The Isotactics Spectrum of Behavioural Equivalences

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Abstract. Process models describe the dynamics of systems and have been established as an essential means in domains of requirements engineering, model-driven development, and system verification. Many application scenarios rely on pairwise comparisons of process models based on behaviours that these models describe, for example, using the well-established behavioural equivalence notion of bisimulation. Bisimulation assumes that every action of a process model corresponds to exactly one action of the other process model, that is, models are defined on the same level of abstraction. In this paper, we study the problem of how to rigorously express and algorithmically compute behavioural equivalence of process models specified at different levels of abstraction. In such process models, sets of actions are related by complex correspondences. We present the equivalence notion of isotactics and propose its spectrum for different configurations of linear time, branching time, interleaving, and concurrent semantics. We demonstrate that isotactics is a generalization of bisimulation and propose an algorithm for deciding isotactics for linear time, interleaving semantics.

Keywords: Behavioural equivalence, isotactics, bisimulation, process semantics, process alignment, process abstraction, model matching

1 Introduction

The dynamics of a system is often described by a process model that defines a set of actions and causal dependencies for their execution. Process models are widely used as requirements artefacts in the design of software systems [11], as implementation models in workflow automation [6,18], and for the verification of a system's behaviour [4]. Various application scenarios rely on assessing behavioural equivalence of two process models, e.g., comparison of computer programs [12] and validation of a system implementation against a specification [8]. It is often the case that process models for which equivalence shall be assessed assume different levels of abstraction when capturing systems' behaviours. The semantic relation on actions of such models cannot be captured using *elementary* correspondences, i.e., as a binary relation between actions of both models, but must rather be expressed using *complex* correspondences between sets of actions. For example, an action in one model may correspond to a set of actions in another model to capture a hierarchical refinement relation. However, once the semantic relation on actions is defined in terms of *complex* correspondences, well-established notions of behavioural equivalence [2,19] are not directly applicable.

A simple example (adapted from [16]) for the addressed setting is given in Fig. 1. Here, actions of two models, i.e., transitions of the respective Petri nets, are related by one elementary correspondence ϵ and four complex correspondences α, β, γ, and δ. For example, transition a of the upper net corresponds to transitions w and x of the lower net. Similarly, transitions x and y of the lower net correspond to transitions e and f of

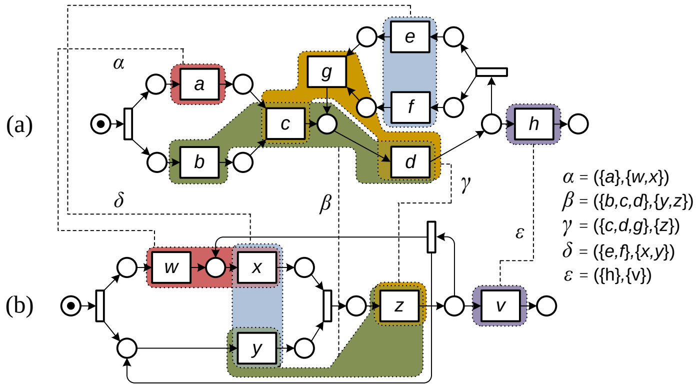

Fig. 1. An alignment between two Petri nets using one elementary correspondence ϵ and four complex correspondences α, β, γ, and δ.

the upper net. Such correspondences on actions of models define an *alignment* relation between the models. Alignments can be defined manually or computed using (semi-) automated techniques devised in the context of process model matching exercises [21,3].

In this paper, we study the *isotactics* notion of behavioural equivalence [16], which does not impose any restrictions on the structure of the alignment relation between process models. Specifically, the contribution of this paper is threefold:

Isotactics for various process semantics (Section 3) We propose a model for the definition of behavioural equivalences in the presence of complex correspondences. We then instantiate this model for four established process semantics [17] to obtain the isotactics spectrum of equivalences, which covers linear time, branching time, interleaving, and concurrent semantics.

Reduction of isotactics to bisimulation (Section 4) We show that the devised isotactics notions are proper generalisations of the well-established behavioural equivalences. That is, if an alignment collapses to a bijection between the actions of two models, isotactics corresponds to bisimulation.

Decidability of isotactics (Section 5) We propose an algorithm that given two aligned process models captured as bounded Petri nets decides whether they are isotactic for linear time, interleaving semantics. To this end, we construct a graph representing pairs of alignable set abstractions and verify the existence of a particular subgraph.

After presenting the above mentioned results, we review related work (Section 6) and draw conclusions (Section 7).

2 Models for Aligned Concurrent Systems

We recall basic notations for Petri nets, behavioural equivalences and alignments.

2.1 Petri Net Syntax and Semantics

Preliminaries. $\mathbb{N}_0$ is the set of non-negative integers. For a set A, $\mathcal{P}(A)$ is the *power set* of A, and $\mathcal{P}_{\geq 1}(A)$ and $\mathcal{P}_{=1}(A)$ denote $\mathcal{P}(A) \setminus \{\emptyset\}$ and $\{\{a\} \mid a \in A\}$, respectively. A binary relation $R \subseteq A \times A$ is an *equivalence relation* if R is reflexive, symmetric, and transitive. $[a]_R := \{b \in A \mid a\ R\ b\}$ is the *equivalence class* of a w.r.t. R, and $A/\!\equiv$ denotes the set $\{[a]_R \mid a \in A\}$ of equivalence classes w.r.t. R. Let B be a set and $f : A \to B$ be a function. Then, $f^\star : \mathcal{P}_{\geq 1}(A) \to \mathcal{P}_{\geq 1}(B)$ is the function with $A' \mapsto \{f(a) \mid a \in A'\}$ for all $A' \in \mathcal{P}_{\geq 1}(A)$. If f is a *bijection* on $A' \subseteq A$ and $B' \subseteq B$, then f^{-1} denotes the function $B \to A$ satisfying $f^{-1}(f(a)) = a$ for all $a \in A$. Let $R \subseteq \mathcal{P}_{=1}(A) \times \mathcal{P}_{=1}(B)$ be a relation. Then, $R_{=1} \subseteq A \times B$ is the relation with $a\ R_{=1}\ b$ iff $\{a\}\ R\ \{b\}$. We write Δ_A for the identity relation on A.

We write $\mathcal{B}(A)$ for the set of all *bags*, or *multisets*, over set A. Let $M \in \mathcal{B}(A)$ and $a \in A$. Then, $M(a)$ is the *multiplicity* of a in M. If $M(a) > 0$, then M *contains* a. We shall use the plus sign '+' and the minus sign '−' to denote the standard operations of union and subtraction of bags, respectively, and the '≤' sign for the comparison of cardinalities of bags. We identify a set $A' \subseteq A$ with the least bag over A containing all elements of A'.

Petri nets. We use the common notation for Petri nets.

Definition 2.1 (Net structure) A *net structure* $N := (P, T, F)$ consists of finite disjoint sets P of *places* and T of *transitions*, and the *flow relation* $F \subseteq (P \times T) \cup (T \times P)$.

Each $x \in P \cup T$ is a *node* of (P, T, F) with *preset* ${}^\bullet x := \{y \mid y\,F\,x\}$ and *postset* $x^\bullet := \{y \mid x\,F\,y\}$. We write N^2 for $(P \cup T)^2$. $M \in \mathcal{B}(P)$ is a *marking* of N. A transition $t \in T$ is *enabled* in M iff $M(p) > 0$ for all $p \in {}^\bullet t$. Let $t \in T$ be enabled in M. Then, $(M, t, (M - {}^\bullet t) + t^\bullet)$ is a *step* of N.

A Petri net is a net structure with a distinct initial marking.

Definition 2.2 (Petri net) A *Petri net*, or a *net*, $N := (P, T, F, M)$ consists of a net structure $N' := (P, T, F)$ and the *initial marking* $M \in \mathcal{B}(P)$ of N'.

The semantics of Petri nets is grounded in firing sequences and reachable markings. A sequence of transitions $w := t_1 \dots t_n \in T^*$, $n \in \mathbb{N}_0$, of a net structure $N := (P, T, F)$ is a *firing sequence* of N iff w is empty or there exists a sequence of markings $M_0 \dots M_n$ such that (M_{i-1}, t_i, M_i) is a step of N for all $i \in [1 \,..\, n]$. Sequence w is also a *firing sequence* of the net $N' := (P, T, F, M_0)$. Markings M_j, $j \in [0 \,..\, n]$, are *reachable markings* in N from M_0 and *reachable markings* in N'.

Semantics. In this paper, we study four different semantics of Petri nets that span two dimensions: *linear time vs. branching time* and *concurrent vs. interleaving* [17]. Branching time refines linear time by considering the points at which decisions are taken. Concurrent semantics refines interleaving semantics by distinguishing between true concurrency and non-determinism. Semantics of a Petri net can be described by a set of *event structures*, cf. [14]. A common approach to define a set of event structures for a Petri net is to first construct a set of *occurrence nets* and then to derive the event structures from the occurrence nets. Due to space considerations, we do not formalise full construction here and limit the discussion to the definition of event structures; for details we refer the reader to [14].

An event structure is commonly defined as a set of events together with three relations on them: causality (<), conflict (#), and concurrency (||). For our purpose, it is more convenient to assume a *behavioural mapping* β that assigns each pair (e, e') of events to the symbol from the set $\{<, >, \#, \|\}$.

Definition 2.3 (Event structure) An *event structure* $\sigma := (E, \Lambda, \beta, \lambda)$ consists of a finite set E of *events*, a set Λ of *labels*, a *behavioural mapping* $\beta : E^2 \to \{<, >, \#, \|\}$, and a *labelling* $\lambda : E \to \Lambda$, such that for all $e_1, e_2, e_3 \in E$ it holds that (i) $\beta(e_1, e_1) = \|$, and (ii) $\beta(e_1, e_2) = \#$ iff $\beta(e_2, e_1) = \#$, and (iii) $\beta(e_1, e_2) = \|$ iff $\beta(e_2, e_1) = \|$, and (iv) $\beta(e_1, e_2) = <$ iff $\beta(e_2, e_1) = >$, and (v) $\beta(e_1, e_2) = \beta(e_2, e_3) = <$ implies $\beta(e_1, e_3) = <$.

If each element in Λ is a set (singleton), then σ is *set-labelled* (*singleton-labelled*). We also recall the notion of an isomorphism for event structures.

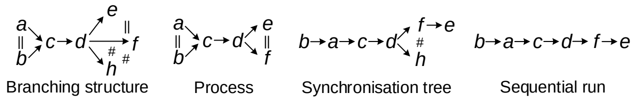

Fig. 2. Event structures representing different semantics of the Petri net in Fig. 1(a).

Definition 2.4 (Event structure isomorphism) Let $\sigma_1 := (E_1, \Lambda_1, \beta_1, \lambda_1)$ and $\sigma_2 := (E_2, \Lambda_2, \beta_2, \lambda_2)$ be two event structures and let $\bowtie \subseteq \Lambda_1 \times \Lambda_2$. A bijection $b : E_1 \to E_2$ is an *event structure isomorphism* from σ_1 to σ_2 w.r.t. $\bowtie$, denoted by $b : \sigma_1 \cong^{\bowtie} \sigma_2$, iff (i) $\beta_1(h, h') = \beta_2(b(h), b(h'))$ for all $h, h' \in E_1$, and (ii) $\lambda_1(h) \bowtie \lambda_2(b(h))$ for all $h \in E_1$.

If $\Lambda_1 = \Lambda_2$ and $\bowtie$ is an identity relation, then $b : \sigma_1 \cong \sigma_2$ denotes $b : \sigma_1 \cong^{\bowtie} \sigma_2$.

The four above mentioned Petri net semantics can be described in terms of event structures defined over *occurrences of transitions*. A *branching structure* is an (branching time/concurrent) event structure obtained by a structure-preserving projection on the events of the branching process [14], i.e., for all e_1, e_2, e_3 it holds that $\beta(e_1, e_2) = >$ and $\beta(e_2, e_3) = \#$ implies $\beta(e_1, e_3) = \#$. A *process* is a (linear time/concurrent) branching structure that prohibits conflict of events, i.e., $\forall e_1, e_2 \in E : \beta(e_1, e_2) \neq \#$. A *sequential run* is a (linear time/interleaving) branching structure that prohibits conflict and concurrency of events, i.e., $\forall e_1, e_2 \in E : \beta(e_1, e_2) \neq \# \wedge \beta(e_1, e_2) \neq \|$. Thus, a sequential run of a Petri net is a total order of events which represent occurrences of transitions in a firing sequence of the Petri net. Finally, a *synchronisation tree* is a prefix condensed representation of a set of sequential runs encoded as a (branching time/interleaving) event structure that prohibits concurrency of events, i.e., $\forall e_1, e_2 \in E : \beta(e_1, e_2) \neq \|$.

