace." It also makes use of the idea of linear regression as a problem about projections. In this case, given a vector y, the goal is to find the subspace induced by $\mathrm{Col}\,\mathbf{X}$ that produced the 'closest' projection vector $\hat{y}$ to the original y.

Question 2a

As discussed in the paper, our face dataset contains cleaned images of faces belonging to different people. Assuming that patterns (faces) from one class (person) are elements of the same subspace, let's try to classify an unknown face using the method presented in the paper. For each class i, we need to:

1. construct the $\mathbf{H}_i$ hat matrix from known faces, being careful to follow the column concatenation step described in the paper to convert an image into its vector representation;
2. calculate the predicted $\hat{y}_i$, the closest vector in $\mathrm{Col}\,\mathbf{X}_i$ to y; and
3. calculate the magnitude of the difference vector between y and $\hat{y}_i$.

You should then be able to make a classification decision.

Notes:

- Use the provided code to download and re-sample the dataset.
- Follow the normalisation step in the paper to ensure the "maximum pixel value is 1".
- Your classifier should have approximately an 80% accuracy.
- Use the image plotting library of matplotlib to display one (or two) correctly classified faces and the known faces.
- Use the image plotting library of matplotlib to display one (or two) incorrectly classified faces and the known faces.

Question 2b - Significant Faces

Select an example of a correctly classified face. Use statsmodels to investigate the most predictive columns (faces) that the model used in this regression:

(i) Which columns (i.e. faces) make the highest contribution to the projection?

(ii) Which columns (i.e. faces) are the least useful in making this projection?

Plot the correctly assigned face, and the two faces from the questions (i) and (ii). What do you notice about these faces?

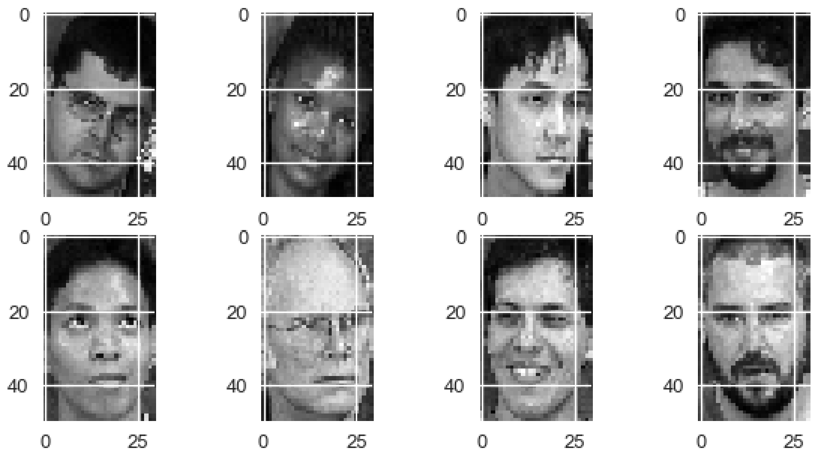
In [63]:
```
from io import BytesIO
from zipfile import ZipFile
import urllib
import os

# Note that you may need to run the following command to install Python Image Library (PIL)
#pip install Pillow
from PIL import Image
import numpy as np
from sklearn.cross_validation import train_test_split

import matplotlib
import matplotlib.pyplot as plt

%matplotlib inline
```

```
D:\ProgramData\Anaconda3\envs\py36\lib\site-packages\sklearn\cross_validation.py:44: DeprecationWarning: This modu
le was deprecated in version 0.18 in favor of the model_selection module into which all the refactored classes and
functions are moved. Also note that the interface of the new CV iterators are different from that of this module.
 This module will be removed in 0.20.
  "This module will be removed in 0.20.", DeprecationWarning)
```

