Interleaving Isotactics - An Equivalence Notion on Abstractions of Dynamic Systems

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Abstract

We study the behavioural equivalence of models that capture dynamic systems. We focus on models that are not related by a bijection over their actions, but by an alignment between sets of their actions. For this setting, we propose *interleaving isotactics* as an equivalence notion based on abstractions that are induced by the alignment. We show that this notion is grounded in established notions of behavioural equivalence, provide a temporal logic characterisation of the properties it preserves, and prove decidability of the respective verification problems.

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1 Introduction

The dynamics of a system is often described by a model that defines a set of actions and causal dependencies for their execution. Such models serve as requirements artefacts in system design [8] or as implementations of process-oriented systems [13].

There exists a broad range of applications that require assessing equivalence of two models, e.g., the validation of a system realisation against a specification [7], or the correct implementation of a reference model [14]. It is often the case that models for which equivalence shall be assessed assume different levels of abstraction when capturing systems. In that case, the semantic correspondence between the actions of such models cannot be captured by a bijection. Rather, actions are *grouped* in either model and the groups of one model are *related* to the groups of another model by means of a binary relation, called *alignment*. For example, an action in one model may correspond to a set of actions in another model to capture a hierarchical refinement relation.

In this paper, we target the question of how to assess behavioural equivalence of two models, given an alignment between groups of their actions. This question translates into verifying equivalence based on abstractions, i.e., on the groups of actions induced by the alignment, rather than the individual actions. This setting is illustrated in Fig. 1, which shows two finite state machines along with an alignment that relates three sets of actions of either model to each other. The alignment thereby defines the abstractions for the verification of equivalence, e.g., a number of occurrences of action a is considered to be equivalent to any number of occurrences of actions s and v. Note that both models show equivalent behaviours modulo these abstractions: in model m_1, actions a and $\{b, c\}$ can occur repeatedly and interleaved, and their occurrence is eventually followed by the occurrence of actions $\{d, e\}$.

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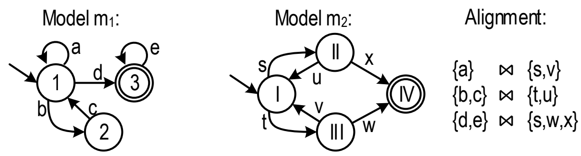

Figure 1 Two finite state machines and an alignment, a relation on sets of their actions.

Model m_2 mirrors this behaviour for the aligned actions. For instance, the run ade of m_1 is mirrored by the run sx of m_2 since a is aligned with s, and both d and e are aligned with x.

Related Work. Behavioural equivalences have been studied for decades, yet, established notions, see [2, 15], are not applicable for the setting outlined above: They impose the assumption of a bijection between the actions of two models. Consequently, the traditional setting of equivalence verification can be seen as a special case of the setting addressed here, requiring that the alignment is of a particular structure.

Alignments that relate groups of actions to each other are related to two types of transformations of behavioural models: on the one hand, action refinement replaces a single action with a set of actions, see place or transition refinements in Petri nets [4]. While refinements that preserve behavioural equivalence have been widely studied [16], these notions do not help to verify if two models show equivalent behaviour under a given alignment. The latter may, for instance, be established automatically using techniques for model matching [3]. On the other hand, action abstraction has been extensively studied in the field of process algebra. In [11], the authors propose vertical bisimulation, which assumes actions of an abstract model to be related to sets of equivalent actions of a detailed model. However, it requires an explicit definition not only of the relation between the actions, but also of their particular occurrences, which limits the applicability of the approach to specific notions of alignments.

Recently, there have been a few attempts to define behavioural equivalence for more general alignments. Specific notions have been proposed for different semantics: interleaving, linear time semantics [17]; interleaving, branching time semantics [18]; and concurrent, linear time semantics [10]. However, all these existing notions are ad-hoc, in the sense that;

I1: It is not known, whether the presented notions are proper generalisations of the well-established behavioural equivalences. That is, if an alignment collapses to a bijection between the actions of two models, does the notion coincides with a known equivalence?

I2: It is not known, what kind of system properties they preserve. That is, if two models show the respective equivalence, is there class of properties (e.g., in terms of temporal logic formulae) that is guaranteed to be preserved?

I3: It is not known, if and under which assumptions any of the notions is decidable.

Contributions. To address the question of how to assess behavioural equivalence of aligned models, we propose the notion of *interleaving isotactics*. It is inspired by earlier ideas on an isotactic notion [10] that, however, suffers from the above mentioned three open issues. Specifically, in this work, we contribute four main results:

C1: A grounding of isotactics in established behavioural equivalences: Trace equivalence and isotactics coincide for simple alignments (bijections over singleton sets of actions) and repetition-free sets of runs.

C2: A Linear Temporal Logic (LTL) characterisation of the system properties preserved by isotactics: given an alignment, we show that tactic-invariant LTL formulae are preserved, regardless of the actual systems.

C3: Decidability of the equivalence: Isotactics is decidable for finite state machines.

C4: Decidability of the logic characterisation: Tactic-invariance of LTL-formulae is decidable.

We proceed by defining formal preliminaries in Section 2. In Section 3, we formalise the setting of alignments of behavioural models and present the notion of isotactics. The main results of this paper are summarised in Section 4. The subsequent sections, Section 5 to Section 7, are devoted to proving the main results. Due to lack of space, the detailed proofs of all results can be found in Appendix A. We close with concluding remarks in Section 8.

2 Formal Framework

Basic Notations. $\mathbb{N}$ denotes the set of all natural numbers including zero. Let $i, j, n \in \mathbb{N}$. Then, by $[i\,..\,j]$ and $[n]$, we denote $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ and $[1\,..\,n]$, respectively. By $\min(i, j)$ and $\max(i, j)$, we denote the minimum and maximum of i and j, respectively. Let A be a set. Then, by $\wp(A)$, $\wp_{>0}(A)$, and $\wp_{=1}(A)$, we denote the power set of A, $\wp(A) \setminus \{\varnothing\}$, and $\{x \in \wp_{>0}(A) \mid \exists\, a \in A : x = \{a\}\}$, respectively. If B is a set, $f : A \to B$ is a function, and $A' \subseteq A$, then by $f(A')$ we denote the set $\{b \in B \mid \exists\, a \in A' : f(a) = b\}$. Let R be a binary relation between A and B. Then, $R^{-1} \coloneqq \{(b, a) \in B \times A \mid (a, b) \in R\}$ denotes the *inverse* of R. Let $\equiv$ be an equivalence relation on A. Then, by $\langle a \rangle_{\equiv}$, where $a \in A$, we denote the equivalence class of A by $\equiv$ that contains a. Moreover, by $A/{\equiv}$ we denote the set of all equivalence classes of A by $\equiv$, i.e., $A/{\equiv} \coloneqq \{x \in \wp_{>0}(A) \mid \exists\, a \in A : \langle a \rangle_{\equiv} = x\}$. We write A^* to denote the set of all (finite) words over A, including the empty word ε. Let $w \coloneqq a_1 \dots a_n \in A^*$ be a word. Then, $w(k) \coloneqq a_k$, where $k \in [n]$, denotes the i-th character of w, and $|w| \coloneqq n$ denotes the length of w. By overloading the notation, we write $[w]$ to denote $[n]$. Let $k \in [w]$, then $w[k\rangle$ denotes the suffix $w(k) \dots w(n)$ of w starting from the k-th character of w. We call w *repetition-free* iff for all $k \in [n-1]$: $w(k) \neq w(k+1)$. We call a set W of words repetition-free, if each of its members $w \in W$ is repetition-free. The concatenation of two words $w = a_1 \dots a_n$ and $w' = a'_1 \dots a'_m$ is defined as $ww' \coloneqq a_1 \dots a_n a'_1 \dots a'_m$.

Traces and Linear Temporal Logic. Let κ be a finite set. A *κ-trace* is an element of $\wp(\kappa)^*$, i.e., a finite sequence of subsets of κ. For $i \in \{1, 2\}$, let κ_i be a set, and let W_i be a set of κ_i-traces. W_1 and W_2 are *trace equivalent* up to a bijection b from κ_1 to κ_2, iff b induces an isomorphism between traces in W_1 and W_2.

▸ **Definition 2.1** (Trace Equivalence). For $i \in \{1, 2\}$, let κ_i be a set, and let W_i be a set of κ_i-traces. Let b be a bijection from κ_1 to κ_2. Then, W_1 and W_2 are *trace equivalent* up to b iff there exists a bijection R from W_1 to W_2 such that for all $w_1 \mathrel{R} w_2$ it holds that (i) $|w_1| = |w_2|$ and (ii) for all $i \in [w_1]$ it holds that $b(w_1(i)) = w_2(i)$.

By $\mathsf{LTL}[\kappa]$, we denote the set of *LTL-Formulae* (without the *next* operator) over a set κ that are given by the following expression:

$$\varphi ::= K \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \,\mathsf{U}\, \varphi, \quad \textit{where} \;\; K \in \kappa.$$

Let $\varphi \in \mathsf{LTL}[\kappa]$ and w be a κ-trace. Then, w *satisfies* φ, written $w \vDash \varphi$, iff:

- $\varphi = K$, $K \in \kappa$, and $K \in w(1)$,
- $\varphi = \neg\psi$, and $w \nvDash \psi$,
- $\varphi = \psi_1 \vee \psi_2$, and there exists $i \in \{1, 2\}$: $w \vDash \psi_i$, or
- $\varphi = \psi_1 \,\mathsf{U}\, \psi_2$, and there exists $k \in [w]$ with $w[k\rangle \vDash \psi_2$ and $w[i\rangle \vDash \psi_1$ for all $i \in [k-1]$.

As usual, conjunction of formulae ($\psi_1 \wedge \psi_2$) and the temporal operators *eventually* $\mathsf{F}\,(\psi)$ and *globally* $\mathsf{G}\,(\psi)$ are defined based on the above operators.

Finite State Machines. We rely on a common notion of finite state machines as follows:

▸ **Definition 2.2** (Finite State Machine). A *finite state machine* (FSM) is a 5-tuple $S \coloneqq (\mathcal{Q}, \Lambda, \rightarrow, q^{ini}, F)$, where $\mathcal{Q}$ is a finite set of *states*, Λ is a set of *labels*, $\mathcal{Q}$ and Λ are disjoint, $\rightarrow \subseteq \mathcal{Q} \times \Lambda \times \mathcal{Q}$ is the *transition relation*, $q^{ini} \in \mathcal{Q}$ is the *initial state*, and $F \subseteq \mathcal{Q}$ is the set of *final states*.

By $p \xrightarrow{\lambda} q$, where $p, q \in \mathcal{Q}$ and $\lambda \in \Lambda$, we denote the fact that $(p, \lambda, q) \in \rightarrow$. We say that S *accepts* a word $w \in \Lambda^*$, iff (i) w is the empty word, i.e., $w = \varepsilon$, and $q^{ini} \in F$, or (ii) there exists a sequence $\rho \in \mathcal{Q}^*$ of states of length $|w| + 1$, such that $\rho(1) = q^{ini}$, $\rho(|w| + 1) \in F$, and for all $i \in [w]$ it holds that $\rho(i) \xrightarrow{w(i)} \rho(i+1)$; we refer to a word that an FSM accepts as its *run*.

By $\mathcal{L}(S)$, we denote the *language* of S, i.e., the set of all words that S accepts. Two FSMs S_1 and S_2 are *language-equivalent* iff $\mathcal{L}(S_1) = \mathcal{L}(S_2)$. Two FSMs are illustrated in Fig. 1, using common notation. For instance, FSM $m_1 \coloneqq (\mathcal{Q}, \Lambda, \rightarrow, q^{ini}, F)$ has three states, $\mathcal{Q} = \{1, 2, 3\}$, with the initial state $q^{ini} = 1$ and the set of final states $F = \{3\}$. Its set of labels is $\Lambda = \{a, b, c, d, e\}$ and the transition relation is $1 \xrightarrow{a} 1$, $1 \xrightarrow{b} 2$, $2 \xrightarrow{c} 1$, $2 \xrightarrow{d} 3$, and $3 \xrightarrow{e} 3$.

3 Equivalence of Aligned Behavioural Models

We consider the dynamics of a system in terms of interleaving, linear time semantics, i.e., as a *set of runs*, where a *run* is a finite sequence over a set of *labels*. A run is produced by a system model, such as an FSM, Petri net, or Turing machine. Against the background of describing the dynamics of systems, a label often represents an action. However, other interpretations are possible: A label could represent a state, a state predicate, a configuration, or a similar concept. The basis for aligning system models comprises:

1. *Grouping* the labels. The groups may overlap, i.e., a label may be a member of more than one group. In addition, one can eliminate labels by not assigning them to any group.
2. *Relating* the groups of one model with the groups of another model by means of a binary relation called *alignment*.