In the remainder, we use the following notations to refer to the different semantics of a Petri net N, each defined as a set of event structures:

- $\mathbb{BC}(N)$ is the set of all branching structures of N,
- $\mathbb{LC}(N)$ is the set of all processes of N,
- $\mathbb{BI}(N)$ is the set of all synchronisation trees of N, and
- $\mathbb{LI}(N)$ is the set of all sequential runs of N.

Fig. 2 shows examples of event structures representing four different semantics of the Petri net in Fig. 1(a) under a projection to a subset of the aligned transitions.

2.2 Behavioural Equivalences

We recall *fully concurrent bisimulation* (CB) [2], an equivalence notion for branching time/concurrent semantics. In contrast to the original definition, we define CB based on *fully concurrent simulation* (CS), which generalizes the classical simulation preorder to concurrent semantics. We define these notions on the level of event structures, and then lift them to Petri nets for each of the four semantics. As a prerequisite, one requires the notion of a *prefix* of an event structure, which induces the notion of *minimality* of an event structure in a given set.

Definition 2.5 (Prefix, minimality) For $i = 1, 2$, let $\sigma_i = (E_i, \Lambda_i, \beta_i, \lambda_i)$ be a an event structure. σ_1 is a *prefix* of σ_2, denoted by $\sigma_2 \sqsupseteq \sigma_1$, iff $E_1 \subseteq E_2$, $\Lambda_1 \subseteq \Lambda_2$, $\beta_1(e, e') = \beta_2(e, e')$ for all $e, e' \in E_1$, $\lambda_1(e) = \lambda_2(e)$ for all $e \in E_1$, and for all $e, e' \in E_2$: if $\beta_2(e, e') = <$ and $e' \in E_1$, then $e \in E_1$. Let Σ be a set of event structures. An event structure $\sigma \in \Sigma$ is *minimal in Σ* iff for all $\sigma' \in \Sigma$: $\sigma \not\sqsupseteq \sigma'$.

Now, we can define CS of a set Σ_1 of event structures by a set Σ_2 of event structures in terms of an *initial CS-relation*. A CS-relation relates the elements of Σ_1 with the elements of Σ_2 by means of isomorphisms, basically requiring that if σ_1 is simulated by σ_2, and σ_1' is a prefix of σ_1 is a prefix of σ_1', then σ_2 is a prefix of some $\sigma_2 \in \Sigma_2$, and σ_2' simulates σ_2.

Definition 2.6 (CS-relation, initial CS-relation) For $i = 1, 2$, let Σ_i be a set of event structures. A set of R of triples is a *CS-relation* from Σ_1 to Σ_2, denoted by $R : \Sigma_1 \lesssim \Sigma_2$, if for all $(\sigma_1, \sigma_2, b) \in R$:

1. $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ and $b : \sigma_1 \cong \sigma_2$.
2. If $\sigma_1' \in \Sigma_1$ with $\sigma_1' \sqsupseteq \sigma_1$ then there exists $(\sigma_1', \sigma_2', b') \in R$:
   (a) $\sigma_2' \sqsupseteq \sigma_2$, and
   (b) $b'(e) = b(e)$ for all events e of σ_1.

R is *initial*, denoted by $R : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$, iff for each minimal element σ_1 of Σ_1, there exists $(\sigma_1, \sigma_2, b) \in R$, such that σ_2 is a minimal element of Σ_2.
Given a CS-relation R, $R^{-1} := \{(\sigma_2, \sigma_1, b^{-1}) \mid (\sigma_1, \sigma_2, b) \in R\}$ is the *reversal* of R. Now, a CB-relation R is a CS-relation, such that R^{-1} is also a CS-relation.

Definition 2.7 (CB-relation) For $i = 1, 2$, let Σ_i be a set of event structures.

1. A set of R of triples is a *CB-relation* on Σ_1 and Σ_2, denoted by $R : \Sigma_1 \sim \Sigma_2$, iff: $R : \Sigma_1 \lesssim \Sigma_2$, and $R^{-1} : \Sigma_2 \lesssim \Sigma_1$.
2. A CB-relation R is an *initial CB-relation*, denoted by $R : \Sigma_1 \sim^{\text{init}} \Sigma_2$, iff $R : \Sigma_1 \sim^{\text{init}} \Sigma_2$, and $R^{-1} : \Sigma_2 \sim^{\text{init}} \Sigma_1$.

We lift CS and CB to Petri nets under one of the four discussed semantics.

Definition 2.8 ((Bi-)similarity of Petri nets) For $i = 1, 2$, let N_i be a Petri net. Let $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$.

- If there exists some R with $R : \mathbb{XY}(N_1) \lesssim^{\text{init}} \mathbb{XY}(N_2)$, then N_2 $\mathbb{XY}$-*simulates* N_1, denoted by $N_1 \lesssim^{\mathbb{XY}} N_2$.
- If there exists some R with $R : \mathbb{XY}(N_1) \sim^{\text{init}} \mathbb{XY}(N_2)$, then N_1 and N_2 are $\mathbb{XY}$-*bisimilar*, denoted by $N_1 \sim^{\mathbb{XY}} N_2$.

It is well-known that CB bisimilarity is stricter than similarity and that the CS and CS each form a hierarchy, where (bi-)similarity on a more refined semantics implies (bi-)similarity on less refined semantics.

2.3 Alignments

An *alignment* relates subsets of transitions of two Petri nets [16,23].

Definition 2.9 (Alignment) An *alignment* $\bowtie$ *between two Petri nets* $N_1 := (P_1, T_1, F_1, M_1)$ and $N_2 := (P_2, T_2, F_2, M_2)$ is a binary relation between $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$, i.e., $\bowtie \subseteq \mathcal{P}(T_1) \times \mathcal{P}(T_2)$.

Elements of an alignment are called *correspondences*. Given alignment $\bowtie$, by $\bowtie[1]$ and $\bowtie[2]$ we denote the *domain* and *range* of $\bowtie$, respectively, i.e., $\bowtie[1] := \{K_1 \in \mathcal{P}(T_1) \mid \exists\, K_2 \in \mathcal{P}(T_2) : K_1 \bowtie K_2\}$ and $\bowtie[2] := \{K_2 \in \mathcal{P}(T_2) \mid \exists\, K_1 \in \mathcal{P}(T_1) : K_1 \bowtie K_2\}$. Next, we define several alignment classes. In Section 4, we show that for some of the proposed classes, the notion of isotactics coincides with well-established notions of behavioural equivalence.

Elementary vs. Complex. Alignments may align singleton and/or arbitrary subsets of transitions. Given an alignment $\bowtie$, if $|K| = 1$ for all $K \in \bowtie[1] \cup \bowtie[2]$, we call $\bowtie$ *elementary*; otherwise we call $\bowtie$ *complex*.

Overlapping vs. Disjoint. Alignments may or may not overlap in the sets of aligned transitions. Given an alignment $\bowtie$, we call $\bowtie$ *overlapping* if there exist $K, K' \in \mathcal{K}$, where $\mathcal{K} \in \{\bowtie[1], \bowtie[2]\}$, with $K \cap K' \neq \emptyset$; otherwise we call $\bowtie$ *disjoint*. Furthermore, $\bowtie$ is *left-unique*, or *injective*, if $K_1 \bowtie K_2$ and $K_1' \bowtie K_2$ implies $K_1 = K_1'$. Similarly, $\bowtie$ is *right-unique*, or *functional*, if $K_1 \bowtie K_2$ and $K_1 \bowtie K_2'$ implies $K_2 = K_2'$.

Total vs. Partial. Alignments may align all or a subset of transitions of two Petri nets. In a *total* alignment, every transition is aligned. We call alignment $\bowtie$ *total*, if $T_1 = \bigcup \bowtie[1]$ and $T_2 = \bigcup \bowtie[2]$; otherwise we call $\bowtie$ *partial*.

The alignment shown in Fig. 1 is complex, overlapping (both injective and functional), and partial.

3 The Isotactics Spectrum

An alignment between two Petri nets defines which transitions are considered to be equivalent. This information imposes certain requirements on the definition of a behavioural equivalence notion. These requirements can be described in terms of the properties of the alignment:

- If an alignment is *complex*, the occurrence of transitions of any subset of the aligned sets of transitions shall be treated as equivalent.
- If an alignment is *overlapping*, the occurrences of transitions that are part of several correspondences shall be treated as equivalent to occurrences of any of the transitions aligned by any of the correspondences.
- If an alignment is *partial*, some transitions are without a counterpart in the aligned net. Hence, one needs to abstract from these transitions, see notions of behaviour inheritance [1], before verifying equivalence.

Isotactics is grounded in the notion of a *tactic* which abstracts from occurrence cardinalities of transitions that are part of a single correspondence. Section 3.1 introduces tactics on event structures. Section 3.2 defines the isotactics spectrum.

3.1 The Notion of a Tactic

Given an event structure σ, and a set $\mathcal{K}$ of non-empty sets of labels of σ, we can derive the *set abstraction* $\alpha_{\mathcal{K}}(\sigma)$ of σ w.r.t. $\mathcal{K}$, a canonically induced set-labelled event structure, which hides all events with labels outside $\bigcup \mathcal{K}$, replaces the label ℓ of an event e with the set of all $K \in \mathcal{K}$ containing ℓ, but preserves the behavioural relations between the remaining events. Hence, $\alpha_{\mathcal{K}}(\sigma)$ is an abstraction in the sense that some events are skipped, and that the actual label of each event is abstracted to a set of sets of labels.

Definition 3.1 (Set abstraction) Let $\sigma := (E, \Lambda, \beta, \lambda)$ be an event structure. Let $\mathcal{K} \subseteq \mathcal{P}_{\geq 1}(\Lambda)$ be sets of labels. The set abstraction of σ with respect to $\mathcal{K}$ is the set-labeled event structure $\alpha_{\mathcal{K}}(\sigma) := (\mathcal{E}, \mathcal{K}, \beta', \lambda')$ with

1. $\mathcal{E} = \{e \in E \mid \exists\, K \in \mathcal{K} : \lambda(e) \in K\}$.
2. $\beta'(e, e') := \beta(e, e')$ for all $e, e' \in \mathcal{E}$, i.e., β' is the restriction of β to $\mathcal{E}$.
3. $\lambda'(e) := \{K \in \mathcal{K} \mid \lambda(e) \in K\}$.

We lift this definition to sets: Let Σ be a set of event structures. Then, $\alpha_{\mathcal{K}}(\Sigma)$ denotes the set $\{\alpha_{\mathcal{K}}(\sigma) \mid \sigma \in \Sigma\}$. In a set abstraction $\alpha_{\mathcal{K}}(\sigma)$, every event is labeled with a set of elements of $\mathcal{K}$. Semantically, every $K \in \mathcal{K}$ describes an abstract event. As an underlying assumption, we have that a repetition or concurrent occurrence of the same abstract event K is indistinguishable from a single occurrence of K. A *tactic* chooses a particular abstract event $K \in L$ for every event with label L, and merges repetitions and self-concurrent occurrences of abstract events.

Definition 3.2 (Tactic) Let $\sigma = (E_\sigma, \Lambda_\sigma, \beta_\sigma, \lambda_\sigma)$ be a set-labelled event structure. An event structure $\theta = (E_\theta, \Lambda_\theta, \beta_\theta, \lambda_\theta)$ is a *tactic* of σ iff each of the following conditions is satisfied:

1. E_θ is a partition of E_σ with $\langle e \rangle_\theta$ denoting the set $M \in E_\theta$ with $e \in M$,
2. $\Lambda_\theta = \bigcup \Lambda_\sigma$ (each label of θ is an element of a label of σ).
3. For all $e_1, e_1', e_2, e_2' \in E_\sigma$: If $\langle e_1 \rangle_\theta = \langle e_1' \rangle_\theta \neq \langle e_2 \rangle_\theta = \langle e_2' \rangle_\theta$, then $\beta_\sigma(e_1, e_2) = \beta_\sigma(e_1', e_2')$.
4. $\beta_\theta(\langle e_1 \rangle_\theta, \langle e_2 \rangle_\theta) = \beta_\sigma(e_1, e_2)$ for all $e_1, e_2 \in E_\sigma$ with $\langle e_1 \rangle_\theta \neq \langle e_2 \rangle_\theta$.
5. $\beta_\theta(\langle e \rangle_\theta, \langle e \rangle_\theta) = \|$ for all $e \in E_\sigma$.
6. $\lambda_\theta(\langle e \rangle_\theta) \in \lambda_\sigma(e)$ for all $e \in E_\sigma$ (λ_θ chooses one label shared by all $e' \in \langle e \rangle_\theta$).
7. For all $e_1, e_2 \in E_\sigma$ with $\langle e_1 \rangle_\theta \neq \langle e_2 \rangle_\theta$ and $\lambda_\theta(\langle e_1 \rangle_\theta) = \lambda_\theta(\langle e_2 \rangle_\theta)$, there exists $e_3 \in E_\sigma \setminus (\langle e_1 \rangle_\theta \cup \langle e_2 \rangle_\theta)$, such that $\beta_\sigma(e_1, e_3) \neq \beta_\sigma(e_2, e_3)$ (maximality).