In [64]:

```python
# starter functions provided to students
def rgb2gray(rgb):
    '''
    function to convert RGB image to gray scale
    accepts 3D numpy array and returns 2D array with same dimensions
    as the first two dimensions of input
    '''

    return np.dot(rgb[...,:3], [0.299, 0.587, 0.114])

def fetch_and_read_data(shape=(50,30)):

    '''
    Function to download image data, store in a local folder (note this is 18.4mb), only download the data when
    the local folder is not present, read in the images, downsample them to the specified shape (default = (50x30)
 (rows x cols))
    and finally split them into a four tuple return object.

    Returns:
        - 1) training image data (i.e. images that should form the predictor matrix in your solution)
        - 2) training image data labels (i.e. labels from 1 to 50 that identify which face (1) belongs to)
        - 3) testing image data (i.e. data that you should use to try and classify - note this forms the predictor
 variable in your regression)
        - 4) testing image data labels (i.e. the labels for (3) - this is to allow you to evaluate your model)


    ___________________
    Aside:
    If you want to change the sampling dimensions of your data, pass the shape = (x,y) argument to the method where
    y is the number of columns and x is the number of rows in the image.
    '''

    if not os.path.exists('./cropped_faces'):
        url = urllib.request.urlopen("http://www.anefian.com/research/GTdb_crop.zip")

        zipfile = ZipFile(BytesIO(url.read()))
        zipfile.extractall()

    data = []
    labels = []

    files = os.listdir('cropped_faces')
    for f in files:
        if '.jpg' in f:
            image = Image.open('cropped_faces/' + f)
            image = image.resize((shape[1], shape[0]))
            data.append(rgb2gray(np.asarray(image)))
            labels.append(int(f.split('_')[0][1:]) - 1)

    data = np.array(data)

    trainX, testX, trainY, testY = train_test_split(data, labels, test_size=0.2, stratify=labels)
    return np.array(trainX), np.array(testX), np.array(trainY), np.array(testY)
```

In [65]:
```
# starter code for the students
train_dataset, test_dataset, train_labels, test_labels = fetch_and_read_data()

# code to plot some of the images
fig, axes = plt.subplots(2,4,figsize=(10,5))
axes = axes.flatten()
[axes[i].imshow(train_dataset[i], cmap='gray') for i in range(len(axes))]
plt.show()
```



2a

In [66]:
```
def image_to_norm_col(image):
    return image.T.reshape(-1)/image.max()
```

In [67]:
```
def get_H(X):
    XT = X.transpose()
    return np.dot(np.dot(X, np.linalg.inv(np.dot(XT, X))), XT)
```

In [68]:
```
class_X_dict = dict()

# Get the total number of classes (people)
# Create the column vector of every image in the same class and store them
# Repeat this process for all different classes
class_set = set(train_labels)
for c in class_set:
    image_class = train_dataset[train_labels==c]

    image_vector_len = image_class[0].shape[0] * image_class[0].shape[1]
    class_size = image_class.shape[0]
    class_X = np.empty((image_vector_len, class_size))

    for i, image in enumerate(image_class):
        class_X[:, i] = image_to_norm_col(image)

    class_X_dict[c] = class_X
```

In [69]:
```
pred_test_labels = []
pred_test_labels_H = []

for image in test_dataset:
    test_vector = image_to_norm_col(image)
    distances = np.zeros(len(class_X_dict))
    distances_H = np.zeros(len(class_X_dict))

    for class_label, class_X in class_X_dict.items():
        # Calculate the l2 norm results using OLS and Hat matrix method
        # Theoretically they should generate the same results
        MLR_result = OLS(test_vector, class_X).fit()
        H = get_H(class_X)
        distances[class_label] = np.linalg.norm(MLR_result.predict(class_X)-test_vector)
        distances_H[class_label] = np.linalg.norm(np.dot(H, test_vector)-test_vector)

    pred_test_labels.append(np.argmin(distances))
    pred_test_labels_H.append(np.argmin(distances_H))

pred_test_labels = np.array(pred_test_labels)
pred_test_labels_H = np.array(pred_test_labels_H)
```