Based thereon, we can assess behavioural equivalence of two models by:

1. *Abstracting* each model based on the groups.
2. *Verifying* whether the abstract models are behaviourally equivalent.

3.1 Groupings and Alignments

This section defines the basic notions of a grouping of a set of labels and an alignment between sets of labels.

▸ **Definition 3.1** (Grouping). A *grouping* of a set of labels Λ is a set $\kappa \subseteq \wp_{>0}(\Lambda)$. A set $K \in \kappa$ is a *κ-group* of Λ.

For example, $\gamma_1 \coloneqq \{\{a\}, \{b, c\}, \{d, e\}\}$ and $\gamma_2 \coloneqq \{\{s, v\}, \{t, u\}, \{s, w, x\}\}$ are groupings of the sets of labels $\Gamma_1 \coloneqq \{a, b, c, d, e\}$ and $\Gamma_2 \coloneqq \{s, t, u, v, w, x\}$, respectively. Given a grouping κ of Λ and a label $\lambda \in \Lambda$, by $\mathcal{G}_\kappa(\lambda)$ we denote the set of all κ-groups that contain λ, i.e., $\mathcal{G}_\kappa(\lambda) \coloneqq \{K \in \kappa \mid \lambda \in K\}$. For example, it holds that $\mathcal{G}_{\gamma_2}(s) = \{\{s, v\}, \{s, w, x\}\}$.

▸ **Definition 3.2** (Alignment). For $i \in \{1, 2\}$, let Λ_i be a set of labels, and let $\kappa_i \subseteq \wp_{>0}(\Lambda_i)$ be a grouping of Λ_i. Then, a relation $\bowtie \subseteq \kappa_1 \times \kappa_2$ is an *alignment* between Λ_1 and Λ_2, written $\bowtie : \Lambda_1 \otimes \Lambda_2$, relating the κ_1-groups of Λ_1 with the κ_2-groups of Λ_2.

For every binary relation $\bowtie \subseteq \wp_{>0}(\Lambda_1) \times \wp_{>0}(\Lambda_2)$, we have $\bowtie : \Lambda_1 \otimes \Lambda_2$, implicitly providing the groupings $\overset{i}{\bowtie} \coloneqq \{K_i \in \wp_{>0}(\Lambda_i) \mid K_1 \bowtie K_2\}$ of Λ_i, $i \in \{1, 2\}$. For example, $\{(\{a\}, \{s, v\}), (\{b, c\}, \{t, u\}), (\{d, e\}, \{s, w, x\})\} \subseteq \gamma_1 \times \gamma_2$ is an alignment between Γ_1 and Γ_2 used in Fig. 1.

3.2 Comparing Traces based on Tactics

Let Λ be a set of labels, and let κ be a grouping of Λ. Every run $\sigma \in \Lambda^*$ induces the κ-trace $w := \mathcal{G}_\kappa(\sigma(1))\ldots\mathcal{G}_\kappa(\sigma(|\sigma|))$. Then, the κ-induced trace of σ is a sequence of sets of κ-groups obtained from w by removing all its elements that are equal to $\varnothing$ without changing the order of the remaining elements.

▸ **Definition 3.3** (Induced Trace). Let Λ be a set of labels, let κ be a grouping of Λ, and let $\mathcal{T}_\kappa : \Lambda^* \to \wp_{>0}(\kappa)^*$ be given by:

- $\mathcal{T}_\kappa(\varepsilon) := \varepsilon$.
- Let $\sigma \in \Lambda^*$, $\lambda \in \Lambda$. Then, $\mathcal{T}_\kappa(\sigma\lambda) := \begin{cases} \mathcal{T}_\kappa(\sigma)\mathcal{G}_\kappa(\lambda) & \text{if } \mathcal{G}_\kappa(\lambda) \neq \varnothing, \\ \mathcal{T}_\kappa(\sigma) & \text{otherwise.} \end{cases}$

Given a run $\sigma \in \Lambda^*$, $\mathcal{T}_\kappa(\sigma)$ is the *κ-induced trace* of σ.

Let Λ be a set of labels, let $\kappa \subseteq \wp_{>0}(\Lambda)$ be a grouping, let $\Sigma \subseteq \Lambda^*$, let $\sigma \in \Sigma$, and let $\varphi \in \mathsf{LTL}[\kappa]$. Then, σ *satisfies* φ w.r.t. κ, denoted by $(\sigma,\kappa) \vDash \varphi$, iff $\mathcal{T}_\kappa(\sigma) \vDash \varphi$. Similarly, Σ *satisfies* φ w.r.t. κ, denoted by $(\Sigma,\kappa) \vDash \varphi$, iff for every $\sigma \in \Sigma$ it holds that $(\sigma,\kappa) \vDash \varphi$.

For example, given the run $\sigma := tvsx$ of model m_2 in Fig. 1, it holds that $\{\{t,u\}\}\{\{s,v\}\}\{\{s,v\},\{s,w,x\}\}\{\{s,w,x\}\}$ is the γ_2-induced trace of σ, where γ_2 is defined in Section 3.1. Moreover, for the example formula $\varphi := (\{s,v\} \vee \{t,u\}) \,\mathsf{U}\, \{s,w,x\}$, we have $(\sigma,\gamma_2) \vDash \varphi$.

We compare traces based on their *tactics*. Let κ be a grouping of a set of labels Λ, and let $w \in \wp_{>0}(\kappa)^*$ be a κ-trace. A *tactic* θ of w selects some $K \in w(i)$ for each index i of w.

▸ **Definition 3.4** (Tactic). A *tactic* of a κ-trace $w \in \wp_{>0}(\kappa)^*$, where κ is a finite set, is a κ-trace $\theta \in \wp_{=1}(\kappa)^*$ such that $|\theta| = |w|$ and $\theta(i) \subseteq w(i)$ for all $i \in [w]$.

For example, the γ_2-induced trace $\mathcal{T}_{\gamma_2}(tvsx)$ of model m_2 in Fig. 1 has two tactics θ_1 and θ_2, with $\theta_1 := \{\{t,u\}\}\{\{s,v\}\}\{\{s,v\}\}\{\{s,w,x\}\}$ and $\theta_2 := \{\{t,u\}\}\{\{s,v\}\}\{\{s,w,x\}\}\{\{s,w,x\}\}$. Each tactic θ induces an equivalence relation $=_\theta$ and a strict partial order $<_\theta$ on the set $[\theta]$ of indices of θ. Two indices are equivalent, if θ selects the same $K \in \kappa$ for them and all indices between them. Two indices are ordered if they are ordered inside θ but are not equivalent.

▸ **Definition 3.5** (Tactic-induced Relations). Let θ be a tactic of a κ-trace $w \in \wp_{>0}(\kappa)^*$, where κ is a finite set. Then, $=_\theta \subseteq [\theta] \times [\theta]$ and $<_\theta \subseteq [\theta] \times [\theta]$ are defined as follows:

1. $i =_\theta j$ iff for all $k,l \in [\theta]$ such that $\min(i,j) \leq k,\ell \leq \max(i,j)$ it holds that $\theta(k) = \theta(\ell)$.
2. $i <_\theta j$ iff $i < j$ and $i \neq_\theta j$.

Obviously, $=_\theta$ is an equivalence relation. We abbreviate $\langle i \rangle_{=_\theta}$ and $[\theta]/{=_\theta}$ as $\langle i \rangle_\theta$ and $[\theta]/_\theta$, respectively. Let $i,j \in [\theta]$, such that $i <_\theta j$. Then, for all $i' \in \langle i \rangle_\theta$ and $j' \in \langle j \rangle_\theta$, it holds that $i' <_\theta j'$. Hence, one can lift $<_\theta$ from $[\theta]$ to $[\theta]/_\theta$: For all $i,j \in [\theta]$, $\langle i \rangle_\theta <_\theta \langle j \rangle_\theta$ iff $i <_\theta j$.

For example, the tactic $\theta_1 := \{\{t,u\}\}\{\{s,v\}\}\{\{s,v\}\}\{\{s,w,x\}\}$ induces the equivalence relation $=_{\theta_1}$ given by $\{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2)\}$; $=_{\theta_1}$ induces three equivalence classes: $\{1\}$, $\{2,3\}$, and $\{4\}$. Moreover, $<_{\theta_1}$ is given by $\{(1,2),(1,3),(1,4),(2,4),(3,4)\}$. Thus, it holds that $\{1\} <_{\theta_1} \{2,3\} <_{\theta_1} \{4\}$.

Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let $w_i \in \wp(\overset{i}{\bowtie})^*$. Intuitively, w_1 and w_2 are *aligned* if there exist tactics θ_1 and θ_2 of w_1 and w_2, respectively, that can be aligned.

▸ **Definition 3.6** (Alignment of Tactics). Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let $w_i \in \wp(\overset{i}{\bowtie})^*$, and let θ_i be a tactic of w_i. Tactics θ_1 and θ_2 are *aligned* by $\bowtie$, denoted by $\theta_1 \bowtie \theta_2$, iff there exists a bijection b from $[\theta_1]/_{\theta_1}$ to $[\theta_2]/_{\theta_2}$ such that:

1. for all $i_1 \in [\theta_1]$ and $i_2 \in [\theta_2]$ it holds that $b(\langle i_1 \rangle_{\theta_1}) = \langle i_2 \rangle_{\theta_2}$ implies $\theta_1(i_1) \bowtie \theta_2(i_2)$, and
2. for all $i,j \in [\theta_1]$ it holds that $\langle i \rangle_{\theta_1} <_{\theta_1} \langle j \rangle_{\theta_1}$ implies $b(\langle i \rangle_{\theta_1}) <_{\theta_2} b(\langle j \rangle_{\theta_1})$.

We say that w_1 and w_2 are *aligned* by θ_1 and θ_2 w.r.t. $\bowtie$, and denote this by $(\theta_1,\theta_2) : w_1 \bowtie w_2$.

For example, consider the γ_1-induced trace $\mathcal{T}_{\gamma_1}(bcad) = \{\{b,c\}\}\{\{b,c\}\}\{\{a\}\}\{\{d,e\}\}$ and the γ_2-induced trace $\mathcal{T}_{\gamma_2}(tvsx) = \{\{t,u\}\}\{\{s,v\}\}\{\{s,v\},\{s,w,x\}\}\{\{s,w,x\}\}$ of models m_1 and m_2 in Fig. 1, respectively. $\mathcal{T}_{\gamma_1}(bcad)$ and $\mathcal{T}_{\gamma_2}(tvsx)$ are aligned by tactics $\theta_1 := \{\{b,c\}\}\{\{b,c\}\}\{\{a\}\}\{\{d,e\}\}$ and $\theta_2 := \{\{t,u\}\}\{\{s,v\}\}\{\{s,w,x\}\}\{\{s,w,x\}\}$ w.r.t. the alignment in Fig. 1; one can construct bijection $b := \{\{1,2\} \mapsto \{1\}, \{3\} \mapsto \{2\}, \{4\} \mapsto \{3,4\}\}$ from $[\theta_1]/_{\theta_1}$ to $[\theta_2]/_{\theta_2}$. In contrast, $\mathcal{T}_{\gamma_1}(bcad)$ and $\mathcal{T}_{\gamma_2}(tw) = \{\{t,u\}\}\{\{s,w,x\}\}$ cannot be aligned. However, $\mathcal{T}_{\gamma_1}(bcd) = \{\{b,c\}\}\{\{b,c\}\}\{\{d,e\}\}$ and $\mathcal{T}_{\gamma_2}(tw)$ can be aligned. Finally, illustrating that alignments abstract from cardinalities, for every $n \in \mathbb{N}$ it holds that $\mathcal{T}_{\gamma_1}(\underbrace{bc\ldots bc}_{n \text{ times}}d)$ can be aligned to $\mathcal{T}_{\gamma_2}(tw)$.

3.3 Tactic Coverage and Isotactics

We compare sets of traces based on *tactic coverage* and *isotactics* relations.

▸ **Definition 3.7** (Tactic Coverage and Isotactics). Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let W_i be a set of $\overset{i}{\bowtie}$-traces.

1. W_1 and W_2 are in the *(interleaving) tactic coverage relation* w.r.t $\bowtie$, denoted by $W_1 \leqslant_{\bowtie} W_2$, iff for every trace $w_1 \in W_1$ there exists a trace $w_2 \in W_2$ such that there exist tactics θ_1 and θ_2 of w_1 and w_2, respectively, such that $(\theta_1,\theta_2) : w_1 \bowtie w_2$.
2. W_1 and W_2 are in the *(interleaving) isotactics relation* w.r.t $\bowtie$, denoted by $W_1 \doteqdot_{\bowtie} W_2$, iff $W_1 \leqslant_{\bowtie} W_2$ and $W_2 \leqslant_{\bowtie^{-1}} W_1$.