We write Θ_σ for the set of all tactics of σ w.r.t. $\mathcal{K}$.

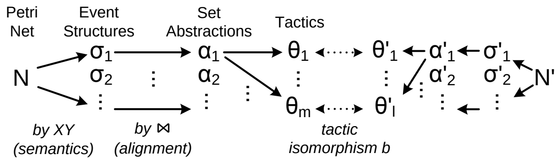

Fig. 4. Schematic representation of how notions of isotactics are defined.

An example of tactics is given in Fig. 3 for one of the event structures introduced earlier. Here, the set $\mathcal{K} = \{\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1\}$ of sets of labels used for the set abstraction is induced by the correspondences defined in Fig. 1. Based on the obtained set abstraction, multiple tactics can be generated, depending on which of the abstract events $\{\beta_1, \gamma_1\}$ is chosen for the events labelled with c and d in the original event structure. Note that labelling both c and d with β_1 will merge the events according to Def. 3.2. Events b and c of the original event structure, however, will not be merged in a tactic labelling both with β_1 since they are differently related to, e.g., event a in the original structure.

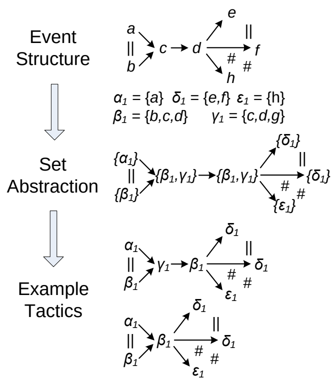

Fig. 3. Example of tactics.

3.2 The Notion of Isotactics

For a given alignment and semantics, we introduce *tactic coverage* (TC) and *isotactics* (IT) as an abstract behavioural preorder and equivalence relation, respectively. As for the notions of CS and CB, we introduce IT based on TC.

Intuitively, a Petri net N_2 covers the tactics of a Petri net N_1 w.r.t. an alignment $\bowtie$ and semantics $\mathbb{XY}$, if for each set abstraction σ_1 of $\mathbb{XY}(N_1)$ there exist a set abstraction σ_2 of $\mathbb{XY}(N_2)$, and tactics $\theta_1 \in \Theta_{\sigma_1}$ and $\theta_2 \in \Theta_{\sigma_2}$, such that θ_1 and θ_2 are isomorphic, see Fig. 4. In analogy to CS, we define TC based on the notion of a *tactic coverage relation* (TC-relation). An element of a TC-relation consists of two set abstractions σ_1 and σ_2, two tactics θ_1 and θ_2, and an isomorphism between the tactics. The requirements on TC-relations are similar to those on CS-relations, which we exploit when studying cases where CS and tactic coverage coincide.

A TC-relation w.r.t. a relation $\bowtie$ on two sets $\varSigma_1$ and $\varSigma_2$ of labelled event structures consists of tuples $(\sigma_1, \sigma_2, \theta_1, \theta_2, b)$, where σ_1 and σ_2 are set labelled event structures from $\varSigma_1$ and $\varSigma_2$ with tactics θ_1 and θ_2, respectively, and b is an isomorphism from θ_1 into θ_2. Similar to the definition of a CS-relation, σ_1 being a prefix of some $\sigma'_1 \in \varSigma_1$ implies the existence of some $(\sigma'_1, \sigma'_2, \theta'_1, \theta'_2, b')$ in the TC-relation, such that σ_2 is a prefix of σ'_2, and for each event e of σ_1: If the restriction of $\langle e \rangle_{\theta'_1}$ to events from σ_1 yields $\langle e \rangle_{\theta_1}$, then the restriction of $b(\langle e \rangle_{\theta'_1})$ to events in σ_2 yields $b(\langle e \rangle_{\theta_1})$.

Definition 3.3 (Tactic Coverage Relation) For $i = 1, 2$, let $\varLambda_i$ be a set, and let $\varSigma_i$ be a set of set-labelled event structures, where each label of each $\sigma \in \varSigma_i$ is a subset of $\varLambda_i$. Let $\bowtie \subseteq \varLambda_1 \times \varLambda_2$. A set R of quintuples is a *tactic coverage relation* (TC-relation) from $\varSigma_1$ to $\varSigma_2$ w.r.t. $\bowtie$, denoted by $R : \varSigma_1 \leqslant_{\bowtie} \varSigma_2$, iff for all $(\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$:

1. For $i = 1, 2$, $\sigma_i = (E_{\sigma_i}, \varLambda_{\sigma_i}, \beta_{\sigma_i}, \lambda_{\sigma_i}) \in \varSigma_i$, $\theta_i \in \Theta_{\sigma_i}$, and $b : \theta_1 \cong^{\bowtie} \theta_2$.
2. If $\sigma'_1 \in \varSigma_1$ and $\sigma'_1 \sqsupseteq \sigma_1$, then there exists $(\sigma'_1, \sigma'_2, \theta'_1, \theta'_2, b') \in R$:
   (a) $\sigma'_2 \sqsupseteq \sigma_2$.
   (b) If $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta'_1} \cap E_{\sigma 1}$ then $b(\langle e \rangle_{\theta_1}) = b'(\langle e \rangle_{\theta'_1}) \cap E_{\sigma 2}$ for each $e \in E_{\sigma 1}$.

Requirement 2b of TC-relations is weaker than requirement 2b of CS-relations (cp. Def. 2.6): The step from θ_1 to θ_1' may involve splitting or joining parts, and if this is the case, then there is no required relation between b and b'. Fig. 5 illustrates such a case: first, there is an isomorphism b defined between tactics of two Petri nets N and N'. Then, extended set abstractions (shown only for N) lead to changed tactics, which require a redefinition of the isomorphism, so that b' is not only an extension of b.

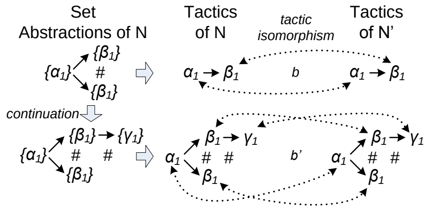

Fig. 5. Tactics isomorphism example.

Now, in analogy to the notions of CS-relations and CB-relations, we define the notions of an *isotactic relation* (IT-relation) based on the notion of TC-relations.

Let $R : \Sigma_1 \lessdot_{\bowtie} \Sigma_2$ be a TC-relation. R is *initial*, denoted by $R : \Sigma_1 \lessdot_{\bowtie}^{\text{init}} \Sigma_2$, iff for each minimal element σ_1 of Σ_1, there exists $(\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$ such that σ_2 is a minimal element of Σ_2. Further, $R^{-1} := \{(\sigma_2, \sigma_1, \theta_2, \theta_1, b^{-1}) \mid (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R\}$ is the *reversal* of R.

Definition 3.4 (Isotactics Relation) For $i = 1, 2$, let Λ_i be a set, and let Σ_i be a set of set-labeled event structures over labels Λ_i. Let $\bowtie \subseteq \Lambda_1 \times \Lambda_2$.

- A set R is an *isotactics relation* (IT-relation) on Σ_1 and σ w.r.t. $\bowtie$, denoted by $R : \Sigma_1 \doteqdot_{\bowtie} \Sigma_2$, iff $R : \Sigma_1 \lessdot_{\bowtie} \Sigma_2$ and $R^{-1} : \Sigma_2 \lessdot_{\bowtie} \Sigma_1$.
- A set R is an *initial IT-relation* on Σ_1 and σ w.r.t. $\bowtie$, denoted by $R : \Sigma_1 \doteqdot_{\bowtie}^{\text{init}} \Sigma_2$, iff $R : \Sigma_1 \lessdot_{\bowtie}^{\text{init}} \Sigma_2$ and $R^{-1} : \Sigma_2 \lessdot_{\bowtie^{-1}}^{\text{init}} \Sigma_1$.

The above notions defined on event structures are lifted to Petri nets as follows:

Definition 3.5 (Tactic Coverage and Isotactics of Petri nets) Let $\bowtie$ be an alignment of two Petri nets N_1 and N_2. Let $\mathbb{X} \in \{\mathbb{L}, \mathbb{B}\}, \mathbb{Y} \in \{\mathbb{I}, \mathbb{C}\}$. For $i = 1, 2$, let $\Sigma_i = \alpha_{\bowtie[i]}(\mathbb{X}\mathbb{Y}(N_i))$.

- N_2 $\mathbb{X}\mathbb{Y}$-*covers the tactics* of N_1 w.r.t. $\bowtie$, denoted by $N_1 \lessdot_{\bowtie}^{\mathbb{X}\mathbb{Y}} N_2$, iff there exists R with $R : \Sigma_1 \lessdot_{\bowtie}^{\text{init}} \Sigma_2$.
- N_1 and N_2 are $\mathbb{X}\mathbb{Y}$-*isotactic* w.r.t. $\bowtie$, denoted by $N_1 \doteqdot_{\bowtie}^{\mathbb{X}\mathbb{Y}} N_2$, iff there exists R with $R : \Sigma_1 \doteqdot_{\bowtie}^{\text{init}} \Sigma_2$.

Turning to the example outlined in Fig. 1, we observe that the Petri nets are $\mathbb{L}\mathbb{C}$-isotactic and $\mathbb{B}\mathbb{C}$-isotactic w.r.t. the illustrated alignment. However, they are not isotactic for any of the interleaving semantics. For instance, the sequential run $x \to y \to x \to z$ of Petri net (b) induces the set abstraction $\{\alpha_2\} \to \{\beta_2, \delta_2\} \to \{\alpha_2, \delta_2\} \to \{\beta_2, \gamma_2\}$ with $\alpha_2 = \{w, x\}$, $\beta_2 = \{y, z\}$, $\gamma_2 = \{z\}$, and $\delta_2 = \{x, y\}$. There is no sequential run of Petri net (a), for which the set abstraction induces a tactic that is isomorphic to any of the tactics induced by $\{\alpha_2\} \to \{\beta_2, \delta_2\} \to \{\alpha_2, \delta_2\} \to \{\beta_2, \gamma_2\}$.

We conclude that, in the general case, isotactics does not induce an equivalence hierarchy for the different semantics, due to the abstraction that is implied by the notion of a tactic. Yet, this changes once alignments coincide with the classical setting of verifying equivalence, as explored in the next section.

4 Relating Isotactics and Known Equivalences

Tactic coverage, and thus isotactics, abstract from cardinalities of events that are part of a single correspondence. Thereby, the degree of abstraction varies with the chosen alignment. If one properly restricts the class of alignments, tactic coverage and isotactics relate very closely to the existing preorder and equivalence notions of simulation and

$$
\begin{array}{ccc}
\lesssim^{\mathbb{XY}} & \Leftarrow & \sim^{\mathbb{XY}} \\
\Downarrow & & \Downarrow \\
\leqslant_{\bowtie}^{\mathbb{XY}} & \Leftarrow & \div_{\bowtie}^{\mathbb{XY}}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\lesssim^{\mathbb{XY}} & \Leftarrow & \sim^{\mathbb{XY}} \\
\Updownarrow & & \Updownarrow \\
\leqslant_{\bowtie}^{\mathbb{XY}} & \Leftarrow & \div_{\bowtie}^{\mathbb{XY}}
\end{array}
$$

Fig. 7. The relations of simulation, bisimulation, tactic coverage, and isotactics w.r.t. a bijective (total, elementary, left-unique, and right-unique) alignment $\bowtie$, and semantics $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$; on the left without further restrictions, on the right with the restriction to repetition-free Petri nets.

bisimulation. In this section, we study alignments $\bowtie$ that are total, elementary, left-unique, and right-unique. That is, $\bowtie$ is of the form

$$\{t\} \bowtie \{h(t)\} \ ,$$

where $t \in T_1$ is a transition of N_1, and h is a bijection on the respective sets T_1 and T_2 of transitions of N_1 and N_2, which is why we refer to $\bowtie$ as a bijective alignment. For such $\bowtie$, simulation and bisimulation imply tactic coverage and isotactics, respectively, as visualized in the left part of Fig. 7. The converse does not generally hold for such $\bowtie$. As a counter example, consider the two nets shown in Fig. 6 and the alignment $\{(\{t_1\}, \{t_2\})\}$. Then, N_1 covers the tactics of N_2 w.r.t. each of the four semantics, because tactic coverage abstracts from the cardinality of t_2-events (N_1 and N_2 are even in isotactics). However, N_1 does not simulate N_2, because N_1 can execute t only once, and not twice.

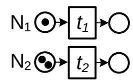

Fig. 6. Counter example.