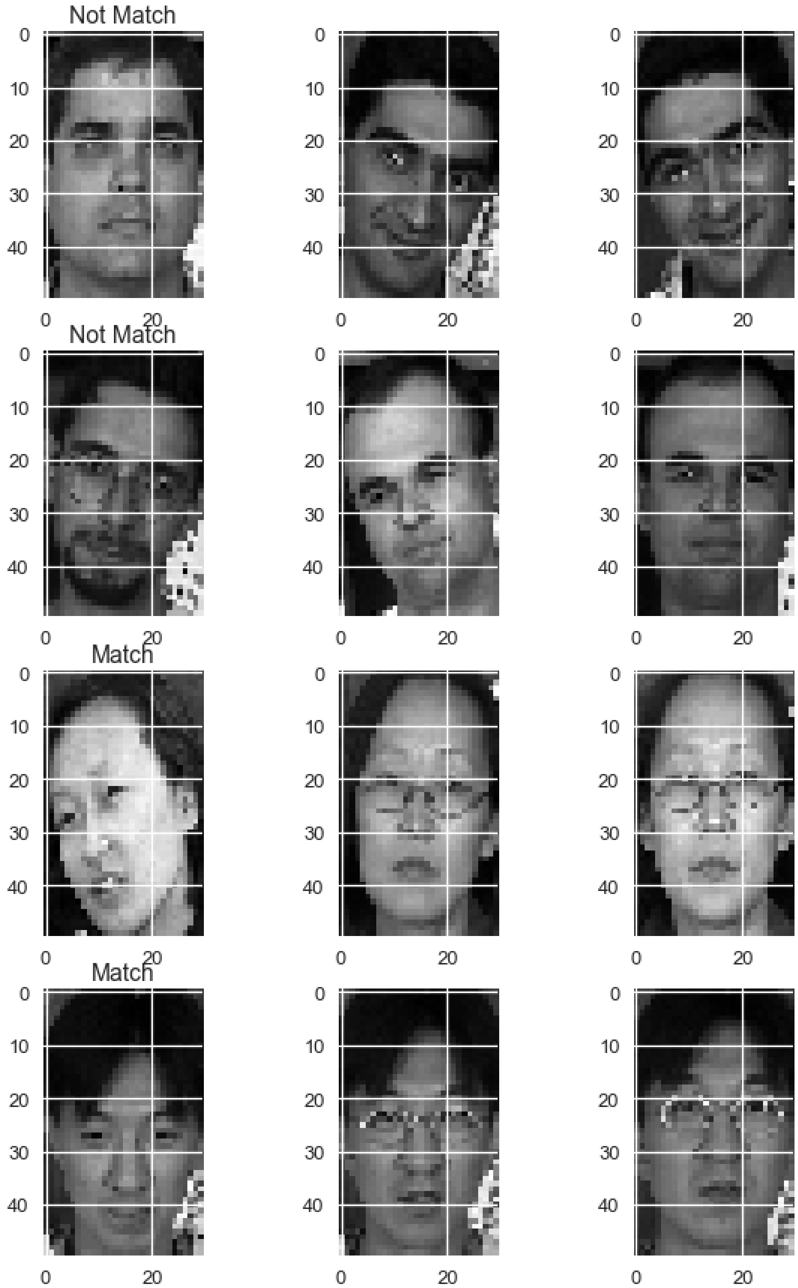
In [70]:
```
print('Compare result from OLS model and Hat Matrix method: %.1f%%' % ((pred_test_labels == pred_test_labels_H).mea
n()*100))
```

```
Compare result from OLS model and Hat Matrix method: 100.0%
```

In [71]:
```
print('Accuracy: %.2f%%' % ((pred_test_labels == test_labels).mean()*100))
```

```
Accuracy: 84.67%
```

In [72]:
```
fig, ax = plt.subplots(4,3,figsize=(10,15))
ax = ax.flatten()

match_labels = pred_test_labels == test_labels
match_labels_not = pred_test_labels != test_labels

ax[0].imshow(test_dataset[match_labels_not][0], cmap='gray')
ax[0].set_title('Not Match')
ax[1].imshow(train_dataset[train_labels==pred_test_labels[match_labels_not][0]][0], cmap='gray')
ax[2].imshow(train_dataset[train_labels==pred_test_labels[match_labels_not][0]][-1], cmap='gray')

ax[3].imshow(test_dataset[match_labels_not][-1], cmap='gray')
ax[3].set_title('Not Match')
ax[4].imshow(train_dataset[train_labels==pred_test_labels[match_labels_not][-1]][0], cmap='gray')
ax[5].imshow(train_dataset[train_labels==pred_test_labels[match_labels_not][-1]][-1], cmap='gray')

ax[6].imshow(test_dataset[match_labels][0], cmap='gray')
ax[6].set_title('Match')
ax[7].imshow(train_dataset[train_labels==pred_test_labels[match_labels][0]][0], cmap='gray')
ax[8].imshow(train_dataset[train_labels==pred_test_labels[match_labels][0]][-1], cmap='gray')

ax[9].imshow(test_dataset[match_labels][-1], cmap='gray')
ax[9].set_title('Match')
ax[10].imshow(train_dataset[train_labels==pred_test_labels[match_labels][-1]][0], cmap='gray')
ax[11].imshow(train_dataset[train_labels==pred_test_labels[match_labels][-1]][-1], cmap='gray')

plt.show()
```