One can compare sets of runs by using their induced traces. For $i \in \{1,2\}$, let Λ_i be a set of labels, let $\Sigma_i \subseteq \Lambda_i^*$ be a set of runs, and let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. We write $\Sigma_1 \leqslant_{\bowtie} \Sigma_2$ and $\Sigma_1 \doteqdot_{\bowtie} \Sigma_2$ to denote the facts that $\mathcal{T}_{\overset{1}{\bowtie}}(\Sigma_1) \leqslant_{\bowtie} \mathcal{T}_{\overset{2}{\bowtie}}(\Sigma_2)$ and $\mathcal{T}_{\overset{1}{\bowtie}}(\Sigma_1) \doteqdot_{\bowtie} \mathcal{T}_{\overset{2}{\bowtie}}(\Sigma_2)$, respectively.

Let Δ be the universe of labels and let $\bowtie$ be an alignment over Δ. If $\bowtie$ is an equivalence relation, then it clearly holds that $\doteqdot_{\bowtie}$ is an equivalence relation.

Considering m_1 and m_2 from Fig. 1, for every γ_1-induced trace of m_1, there exists some alignable γ_2-induced trace of m_2, and vice versa. Hence, it holds that $\mathcal{L}(m_1) \doteqdot_{\bowtie} \mathcal{L}(m_2)$.

3.4 Alignments and LTL

Every tactic θ of a κ-trace w is a κ-trace. Thus, every formula $\varphi \in \mathsf{LTL}[\kappa]$ can be evaluated in both w and θ. For some φ, the truth value of φ is the same for w and each of its tactics θ. In this case, we call φ *tactic-invariant.*

▸ **Definition 3.8** (Tactic-Invariant LTL-Formula). Let Λ be a set of labels and let $\kappa \subseteq \wp_{>0}(\Lambda)$ be a grouping of Λ. An LTL-formula $\varphi \in \mathsf{LTL}[\kappa]$ is *tactic-invariant* w.r.t. Λ and κ, iff for every $w \in \mathcal{T}_\kappa(\Lambda^*)$ and every tactic θ of w it holds that $w \vDash \varphi$ iff $\theta \vDash \varphi$.

For instance, consider the alignment $\bowtie$ from Fig. 1, and the following $\mathsf{LTL}[\overset{2}{\bowtie}]$-formulae:

$$\varphi_1 := \{t,u\} \,\mathsf{U}\, (\{s,v\} \vee \{s,w,x\}), \quad \varphi_2 := \neg\{s,v\} \vee \neg\{s,w,x\}, \quad \text{and} \quad \varphi_3 := \neg\varphi_2.$$

Clearly, φ_1 is tactic-invariant: The satisfying $\overset{2}{\bowtie}$-induced traces are of the form $w_1w_2w_3$ where w_1 matches $\{\{t,u\}\}^*$, w_2 is a non-empty subset of $\{\{s,v\},\{s,w,x\}\}$, and w_3 is an arbitrary $\overset{2}{\bowtie}$-induced trace. Considering the tactics of such traces, they are of the form $\theta_1\theta_2\theta_3$, where $\theta_1 = w_1$ because w_1 is a sequence of singleton sets, and θ_2 always satisfies $\{s,v\} \vee \{s,w,x\}$. Similarly, we argue that every φ-satisfying tactic of an arbitrary $\overset{2}{\bowtie}$-induced trace w is of the form $\theta_1\theta_2\theta_3$, and we can show that w is then also of the form $w_1w_2w_3$ as above, yielding satisfaction of φ_1. In contrast to that, φ_2 and φ_3 are not tactic-invariant: As proof, we take the $\overset{2}{\bowtie}$-induced trace $w = \{\{s,v\},\{s,w,x\}\}$ and tactic $\theta = \{\{s,v\}\}$ of w. Then, $w \nvDash \varphi_2$ but $\theta \vDash \varphi_2$, and $w \vDash \varphi_3$ and $\theta \nvDash \varphi_3$.

Given an alignment $\bowtie$ between sets κ_1 and κ_2, every formula $\varphi_1 \in \mathsf{LTL}[\kappa_1]$ is *aligned* to a similar formula $\varphi_2 \in \mathsf{LTL}[\kappa_2]$, viz. φ_1 and φ_2 are aligned if φ_1 and φ_2 have the same structure, and φ_2 replaces each disjunction of atomic propositions from φ_1 with an aligned disjunction of atomic propositions.

▸ **Definition 3.9** (Alignment of LTL-Formulae). Let $\bowtie$ be an alignment. LTL-Formulae $\varphi_1 \in \mathsf{LTL}[\overset{1}{\bowtie}]$ and $\varphi_2 \in \mathsf{LTL}[\overset{2}{\bowtie}]$ are aligned by $\bowtie$, denoted by $\varphi_1 \bowtie \varphi_2$, iff for every $i \in \{1,2\}$ at least one of the following holds:

- $\varphi_i = \bigvee_{K_i \in \mathcal{K}_i} K_i$, where $\mathcal{K}_i \subseteq \overset{i}{\bowtie}$ and $(\mathcal{K}_1 \times \overset{2}{\bowtie}) \cap \bowtie = (\overset{1}{\bowtie} \times \mathcal{K}_2) \cap \bowtie$.
- $\varphi_i = \neg\psi_i$ and $\psi_1 \bowtie \psi_2$.
- $\varphi_i = \psi_i * \psi_i'$, where $* \in \{\vee, \mathsf{U}\}$, $\psi_1 \bowtie \psi_2$, and $\psi_1' \bowtie \psi_2'$.

Given formulae $\psi_1 := (\{a\} \vee \{b,c\}) \,\mathsf{U}\, \{d,e\}$ and $\psi_2 := (\{s,v\} \vee \{t,u\}) \,\mathsf{U}\, \{s,w,x\}$ for models m_1 and m_2 of Fig. 1, respectively, it holds that $\psi_1 \bowtie \psi_2$.

4 Main Results

This section summarizes the main results of this paper. For each proposed formal statement, we refer to the Lemmas which prove the statement in the subsequent sections.

Once an alignment collapses to a bijection between the labels of two sets of runs, isotactics can be grounded in well-established notions of behavioural equivalence for repetition-free sets of runs. Furthermore, isotactics preserves LTL-formulae that have the same truth value in all tactics of a trace, i.e., that are tactic-invariant.

▸ **Theorem 4.1** (Trace equivalence and isotactics for simple alignments and repetition-free runs). *For $i \in \{1,2\}$, let Λ_i be a set of labels and let $\Sigma_i \subseteq \Lambda_i^*$ be repetition-free. Let b be a bijection from Λ_1 to Λ_2, and let $\bowtie := \{(\{\lambda\},\{b(\lambda)\}) \mid \lambda \in \Lambda_1\}$. Then, the following statements are equivalent:*

1. *$\mathcal{T}_{\overset{1}{\bowtie}}(\Sigma_1)$ and $\mathcal{T}_{\overset{2}{\bowtie}}(\Sigma_2)$ are trace equivalent up to b.*
2. *$\Sigma_1 \div_{\bowtie} \Sigma_2$.*

Proof. Follows from Lemmas 5.1 and 5.2. ◂

▸ **Theorem 4.2** (Tactic-invariant LTL-formulae are preserved). *Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let $\Sigma_i \subseteq \Lambda_i^*$ and let $\varphi_i \in \mathsf{LTL}[\overset{i}{\bowtie}]$ be tactic-invariant w.r.t. Λ_i and $\overset{i}{\bowtie}$. Let $\varphi_1 \bowtie \varphi_2$ and let $\Sigma_1 \div_{\bowtie} \Sigma_2$. Then, it holds that $(\Sigma_1, \overset{1}{\bowtie}) \vDash \varphi_1$ iff $(\Sigma_2, \overset{2}{\bowtie}) \vDash \varphi_2$.*

Proof. Follows from Definition 3.7 and Lemma 6.1. ◂

Given an LTL-formula, it is decidable whether it is tactic-invariant. Moreover, for two FSMs, it is decidable whether their languages are in the interleaving isotactics relation.

▸ **Theorem 4.3** (Tactic-invariance of LTL-formula is decidable). *Let Λ be a set of labels, let $\kappa \subseteq \wp_{>0}(\Lambda)$ be a grouping, and let $\varphi \in \mathsf{LTL}[\kappa]$. Then, the following problem is decidable: To decide whether φ is tactic-invariant w.r.t. Λ and κ.*

Proof. Follows from Lemmas 6.2 and 6.3. ◂

▸ **Theorem 4.4** (Isotactics is decidable for FSMs). *For $i \in \{1,2\}$, let Λ_i be a set of labels, S_i be an FSM over Λ_i. Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. Then, the following problem is decidable: To decide whether $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ are in the interleaving isotactics relation w.r.t $\bowtie$, i.e., to decide whether it holds that $\mathcal{L}(S_1) \div_{\bowtie} \mathcal{L}(S_2)$.*

Proof. Follows from Lemmas 7.7 and 7.8. ◂

5 Conditional Coincidence with Trace Equivalence

We prove two statements that justify the conditional coincidence of trace equivalence and isotactics stated in Theorem 4.1. Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment such that $\bowtie$ is a bijection from the singletons over Λ_1 to the singletons over Λ_2. In the proofs, we exploit the fact that every trace $w := \mathcal{T}_{\overset{i}{\bowtie}}(\sigma)$, where $i \in \{1,2\}$ and $\sigma \in \Lambda_i^*$, is a sequence of singletons and, thus, has exactly one tactic, namely w. First, we show that trace equivalence implies isotactics.

▸ **Lemma 5.1.** *For $i \in \{1,2\}$, let Λ_i be a set of labels and let $\Sigma_i \subseteq \Lambda_i^*$. Let b be a bijection from Λ_1 to Λ_2, and let $\bowtie := \{(\{\lambda\},\{b(\lambda)\}) \mid \lambda \in \Lambda_1\}$. If $\mathcal{T}_{\overset{1}{\bowtie}}(\Sigma_1)$ and $\mathcal{T}_{\overset{2}{\bowtie}}(\Sigma_2)$ are trace equivalent up to b, then it holds that $\Sigma_1 \div_{\bowtie} \Sigma_2$.*

Next, we demonstrate the converse of Lemma 5.1 for the case when Σ_1 and Σ_2 are repetition-free. If Σ_1 and Σ_2 are repetition-free, then the traces induced by the runs in Σ_1 and Σ_2 are repetition-free as well. Thus, every equivalence class of an equivalence relation induced by a tactic of any of the induced traces is a singleton.

▸ **Lemma 5.2.** *For $i \in \{1,2\}$, let Λ_i be a set of labels and let $\Sigma_i \subseteq \Lambda_i^*$ be repetition-free. Let b be a bijection from Λ_1 to Λ_2, and let $\bowtie := \{(\{\lambda\},\{b(\lambda)\}) \mid \lambda \in \Lambda_1\}$. If $\Sigma_1 \div_{\bowtie} \Sigma_2$, then it holds that $\mathcal{T}_{\overset{1}{\bowtie}}(\Sigma_1)$ and $\mathcal{T}_{\overset{2}{\bowtie}}(\Sigma_2)$ are trace equivalent up to b.*

6 Property Preservation

Below, we prove a statement using which one can justify Theorem 4.2. In particular, we demonstrate that given two aligned traces w_1 and w_2, it holds that $w_1 \models \varphi_1$ iff $w_2 \models \varphi_2$, where φ_1 and φ_2 are two aligned tactic-invariant LTL-formulae.

▸ **Lemma 6.1.** *Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let $\sigma_i \in \Lambda_i^*$, $w_i := \mathcal{T}_{\overset{i}{\bowtie}}(\sigma_i)$, θ_i be a tactic of w_i, and let $\varphi_i \in \mathsf{LTL}[\overset{i}{\bowtie}]$ be tactic-invariant w.r.t. Λ_i and $\overset{i}{\bowtie}$. Let φ_1 and φ_2 be aligned by $\bowtie$, i.e., $\varphi_1 \bowtie \varphi_2$, and let $(\theta_1,\theta_2) : w_1 \bowtie w_2$. Then, it holds that $w_1 \models \varphi_1$ iff $w_2 \models \varphi_2$.*

For instance, consider the alignment $\bowtie$ from Fig. 1, the $\mathsf{LTL}[\overset{1}{\bowtie}]$-formulae ψ_1, ψ_2 and ψ_3, and $\mathsf{LTL}[\overset{2}{\bowtie}]$-formulae φ_1, φ_2 and φ_3:

$$\begin{aligned}
&\psi_1 := \{b,c\}\,\mathsf{U}\,(\{a\} \lor \{d,e\}), && \psi_2 := \neg\{a\} \lor \neg\{d,e\}, && \psi_3 := \neg\psi_2,\\
&\varphi_1 := \{t,u\}\,\mathsf{U}\,(\{s,v\} \lor \{s,w,x\}), && \varphi_2 := \neg\{s,v\} \lor \neg\{s,w,x\}, && \varphi_3 := \neg\varphi_2.
\end{aligned}$$

Then, for all $i \in [3]$, we have $\psi_i \bowtie \varphi_i$. Because the sets in $\overset{1}{\bowtie}$ are disjoint, ψ_1, ψ_2 and ψ_3 are tactic-invariant. As explained in Section 3.4, φ_1 is tactic-invariant, but φ_2 and φ_3 are not. Therefore, ψ_1 is preserved as φ_1: Let w_1 be a $\overset{1}{\bowtie}$-induced trace of the form $\{\{b,c\}\}^*(\{\{a\}\}|\{\{d,e\}\})$. Then, $w_1 \models \psi_1$. Assume now some w_2 alignable to w_1. Then, w_2 has the form $\{\{t,u\}\}^*(\{\{s,v\}\}|\{\{s,w,x\}\}|\{\{s,v\},\{s,w,x\}\})^+$, and thus $w_2 \models \varphi_1$. In contrast to that, consider the $\overset{1}{\bowtie}$-induced trace $w_1' = \{\{a\}\}$ and the alignable $\overset{2}{\bowtie}$-induced trace $w_2' = \{\{s,v\},\{s,w,x\}\}$: Then, $w_1' \models \psi_2$ but $w_2' \not\models \psi_2$, and $w_1' \not\models \psi_3$ but $w_2' \models \psi_3$.

We decide tactic-invariance by reducing it to checking language equivalence, exploiting that every LTL-formula can be translated into an according FSM. Let $\kappa \subseteq \wp_{>0}(\Lambda)$ be a grouping. The main idea is to translate some $\varphi \in \mathsf{LTL}[\kappa]$ to an FSM S_φ. Then, we can intersect S_φ with the sets of traces and tactics over κ, respectively, and compare the results. For the remainder of this section, we fix a grouping κ and a formula $\varphi \in \mathsf{LTL}[\kappa]$. For some κ-trace w, we write $\mathrm{Tactics}(w)$ for the set of all tactics of w.

As a starting point for the decidability proof, we introduce the helper sets $W \coloneqq \{w \in \mathcal{T}_\kappa(\Lambda^*) \mid w \vDash \varphi\}$ and $\Theta \coloneqq \{\theta \in \text{Tactics}(w) \mid w \in W, \theta \vDash \varphi\}$, denoting the sets of all φ-satisfying κ-induced traces, and φ-satisfying tactics of κ-induced traces, respectively. For the purpose of our decision procedure, a given tactic itself does not carry sufficient information about its "origin": We can have two κ-induced traces w, w' with common tactics, that is, $\text{Tactics}(w) \cap \text{Tactics}(w') \neq \varnothing$. For instance, the $\overset{2}{\bowtie}$-induced traces $w = \{\{t,u\}\}\{\{s,v\},\{s,w,x\}\}$ and $w' = \{\{t,u\}\}\{\{s,v\}\}$ have the common tactic $\theta = w'$. Therefore, we encode an original trace into the tactic as follows: For each tactic $\theta \in \text{Tactics}(w)$ of some κ-induced trace w, we define the sequence $\theta_w \in (\wp_{=1}(\kappa) \times \wp_{>0}(\kappa))^*$ of pairs of sets of groups by $\theta_w(i) \coloneqq (\theta(i), w(i))$ for all $i \in [w]$. That is, each $\theta_w(i)$ contains the chosen group of the particular tactic, and the set of groups this group has been chosen from. For instance, for $\overset{2}{\bowtie}$-induced trace $w = \{\{t,u\}\}\{\{s,v\},\{s,w,x\}\}$ and tactic $\theta = \{\{t,u\}\}\{\{s,v\}\}$, we have $\theta_w = (\{\{t,u\}\},\{\{t,u\}\})(\{\{s,v\}\},\{\{s,v\},\{s,w,x\}\})$.

Based on the notion of θ_w, we define the sets $\hat{\Theta} \coloneqq \{\theta_w \mid w \in W, \theta \in \text{Tactics}(w)\}$ and $\hat{\Theta}' \coloneqq \{\theta_w \mid \theta \in \Theta, w \in \mathcal{T}_\kappa(\Lambda^*)\}$ built from W and Θ, respectively. That is, $\hat{\Theta}$ contains the enriched versions of all tactics of φ-satisfying traces, and $\hat{\Theta}'$ contains the enriched versions of all φ-satisfying tactics of arbitrary traces.

Now we show that tactic-invariance coincides with equality of $\hat{\Theta}$ and $\hat{\Theta}'$: If φ is tactic-invariant, then for every trace w and tactic $\theta \in \text{Tactics}(w)$, it holds $w \in W$ iff $\theta \in \Theta$. Otherwise, there exist a trace w and a tactic $\theta \in \text{Tactics}(w)$ with $w \in W$ iff $\theta \notin \Theta$.

▸ **Lemma 6.2.** $\hat{\Theta} = \hat{\Theta}'$ *iff* φ *is tactic-invariant w.r.t.* Λ *and* κ.

For example, consider the $\overset{2}{\bowtie}$-induced trace $w = \{\{s,v\},\{s,w,x\}\}$ and tactic $\theta = \{\{s,v\}\}$ of w yielding $\theta_w = (\{\{s,v\}\},\{\{s,v\},\{s,w,x\}\})$. As mentioned above, $w \nvDash \varphi_2$ but $\theta \vDash \varphi_2$, and $w \vDash \varphi_3$ and $\theta \nvDash \varphi_3$. Taking $\varphi = \varphi_2$, we find $\theta_w \notin \hat{\Theta}$ (because $w \notin W$) but $\theta_w \in \hat{\Theta}'$ (because $\theta \vDash \varphi_2$). Similarly, for $\varphi = \varphi_3$, we find $\theta_w \in \hat{\Theta}$ and $\theta_w \notin \hat{\Theta}'$.

Now, we reduce deciding $\hat{\Theta} = \hat{\Theta}'$ to language equivalence of FSMs, thus showing decidability of the equality—and thus also tactic invariance. The idea is to construct FSMs $\hat{S}_\Theta$ and $\hat{S}'_\Theta$ accepting $\hat{\Theta}$ and $\hat{\Theta}'$, respectively. To this end, we first construct FSMs S_W and S_Θ accepting W and Θ, respectively; exploiting that φ can be encoded as an FSM. Then, we unfold the transitions of S_W and S_Θ to obtain $\hat{S}_\Theta$ and $\hat{S}'_\Theta$, respectively.

▸ **Lemma 6.3.** *It is decidable whether* $\hat{\Theta}$ *and* $\hat{\Theta}'$ *are equal sets.*

For formula φ_1, Appendix B.1 exemplifies the construction of S_φ, S_W, S_Θ, $\hat{S}_\Theta$, and $\hat{S}'_\Theta$.

Deciding tactic-invariance of φ requires at most exponential space w.r.t. to the size of φ and $\bowtie$: The FSM S_φ can be computed in EXPTIME and has exponential size w.r.t. φ and $\bowtie$. The FSMs $\hat{S}_\Theta$ and $\hat{S}'_\Theta$ can be computed from S_φ in polynomial time; the resulting FMSs having exponential size. Finally, the equivalence check requires polynomial space w.r.t. the size of the exponential-sized FSMs.

7 Deciding Isotactics

We show the decidability of isotactics for languages produced by FSMs S_1 and S_2 w.r.t. an alignment $\bowtie : \Lambda_1 \otimes \Lambda_2$. The idea of our approach is to compute a product $\mathcal{W}(S_1, S_2, \bowtie)$ of S_1 and S_2 w.r.t. $\bowtie$, called *witness graph*, and then to reduce deciding isotactics to two language equivalence checks between $\mathcal{W}(S_1, S_2, \bowtie)$ and each of the two FSMs.

To simplify subsequent discussions, we assume that S_1 and S_2 are *deterministic* w.r.t. $\overset{1}{\bowtie}$ and $\overset{2}{\bowtie}$, respectively. Here, determinism is defined on the level of the grouping. In addition, determinism requires the absence of labels which do not participate in any group.

▸ **Definition 7.1** (Deterministic FSM). An FSM $S := (\mathcal{Q}, \Lambda, \rightarrow, q^{ini}, F)$ is *deterministic* w.r.t. a grouping κ of Λ, iff for all states $q \in \mathcal{Q}$ and for all transitions $q \xrightarrow{\lambda_1} q_1$, $q \xrightarrow{\lambda_2} q_2$ of S it holds that: (i) $\mathcal{G}_\kappa(\lambda_1) \neq \varnothing$, and (ii) $\mathcal{G}_\kappa(\lambda_1) = \mathcal{G}_\kappa(\lambda_2)$ implies $q_1 = q_2$.

The FSMs m_1 and m_2 in Fig. 1 are deterministic w.r.t. to $\overset{1}{\bowtie}$ and $\overset{2}{\bowtie}$, respectively. A counter example for determinism (w.r.t. $\overset{1}{\bowtie}$) would be if m_1 had an additional state 4 with $1 \xrightarrow{c} 4$.

Given an FSM S and a grouping, one can always construct a deterministic FSM w.r.t the grouping that describes the same set of induced traces as S. The construction can be accomplished by applying a power set construction [12].

▸ **Lemma 7.2.** *Let $S := (\mathcal{Q}, \Lambda, \rightarrow, q^{ini}, F)$ be an FSM, and let κ be a grouping of Λ. There exists an FSM S' such that: (i) $\mathcal{T}_\kappa(\mathcal{L}(S)) = \mathcal{T}_\kappa(\mathcal{L}(S'))$, and (ii) S' is deterministic w.r.t. κ.*

For the remainder of this section, we fix an alignment $\bowtie : \Lambda_1 \otimes \Lambda_2$, and two FSMs $S_i := (\mathcal{Q}_i, \Lambda_i, \rightarrow_i, q_i^{ini}, F_i)$, $i \in \{1,2\}$, such that S_i is deterministic w.r.t. $\overset{i}{\bowtie}$.

The idea behind the *witness graph* $\mathcal{W}(S_1, S_2, \bowtie)$ is to construct a finite representation of all possible ways to align the traces of S_1 with the traces of S_2. To this end, we build a product of S_1 and S_2 where each product state (q_1, q_2) is additionally distinguished by a set M of *possible matches*. Here, a *match* is a pair (K_1, K_2) of two aligned groups K_1 and K_2, indicating that there exist traces w_1 and w_2 yielding q_1 and q_2, respectively, which can be aligned by some tactics θ_1 and θ_2 satisfying $\theta_i(|\theta_i|) = \{K_i\}$, for $i \in \{1,2\}$. The edges of $\mathcal{W}(S_1, S_2, \bowtie)$ are labelled with pairs $(\mathcal{K}_1, \mathcal{K}_2)$, where each $\mathcal{K}_i$, $i \in \{1,2\}$, is a set of groups. A non-empty set of groups $\mathcal{K}_i$ indicates an action of S_i which is abstracted by $\overset{i}{\bowtie}$ to $\mathcal{K}_i$. In contrast to that, $\mathcal{K}_i = \varnothing$ indicates that S_i did not 'move'.

To simplify further discussions, we introduce the notation $M + (\mathcal{K}_1, \mathcal{K}_2)$ for a set M of matches and a pair $(\mathcal{K}_1, \mathcal{K}_2)$ of sets of groups. Let $M \subseteq \bowtie$ and let $\mathcal{K}_i \subseteq \overset{i}{\bowtie}$, $i \in \{1,2\}$, such that $\mathcal{K}_1 \cup \mathcal{K}_2 \neq \varnothing$. Then, $M + (\mathcal{K}_1, \mathcal{K}_2)$ is defined as follows:

$$M + (\mathcal{K}_1, \mathcal{K}_2) := \begin{cases} \{(G_1, G_2) \in (\mathcal{K}_1 \times \mathcal{K}_2) \cap \bowtie \mid (G_1, G_2) \notin M\} & \text{if } \mathcal{K}_1 \neq \varnothing \wedge \mathcal{K}_2 \neq \varnothing, \\ \{(G_1, G_2) \in M \mid \exists\, i \in \{1,2\} : G_i \in \mathcal{K}_i\} & \text{otherwise.} \end{cases}$$

Consider m_1, m_2, and $\bowtie$ from Fig. 1, and let $M := \{(\{b,c\},\{t,u\}), (\{d,e\},\{s,w,x\})\}$. Then, $M + (\{\{a\},\{b,c\}\}, \{\{s,v\},\{s,w,x\}\}) = \{(\{a\},\{s,v\})\}$, $M + (\{\{b,c\}\}, \varnothing) = \{(\{b,c\},\{t,u\})\}$, and $M + (\{\{b,c\}\}, \{\{t,u\}\}) = \varnothing$.

Next, we define the *witness graph* $\mathcal{W}(S_1, S_2, \bowtie)$ based on the possible transitions in S_1 and S_2. We start from the initial state, and the empty set of matches. Then, we add nodes and edges according to the respective transition relations and the alignment.

▸ **Definition 7.3** (Witness Graph). The *witness graph* $\mathcal{W}(S_1, S_2, \bowtie)$ of S_1 and S_2 w.r.t. $\bowtie$ is the least edge-labelled graph (V, E), where $V \subseteq \mathcal{Q}_1 \times \mathcal{Q}_2 \times \wp(\bowtie)$ and $E \subseteq V \times \wp(\overset{1}{\bowtie}) \times \wp(\overset{2}{\bowtie}) \times V$, s.t.:

1. $(q_1^{ini}, q_2^{ini}, \varnothing) \in V$.
2. Let $v = (q_1, q_2, M) \in V$.
   a. For $i \in \{1,2\}$, let $q_i \xrightarrow{\lambda_i}_i q_i'$ be a transition of S_i, $\mathcal{K}_i = \mathcal{G}_{\overset{i}{\bowtie}}(\lambda_i)$, and $M' = M + (\mathcal{K}_1, \mathcal{K}_2)$, such that $M' \neq \varnothing$. Then, it holds that $v' := (q_1', q_2', M') \in V$ and $(v, \mathcal{K}_1, \mathcal{K}_2, v') \in E$.
   b. Let $i, j \in \{1,2\}$, $i \neq j$. Let $q_i \xrightarrow{\lambda_i}_i q_i'$ be a transition of S_i, $\mathcal{K}_i = \mathcal{G}_{\overset{i}{\bowtie}}(\lambda_i)$, $\mathcal{K}_j = \varnothing$, $q_j' = q_j$, and $M' = M + (\mathcal{K}_1, \mathcal{K}_2)$, such that $M' \neq \varnothing$. Then, it holds that $v' := (q_1', q_2', M') \in V$ and $(v, \mathcal{K}_1, \mathcal{K}_2, v') \in E$.

Let $e = (v, \mathcal{K}_1, \mathcal{K}_2, v') \in E$. For $i \in \{1,2\}$, $e[i] := \mathcal{K}_i$ and $e[i]_{\neq\varnothing} := \begin{cases} e[i] & e[i] \neq \varnothing, \\ \varepsilon & \text{otherwise.} \end{cases}$

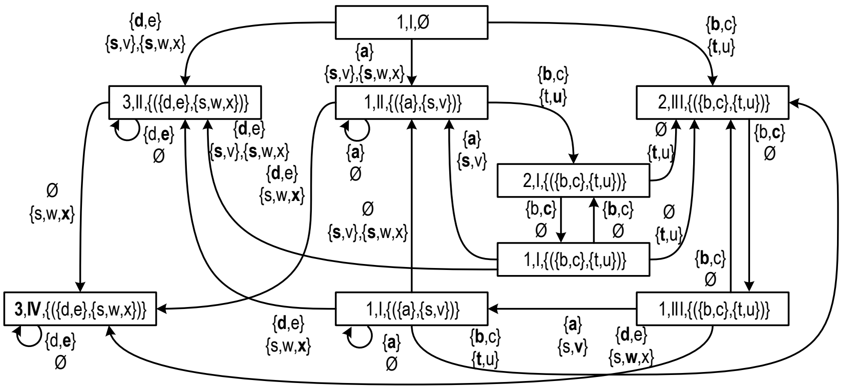

Figure 2 Witness graph for the FSMs and the alignment shown in Fig. 1.

Let $v_0 \dots v_n \in V^*$ such that $v_0 = (q_1^{ini}, q_2^{ini}, \varnothing)$. Let $\pi = e_1 \dots e_n \in E^*$ such that $\pi(i) = (v_{i-1}, \mathcal{K}_1^i, \mathcal{K}_2^i, v_i)$ for all $i \in [n]$. Then, π is a *path* of $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in v_n. We define $\pi[i] \coloneqq \pi(1)[i] \dots \pi(n)[i]$, and $\pi[i]_{\neq\varnothing} \coloneqq \pi(1)[i]_{\neq\varnothing} \dots \pi(n)[i]_{\neq\varnothing}$, where $i \in \{1,2\}$.

Fig. 2 exemplifies the notion of the witness graph $\mathcal{W}(m_1, m_2, \bowtie)$ for the FSMs m_1 and m_2, and the alignment $\bowtie$ given in Fig. 1. In the proposed notation, the outmost round brackets in the node labels are omitted. For example, the topmost node in the figure is $(1, \mathrm{I}, \varnothing)$, but we write $1, \mathrm{I}, \varnothing$. Note that edge labels are written in two lines (the first element is written above the second one). We also omit the outmost braces when depicting each of the non-empty elements of the label. For example, the edge from $(1, \mathrm{I}, \varnothing)$ to $(1, \mathrm{II}, \{(\{a\},\{s,v\})\})$ has label $(\{\{a\}\}, \{\{s,v\},\{s,w,x\}\})$. However, in the figure, the corresponding edge is labelled with $\{a\}$ written above $\{s,v\},\{s,w,x\}$. Labels of the respective transitions in m_1 and m_2 are printed in bold in the edge labels of the witness graph. Finally, if both m_1 and m_2 are in a final state, the state names are also put in bold in the graph.

For example, Rule 2.a. in Def. 7.3 produces the edge labelled $(\{\{a\}\}, \{\{s,v\},\{s,w,x\}\})$ from node $(1, \mathrm{I}, \varnothing)$ to node $(1, \mathrm{II}, \{(\{a\},\{s,v\})\})$: $1 \xrightarrow{a}_{m_1} 1$, $\mathrm{I} \xrightarrow{s}_{m_2} \mathrm{II}$, $\mathcal{G}_{\bowtie}^{1}(a) = \{\{a\}\}$, $\mathcal{G}_{\bowtie}^{2}(s) = \{\{s,v\},\{s,w,x\}\}$, and $\varnothing + (\mathcal{G}_{\bowtie}^{1}(a), \mathcal{G}_{\bowtie}^{2}(s)) = \{(\{a\},\{s,v\})\}$. Rule 2.b. produces the edge labelled $(\{\{b,c\}\}, \varnothing)$ from node $(2, \mathrm{III}, \{(\{b,c\},\{t,u\})\})$ to $(1, \mathrm{III}, \{(\{b,c\},\{t,u\})\})$: $2 \xrightarrow{c}_{m_1} 1$, $\mathcal{G}_{\bowtie}^{1}(c) = \{\{b,c\}\}$, and $\{(\{b,c\},\{t,u\})\} + (\mathcal{G}_{\bowtie}^{1}(c), \varnothing) = \{(\{b,c\},\{t,u\})\}$. The sequence π of edges with respective labels $(\{\{a\}\}, \{\{s,v\},\{s,w,x\}\})$, $(\{\{b,c\}\}, \{\{t,u\}\})$, $(\{\{b,c\}\}, \varnothing)$ and $(\varnothing, \{\{t,u\}\})$ from node $(1, \mathrm{I}, \varnothing)$ to node $(2, \mathrm{III}, \{(\{b,c\},\{t,u\})\})$ is a path with $\pi[2] = \{\{s,v\},\{s,w,x\}\}\{\{t,u\}\}\varnothing\{\{t,u\}\}$ and $\pi[2]_{\neq\varnothing} = \{\{s,v\},\{s,w,x\}\}\{\{t,u\}\}\{\{t,u\}\}$. We sketch an example producing non-singleton sets of matches in Appendix B.2.

A pair $(w_1, w_2) \in \wp(\overset{1}{\bowtie})^* \times \wp(\overset{2}{\bowtie})^*$ is *realisable* iff there exists a path of $\mathcal{W}(S_1, S_2, \bowtie)$ that represents w_1 and w_2, possibly also containing $\varnothing$.

▸ **Definition 7.4** (Realisable). For $i \in \{1,2\}$, let $w_i \in \wp(\overset{i}{\bowtie})^*$. Then, (w_1, w_2) is *realisable* in $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in a node v iff there exists a path π of $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in v, such that $\pi[i]_{\neq\varnothing} = w_i$, $i \in \{1,2\}$.

For example, let $w_1 \coloneqq \mathcal{T}_{\bowtie}^{1}(bca) = \{\{b,c\}\}\{\{b,c\}\}\{\{a\}\}$ and $w_2 \coloneqq \mathcal{T}_{\bowtie}^{2}(tv) = \{\{t,u\}\}\{\{s,v\}\}$. Then, (w_1, w_2) is realisable in the graph in Fig. 2, resulting in node $(1, \mathrm{I}, \{(\{a\},\{s,v\})\})$. In contrast, the pair of traces $(\mathcal{T}_{\bowtie}^{1}(bc), w_2)$ is not realisable.

We now show that realisability of (w_1, w_2) implies that w_1 and w_2 can be aligned.

▸ **Lemma 7.5.** *For $i \in \{1,2\}$, let $w_i \in \wp(\overset{i}{\bowtie})^*$. Let (w_1, w_2) be realisable in $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in a node $v = (q_1, q_2, M)$. For each $(K_1, K_2) \in M$, there exist a tactic θ_1 of w_1 and a tactic θ_2 of w_2, such that $(\theta_1, \theta_2) : w_1 \bowtie w_2$ and for all $i \in \{1,2\}$: $|w_i| > 0 \Rightarrow \theta_i(|w_i|) = \{K_i\}$.*

Because (w_1, w_2) is realizable, where $w_1 := \mathcal{T}^1_{\bowtie}(bca) = \{\{b,c\}\}\{\{b,c\}\}\{\{a\}\}$ and $w_2 := \mathcal{T}^2_{\bowtie}(tv) = \{\{t,u\}\}\{\{s,v\}\}$, w_1 and w_2 can be aligned. To justify this fact, one can use tactics $\theta_1 := w_1$ and $\theta_2 := w_2$, and the bijection $\{\{1,2\} \mapsto \{1\}, \{3\} \mapsto \{2\}\}$ from $[\theta_1]/_{\theta_1}$ to $[\theta_2]/_{\theta_2}$.

We now show the converse, i.e., if w_1 and w_2 can be aligned, then (w_1, w_2) is realisable. The idea of the proof is to construct a path that justifies that (w_1, w_2) is indeed realisable based on alignable tactics θ_1 and θ_2 in the order of the aligned equivalence classes: Rule 2.a. creates an edge for every fresh pair of aligned equivalence classes, while Rule 2.b. creates the edges for the remaining indices in these equivalence classes.

▸ **Lemma 7.6.** *For $i \in \{1,2\}$, let σ_i be a prefix of some word in $\mathcal{L}(S_i)$, and $w_i = \mathcal{T}^i_{\bowtie}(\sigma_i)$. Let for $i \in \{1,2\}$, θ_i be a tactic of w_i, such that $(\theta_1, \theta_2) : w_1 \bowtie w_2$. Then, (w_1, w_2) is realisable resulting in some node $v = (q_1, q_2, M)$ with $|w_1| > 0 \Rightarrow |w_2| = 0 \wedge (\theta_1(|w_1|), \theta_2(|w_2|)) \in M$.*

To give an example, we again consider $w_1 := \mathcal{T}^1_{\bowtie}(bca) = \{\{b,c\}\}\{\{b,c\}\}\{\{a\}\}$ and $w_2 := \mathcal{T}^2_{\bowtie}(tv) = \{\{t,u\}\}\{\{s,v\}\}$, which can be aligned by tactics $\theta_1 := w_1$ and $\theta_2 := w_2$, and bijection $\{\{1,2\} \mapsto \{1\}, \{3\} \mapsto \{2\}\}$. One can construct a path that justifies that (w_1, w_2) is realisable by using the bijection: $\{1,2\} \mapsto \{1\}$ yields that one must start by first taking edge $(\theta_1(1), \theta_2(1))$ followed by edge $(\theta_1(2), \varnothing)$. Then, $\{3\} \mapsto \{2\}$ yields edge $(\theta_1(3), \theta_2(2))$.

Intuitively, for $i, j \in \{1,2\}$, $i \neq j$, we can conceive $\mathcal{W}(S_1, S_2, \bowtie)$ as an FSM, where for each edge labelled $(\mathcal{K}_1, \mathcal{K}_2)$, we only consider the action $\mathcal{K}_i$ of S_i, i.e., we omit the actions of S_j. Let Π be the set of all paths of $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in a node (q_1, q_2, M), where $q_1 \in F_1$ and $q_2 \in F_2$. For $i \in \{1,2\}$, $\mathcal{L}_i(\mathcal{W}(S_1, S_2, \bowtie)) := \{\pi[i]_{\neq\varnothing} \mid \pi \in \Pi\}$. Next, we reduce the problem of deciding isotactics of $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ to two language equivalence checks.

▸ **Lemma 7.7.** *The following statements are equivalent:*
1. *$\mathcal{L}_1(\mathcal{W}(S_1, S_2, \bowtie)) = \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$ and $\mathcal{L}_2(\mathcal{W}(S_1, S_2, \bowtie)) = \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$.*
2. *$\mathcal{L}(S_1) \doteqdot_{\bowtie} \mathcal{L}(S_2)$.*

Because language equivalence is decidable for FSMs [12], we can also decide the first proposition of Lemma 7.7 by transforming $\mathcal{W}(S_1, S_2, \bowtie)$ into two FSMs: one with language $\mathcal{L}_1(\mathcal{W}(S_1, S_2, \bowtie))$ and the other with language $\mathcal{L}_2(\mathcal{W}(S_1, S_2, \bowtie))$. The transformation basically comprises the projection of the edge labels to the i-th component, $i \in \{1,2\}$, and the subsequent "removal" of $\varnothing$-transitions.

▸ **Lemma 7.8.** *For $i \in \{1,2\}$, the following problem is decidable: To decide whether it holds that $\mathcal{L}_i(\mathcal{W}(S_1, S_2, \bowtie)) = \mathcal{T}^i_{\bowtie}(\mathcal{L}(S_i))$.*

We conclude that deciding isotactics is in EXSPACE: if the FSMs are deterministic, $\mathcal{W}(S_1, S_2, \bowtie)$ has at most $|Q_1| \cdot |Q_2| \cdot 2^{|\bowtie|}$ nodes; otherwise, determinisation of the FSMs yields a witness graph with at most $2^{|Q_1||Q_2||\bowtie|}$ nodes. $\mathcal{W}(S_1, S_2, \bowtie)$ can be computed in EXPTIME. Deciding language equivalence requires polynomial space in the size of the FSMs.

8 Concluding Remarks

Having introduced the notion of interleaving isotactics to assess behavioural equivalence of aligned models, we reflect on potential causes for non-equivalence. First and foremost, an action that is part solely of one model may lead to non-equivalence. Formally, in our model, the respective label would be part of the alignment, which relates two groups containing this label to each other. Yet, occurrences of this label are limited to runs of one of the system models and will not be mirrored by any run of the other model. For such a setting, notions of behaviour inheritance have been defined [1]: A label occurring in only one model is either *blocked* or *hidden* before verifying equivalence. These techniques, originally defined in the context of branching bisimulation, can be directly lifted to the introduced isotactics notion.

For system models that are not isotactic, it is often relevant to quantify the discrepancies in their behaviour. To this end, measures for behavioural similarity have been proposed for different semantics of system models, see [5, 6]. We foresee that the presented notion of isotactics may also be exploited for the definition of such measures, quantifying for instance the ratio of aligned groups of labels, for which the behavioural projections are equivalent.

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A Proofs

▸ **Lemma 5.1.** *For $i \in \{1,2\}$, let Λ_i be a set of labels and let $\Sigma_i \subseteq \Lambda_i^*$. Let b be a bijection from Λ_1 to Λ_2, and let $\bowtie := \{(\{\lambda\},\{b(\lambda)\}) \mid \lambda \in \Lambda_1\}$. If $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ and $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$ are trace equivalent up to b, then it holds that $\Sigma_1 \doteqdot_{\bowtie} \Sigma_2$.*

Proof. According to Def. 2.1, there exists a bijection R from $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ to $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$ such that for all $(w_1,w_2) \in R$ it holds that (i) $|w_1| = |w_2|$, and (ii) for all $i \in [w_1]$ it holds that $b(w_1(i)) = w_2(i)$. Let $(w_1,w_2) \in R$. Because $\bowtie \subseteq \wp_{=1}(\Lambda_1) \times \wp_{=1}(\Lambda_2)$, for all $i \in [w_1]$ it holds that $|w_1(i)| = 1 = |w_2(i)|$. Therefore, for $i \in \{1,2\}$, w_i is the only tactic of w_i. Note that for all $i \in [w_1]$, it also holds that $w_1(i) \bowtie w_2(i)$ and, thus, $[w_1]/w_1 = [w_2]/w_2$. Then, it trivially holds that $(w_1,w_2) : w_1 \bowtie w_2$, cf. Def. 3.6; one can use the identity relation on $[w_1]/w_1$ as a bijection from $[w_1]/w_1$ to $[w_2]/w_2$ to justify this fact. Because R is a bijection, we get $\Sigma_1 \leqslant_{\bowtie} \Sigma_2$. Because R and $\bowtie$ are bijections, we get $\Sigma_2 \leqslant_{\bowtie^{-1}} \Sigma_1$; note that $\bowtie$ is a bijection because b is a bijection. Thus, it holds that $\Sigma_1 \doteqdot_{\bowtie} \Sigma_2$. ◂

▸ **Lemma 5.2.** *For $i \in \{1,2\}$, let Λ_i be a set of labels and let $\Sigma_i \subseteq \Lambda_i^*$ be repetition-free. Let b be a bijection from Λ_1 to Λ_2, and let $\bowtie := \{(\{\lambda\},\{b(\lambda)\}) \mid \lambda \in \Lambda_1\}$. If $\Sigma_1 \doteqdot_{\bowtie} \Sigma_2$, then it holds that $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ and $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$ are trace equivalent up to b.*

Proof. Because $\bowtie$ is a relation on singletons, for $i \in \{1,2\}$, $w_i \in \mathcal{T}_{\bowtie}^{i}(\Sigma_i)$ is the only tactic of w_i. Let $R := \{(x,y) \in \mathcal{T}_{\bowtie}^{1}(\Sigma_1) \times \mathcal{T}_{\bowtie}^{2}(\Sigma_2) \mid (x,y) : x \bowtie y\}$. Let $(w_1,w_2) \in R$. Because Σ_1 and Σ_2 are repetition-free and because $\bowtie$ is a bijection, it holds that w_1 and w_2 are repetition-free; note that $\bowtie$ is a bijection because b is a bijection. Therefore, for $i \in \{1,2\}$, it holds that $[w_i]/w_i = \wp_{=1}([w_i])$. Hence, $|w_1| = |w_2|$ and for all $k \in [w_1]$ it holds that $w_1(k) \bowtie w_2(k)$ and, thus, $b(w_1(k)) = w_2(k)$. Next, we show that R is a bijection from $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ to $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$. Let $(w_1,w_2),(w_1',w_2') \in R$ and $i,j \in \{1,2\}$, $i \neq j$. Let us assume that $w_i = w_i'$. As shown above, it holds that $|w_i| = |w_j|$ and $|w_i'| = |w_j'|$. Then, $w_i = w_i'$ yields $|w_j| = |w_j'|$. Additionally, for every $k \in [w_i]$ it holds that $w_1(k) \bowtie w_2(k)$ and $w_1'(k) \bowtie w_2'(k)$. Because $w_i = w_i'$ and $\bowtie$ is a bijection, $w_j(k) = w_j'(k)$, $k \in [w_j]$. Hence, it holds that $w_j = w_j'$. Finally, because $\Sigma_1 \leqslant_{\bowtie} \Sigma_2$ and $\Sigma_2 \leqslant_{\bowtie^{-1}} \Sigma_1$, for $i \in \{1,2\}$, it holds that for all $\sigma \in \Sigma_i$ there exists $(w_1,w_2) \in R$ such that $\mathcal{T}_{\bowtie}^{i}(\sigma) = w_i$. Thus, R is a bijection from $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ to $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$ that justifies the fact that $\mathcal{T}_{\bowtie}^{1}(\Sigma_1)$ and $\mathcal{T}_{\bowtie}^{2}(\Sigma_2)$ are trace equivalent up to b. ◂

▸ **Lemma 6.1.** *Let $\bowtie : \Lambda_1 \otimes \Lambda_2$ be an alignment. For $i \in \{1,2\}$, let $\sigma_i \in \Lambda_i^*$, $w_i := \mathcal{T}_{\bowtie}^{i}(\sigma_i)$, θ_i be a tactic of w_i, and let $\varphi_i \in \mathsf{LTL}[\overset{i}{\bowtie}]$ be tactic-invariant w.r.t. Λ_i and $\overset{i}{\bowtie}$. Let φ_1 and φ_2 be aligned by $\bowtie$, i.e., $\varphi_1 \bowtie \varphi_2$, and let $(\theta_1,\theta_2) : w_1 \bowtie w_2$. Then, it holds that $w_1 \models \varphi_1$ iff $w_2 \models \varphi_2$.*

Proof. For $i \in \{1,2\}$, it holds that $\theta_i \models \varphi_i$ iff $w_i \models \varphi_i$, because φ_i is tactic-invariant. We show that $\theta_1 \models \varphi_1$ iff $\theta_2 \models \varphi_2$ by the structural induction on φ_1. Let $\varphi_1 := \bigvee_{K_1 \in \mathcal{K}_1} K_1$ and $\varphi_2 := \bigvee_{K_2 \in \mathcal{K}_2} K_2$, where $\mathcal{K}_1 \subseteq \overset{1}{\bowtie}$ and $\mathcal{K}_2 \subseteq \overset{2}{\bowtie}$ such that $(\mathcal{K}_1 \times \overset{2}{\bowtie}) \cap \bowtie = (\overset{1}{\bowtie} \times \mathcal{K}_2) \cap \bowtie$. Then, for $j \in \{1,2\}$, $\theta_j \models \varphi_j$ iff $\theta_j(1) \cap \mathcal{K}_j \neq \varnothing$. Because $\theta_1(1) \bowtie \theta_2(1)$, $\theta_1(1) \cap \mathcal{K}_1 \neq \varnothing$ iff $\theta_2(1) \cap \mathcal{K}_2 \neq \varnothing$ and, thus, it holds that $\theta_1 \models \varphi_1$ iff $\theta_2 \models \varphi_2$. Let $\varphi_1 := \neg\psi_1$ and $\varphi_2 := \neg\psi_2$ such that $\psi_1 \bowtie \psi_2$. Then, for $k \in \{1,2\}$, it holds that $\theta_k \models \psi_k$ iff $\theta_k \not\models \varphi_k$. Using the inductive assumption, we conclude that $\theta_1 \models \varphi_1$ iff $\theta_2 \models \varphi_2$. Let $\varphi_1 := \psi_1 \vee \psi_1'$ and $\varphi_2 := \psi_2 \vee \psi_2'$ such that $\psi_1 \bowtie \psi_2$ and $\psi_1' \bowtie \psi_2'$. Then, for $k \in \{1,2\}$, it holds that $\theta_k \models \varphi_k$ iff $\theta_k \models \psi_k$ or $\theta_k \models \psi_k'$. Again, using the inductive assumption, we conclude that $\theta_1 \models \varphi_1$ iff $\theta_2 \models \varphi_2$. Let $\varphi_1 := \psi_1 \mathsf{U} \psi_1'$ and $\varphi_2 := \psi_2 \mathsf{U} \psi_2'$ such that $\psi_1 \bowtie \psi_2$ and $\psi_1' \bowtie \psi_2'$. Then, for $k \in \{1,2\}$, it holds that $\theta_k \models \varphi_k$ iff there exists $\iota_k \in [\theta_k]$ such that $\theta_k[\iota_k\rangle \models \psi_k'$ and $\theta_k[\eta\rangle \models \psi_k$ for all $\eta \in [\iota_k - 1]$. By definition, for $k \in \{1,2\}$ and $\iota \in [\theta_k - 1]$ it holds that $\iota =_{\theta_k} \iota + 1$ implies $\theta_k(\iota) = \theta_k(\iota + 1)$. Because of the stutter equivalence of LTL-formulae without the next operator [9], $\iota =_{\theta_k} \iota+1$ also implies that $\theta_k[\iota\rangle$ and $\theta_k[\iota+1\rangle$ satisfy exactly the same LTL-formulae. Applying the inductive assumption then yields $\theta_1 \vDash \varphi_1$ iff $\theta_2 \vDash \varphi_2$. Finally, because ψ_1 and ψ_2 are tactic-invariant, it holds that $w_1 \vDash \varphi_1$ iff $w_2 \vDash \varphi_2$. ◂

▸ **Lemma 6.2.** *$\hat{\Theta} = \hat{\Theta}'$ iff φ is tactic-invariant w.r.t. Λ and κ.*

Proof. We show both directions separately.

1. "1.⇒2.": Let $w \in \mathcal{T}_\kappa(\Lambda^*)$ and $\theta \in \text{Tactics}(w)$. Assume first $w \vDash \varphi$. Then, $w \in W$, and thus $\theta_w \in \hat{\Theta}$. Then, by assumption, $\theta \in \hat{\Theta}'$. Hence, $\theta \in \Theta$. Therefore, $\theta \vDash \varphi$. Symmetrically, if $\theta \vDash \varphi$, then $\theta \in \Theta$, and thus also $\theta_w \in \hat{\Theta}'$. Then, by assumption, $\theta_w \in \hat{\Theta}$. Then, $w \in W$, and therefore $w \vDash \varphi$.
2. "2.⇒1.": Assume first $\theta_w \in \hat{\Theta}$. Then, $w \in \mathcal{T}_\kappa(\Lambda^*)$ with $w \vDash \varphi$, and $\theta \in \text{Tactics}(w)$ of w. Because φ is tactic-invariant, $\theta \vDash \varphi$. Thus, $\theta \in \Theta$, and $\theta_w \in \hat{\Theta}'$. Now, assume $\theta_w \in \hat{\Theta}'$. Then, there exists a tactic $\theta \in \Theta$ of some word w, such that $\theta \vDash \varphi$. Because φ is tactic-invariant, $w \vDash \varphi$. Therefore, $\theta_w \in \hat{\Theta}'$.

◂

▸ **Lemma 6.3.** *It is decidable whether $\hat{\Theta}$ and $\hat{\Theta}'$ are equal sets.*

Proof. We construct an FSM S_φ from φ, such that $\mathcal{L}(S_\varphi)$ is the set of all traces satisfying φ. Then, we construct FSMs $\underline{S}$ and $\overline{S}$ accepting all $w \in \mathcal{T}_\kappa(\Lambda^*)$ with $w(i) \neq \emptyset$ for all $i \in [w]$, and $\theta \in \text{Tactics}(w)$ and $\overline{S}$ accepts w, respectively. Let S_W be the intersection of S_φ and $\underline{S}$, and let S_Θ be the intersection of S_φ and $\overline{S}$. We construct the FSM $\hat{S}_\Theta$ from S_W by replacing each transition $q \xrightarrow{\mathcal{K}}_{S_W} q'$ by a transition $q \xrightarrow{(K,\mathcal{K})}_{\hat{S}_\Theta} q'$ for each $K \in \mathcal{K}$. Then, we construct the FSM $\hat{S}'_\Theta$ from S_Θ by replacing each transition $q \xrightarrow{\{K\}}_{S_\Theta} q'$ by a transition $q \xrightarrow{(K,\mathcal{K})}_{\hat{S}_{\Theta'}} q'$ for each $\mathcal{K} \in \mathcal{G}_\kappa(\Lambda)$ with $K \in \mathcal{K}$. Obviously, the languages of $\hat{S}_\Theta$ and $\hat{S}'_\Theta$ are $\hat{\Theta}$ and $\hat{\Theta}'$, respectively. Thus, we decide $\hat{\Theta} = \hat{\Theta}'$ by deciding equivalence of $\hat{S}_\Theta$ and $\hat{S}'_\Theta$. ◂

▸ **Lemma 7.5.** *For $i \in \{1,2\}$, let $w_i \in \wp(\overset{i}{\bowtie})^*$. Let (w_1, w_2) be realisable in $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in a node $v = (q_1, q_2, M)$. For each $(K_1, K_2) \in M$, there exist a tactic θ_1 of w_1 and a tactic θ_2 of w_2, such that $(\theta_1, \theta_2) : w_1 \bowtie w_2$ and for all $i \in \{1,2\}$: $|w_i| > 0 \Rightarrow \theta_i(|w_i|) = \{K_i\}$.*

Proof by induction over the length of the realising path. Because (w_1, w_2) is realisable, there exists a path π of $\mathcal{W}(S_1, S_2, \bowtie)$ resulting in v, such that for $i \in \{1,2\}$, the restriction of $\pi[i]$ to non-empty sets yields w_i. Let $|\pi| = 0$. Then, $w_1 = w_2 = \varepsilon$, trivially satisfying the requirements. Let $|\pi| = 1$. Inspecting the rules 2.a. and 2.b. of Def. 7.3, we find that only a) can produce an edge from the node $(q_1^{ini}, q_2^{ini}, \emptyset)$. Hence, for $i \in \{1,2\}$, $\pi[i] = \mathcal{K}_i \neq \emptyset$, and thus $w_i = \mathcal{K}_i$. Additionally, $M = (\mathcal{K}_1 \times \mathcal{K}_2) \cap \bowtie$. Hence, for each $(G_1, G_2) \in M$, there also exists the tactics $\theta_i = \{G_i\}$ and b proving $(\theta_1, \theta_2) : w_1 \bowtie w_2$. Now, let $|\pi| = n \geq 2$, and the proposition hold for all realising paths π' with $|\pi'| < n$. Let the penultimate node of π be $\hat{v} = (\hat{q}_1, \hat{q}_2, \hat{M})$.

1. Assume that the last edge of π is produced by rule 2.a. and labelled $\mathcal{K}_1, \mathcal{K}_2$. Then, for $i \in \{1,2\}$, $w_i(|w_i|) = \mathcal{K}_i$, and $M = \{(K_1, K_2) \in (\mathcal{K}_1 \times \mathcal{K}_2) \cap \bowtie \mid (K_1, K_2) \notin \hat{M}\} \neq \emptyset$. Let $\hat{w}_i = w_i(1) \dots w_i(|w_i| - 1)$. Let $(K_1, K_2) \in M$. Then, $(\hat{w}_1, \hat{w}_2)$ is realisable resulting in $\hat{v}$ by the prefix of π with length $n - 1 > 0$. Therefore, for each $(\hat{K}_1, \hat{K}_2) \in \hat{M}$, and $i \in \{1,2\}$, there exist a tactic $\hat{\theta}_i$ of $\hat{w}_i$ with $\hat{\theta}_i(|\hat{w}_i|) = \hat{K}_i$, and $\hat{b}$ proving $(\hat{\theta}_1, \hat{\theta}_2) : \hat{w}_1 \bowtie \hat{w}_2$. We first observe that $(K_1, \hat{K}_2), (\hat{K}_1, K_2) \in \hat{M}$ would imply $(K_1, K_2) \in \hat{M}$. Therefore, there exists $(\hat{K}_1, \hat{K}_2) \in \hat{M}$, such that $K_1 \neq \hat{K}_1$ and $K_2 \neq \hat{K}_2$. Then, for $i \in \{1,2\}$, $\hat{\theta}_i(|\hat{w}_i|) \neq \{K_i\}$. Now,

set $\theta_i = \hat{\theta}_i\{K_i\}$. Then, $[w_i]/\theta_i = [\hat{w}_i]/\hat{\theta}_i \cup \{\{|w_i|\}\}$, that is, θ_i adds another equivalence class "to the end" of $\hat{\theta}_i$. Let $b = \hat{b} \cup \{(|w_1|, |w_2|)\}$. Because b proves $(\hat{\theta}_1, \hat{\theta}_2) : \hat{w}_1 \bowtie \hat{w}_2$, and $K_1 \bowtie K_2$, $\hat{b}$ proves $(\theta_1, \theta_2) : w_1 \bowtie w_2$.

2. Assume that the last edge of π is produced by rule 2.b. and labelled $(\mathcal{K}_1, \mathcal{K}_2)$. Then, there exist $i, j \in \{1,2\}$, $i \neq j$, $\lambda_i \in \Lambda_i$, $\mathcal{K}_i = \mathcal{G}_{\bowtie}^{i}(\lambda_i)$, such that $\hat{q}_i \xrightarrow{\lambda_i}_i q_i$, and $M = \{(K_1, K_2) \in \hat{M} \mid \exists\, i \in \{1,2\} : K_i \in \mathcal{K}_i\}$. Let $\hat{w}_i = w_i(1) \dots w_i(|w_i| - 1)$. Let $\hat{w}_j = w_j$. Then, $(\hat{w}_1, \hat{w}_2)$ is realisable resulting in $\hat{v}$ by the prefix of π with length $n - 1 > 0$. Therefore, for $k \in \{1,2\}$, and each $(K_1, K_2) \in \hat{M}$ (thus also $(K_1, K_2) \in M$), there exist a tactic $\hat{\theta}_k$ of $\hat{w}_k$ with $\hat{\theta}_k(|\hat{w}_k|) = K_i$, $\hat{\theta}_j(|\hat{w}_j|) = \hat{G}_j$, and $\hat{b}$ proving $(\hat{\theta}_1, \hat{\theta}_2) : \hat{w}_1 \bowtie \hat{w}_2$. Let $\theta_i = \hat{\theta}_i\{K_i\}$, $\hat{X} = \langle|\hat{w}_i|\rangle_{\hat{\theta}_i}$ be the "last" equivalence class in $\hat{\theta}_i$, and $X = \hat{X} \cup \{|w_i|\}$. Then, $[w_i]/\theta_i = [\hat{w}_i]/\hat{\theta}_i \setminus \{\hat{X}\} \cup \{X\}$, that is, the "last" equivalence class of θ_i is the union of the last equivalence class of $\hat{\theta}_i$ and $\{|w_i|\}$. Let $\theta_j = \hat{\theta}_j$, $\hat{Y} = \langle|\hat{w}_j|\rangle_{\hat{\theta}_j}$ and $b = \hat{b} \setminus \{(\hat{X}, \hat{Y})\} \cup \{(X, \hat{Y})\}$. Then, b proves $(\theta_1, \theta_2) : w_1 \bowtie w_2$.

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▸ **Lemma 7.6.** *For $i \in \{1,2\}$, let σ_i be a prefix of some word in $\mathcal{L}(S_i)$, and $w_i = \mathcal{T}_{\bowtie}^{i}(\sigma_i)$. Let for $i \in \{1,2\}$, θ_i be a tactic of w_i, such that $(\theta_1, \theta_2) : w_1 \bowtie w_2$. Then, (w_1, w_2) is realisable resulting in some node $v = (q_1, q_2, M)$ with $|w_1| > 0 \Rightarrow |w_2| = 0 \wedge (\theta_1(|w_1|), \theta_2(|w_2|)) \in M$.*

Proof by induction over the number of equivalence classes in the tactics. Let w_1, w_2 be traces with respective tactics θ_1, θ_2, such that $(\theta_1, \theta_2) : w_1 \bowtie w_2$. Let $n = |[w]_1/\theta_1|$. We observe $n = |[w_2]/\theta_2|$ because there exists a bijection b aligning θ_1 and θ_2.

1. Let $n = 0$. Then, $w_1 = w_2 = \varepsilon$. Then, the empty path proves realisability of (w_1, w_2).
2. Let $n = 1$. Then, for $i \in \{1,2\}$, there exists a label λ_i and a state q_i with $q_i^{ini} \xrightarrow{\lambda_i}_{S_i} q_i$, $w_i = \mathcal{T}_{\bowtie}^{i}(\lambda_i) = \mathcal{G}_{\bowtie}^{i}(\lambda_i)$, and $K_i \in \mathcal{G}_{\bowtie}^{i}(\lambda_i)$. Let $M = \varnothing + (\mathcal{G}_{\bowtie}^{1}(\lambda_1), \mathcal{G}_{\bowtie}^{2}(\lambda_2))$. From $(\theta_1, \theta_2) : w_1 \bowtie w_2$, we get $K_1 \bowtie K_2$ and $(K_1, K_2) \in M$. Hence, $M \neq \varnothing$, and rule 2.a. produces an edge labelled $(\mathcal{G}_{\bowtie}^{1}(\lambda_1), \mathcal{G}_{\bowtie}^{2}(\lambda_2))$ from $(q_1^{ini}, q_2^{ini}, \varnothing)$ to node (q_1, q_2, M) in $\mathcal{W}(S_1, S_2, \bowtie)$.
3. Let $n > 1$ and assume that the lemma holds for all alignable traces and respective tactics with $n - 1$ equivalence classes. We first introduce an auxiliary notation for this part of the proof: Let w be some trace with $|w| > 1$ and θ be a tactic of w. Let $\mathrm{cut}(\theta)$ denote the maximal index in $[w]$ with $\theta(i) \neq \theta(|w|)$. We observe $1 \leq \mathrm{cut}(\theta) < |w|$. Now, for $i \in \{1,2\}$, let $\hat{w}_i = w_i(1) \dots w_i(\mathrm{cut}(\theta_i))$ and $\hat{\theta}_i = \theta_i(1) \dots \theta_i(\mathrm{cut}(\theta_i))$. Let $\hat{b} = b \setminus \{(\langle|w_1|\rangle_{\theta_1}, \langle|w_2|\rangle_{\theta_2})\}$. Then, $\hat{b}$ proves $(\hat{\theta}_1, \hat{\theta}_2) : \hat{w}_1 \bowtie \hat{w}_2$, and we observe $|[\hat{w}_i]/\theta_i| = n - 1$ for $i \in \{1,2\}$. Let for $i \in \{1,2\}$, $\hat{\theta}_i(\mathrm{cut}(\theta_i)) = \{\hat{K}_i\}$. By assumption, $(\hat{w}_1, \hat{w}_2)$ is realisable resulting in a node $(\hat{q}_1, \hat{q}_2, \hat{M})$ with $(\hat{K}_1, \hat{K}_2) \in \hat{M}$. By assumption, for $i \in \{1,2\}$, there exists a label λ_i, and a transition $\hat{q}_i \xrightarrow{\lambda_i} q_i$ with $\mathcal{G}_{\bowtie}^{i}(\lambda_i) = w_i(\mathrm{cut}(\theta_i) + 1)$ and $K_i = \theta_i(\mathrm{cut}(\theta_i) + 1) \in \mathcal{G}_{\bowtie}^{i}(\lambda_i)$. We distinguish the cases $(K_1, K_2) \notin \hat{M}$ and $(K_1, K_2) \in \hat{M}$.
   - **a.** Let $(K_1, K_2) \notin \hat{M}$. Let $M = \hat{M} + (\mathcal{G}_{\bowtie}^{1}(\lambda_1), \mathcal{G}_{\bowtie}^{2}(\lambda_2))$. Because $K_1 \bowtie K_2$, $(K_1, K_2) \in M$, and hence $M \neq \varnothing$. Therefore, rule 2.a. produces an edge labelled $(\mathcal{G}_{\bowtie}^{1}(\lambda_1), \mathcal{G}_{\bowtie}^{2}(\lambda_2))$ from node $(\hat{q}_1, \hat{q}_2, \hat{M})$ to the node (q_1, q_2, M).
   - **b.** Let $(K_1, K_2) \in \hat{M}$. Let $M' = \hat{M} + (\mathcal{G}_{\bowtie}^{1}(\lambda_1), \varnothing)$ and $M = M' + (\varnothing, \mathcal{G}_{\bowtie}^{2}(\lambda_2))$. Then, $(K_1, K_2) \in M' \cap M$, and hence $M' \neq \varnothing \neq M$. From $M' \neq \varnothing$ and $M \neq \varnothing$, we conclude that rule 2.b. produces an edge labelled $(\mathcal{G}_{\bowtie}^{1}(\lambda_1), \varnothing)$ from $(\hat{q}_1, \hat{q}_2, \hat{M})$ to $(q_1, \hat{q}_2, M')$, and an edge labelled $(\varnothing, \mathcal{G}_{\bowtie}^{2}(\lambda_2))$ from $(q_1, \hat{q}_2, M')$ to (q_1, q_2, M).

   If for $i \in \{1,2\}$, $\mathrm{cut}(\theta_i) = |w_i| - 1$, we are finished. Otherwise, rule 2.b. produces the remaining edges: We first add the edges for the remaining elements of w_1, and then for the remaining elements of w_2, resulting in edges labelled

$$(\,w_1(\mathrm{cut}(\theta_i)+2),\varnothing\,)\ldots(\,w_1(|w_i|),\varnothing\,)\ \ (\varnothing,w_2(\mathrm{cut}(\theta_i)+2)\,)\ldots(\,\varnothing,w_2(|w_2|)\,)$$

and nodes $(q_1^1,q_2,M_1)\ldots(q_1^\ell,q_2,M_\ell)(q_1^\ell,q_2^1,M_{\ell+1})\ldots(q_1^\ell,q_2^m,M_{\ell+m})$, where for each $1 \le j \le \ell+m$, we exploit $(K_1,K_2)\in M_j$.

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▸ **Lemma 7.7.** *The following statements are equivalent:*
1. $\mathcal{L}_1(\mathcal{W}(S_1,S_2,\bowtie)) = \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$ *and* $\mathcal{L}_2(\mathcal{W}(S_1,S_2,\bowtie)) = \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$.
2. $\mathcal{L}(S_1) \doteqdot_{\bowtie} \mathcal{L}(S_2)$.

Proof. We show both directions separately.

1. "$1 \Rightarrow 2$": We show: For each $w_1 \in \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$, there exists $w_2 \in \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$, and tactics θ_1 and θ_2 of w_1 and w_2, respectively, such that there is b proving $(\theta_1,\theta_2): w_1 \bowtie w_2$. From $w_1 \in \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$ and the assumption, we get $w_1 \in \mathcal{L}_1(\mathcal{W}(S_1,S_2,\bowtie))$. Hence, there exists a path π of $\mathcal{W}(S_1,S_2,\bowtie)$, such that $\pi[i]_{\neq\varnothing} = w_1$, resulting in a node (q_1,q_2,M) satisfying $q_1 \in F_1$ and $q_2 \in F_2$. Let $w_2 = \pi[2]_{\neq\varnothing}$. Then, by definition, (w_1,w_2) are realisable. Applying Lemma 7.5 yields the existence of required θ_1, θ_2 and b. It now remains to be shown that $w_2 \in \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$, which follows directly from the assumption.
2. "$2 \Rightarrow 1$":
   - We show $\mathcal{L}_1(\mathcal{W}(S_1,S_2,\bowtie)) \subseteq \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$. Let $w_1 \in \mathcal{L}_1(\mathcal{W}(S_1,S_2,\bowtie))$. Hence, there exists a path π of $\mathcal{W}(S_1,S_2,\bowtie)$, such that $\pi[1]_{\neq\varnothing} = w_1$, resulting in a node (q_1,q_2,M) satisfying $q_1 \in F_1$ and $q_2 \in F_2$. Now, we can show by induction, that there exists an accepting path of S_1 accepting some σ_1 with $\mathcal{T}^1_{\bowtie}(\sigma_1) = w_1$.
   - We show $\mathcal{L}_1(\mathcal{W}(S_1,S_2,\bowtie)) \supseteq \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$ and $\mathcal{L}_2(\mathcal{W}(S_1,S_2,\bowtie)) \supseteq \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$. Let $w_1 \in \mathcal{T}^1_{\bowtie}(\mathcal{L}(S_1))$. Then, by assumption, there exists $w_2 \in \mathcal{T}^2_{\bowtie}(\mathcal{L}(S_2))$, θ_1, θ_2 and b proving $(\theta_1,\theta_2): w_1 \bowtie w_2$. Hence, by Lemma 7.6, (w_1,w_2) is realisable, yielding some node (q_1,q_2,M). Hence, there exists a path π with $\pi[1]_{\neq\varnothing} = w_1$ and $\pi[2]_{\neq\varnothing} = w_2$. Because for $i \in \{1,2\}$, $w_i \in \mathcal{T}^i_{\bowtie}(\mathcal{L}(S_i))$, it holds that $q_i \in F_i$. Hence, for $i \in \{1,2\}$, $w_i \in \mathcal{L}_i(\mathcal{W}(S_1,S_2,\bowtie))$.

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▸ **Lemma 7.8.** *For* $i \in \{1,2\}$, *the following problem is decidable: To decide whether it holds that* $\mathcal{L}_i(\mathcal{W}(S_1,S_2,\bowtie)) = \mathcal{T}^i_{\bowtie}(\mathcal{L}(S_i))$.

Proof. $\mathcal{W}(S_1,S_2,\bowtie)$ is obviously finite and computable. One can construct an FSM S accepting $\mathcal{L}_i(\mathcal{W}(S_1,S_2,\bowtie))$, $i \in \{1,2\}$, by following this procedure:

1. Each node of $\mathcal{W}(S_1,S_2,\bowtie)$ is a state of S.
2. For each edge $(v,\mathcal{K}_1,\mathcal{K}_2,v')$ of $\mathcal{W}(S_1,S_2,\bowtie)$ introduce a transition $v \xrightarrow{K_i}_S v'$,
3. Set the node $(q_1^{ini},q_2^{ini},\varnothing)$ as the initial state.
4. A state (q_1,q_2,M) is a final state of S iff $q_1 \in F_1$ and $q_2 \in F_2$.
5. Finally, remove all $\varnothing$-transitions with a standard ε-removal algorithm [12].

We decide $\mathcal{L}_i(\mathcal{W}(S_1,S_2,\bowtie)) = \mathcal{T}^i_{\bowtie}(\mathcal{L}(S_i))$ by checking language equivalence of S and S_i. ◂

B Additional Examples

B.1 Example for Lemma 6.2

This section introduces an example for the decision procedure explained in the proof of Lemma 6.2. We consider the example alignment $\bowtie$ given by Fig. 1, and the LTL[$\overset{2}{\bowtie}$]-formula

$\varphi = \{t,u\} \cup (\{s,v\} \vee \{s,w,x\})$. Then, S_φ consists of an initial state q_1, a final state q_2, for each $\mathcal{K} \subseteq \overset{2}{\bowtie}$ a transition $q_2 \xrightarrow{\mathcal{K}} q_2$ (in the following abbreviated as $q_2 \xrightarrow{*} q_2$), and for each $\mathcal{K} \subseteq \overset{2}{\bowtie}$ satisfying $\{s,v\} \in \mathcal{K}$ implies $\{s,w,x\} \notin \mathcal{K}$ a transition $q_1 \xrightarrow{\mathcal{K}} q_2$, namely:

$$q_1 \xrightarrow{\varnothing} q_2, \quad q_1 \xrightarrow{\{\{t,u\}\}} q_2, \quad q_1 \xrightarrow{\{\{s,v\}\}} q_2,$$
$$q_1 \xrightarrow{\{\{s,v\},\{t,u\}\}} q_2, \; q_1 \xrightarrow{\{\{s,w,x\}\}} q_2, \; q_1 \xrightarrow{\{\{s,w,x\},\{t,u\}\}} q_2 \; .$$

S_W has the same initial and final states, and the following transitions:

$$q_1 \xrightarrow{\{\{t,u\}\}} q_2, \; q_1 \xrightarrow{\{\{s,v\}\}} q_2, \; q_1 \xrightarrow{\{\{s,w,x\}\}} q_2,$$
$$q_2 \xrightarrow{\{\{t,u\}\}} q_2, \; q_2 \xrightarrow{\{\{s,v\}\}} q_2, \; q_2 \xrightarrow{\{\{s,w,x\}\}} q_2, \; q_2 \xrightarrow{\{\{s,v\},\{s,w,x\}\}} q_2 \; .$$

S_Θ has the same initial and final states, and the following transitions:

$$q_1 \xrightarrow{\{\{t,u\}\}} q_2, \; q_1 \xrightarrow{\{\{s,v\}\}} q_2, \; q_1 \xrightarrow{\{\{s,w,x\}\}} q_2,$$
$$q_2 \xrightarrow{\{\{t,u\}\}} q_2, \; q_2 \xrightarrow{\{\{s,v\}\}} q_2, \; q_2 \xrightarrow{\{\{s,w,x\}\}} q_2 \; .$$

$\hat{S}_\Theta$ has the initial and final states, and the following transitions:

$$q_1 \xrightarrow{(\{\{t,u\}\},\{\{t,u\}\})} q_2, \quad q_1 \xrightarrow{(\{\{s,v\}\},\{\{s,v\}\})} q_2, \quad q_1 \xrightarrow{(\{\{s,w,x\}\},\{\{s,w,x\}\})} q_2,$$
$$q_2 \xrightarrow{(\{\{t,u\}\},\{\{t,u\}\})} q_2, \quad q_2 \xrightarrow{(\{\{s,v\}\},\{\{s,v\}\})} q_2, \quad q_2 \xrightarrow{(\{\{s,v\}\},\{\{s,v\},\{s,w,x\}\})} q_2,$$
$$q_2 \xrightarrow{(\{\{s,w,x\}\},\{\{s,w,x\}\})} q_2, \; q_2 \xrightarrow{(\{\{s,w,x\}\},\{\{s,v\},\{s,w,x\}\})} q_2 \; .$$

$\hat{S}'_\Theta$ evolves from $\hat{S}_\Theta$ by adding the following transitions:

$$q_1 \xrightarrow{(\{\{s,v\}\},\{\{s,v\},\{s,w,x\}\})} q_2, \; q_1 \xrightarrow{(\{\{s,w,x\}\},\{\{s,v\},\{s,w,x\}\})} q_2.$$

Obviously, $\hat{S}'_\Theta$ accepts the language of $\hat{S}_\Theta$, and additionally all sequences starting with $(\{\{s,v\}\},\{\{s,v\},\{s,w,x\}\})$ and $(\{\{s,w,x\}\},\{\{s,v\},\{s,w,x\}\})$, respectively. Therefore, φ is not tactic-invariant.

B.2 Example for Non-Singleton Matches in Witness Graphs

We sketch an example producing non-singleton sets of matches: For $i \in \{1,2\}$, let K_i and K'_i be groups with $\lambda_i \in K_i \cap K'_i$. Let $\bowtie'$ be an alignment with $K_1 \bowtie' K_2$ and $K'_1 \bowtie K'_2$. For $i \in \{1,2\}$, let S_i be an FSM accepting the word λ_i. Then, $\mathcal{W}(S_1, S_2, \bowtie')$ has an edge labelled $(\{K_1, K'_1\}, \{K_2, K'_2\})$ from the initial node to a node with matches $\{(K_1, K_2), (K'_1, K'_2)\}$.