Narrowing the class of models under study to *repetition-free* Petri nets yields the coincidence of tactic coverage and isotactics with simulation and bisimulation: If N_1 and N_2 are repetition-free, then tactic coverage and isotactics imply simulation and bisimulation, respectively, as visualised in the right part of Fig. 7.

We proceed as follows: In Section 4.1, we prove prerequisite properties for singleton labelled event structures, and construct a TC-relation from a given CS-relation in Section 4.2. Then, we combine the results in Section 4.3, showing that simulation and bisimulation imply tactic coverage and isotactics, respectively. We study the converse for the class of repetition-free Petri nets in Section 4.4.

4.1 Singleton Labelled Event Structures

For an elementary, left-unique and right-unique alignment $\bowtie$, the set abstraction $\alpha_{\bowtie[i]}(\sigma)$ of an event structure σ w.r.t. $\bowtie[i]$ ($i = 1, 2$) is a singleton labelled event structure: Every event occurs in at most one element a of $\bowtie[i]$, and is hence labelled $\{a\}$ in $\alpha_{\bowtie[i]}(\sigma)$. Hence, we prove some properties for singleton labelled event structures, which are useful when relating isotactics to known equivalences.

Let b be a bijection on the events of two singleton labelled event structures σ_1 and σ_2, and $\bowtie$ be a bijection on the labels of σ_1 and σ_2. Let θ_1 and θ_2 be tactics of σ_1 and σ_2, respectively. We observe:

- b can be "lifted" to a relation on the sets of events of θ_1 and θ_2: Each event of θ_i is a set of events of σ_i, so that $b^\star$ relates the sets of events of θ_1 and θ_2.
- $\bowtie$ can be "lifted" to a relation on the labels of θ_1 and θ_2: Each event of θ_i is labelled with some ℓ, such that $\{\ell\}$ is a label occurring in σ_i. Because σ_i is singleton labelled, $\bowtie_{=1}$ relates the labels of θ_1 and θ_2.

In particular, the notion of a tactic is compatible with the notion of an isomorphism as follows: If b is an isomorphism from a singleton-labelled event structure σ_1 to a singleton labelled event structure σ_2 w.r.t. a bijection $\bowtie$, and θ_1 is a tactic of σ_1, then b induces a tactic θ_2 of σ_2, such that θ_1 and θ_2 are isomorphic:

Lemma 4.1 *For $i = 1, 2$, let $\sigma_i = (E_i, \Lambda_i, \beta_i, \lambda_i)$ be singleton labelled event structures. Let $\bowtie \subseteq \Lambda_1 \times \Lambda_2$ be a bijection, $b : \sigma_1 \cong^{\bowtie} \sigma_2$, and $\theta_1 = (E_1', \Lambda_1', \beta_1', \lambda_1') \in \Theta_{\sigma_1}$. Let $\theta_2 = (E_2', \Lambda_2', \beta_2', \lambda_2')$ with $E_2' = \{b^\star(\langle e \rangle_{\theta_1}) \mid e \in E_1\}$, $\Lambda_2' = \bigcup \Lambda_2$, $\beta_2'(b^\star(\langle e \rangle_{\theta_1}), b^\star(\langle e' \rangle_{\theta_1})) = \beta_1'(\langle e \rangle_{\theta_1}, \langle e' \rangle_{\theta_1})$, and λ_2', such that $\{\lambda_2'(b^\star(\langle e \rangle_{\theta_1}))\} \bowtie \{\lambda_1'(\langle e \rangle_{\theta_1})\}$. Then, $\theta_2 \in \Theta_{\sigma_2}$, and $b^\star : \theta_1 \cong^{\bowtie_{=1}} \theta_2$.*

Proof. The proof can be found as Proof 1 in the Appendix.

Next, we show that a singleton labelled event structure has exactly one tactic. Here, the key is to prove that the partition of the set of events of σ is distinct in the sense that any other partition cannot be the set of events of a tactic.

Lemma 4.2 *Let $\sigma = (E_\sigma, \Lambda_\sigma, \beta_\sigma, \lambda_\sigma)$ be a singleton-labelled event structure. For $i = 1, 2$, let $\theta_i = (E_i, \Lambda_i, \beta_i, \lambda_i) \in \Theta_\sigma$. Then, $\theta_1 = \theta_2$.*

Proof. By Def. 3.2, $\Lambda_1 = \Lambda_2$. For $i = 1, 2$ and $m \in E_i$, let $\lambda_i(m) = \{\ell_m\}$. Assume $E_1 = E_2$. Then, $\theta_1 = \theta_2$, because β_1 and β_2 are directly induced by β_σ, and $\lambda_1(M) = \lambda_2(M) = \ell_m$ for all $M \in E_1 = E_2$ and $m \in M$. Thus, we show $E_1 = E_2$. Assume that $E_1 \neq E_2$. Then, there exist $e_1, e_2 \in E_\sigma$, such that $(*)$ $\langle e_1 \rangle_{\theta_1} \neq \langle e_2 \rangle_{\theta_1}$ but $(**)$ $\langle e_1 \rangle_{\theta_2} = \langle e_2 \rangle_{\theta_2}$. From σ being singleton labelled and $(**)$, we know $\lambda_\sigma(e_1) = \lambda_\sigma(e_2) = \lambda_1(\langle e_1 \rangle_{\theta_1} = \lambda_1(\langle e_2 \rangle_{\theta_1}) = \lambda_2(\langle e_1 \rangle_{\theta_2}) = \lambda_2(\langle e_2 \rangle_{\theta_2})$. From $(*)$, we get that there exists $e_3 \in E_\sigma \smallsetminus (\langle e_1 \rangle_{\theta_1} \cup \langle e_2 \rangle_{\theta_1})$ with $\beta_\sigma(e_1, e_3) \neq \beta_\sigma(e_2, e_3)$. From $(**)$, we get that $e_3 \in \langle e_1 \rangle_{\theta_2} = \langle e_2 \rangle_{\theta_2}$. From σ being singleton labelled, we get $\lambda_1(\langle e_1 \rangle_{\theta_1}) = \lambda_1(\langle e_2 \rangle_{\theta_1}) = \lambda_1(\langle e_3 \rangle_{\theta_1})$. Hence, there exists $e_4 \in E_\sigma \smallsetminus (\langle e_1 \rangle_{\theta_1} \cup \langle e_3 \rangle_{\theta_1})$ with $\beta_\sigma(e_1, e_4) \neq \beta_\sigma(e_3, e_4)$. Now, we can apply the same argument as before, and conclude: $e_4 \in \langle e_1 \rangle_{\theta_2} = \langle e_2 \rangle_{\theta_2} = \langle e_3 \rangle_{\theta_2}$. From σ being finite, we can only repeat this step finitely many times, resulting in a contradiction. ▫

Based on this result, we introduce the following notation: If σ is a set-labelled event structure with $\Theta_\sigma = \{\theta\}$, then we define $\Theta(\sigma) := \theta$. Then, from Lemma 4.1 and Lemma 4.2, we get:

Corollary 4.3 *For $i = 1, 2$, let Λ_i be a set, and let σ_i be a singleton-labelled event structure over labels Λ_i. Let $\bowtie$ be a bijection on Λ_1 and Λ_2. If $b : \sigma_1 \cong^{\bowtie} \sigma_2$, then $b^\star : \Theta(\sigma_1) \cong^{\bowtie_{=1}} \Theta(\sigma_2)$.*

So far, we have proven that an isomorphism on singleton labelled event structures induces an isomorphism on their distinct respective tactics.

4.2 From CS-Relations to TC-Relations

Next, we construct a TC-relation $\underline{R}$ for a given CS-relation R on two sets of singleton labelled event structures for the case that the alignment $\bowtie$ is the identity relation. We show that $\underline{R}$ is initial if R is initial, and that $\underline{R}$ is an IT-relation if R is a CB-relation. As the final result, we have that $\underline{R}$ is an initial IT-relation if R is a initial CB-relation.

We begin with constructing $\underline{R}$ from R and showing that $\underline{R}$ is a TC-relation. The clue is to exploit that the considered event structures are singleton labelled, and thus have a unique tactic. Then, the remainder of the proof is to show that the property 2b of CS-relations (cp. Def. 2.6) induces the respective property of TC-relations (cp. Def. 3.3).

Lemma 4.4 *For $i = 1, 2$, let Σ_i be a set of singleton labelled event structures. Let Λ be the set of all labels occurring in $\Sigma_1 \cup \Sigma_2$. Let $\bowtie = \Delta_\Lambda$. Let $R : \Sigma_1 \lesssim \Sigma_2$. Then, there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$.*

Proof. Let $\underline{R}$ be the set of all $(\sigma_1, \sigma_2, \theta_1, \theta_2, b^\star)$ with $(\sigma_1, \sigma_2, b) \in R$ and $\theta_i = \Theta(\sigma_i)$. We show $R : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$. Let $(\sigma_1, \sigma_2, \theta_1, \theta_2, b^\star) \in \underline{R}$, and $\sigma_i = (E_i, \Lambda_i, \beta_i, \lambda_i)$. By definition of $\underline{R}$, $\theta_i = \Theta(\sigma_i) \in \Theta_{\sigma_i}$. By Corollary 4.3, $b^\star : \theta_1 \cong^{\bowtie=1} \theta_2$. Let $\sigma_1' \in \Sigma_1$ and $\sigma_1' \sqsupseteq \sigma_1$. From $(\sigma_1, \sigma_2, b) \in R$ and $R : \Sigma_1 \lesssim \Sigma_2$, we conclude: There exists $(\sigma_1', \sigma_2', b') \in R$, such that $\sigma_2' \sqsupseteq \sigma_2$, $b' : \sigma_1' \cong \sigma_2'$, and $b(e) = b'(e)$ for each $e \in E_1$. By definition of $\underline{R}$, $(\sigma_1', \sigma_2', \theta_1', \theta_2', b'^\star) \in \underline{R}$ with $\theta_i' = \Theta(\sigma_i')$.

(a) $\sigma_2' \sqsupseteq \sigma_2$ holds as shown above.

(b) Let $e \in E_1$ with $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta_1'} \cap E_1$. We show: $b^\star(\langle e \rangle_{\theta_1}) = b'^\star(\langle e \rangle_{\theta_1'}) \cap E_2$. By definition of $b^\star$, we have: $b^\star(\langle e \rangle_{\theta_1}) = \{b(e') \mid e' \in \langle e \rangle_{\theta_1}\}$. Applying the assumption $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta_1'} \cap E_1$, we get:

$$\{b(e') \mid e' \in \langle e \rangle_{\theta_1}\} = \{b(e') \mid e' \in \langle e \rangle_{\theta_1'} \cap E_1\} = \{b(e') \mid e' \in \langle e \rangle_{\theta_1'}, e' \in E_1\} \quad .$$

Applying $b(e) = b'(e)$ for all $e \in E_1$, we get:

$$\begin{aligned} \{b(e') \mid e' \in \langle e \rangle_{\theta_1'}, e' \in E_1\} \quad &= \{b'(e') \mid e' \in \langle e \rangle_{\theta_1'}, e' \in E_1\} \\ = \{b'(e') \mid e' \in \langle e \rangle_{\theta_1'}, b'(e') \in b^\star(E_1)\} &= \{b'(e') \mid e' \in \langle e \rangle_{\theta_1'}, b'(e') \in E_2\} \\ = \{b'(e') \mid e' \in \langle e \rangle_{\theta_1'}\} \cap E_2 \quad . & \end{aligned}$$

Finally, by definition of $b'^\star$: $\{b'(e') \mid e' \in \langle e \rangle_{\theta_1'}\} \cap E_2 = b'^\star(\langle e \rangle_{\theta_1'}) \cap E_2$.

□

Using the same construction, we show that an initial CS-relation induces an initial TC-relation:

Lemma 4.5 *For $i = 1, 2$, let Σ_i be a set of singleton labelled event structures. Let Λ be the set of all labels occurring in $\Sigma_1 \cup \Sigma_2$. Let $\bowtie = \Delta_\Lambda$. Let $R : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$. Then, there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \leqslant_{\bowtie}^{\text{init}} \Sigma_2$.*

Proof. The proof can be found as Proof 2 in the Appendix.

Similarly, we show that an CB-relation induces an IT-relation. In combination with Lemma 4.5, we get that a initial CB-relation induces an initial IT-relation:

Lemma 4.6 *For $i = 1, 2$, let Σ_i be a set of singleton labelled event structures. Let Λ be the set of all labels occurring in $\Sigma_1 \cup \Sigma_2$. Let $\bowtie = \Delta_\Lambda$. Let $R : \Sigma_1 \sim^{\text{init}} \Sigma_2$. Then, there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \doteqdot_{\bowtie}^{\text{init}} \Sigma_2$.*

Proof. The proof can be found as Proof 3 in the Appendix.

4.3 Simulation Implies Tactic Coverage for Bijective Alignments

We now show that for a bijective alignment between the transitions of N_1 and N_2, respectively simulation (bisimulation) between N_1 and N_2 implies tactic coverage (isotactics) between N_1 and N_2. Without loss of generality, we assume that N_1 and N_2 have the same transitions and that the alignment is derived from the identity relation, i.e., $\{t\} \bowtie \{t\}$ for all transitions. First, we show that in case of a bijective alignment, the set abstraction is isomorphic to its original:

Lemma 4.7 *Let $\sigma = (E_\sigma, \Lambda_\sigma, \beta_\sigma, \lambda_\sigma)$ be an event structure. Let $c = \{(\ell, \{\ell\}) \mid \ell \in \Lambda_\sigma\}$. Then, $\Delta_{E_\sigma} : \sigma \cong^c \alpha_{c[2]}(\sigma)$.*

Proof. Follows directly from Def. 2.4 and Def. 3.1. □

Now, we show that $N_1 \lesssim^{\mathbb{XY}} N_2$ implies $N_1 \leqslant_{\bowtie}^{\mathbb{XY}} N_2$, where N_1 and N_2 are Petri nets with the same sets of transitions.

Lemma 4.8 *For $i = 1, 2$, let $N_i = (P_i, T_i, F_i, m_i)$ be a Petri net with $T_1 = T_2$. Let $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$. Let $\bowtie = \Delta_{\mathcal{P}_{=1}(T_1)}$. Let $\Sigma_i = \alpha_{\bowtie[i]}(\mathbb{XY}(N_i))$. If $N_1 \lesssim^{\mathbb{XY}} N_2$, then there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$.*

Proof. By Def. 2.8, there exists R with $R : \mathbb{XY}(N_1) \lesssim^{\text{init}} \mathbb{XY}(N_2)$. Let $\underline{R}$ be the set of all $(\alpha_{\bowtie[1]}(\sigma_1), \alpha_{\bowtie[2]}(\sigma_2), b)$ with $(\sigma_1, \sigma_2, b) \in R$. We apply Lemma 4.7 to show $\underline{R} : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$. To this end, let $(\underline{\sigma}_1, \underline{\sigma}_2, b) \in \underline{R}$ with $\underline{\sigma}_i = \alpha_{\bowtie[i]}(\sigma_i)$ for $i = 1, 2$. By Lemma 4.7, we have $\Delta : \sigma_i \cong^{c_i} \underline{\sigma_i}$ with $c_i = \{(t, \{t\} \mid t \in T_i\}$. Then, $b : \underline{\sigma_1} \cong^{\bowtie} \underline{\sigma_2}$. Let $\underline{\sigma_1}' \in \Sigma_1$ with $\underline{\sigma_1}' \sqsupseteq \underline{\sigma_1}$ and $\underline{\sigma_1}' = \alpha_{\bowtie[1]}(\sigma_1')$. By Lemma 4.7, we have $b_1' : \sigma_1 \cong^{c_1'} \underline{\sigma_1}'$ with $c(t) = \{t\}$ and $b_i'(e) = e$, and thus $\sigma_1' \sqsupseteq \sigma_1$. By Def. 2.6, there exists $(\sigma_1', \sigma_2', b') \in R$, such that $\sigma_2' \sqsupseteq \sigma_2$ and $b'(e) = b(e)$ for each event e of σ_1. By definition of $\underline{R}$, $(\alpha_{\bowtie[1]}(\sigma_1'), \alpha_{\bowtie[2]}(\sigma_2'), b) \in \underline{R}$. Hence, $\underline{R} : \Sigma_1 \lesssim \Sigma_2$. Let $\alpha_{\bowtie[1]}(\sigma_1)$ be minimal in Σ_1. Applying Lemma 4.7, we conclude: σ_1 is minimal in $\mathbb{XY}(N_1)$. By Def. 2.6, there exists $(\sigma_1, \sigma_2, b) \in R$, such that σ_2 is minimal in $\mathbb{XY}(N_2)$. Hence, $(\alpha_{\bowtie[1]}(\sigma_1), \alpha_{\bowtie[2]}(\sigma_2), b) \in \underline{R}$. Applying Lemma 4.7, we observe that $\alpha_{\bowtie[2]}(\sigma_2)$ is minimal in Σ_2. Hence, $\underline{R}$ is initial. □

Theorem 4.9 *For $i = 1, 2$, let $N_i = (P_i, T_i, F_i, m_i)$ be a Petri net with $T_1 = T_2$. Let $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$. Let $\bowtie = \Delta_{\mathcal{P}_{=1}(T_1)}$. Then, $N_1 \lesssim^{\mathbb{XY}} N_2 \Rightarrow N_1 \lessdot^{\mathbb{XY}}_{\bowtie} N_2$.*

Proof. By Def. 2.8, there exists R with $R : \mathbb{XY}(N_1) \lesssim^{\text{init}} \mathbb{XY}(N_2)$ with $\Sigma_i = \alpha_{\bowtie[i]}(\mathbb{XY}(N_i))$. By Lemma 4.8, there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$. By Lemma 4.5, there exists $\underline{\underline{R}}$ with $\underline{\underline{R}} : \Sigma_1 \lessdot^{\text{init}}_{\bowtie} \Sigma_2$, and thus $N_1 \lessdot^{\mathbb{XY}}_{\bowtie} N_2$. □

We now turn from simulation and tactic coverage to bisimulation and isotactics, respectively: $N_1 \sim^{\mathbb{XY}} N_2$ implies $N_1 \doteqdot^{\mathbb{XY}}_{\bowtie} N_2$.

Theorem 4.10 *For $i = 1, 2$, let $N_i = (P_i, T_i, F_i, m_i)$ be a Petri net with $T_1 = T_2$. Let $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$. Let $\bowtie = \Delta_{\mathcal{P}_{=1}(T_1)}$. Then, $N_1 \sim^{\mathbb{XY}} N_2 \Rightarrow N_1 \doteqdot^{\mathbb{XY}}_{\bowtie} N_2$.*

Proof. The proof can be found as Proof 4 in the Appendix, and is mostly analogous to the proof of Thm. 4.9.

4.4 The Converse for Repetition-Free Petri Nets

As shown above by the counter example in Fig. 6, tactic coverage (isotactics) does not imply simulation (bisimulation) in the general case, since it abstracts from occurrence cardinalities of transitions. We thus introduce the notion of repetition-free Petri nets and show that for this class, the implications hold true.

Intuitively, a Petri net is *repetition-free* if the same transition cannot fire two times in a row. Formally, this excludes the existence of a marking M such that t can fire from both marking M, and the resulting marking of firing t in M. Note that this does not require that t is enabled twice in M and, thus, deviates from the notion of self-concurrency defined in [2].

Definition 4.11 (Repetition-free Petri nets) A Petri net $N = (P, T, F, M)$ is *repetition-free* iff for each marking M' which is reachable in N, and every transition $t \in T$ of N: If M' enables t, then $(M' - {}^\bullet t) + t^\bullet$ does not enable t.

Studying the semantics of repetition-free Petri nets, we observe that, under bijective alignments as discussed before, the following holds true: the tactic of each set abstraction is a singleton set.

Lemma 4.12 *Let $N = (P, T, F, M)$ be a repetition-free Petri net, $\bowtie = \Delta_{\mathcal{P}_{=1}(T)}$, $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$, and $\sigma \in \mathbb{XY}(N)$. Let $\theta = \Theta(\alpha_{\bowtie[1]}(\sigma))$. Let e be an event of σ. Then, $|\langle e \rangle_\theta| = 1$.*

Proof. Let $\sigma = (E, \Lambda, \beta, \lambda)$, $e \neq e' \in E$, and $\lambda(e) = \lambda(e')$. Because $\bowtie = \Delta_{\mathcal{P}_{=1}(T)}$, we get $\alpha_{\bowtie[1]}(\sigma) = (E, \mathcal{P}_{=1}(\Lambda), \beta, e \mapsto \{\lambda(e)\})$. We distinguish based on $\beta(e, e')$.

- Let $\beta(e, e') = <$. Because N is repetition-free, there exists e'' with $\beta(e, e'') = \beta(e'', e') = <$, and $\lambda(e'') \neq \lambda(e)$. Hence, $e'' \notin \langle e \rangle_\theta$ and $e'' \notin \langle e' \rangle_\theta$. From $\beta(e, e'') = \beta(e'', e')$, we get $\langle e \rangle_\theta \neq \langle e' \rangle_\theta$. The case $\beta(e, e') = >$ is symmetric.
- Let $\beta(e, e') = \|$. Because N is repetition-free, at least one of the two events e and e' are not minimal w.r.t. causality. Hence, at least one of the two, say e, has a direct predecessor e'' w.r.t. causality. Because N is repetition-free, e'' cannot be a direct predecessor of e' w.r.t. causality. From $\beta(e, e') = \|$, we conclude: e' cannot be a successor of e'' w.r.t. causality. Hence, $\langle e \rangle_\theta \neq \langle e' \rangle_\theta$.
- Let $\beta(e, e') = \#$. Because $\sigma \in \mathbb{XY}(N)$, there exists e'' with $\beta(e'', e) = <$ or $\beta(e'', e') = <$. Let $\beta(e'', e) = <$ (the case $\beta(e'', e') = <$ is analogous). Because $\beta(e, e) \neq \#$, $\beta(e'', e') \neq <$. Hence, $\langle e \rangle_\theta \neq \langle e' \rangle_\theta$. ▫

We exploit this structural property of a tactic and to transform every isomorphism on tactics to an isomorphism on the set abstractions. As a prerequisite, we show that we can obtain a bijection on the respective events of two set abstractions from a bijection on the respective events of two tactics.

Lemma 4.13 *For $i = 1, 2$, let σ_i be a singleton labelled event structure with events E_i, and $\theta_i = \Theta(\sigma_i)$. Let $\bowtie$ be a relation, and $b : \theta_1 \cong^{\bowtie} \theta_2$, such that $|\langle e \rangle_{\theta_1}| = |b(\langle e \rangle_{\theta_1})|$ for all $e \in E_1$.*

1. *There exists a bijection b_e on $\langle e \rangle_{\theta_1}$ and $b(\langle e \rangle_{\theta_1})$ for each $e \in E_1$.*
2. *There exists a bijection c on E_1 and E_2 with $c(e) \in b(\langle e \rangle_{\theta_1})$ for each $e \in E_1$.*

Proof. The proof can be found as Proof 5 in the Appendix.

Hence, we can transform an isomorphism on tactics to one on set abstractions.

Lemma 4.14 *For $i = 1, 2$, let σ_i be a singleton labelled event structure with events E_i and label set Λ_i, and $\theta_i = \Theta(\sigma_i)$. Let $\Lambda_1 = \Lambda_2$. Let $\bowtie = \Delta_{\Lambda_1}$, and $b : \theta_1 \cong^{\bowtie_{=1}} \theta_2$, such that $|\langle e \rangle_{\theta_1}| = |b(\langle e \rangle_{\theta_1})|$ for all $e \in E_1$. Let c be a bijection on E_1 and E_2, such that $c(e) \in b(\langle e \rangle_{\theta_1})$ for each $e \in E_1$. Then, $(\forall e \in E_1\text{: } |\langle e \rangle_{\theta_1}| = 1) \Rightarrow c : \sigma_1 \cong^{\bowtie} \sigma_2$.*

Proof. The proof can be found as Proof 6 in the Appendix.

Next, we need the argument that there always exists a *maximal TC-relation* w.r.t. the subset relation, i.e., a TC-relation such that every TC-relation is a subset. To this end, we show that the union of two TC-relations is again a TC-relation, and that the empty set is a (not necessarily initial) TC-relation.

Lemma 4.15 *For $i = 1, 2$, let Σ_i be a set of set-labelled event structures, and $\bowtie$ be a relation. Then, there exists a set $R_{\max}$, such that*

1. *$R_{\max} : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$.*
2. *For all R' with $R' : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$: $R' \subseteq R_{\max}$.*

Proof. The proof can be found as Proof 7 in the Appendix.

Now, we can apply the existence of a maximal TC-relation to show that there always exists a CS-relation on two sets of singleton labelled event structures. This CS-relation is the empty set if the maximal TC-relation is the empty set.

Lemma 4.16 *For $i = 1,2$, let Σ_i be a set of singleton labelled event structures. Let Λ be the set of labels occurring in $\Sigma_1 \cup \Sigma_2$. Let $\bowtie = \Delta_\Lambda$. For each $\sigma \in \Sigma_1 \cup \Sigma_2$, and event e of σ, let $|\langle e \rangle_{\Theta(\sigma)}| = 1$. Let $R : \Sigma_1 \leqslant_\bowtie \Sigma_2$, such that $R' : \Sigma_1 \leqslant_\bowtie \Sigma_2$ implies $R' \subseteq R$. Then, there exists $\underline{R}$, such that $\underline{R} : \Sigma_1 \lesssim \Sigma_2$.*

Proof. For each $r = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$, let $c_r = b_{=1}$. We construct $\underline{R}$ from R as follows: Let $\underline{R}$ be the set of all $(\sigma_1, \sigma_2, c_r)$, such that $r = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$. Let $r = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$ and $\underline{r} = (\sigma_1, \sigma_2, c_r) \in \underline{R}$. From $R : \Sigma_1 \leqslant_\bowtie \Sigma_2$, we get $b : \theta_1 \cong^{\bowtie_{=1}} \theta_2$. By Lemma 4.13 and the definition of c_r, we have $c_r(e) \in b(\langle e \rangle_{\theta_1})$ for all $e \in E_1$. Then, Lemma 4.14 yields $c_r : \sigma_1 \cong \sigma_2$. Let $\sigma_1' \in \Sigma_1$ and $\sigma_1' \sqsupseteq \sigma_1$. Because R is a TC-relation, there exists $r' = (\sigma_1', \sigma_2', \theta_1', \theta_2', b') \in R$, such that $\sigma_2' \sqsupseteq \sigma_2$, and for all $e \in E_1$: $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta_1'} \cap E_1$ implies $b(\langle e \rangle_{\theta_1}) = b'(\langle e \rangle_{\theta_1'}) \cap E_2$. Let $\underline{r'} = (\sigma_1', \sigma_2, c_{r'})$. By definition of $\underline{R}$, $\underline{r'} \in \underline{R}$. Let $e \in E_1$. We show $c_r(e) = c_{r'}(e)$. By assumption, $|\langle e \rangle_{\theta_1}| = |\langle e \rangle_{\theta_1'}| = 1$. By definition of $\langle e \rangle_{\theta_1}$ and $\langle e \rangle_{\theta_1'}$, $e \in \langle e \rangle_{\theta_1}$ and $e \in \langle e \rangle_{\theta_1'}$. Hence, $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta_1'} = \langle e \rangle_{\theta_1'} \cap E_1$. Then, $b(\langle e \rangle_{\theta_1}) = b'(\langle e \rangle_{\theta_1'}) \cap E_2 = b'(\langle e \rangle_{\theta_1'})$. By definition of c_r and $c_{r'}$, we conclude $c_r(e) = c_{r'}(e)$. ▫

We observe that $\underline{R}$ is initial if R is initial, and that $\underline{R}$ is a CB-relation if R is an IT-relation. Based thereon, we prove that tactics coverage and isotactics imply simulation and bisimulation, respectively, for the simplified setting of repetition-free Petri nets under a bijective alignment. Applying Thm. 4.9 and Thm. 4.10 yields the full coincidence of tactics coverage and isotactics with simulation and bisimulation, respectively, for this simplified setting.

Theorem 4.17 *For $i = 1,2$, let $N_i = (P_i, T_i, F_i, m_i)$ be a repetition-free Petri net with $T_1 = T_2$. Let $\mathbb{XY} \in \{\mathbb{BC}, \mathbb{BI}, \mathbb{LC}, \mathbb{LI}\}$. Let $\bowtie = \Delta_{\mathcal{P}_{=1}(T_1)}$.*
1. *$N_1 \lesssim^{\mathbb{XY}} N_2$ iff $N_1 \leqslant_\bowtie^{\mathbb{XY}} N_2$.*
2. *$N_1 \sim^{\mathbb{XY}} N_2$ iff $N_1 \doteqdot_\bowtie^{\mathbb{XY}} N_2$.*

Proof. The "⇒"-parts of 1. and 2. are Thm. 4.9 and Thm. 4.10, repectively. Let $\Sigma_i = \alpha_{\bowtie[i]}(\mathbb{XY}(N_i))$. For the "⇐"-part of 1., we observe that there exists R with $R : \Sigma_1 \leqslant_\bowtie^{\text{init}} \Sigma_2$. By Lemma 4.12, we can apply Lemma 4.16 to show that there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \lesssim \Sigma_2$. Inspecting the construction of $\underline{R}$, we find $\underline{R} : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$ because R is initial. Now, applying Lemma 4.7 as in the proof of Lemma 4.8, we conclude $N_1 \lesssim^{\mathbb{XY}} N_2$. For the "⇐"-part of 2., we can apply the same arguments, and additionally apply the arguments in the proof of Lemma 4.16 to show that $\underline{R}^{-1} : \Sigma_2 \lesssim^{\text{init}} \Sigma_1$. Again, applying Lemma 4.7 as in the proof of Lemma 4.8, we conclude $N_1 \sim^{\mathbb{XY}} N_2$. ▫

5 Deciding Linear Time, Interleaving Isotactics

We show that isotactics is decidable for linear time, interleaving semantics. Below, we fix two bounded Petri nets N_1 and N_2 and an alignment $\bowtie$ between N_1 and N_2. The idea is to reduce the problem of deciding $N_1 \doteqdot_\bowtie^{\mathbb{LI}} N_2$ to (1) computing the *witness graph* $\mathcal{G}_\bowtie(N_1, N_2)$ of N_1 and N_2 w.r.t. $\bowtie$, and (2) deciding whether $\mathcal{G}_\bowtie(N_1, N_2)$ contains a *complete subgraph*. Intuitively, $\mathcal{G}_\bowtie(N_1, N_2)$ is an edge-labelled, directed graph representing pairs of alignable set abstractions, and a subgraph G of $\mathcal{G}_\bowtie(N_1, N_2)$ is complete if it preserves continuation of set abstractions: $\mathcal{G}_\bowtie(N_1, N_2)$ contains a complete subgraph, iff every set abstraction of N_1 can be aligned to some set abstraction of N_2, and vice versa. Each path yields a pair (σ_1, σ_2) of alignable set abstractions of N_1 and N_2, respectively. Each node represents the respective sets of markings that are reachable by firing sequences yielding σ_1 and σ_2. Those sets are singleton if $\bowtie$ is total and elementary. However, in general, different firing sequences may yield the same set abstraction.

Formally, each node p of $\mathcal{G}_{\bowtie}(N_1, N_2)$ consists of a pair $(\mathcal{M}_1, \mathcal{M}_2)$ of sets of markings of N_1 and N_2, together with the labels of the respective last event of σ_1 and σ_2. Each edge is labelled by a pair (Y_1, Y_2), where either (a) Y_1 and Y_2 are an element of $\bowtie[1]$ and $\bowtie[2]$, respectively, or (b) Y_i is an element of $\bowtie[i]$ and $Y_j = \varnothing$ ($1 \leq i \neq j \leq 2$). Thereby, case (a) means that σ_1 and σ_2 are continued by alignable events labelled Y_1 and Y_2. Case (b) means that σ_i is continued by an event labelled Y_i but σ_j stays the same, where Y_i shares a label with the last event in σ_i and is alignable to the last event in σ_j.

We introduce some preliminary notions: For $i = 1, 2$, let $\mathcal{M}$ be a set of markings of N_i, t be a transition of N_i and κ be a set of transitions of N_i. Then, $\mathcal{M}^{\bowtie}$ is the set of markings reachable from $\mathcal{M}$ in N_i by only firing transitions t with $t \notin K$ for all $K \in \bowtie[i]$, $\kappa_t := \{K \mid t \in K \in \bowtie[i]\}$, and $\mathcal{M} + \kappa := \{M' \mid M \xrightarrow{t} M', M \in \mathcal{M}, \kappa_t = \kappa\}$.

Definition 5.1 (Witness Graph) The *witness graph* $\mathcal{G}_{\bowtie}(N_1, N_2)$ of two Petri nets N_1 and N_2 w.r.t. an alignment $\bowtie$ is the least edge-labelled graph satisfying:

1. Let M_i be the initial marking of N_i. Then, $(\{M_1\}^{\bowtie}, \{M_2\}^{\bowtie}, \varnothing, \varnothing)$ is a node of $\mathcal{G}_{\bowtie}(N_1, N_2)$.
2. Let $p = (\mathcal{M}_1, \mathcal{M}_2, \kappa_1, \kappa_2)$ be a node of $\mathcal{G}_{\bowtie}(N_1, N_2)$. For $i = 1, 2$, let $\kappa'_i \subseteq \bowtie[i]$
   (a) For $i = 1, 2$, let $\mathcal{M}_i + \kappa_i \neq \varnothing$ and $\mathcal{M}'_i = (\mathcal{M} + \kappa_i)^{\bowtie}$. Let $(K_1, K_2) \in (\kappa'_1 \times \kappa'_2) \cap \bowtie$, such that $K_1 \in \kappa_1$ iff $K_2 \in \kappa_2$. Then, there is an edge labelled (Y_1, Y_2) from p to $(\mathcal{M}'_1, \mathcal{M}'_2, \kappa_1, \kappa_2)$.
   (b) Let $i \neq j \in \{1, 2\}$. Let $\mathcal{M}_i + \kappa_i \neq \varnothing$. Let $\mathcal{M}'_i = (\mathcal{M}_i + \kappa)^{\bowtie}$ and $\mathcal{M}'_j = \mathcal{M}_j$. Let $((\kappa_i \cap \kappa'_i) \times \kappa_j) \cap \bowtie \neq \varnothing$. Let $Y_i = \kappa_i$ and $Y_j = \varnothing$. Then, there is an edge labelled (Y_1, Y_2) from p to $(\mathcal{M}'_1, \mathcal{M}'_2, Y_1, Y_2)$.

Now, we introduce the notion of *realisability*: A pair (σ_1, σ_2) of set abstractions is realisable, if there exists a path in $\mathcal{G}_{\bowtie}(N_1, N_2)$ representing (σ_1, σ_2).

Definition 5.2 (Realisability) For $i = 1, 2$, let $\sigma_i = e_i^1 \dots e_i^{n_i}$ be a set abstraction of N_i w.r.t. $\bowtie$. Let $Y = (Y_1^1, Y_2^1) \dots (Y_1^n, Y_2^n)$ be a path of $\mathcal{G}_{\bowtie}(N_1, N_2)$ starting in the initial node and resulting in a node p, where the projection of $Y_i^1 \dots Y_i^n$ to $\bowtie[i]$ yields $\lambda(e_i^1) \dots \lambda(e_i^{n_i})$ for each $1 \leq k \leq n_i$ for $i = 1, 2$. Then, Y *realises* (σ_1, σ_2), and (σ_1, σ_2) is *realisable yielding* p.

Because of rule 2a), (σ_1, σ_2) may be realised by different paths, but those yield the same node. We now show coincidence of realisability and the existence of isomorphic tactics:

Lemma 5.3 *Let (σ_1, σ_2) be realisable yielding a node $(., ., \kappa_1, \kappa_2)$. Then, for $(K_1, K_2) \in (\kappa_1 \times \kappa_2) \cap \bowtie$, there exist $\theta_1 \in \Theta_{\sigma_1}, \theta_2 \in \Theta_{\sigma_2}$, and b, such that the label of the last element of θ_1 is K_1, and b is an isomorphism from θ_1 into θ_2 w.r.t. $\bowtie$.*

Proof. We show the lemma by induction over the length of the realising path.

Let $Y = (Y_1, Y_2)$ be the realising path. Then, σ_1 and σ_2 consist of only one event e_1 and e_2, respectively. Because Y starts at the initial state, only rule 2a) can have created Y. Let $K_1 \in \kappa_1, K_2 \in \kappa_2$, s.t. $K_1 \bowtie K_2$. Then, there exists the trivial tactics with only label K_1 and K_2, respectively, with the trivial isomorphism.

Let $Y = (Y_1^1, Y_2^1) \dots (Y_1^n, Y_2^n)$ realise (σ_1, σ_2) yielding $(., ., \kappa_1, \kappa_2)$. Let $Y' = Y(Y_1, Y_2)$.

Let (Y_1, Y_2) be created by rule 2a). For $i = 1, 2$, let σ'_i be the result of appending a Y_i-labelled event e_i to σ_i. Then, Y' realises (σ_1, σ_2) yielding a node $(., ., Y_1, Y_2)$. By rule 2a), there exists $(K'_1, K'_2) \in (\kappa'_1 \times \kappa'_2) \cap \bowtie$, such that $K_1 \in \kappa_1$ iff $K_2 \in \kappa_2$. Let $K_1 \in \kappa_1$. Then, $K_2 \in \kappa_2$. By induction, there exists a pair (θ_1, θ_2) of tactics for (σ_1, σ_2) and b, such that there exists an isomorphism b from θ_1 into θ_2, and the label of the last element of θ_1 is K_1. Then, we can inject e_i into the last event of θ_i. Then, (θ'_1, θ'_2) is a pair of tactics for (σ'_1, σ'_2), and b can be also be canonically extended to an isomorphism from θ'_1 into θ'_2. Finally, the last element of each θ'_i is still labelled K_i. Let $K_1 \notin \kappa_1$. Then, $K_2 \notin \kappa_2$. Let (θ_1, θ_2) be an arbitrary pair of tactics for (σ_1, σ_2). For $i = 1, 2$, we extend θ_i to θ'_i by adding the event $\{e_i\}$ labelled K_i to the end of θ_i. Then, (θ'_1, θ'_2) is a pair of tactics for (σ'_1, σ'_2), and b can be also be canonically extended to an isomorphism from θ'_1 into θ'_2. Finally, the last element of each θ'_i is now labelled K_i.

Let (Y_1, Y_2) be created by rule 2b). Let $i \neq j \in \{1, 2\}$, such that $Y_i \neq \emptyset$. Let σ'_i be an event structure resulting from appending a Y_i-labelled event e to σ_i, and $\sigma'_j = \sigma_j$. Let $\kappa'_i = Y_i$ and $\kappa'_j = \kappa_j$. Then, Y' realises (σ'_1, σ'_2) yielding a node $(.,., \kappa'_1, \kappa'_2)$. By rule 2b), there exists $K_i \in ((\kappa_i \cap \kappa'_i) \times \kappa_j) \cap \bowtie$. By induction, there exists a pair (θ_1, θ_2) of tactics for (σ_1, σ_2) and b, such that there exists an isomorphism b from θ_1 into θ_2, and the label of the last element of θ_1 is K_1. Now, let θ_i be the event structure resulting from injecting e into the last event of θ_i. Let $\theta'_j = \theta_j$. Then, (θ'_1, θ'_2) is a pair of tactics for (σ'_1, σ'_2), and b can be also be canonically extended to an isomorphism from θ'_1 into θ'_2. Finally, the last elements of θ'_i and θ'_j are still labelled K_i and K_j, respectively. ▫

Lemma 5.4 *Let σ_1, σ_2 be set abstractions, $\theta_1 \in \Theta_{\sigma_1}, \theta_2 \in \Theta_{\sigma_2}$, and b, such that for $i = 1, 2$, the label of the last element of θ_1 is K_1, and b is an isomorphism from θ_1 into θ_2 w.r.t. $\bowtie$. Then, (σ_1, σ_2) is realisable, yielding a node $(.,., \kappa_1, \kappa_2)$ with $K_i \in \kappa_i$ for $i = 1, 2$.*

Proof. For $i = 1, 2$, let $\sigma_i = e_i^1 \dots e_i^{n_i}$. We find the realising path as follows: For $n_1 = n_2 = 1$, we choose the edge $(\lambda(e_1^1), \lambda(e_2^1))$, which is produced by rule 2a), and results in a node $(.,., \lambda(e_1^1), \lambda(e_2^1))$. The parts $\langle e_1^1 \rangle_{\theta_1}$ and $\langle e_2^1 \rangle_{\theta_2}$ must be aligned and thus there must be corresponding labels in $\lambda(e_1^1)$ and $\lambda(e_2^1)$. For $n_1 + n_2 > 2$, we continue along σ_1 and σ_2 as follows: If $\langle e_1^k \rangle_{\theta_1} = \langle e_1^{k+1} \rangle_{\theta_1}$, we append $(\emptyset, \lambda(e_1^{k+1})$. This is possible, since $\langle e_1^k \rangle_{\theta_1} = \langle e_1^{k+1} \rangle_{\theta_1}$ implies that there exists a common label in e_1^k, e_1^{k+1}, which is aligned to the label of $b(\langle e_1^k \rangle_{\theta_1})$. If $\langle e_1^k \rangle_{\theta_1} \neq \langle e_1^{k+1} \rangle_{\theta_1}$, then also $b(\langle e_1^k \rangle_{\theta_1}) \neq b(\langle e_1^{k+1} \rangle_{\theta_1})$. Hence, we first append corresponding edges $(\emptyset, \dots)$ for the events before the minimal element e in $b(\langle e_1^{k+1} \rangle_{\theta_1})$, and then add the edge $(\lambda(e_1^k), \lambda(e))$. This is possible, since e_1^k and e must have aligned labels. ▫

We introduce the notion of a *complete subgraph*, and show the coincidence of isotactics with the existence a complete subgraph of $\mathcal{G}_{\bowtie}(N_1, N_2)$.

Definition 5.5 (Complete Subgraph) A subgraph G of $\mathcal{G}_{\bowtie}(N_1, N_2)$ is *complete* iff:

1. G contains the initial node, and
2. for every $i \in \{1, 2\}$, and node p of G: If p i-enables κ, then there starts a path $(Y_1^1, Y_2^1) \dots (Y_1^n, Y_2^n)(Y_1, Y_2)(Y_1^{n+1}, Y_2^{n+1}) \dots (Y_1^m, Y_2^m)$ in $\mathcal{G}_{\bowtie}(N_1, N_2)$ from p resulting in a node p' of G with $Y_i = \kappa$, and $Y_i^k = \emptyset$ for all $1 \le k \le m$.

Lemma 5.6 *$\mathcal{G}_{\bowtie}(N_1, N_2)$ has a complete subgraph, iff $N_1 \doteqdot_{\bowtie}^{\mathbb{LI}} N_2$.*

Proof. Let G be a complete subgraph of $\mathcal{G}_{\bowtie}(N_1, N_2)$. We define the set R_G as follows: Let Y be a path of G realising (σ_1, σ_2) yielding a node $(\dots, \dots, \kappa_1, \kappa_2)$. Then, by Lemma 5.3 for every $K_1, K_2 \in (\kappa_1 \times \kappa_2) \cap \bowtie$, there exist θ_1, θ_2, b, such that θ_1 and θ_2 are tactics of σ_1, σ_2 ending in K_1 and K_2, respectively, and b is an isomorphism from θ_1 into θ_2. Add $\widehat{Y}_{\theta_1, \theta_2} \coloneqq (\sigma_1, \sigma_2, \theta_1, \theta_2, b)$ to R_G. We show that R_G is an isotactics relation. Let $\widehat{Y}_{\theta_1, \theta_2} = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R_G$. Let $\sigma_1 e$ be a set abstraction. Then, by completeness of G, there exists an outgoing path $Y' = (Y_1^1, Y_2^1) \dots (Y_1^n, Y_2^n)(Y_1, Y_2)$ from p with $Y_1^k = \emptyset$ for $1 \le k \le n$, and $Y_1 = \lambda(e)$. Then, YY' realises some $(\sigma_1 e, \sigma'_2)$ with $\sigma'_2 \sqsupseteq \sigma_2$. Therefore, there exists $\widehat{YY'}_{\theta'_1, \theta'_2} = (\sigma_1 e, \sigma'_2, \theta'_1, \theta'_2, b') \in R_G$. Inspecting the proof of Lemma 5.4, we find that b is always canonically extended.

Let R be an initial isotactics relation. Let G_R be some subgraph containg exactly the following nodes: (1) the initial node. (2) Node $\widehat{x} := p$ for each $x = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$, such that (σ_1, σ_2) is realisable yielding node p. We show that G_R is complete. First, by (1), the initial node p_0 is contained in G_R. Let p_0 i-enable κ_i. Then, there is the minimal set abstraction $\sigma_i = e_i$ of N_i with $\lambda(e_i) = \kappa_i$. By initiality of R, there exists an initial σ_j of N_j with $(\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$. Then, $\sigma_j = e_j$ with $\lambda(e_j) = \kappa_j$, and there is $(K_1, K_2) \in (\kappa_1 \times \kappa_2) \cap \bowtie$. Following rule 2a), there is an outgoing edge from p_0 labelled (Y_1, Y_2), where $Y_i = \kappa_i$. Now, let $p = \widehat{x}$ for $x = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$. Let p i-enable κ. Then, $\sigma_i' = \sigma_i e$ with $\lambda(e) = \kappa$ is a set abstraction. Hence, there exists $x' = (\sigma_1', \sigma_2', \theta_1', \theta_2', b') \in R$ with $\sigma_j' \sqsupseteq \sigma_j$ ($j \neq i \in \{1,2\}$) and b being an isomorphism from θ_1' into θ_2'. Therefore, by Lemma 5.4, (σ_1', σ_2') is realisable by some path $Y = (Y_1^1, Y_2^1) \dots (Y_1^n, Y_2^n)$ yielding $p' := \widehat{x'}$. By definition, the projection of $Y_k^1 \dots Y_k^n$ ($k = 1, 2$) to non-empty sets yields σ_i'. Therefore, there exists a prefix $(Y_1^1, Y_2^1) \dots (Y_1^m, Y_2^m)$ resulting in p. Inspecting $Y_i^{m+1} \dots Y_i^n$, we find that it contains κ exactly once, and all other labels have to be $\varnothing$. Thus, there exists the required outgoing path from p to some node in G_R, namely p'. ▫

Therefore, deciding isotactics is equivalent to checking the existence of a complete subgraph of $\mathcal{G}_{\bowtie}(N_1, N_2)$, from which we conclude decidability of isotactics.

Theorem 5.7 *Isotactics is decidable for bounded Petri nets.*

Proof. Since N_1 and N_2 are bounded, $\mathcal{G}_{\bowtie}(N_1, N_2)$ is finite and can be computed. By Lemma 5.6, we decide isotactics by checking each subgraph G of $\mathcal{G}_{\bowtie}(N_1, N_2)$ for completeness. This requires inspecting the marking sets in each node p, finding enabled κ, and checking whether there exists a witness path of $\mathcal{G}_{\bowtie}(N_1, N_2)$ starting at p in G and resulting in some node in G. As Q and $\bowtie$ are finite, this is feasible. ▫

Regarding complexity of deciding isotactics, we can give EXPSPACE as a lower bound for complexity: By introducing a delimiter symbol between transition labels, we can reduce language equivalence for labelled, bounded Petri nets to isotactics, which is a EXPSPACE-complete problem [9]. This also means that isotactics is undecidable for unbounded Petri nets, because language inclusion is well-known to be undecidable for labelled, unbounded Petri nets [7].

6 Related Work

There have been a few attempts to define notions of behavioural equivalence for aligned process models. For interleaving semantics, our earlier work outlined how to formulate equivalence in the presence of complex correspondences under linear time semantics [22] and branching time semantics [23]. Unlike the isotactics spectrum presented in this work, however, the existing notions are not applicable for overlapping correspondences and lack a generic foundation that is independent of the chosen process semantics. An initial notion of isotactics was presented for linear time, partial order semantics in [16], yet lacking results on the reduction to well-established equivalences and decidability as they are delivered in this work.

In this work, we considered only total alignments. To cope with partial alignments, it was argued that actions can be *blocked* or *hidden* before verifying equivalence, which yields notions of behaviour inheritance [1]. This approach, originally defined for branching bisimulation, can directly be lifted to the introduced isotactics notions, making them applicable for partially aligned models.

Behavioural comparison of process models that are aligned by complex, but disjoint sets of actions relates to action refinement, e.g., place or transition refinement in Petri nets. Refinements that preserve behavioural equivalence have been widely studied, see [20]. While one can expect to lift these results to isotactics, it also interesting to investigate which types of non-hierarchical refinements—inducing complex, overlapping correspondences—preserve isotactics.

For process models that are not isotactic, measures for behavioural similarity can be a useful means to quantify the deviation. Such measures have been proposed for different process semantics, see, for instance [5,13].

7 Conclusion

This paper presented a spectrum of isotactics notions, which are proper generalisations of the well-established behavioural equivalences. In addition, we showed decidability of isotactics for linear time, interleaving semantics.

Isotactics notions enable behavioural comparison for a broader class of process model pairs, compared to established notions of equivalence. This is relevant in particular for non-hierarchical transformations of process models that induce complex, overlapping alignments. In work on iterative approaches to system design and multi-viewpoint modelling, it has been noted that such non-hierarchical refinements are inevitable [15,10]. Isotactics may be interpreted as a correctness criterion for the definition of non-hierarchical refinements. As such, they may pave the way for more expressive approaches to design complex systems.

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Appendix

Proof 1 (Proof of Lemma 4.1) *To simplify notation, we define $\langle b(e)\rangle_{\theta_2} := b^\star(\langle e\rangle_{\theta_1})$), which would be undefined unless θ_2 is proven to be a tactic of σ_2 (as done below). We first check all the conditions for a tactic based on Def. 3.2, and then show $b^\star : \theta_1 \cong^{\bowtie_{=1}} \theta_2$.*

1. *Here, we observe that $b^\star$ is a bijection on E'_1 and E'_2: By definition, $b^\star(\langle e\rangle_{\theta_1}) \in E'_2$ for each $e \in E_1$, and $E'_1 = \{\langle e\rangle_{\theta_1} \mid e \in E_1\}$. Let $\langle b(e)\rangle_{\theta_2}, \langle b(e')\rangle_{\theta_2} \in E'_2$ and $\langle b(e)\rangle_{\theta_2} \neq \langle b(e')\rangle_{\theta_2}$. By definition, $\langle b(e)\rangle_{\theta_2} = b^\star(\langle e\rangle_{\theta_1})$ and $\langle b(e')\rangle_{\theta_2} = b^\star(\langle e'\rangle_{\theta_1})$. Then, $b^\star(\langle e\rangle_{\theta_1}) \neq b^\star(\langle e'\rangle_{\theta_1})$ and $\langle e\rangle_{\theta_1} \neq \langle e'\rangle_{\theta_1}$. Because E'_1 is a partition of E_1, and $b^\star$ is a bijection on E'_1 and $E'_2 = \{b^\star(M) \mid M \in E_1\}$, we also have that E'_2 is a partition of E_2.*
2. *Obvious.*
3. *Let $\langle b(e_1)\rangle_{\theta_2} = \langle b(e'_1)\rangle_{\theta_2} \neq \langle b(e_2)\rangle_{\theta_2} = \langle b(e'_2)\rangle_{\theta_2}$. From Def. 2.4, we get $\langle e_1\rangle_{\theta_1} = \langle e'_1\rangle_{\theta_1} \neq \langle e_2\rangle_{\theta_1} = \langle e'_2\rangle_{\theta_1}$, and by Def. 3.2 $\beta'_1(e_1,e_2) = \beta'_1(e'_1,e'_2)$. Now, by definition of β'_2, $\beta'_2(\langle b(e_1)\rangle_{\theta_2}, \langle b(e_2)\rangle_{\theta_2}) = \beta'_1(e_1,e_2) = \beta'_1(e'_1,e'_2) = \beta'_2(\langle b(e'_1)\rangle_{\theta_2}, \langle b(e'_2)\rangle_{\theta_2})$.*
4. *Let $\langle b(e_1)\rangle_{\theta_2} \neq \langle b(e_2)\rangle_{\theta_2}$. By definition of β'_2, $\langle e_1\rangle_{\theta_1} \neq \langle e_2\rangle_{\theta_1}$. Because θ_1 is a tactic, we get $\beta'_1(\langle e_1\rangle_{\theta_1}, \langle e_2\rangle_{\theta_1}) = \beta_1(e_1,e_2)$. Because b is an isomorphism, we have $\beta_1(e_1,e_2) = \beta_2(b(e_1), b(e_2))$.*
5. *Follows from Def. 3.2 and the definition of β'_2.*
6. *Let $b(e') \in \langle b(e)\rangle_{\theta_2}$. By definition, $\lambda'_2(\langle b(e)\rangle_{\theta_2}) \in \bowtie(\{\lambda'_1(\langle e\rangle_{\theta_1})\})$, and thus $\{\lambda'_2(\langle b(e)\rangle_{\theta_2})\} = \bowtie(\{\lambda'_1(\langle e\rangle_{\theta_1})\})$ By Def. 3.2, $\lambda'_1(\langle e\rangle_{\theta_1}) \in \lambda_1(e')$. Since σ_1 is singleton labelled, $\lambda_1(e') = \{\lambda'_1(\langle e\rangle_{\theta_1})\}$. By Def. 2.4, and $\bowtie$ being a bijection, $\lambda_2(b(e')) = \bowtie(\lambda_1(e'))$. Hence, $\lambda_2(b(e')) = \bowtie(\{\lambda'_1(\langle e\rangle_{\theta_1})\}) = \{\lambda'_2(\langle b(e)\rangle_{\theta_2})\}$.*
7. *Let $e_1, e_2 \in E_1$ with $\langle b(e_1)\rangle_{\theta_2} \neq \langle b(e_2)\rangle_{\theta_2}$ and $\lambda'_2(\langle b(e_1)\rangle_{\theta_2}) = \lambda'_2(\langle b(e_2)\rangle_{\theta_2})$. By definition of E'_2, $\langle e_1\rangle_{\theta_1} \neq \langle e_2\rangle_{\theta_1}$. By definition of λ'_2, and $\bowtie$ being a bijection, $\lambda'_2(\langle e_1\rangle_{\theta_1}) = \lambda'_2(\langle e_2\rangle_{\theta_1})$. By Def. 3.2, there exists e_3 satisfying $e_3 \notin (\langle e_1\rangle_{\theta_1} \cup \langle e_2\rangle_{\theta_1})$ and $\beta_1(e_1,e_3) \neq \beta_1(e_2,e_3)$. By definition of E'_2, $e_3 \notin (\langle b(e_1)\rangle_{\theta_2} \cup \langle b(e_2)\rangle_{\theta_2})$. By Def. 2.4, $\beta_2(b(e_1), b(e_3)) \neq \beta_2(b(e_2), b(e_3))$.*

As shown before, b is a bijection. By definition of β'_2, it holds that $\beta'_2(\langle b(e)\rangle_{\theta_2}, \langle b(e')\rangle_{\theta_2}) = \beta'_1(\langle e\rangle_{\theta_1}, \langle e'\rangle_{\theta_1})$. $\lambda'_2(\langle b(e)\rangle_{\theta_2}) \in \bowtie(\{\lambda'_1(\langle e\rangle_{\theta_1})\})$ implies $\{\lambda'_2(\langle b(e)\rangle_{\theta_2})\} \bowtie \{\lambda'_1(\langle e\rangle_{\theta_1})\}$, which implies $\lambda'_2(\langle b(e)\rangle_{\theta_2}) \bowtie_{=1} \lambda'_1(\langle e\rangle_{\theta_1})$. ◻

Proof 2 (Proof of Lemma 4.5) *We construct $\underline{R}$ analogously to the proof of Lemma 4.4. Let σ_1 be minimal in Σ_1. Applying $R : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$, we obtain that there exists some minimal element σ_2 of Σ_2, and some b, such that $(\sigma_1, \sigma_2, b) \in R$. By definition of $\underline{R}$, we get: $(\sigma_1, \sigma_2, \Theta(\sigma_1), \Theta(\sigma_2), b^\star) \in \underline{R}$.* $\square$

Proof 3 (Proof of Lemma 4.6) *We construct $\underline{R}$ analogously to the proof of Lemma 4.4. Now, by Lemma 4.4 and $R : \Sigma_1 \sim \Sigma_2$, we obtain $R : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$. We show: $R : \Sigma_1 \doteqdot_{\bowtie} \Sigma_2$ by showing $R^{-1} : \Sigma_2 \leqslant_{\bowtie^{-1}} \Sigma_1$. From $R : \Sigma_1 \sim \Sigma_2$, we know $R^{-1} : \Sigma_2 \sim \Sigma_1$. Then, applying Lemma 4.4, we get $R^{-1} : \Sigma_2 \leqslant_{\bowtie} \Sigma_1$. Obviously, $\bowtie = \bowtie^{-1}$, and therefore $R^{-1} : \Sigma_2 \leqslant_{\bowtie^{-1}} \Sigma_1$. Now, we apply Lemma 4.5, to conclude $R : \Sigma_1 \leqslant^{\text{init}}_{\bowtie} \Sigma_2$, $R^{-1} : \Sigma_2 \leqslant^{\text{init}}_{\bowtie^{-1}} \Sigma_1$, and finally $R : \Sigma_1 \doteqdot^{\text{init}}_{\bowtie} \Sigma_2$.* $\square$

Proof 4 (Proof of Thm. 4.10) *Let $\Sigma_i = \alpha_{\bowtie[i]}(\mathbb{XY}(N_i))$. By Def. 2.8, there exists R with $R : \mathbb{XY}(N_1) \lesssim^{\text{init}} \mathbb{XY}(N_2)$ and $R^{-1} : \mathbb{XY}(N_2) \lesssim^{\text{init}} \mathbb{XY}(N_1)$. By Lemma 4.8, there exists $\underline{R}$ with $\underline{R} : \Sigma_1 \lesssim^{\text{init}} \Sigma_2$ and $\underline{R}^{-1} : \Sigma_2 \lesssim^{\text{init}} \Sigma_1$. By Lemma 4.6, there exists $\underline{\underline{R}}$ with $\underline{\underline{R}} : \Sigma_1 \leqslant^{\text{init}}_{\bowtie} \Sigma_2$, and thus $N_1 \doteqdot^{\mathbb{XY}}_{\bowtie} N_2$.* $\square$

Proof 5 (Proof of Lemma 4.13)

1. *Follows from $\langle e \rangle_{\theta_1}$ and $b(\langle e \rangle_{\theta_1})$ being both finite and $|\langle e \rangle_{\theta_1}| = |b(\langle e \rangle_{\theta_1})|$.*
2. *Let b_e be a bijection on $\langle e \rangle_{\theta_1}$ and $b(\langle e \rangle_{\theta_1})$ for each $e \in E_1$. Let $c = \bigcup_{e \in E_1} b_e$. Then, c is a bijection on E_1 and E_2, because $\{\langle e \rangle_{\theta_1} \mid e \in E_1\}$ is a partition of E_1, and $\{b(\langle e \rangle_{\theta_1}) \mid e \in E_1\}$ is a partition of E_2.* $\square$

Proof 6 (Proof of Lemma 4.14) *Let $e, e' \in E_1$.*

- *We show: $\beta_{\sigma_1}(e, e') = \beta_{\sigma_2}(c(e), c(e'))$. If $e = e'$, then $\beta_{\sigma_1}(e, e') = \beta_{\sigma_1}(e, e) = \| = \beta_{\sigma_2}(c(e), c(e))$. Let $e \neq e'$. Then, $\langle e \rangle_{\theta_1} \neq \langle e' \rangle_{\theta_1}$ because $|\langle e \rangle_{\theta_1}| = 1$. Then, $\beta_{\sigma_1}(e, e') = \beta_{\theta_1}(\langle e \rangle_{\theta_1}, \langle e' \rangle_{\theta_1}) = \beta_{\theta_2}(b(\langle e \rangle_{\theta_1}), b(\langle e \rangle_{\theta_1}))$. From $c(e) \in b(\langle e \rangle_{\theta_1})$ and $c(e') \in b(\langle e' \rangle_{\theta_1})$, we get: $\beta_{\theta_2}(b(\langle e \rangle_{\theta_1}), b(\langle e \rangle_{\theta_1})) = \beta_{\sigma_2}(c(e), c(e'))$.*
- *We show: $\lambda_{\sigma_1}(e) \bowtie \lambda_{\sigma_2}(c(e))$. By Def. 3.2, $\lambda_{\theta_1}(\langle e \rangle_{\theta_1}) \in \lambda_{\sigma_1}(e)$. Because σ_1 is singleton-labelled, $\{\lambda_{\theta_1}(\langle e \rangle_{\theta_1})\} = \lambda_{\sigma_1}(e)$. By Def. 2.4, $\{\lambda_{\theta_1}(\langle e \rangle_{\theta_1})\} \bowtie \{\lambda_{\theta_2}(b(\langle e \rangle_{\theta_1}))\}$. Because $c(e) \in b(\langle e \rangle_{\theta_1})$, θ_2 is a tactic and σ_2 is singleton labelled, we have $\lambda_{\sigma_2}(c(e)) = \{\lambda_{\theta_2}(b(\langle e \rangle_{\theta_1}))\}$. Thus, $\lambda_{\sigma_1}(e) = \{\lambda_{\theta_1}(\langle e \rangle_{\theta_1})\} \bowtie \{\lambda_{\theta_2}(b(\langle e \rangle_{\theta_1}))\} = \lambda_{\sigma_2}(c(e))$.* $\square$

Proof 7 (Proof of Lemma 4.15) *First, we observe $\varnothing : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$. Hence, there always exists an isotactics relation. Now, we show that any union of TC-relations is a TC-relation. Let J be a set, and R_j be sets, such that $R_j : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$ for all $j \in J$. Let $R = \bigcup_{j \in J} R_j$. Let $r = (\sigma_1, \sigma_2, \theta_1, \theta_2, b) \in R$. Then, $r \in R_j$ for some $j \in J$. Thus, $\theta_j \in \Theta_{\sigma_j}$ and $b : \theta_1 \cong^{\bowtie} \theta_2$. Let E_1 be the events of σ_1. Let $\sigma_1' \in \Sigma_1$ and $\sigma_1' \sqsupseteq \sigma_1$. From $r \in R_j$, we get: There exists $r' = (\sigma_1', \sigma_2', \theta_1', \theta_2', b') \in R_j$ with $\sigma_2' \sqsupseteq \sigma_2$ and $\langle e \rangle_{\theta_1} = \langle e \rangle_{\theta_1'} \cap E_1$ implies $b(\langle e \rangle_{\theta_1}) = b(\langle e \rangle_{\theta_1'}) \cap E_2$ for all $e \in E_1$. From $R_j \subseteq R$ and $r' \in R_j$, we get $r' \in R$. Hence, $R : \Sigma_1 \leqslant_{\bowtie} \Sigma_2$. Finally, we set $R_{\max} := \bigcup_{R : \Sigma_1 \leqslant_{\bowtie} \Sigma_2} R$ the union of all TC-relations.* $\square$