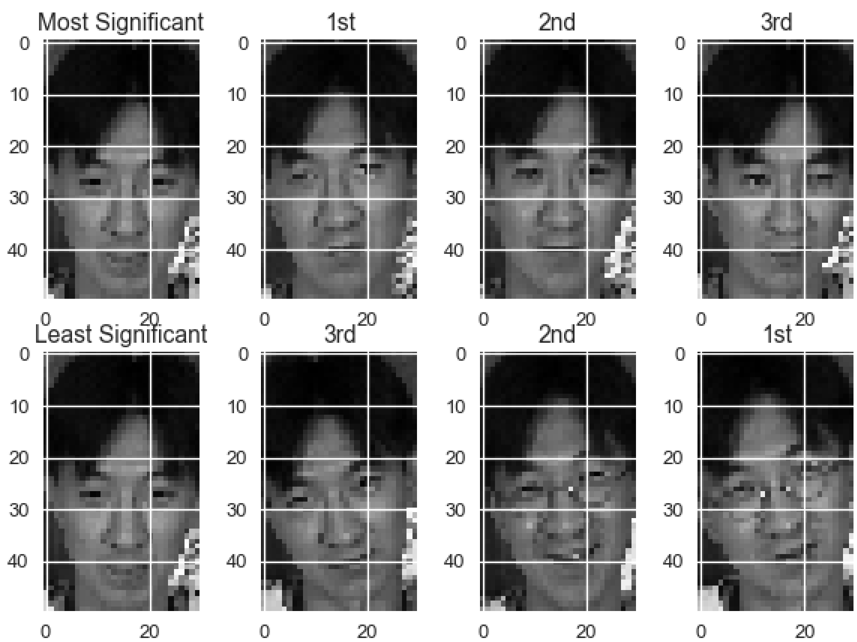
2b

In [73]:
```
match_image = test_dataset[match_labels][-1]
match_class = test_labels[match_labels][-1]
```

In [75]:
```
match_y = image_to_norm_col(match_image)
match_X = class_X_dict[match_class]

MLR_result = OLS(match_y, match_X).fit()

coefs_df = pd.Series(np.abs(MLR_result.params))
```

In [76]:
```
coef_ind_sort = np.array(coefs_df.sort_values().index)
max_3_ind = coef_ind_sort[-3:]
min_3_ind = coef_ind_sort[:3]
```

In [77]:
```
fig, ax = plt.subplots(2,4,figsize=(10,7))
ax = ax.flatten()

ax[0].imshow(match_image, cmap='gray')
ax[0].set_title('Most Significant')
ax[1].imshow(train_dataset[train_labels==match_class][max_3_ind[0]], cmap='gray')
ax[1].set_title('1st')
ax[2].imshow(train_dataset[train_labels==match_class][max_3_ind[1]], cmap='gray')
ax[2].set_title('2nd')
ax[3].imshow(train_dataset[train_labels==match_class][max_3_ind[2]], cmap='gray')
ax[3].set_title('3rd')

ax[4].imshow(match_image, cmap='gray')
ax[4].set_title('Least Significant')
ax[5].imshow(train_dataset[train_labels==match_class][min_3_ind[0]], cmap='gray')
ax[5].set_title('3rd')
ax[6].imshow(train_dataset[train_labels==match_class][min_3_ind[1]], cmap='gray')
ax[6].set_title('2nd')
ax[7].imshow(train_dataset[train_labels==match_class][min_3_ind[2]], cmap='gray')
ax[7].set_title('1st')

plt.show()
```



The most significant contribution comes from the images whose head shot has similar angle with the test one.

In [ ]: