

PPT 14

ALJABAR LINEAR



Lesson Outline



INNER PRODUCT

BASIS ORTHONORMAL

GRAM SCHMIT

KOORDINAT BASIS BARU



INNER PRODUCT



INNER PRODUCT??

Hasil kali dalam pada ruang vektor nyata V adalah suatu fungsi $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ yang mengaitkan suatu bilangan real $\langle u, v \rangle$ pada setiap pasangan vektor u dan v dalam V dan memenuhi aksioma-aksioma berikut untuk semua vektor $u, v, w \in V$ dan semua skalar $k \in \mathbb{R}$

1. $\langle u, v \rangle = \langle v, u \rangle$ [aksiom simetris]
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ [aksiom penjumlahan]
3. $\langle ku, v \rangle = k\langle u, v \rangle$ [aksiom homogenitas]
4. $\langle v, v \rangle \geq 0$ dan $\langle v, v \rangle = 0$ jika dan hanya jika $v = 0$ [aksiom positif]



$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$$
$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

karena aksioma $\langle u, v \rangle$ memenuhi perkalian dot dari 2 vektor, maka:

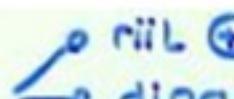
$$\langle u, v \rangle = u \cdot v$$

$$= U_1 \cdot V_1 + U_2 \cdot V_2 + \dots + U_n \cdot V_n$$



Namun, dalam beberapa aplikasi, kita mungkin perlu memodifikasi hasil kali dalam ini dengan memberi BOBOT berbeda pada setiap komponen. Misalnya, mungkin ada kasus di mana beberapa elemen dari vektor memiliki pengaruh lebih besar atau lebih kecil dibandingkan yang lain.

Oleh karena itu, kita memperkenalkan bobot w_1, w_2, \dots, w_n

misal w_1, w_2, \dots, w_n  \oplus dinamakan bobot
atau:
 $\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \dots + w_n u_n v_n$
dinamakan \rightarrow hasil kali dalam Euclidis
yang diboboti dengan bobot w_1, w_2, \dots, w_n

contoh 47: $\begin{cases} \text{lebih fleksibel dibanding } \bar{u}, \bar{v} \\ \langle \bar{u}, \bar{v} \rangle \text{ ~dapat } \in \text{ bilangan matik, pers. Uniter} \end{cases}$
 $\bar{u} = (u_1, u_2)$ } \rightarrow vektor $\in \mathbb{R}^2$, buktikan!
 $\bar{v} = (v_1, v_2)$ } bahwa hasil kali dalam Euclidir
 yg dibuktikan :

$$\langle \bar{u}, \bar{v} \rangle = 3u_1v_1 + 2u_2v_2$$

menyenuhi 4 aksiom pd hal 175
 \rightarrow polanya sama

$$\begin{aligned} ① \text{ ruas kiri} &= \langle \bar{u}, \bar{v} \rangle = 3u_1v_1 + 2u_2v_2 \\ \text{ruas kanan} &= \langle \bar{v}, \bar{u} \rangle = 3u_1v_1 + 2u_2v_2 \\ \Rightarrow \text{ruas kiri} &= \text{ruas kanan, mk aksioma 1 terpenuhi} \end{aligned}$$

$$② \langle \bar{u} + \bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$$

Kiri $\Rightarrow \langle \bar{u} + \bar{v}, \bar{w} \rangle = 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2$

$$\begin{aligned} &= 3u_1w_1 + 3v_1w_1 + 2u_2w_2 + 2v_2w_2 \\ &= 3u_1w_1 + 2u_2w_2 + 3v_1w_1 + 2v_2w_2 \end{aligned}$$

Kanan $\Rightarrow \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$
 $= 3u_1w_1 + 2u_2w_2 + 3v_1w_1 + 2v_2w_2$

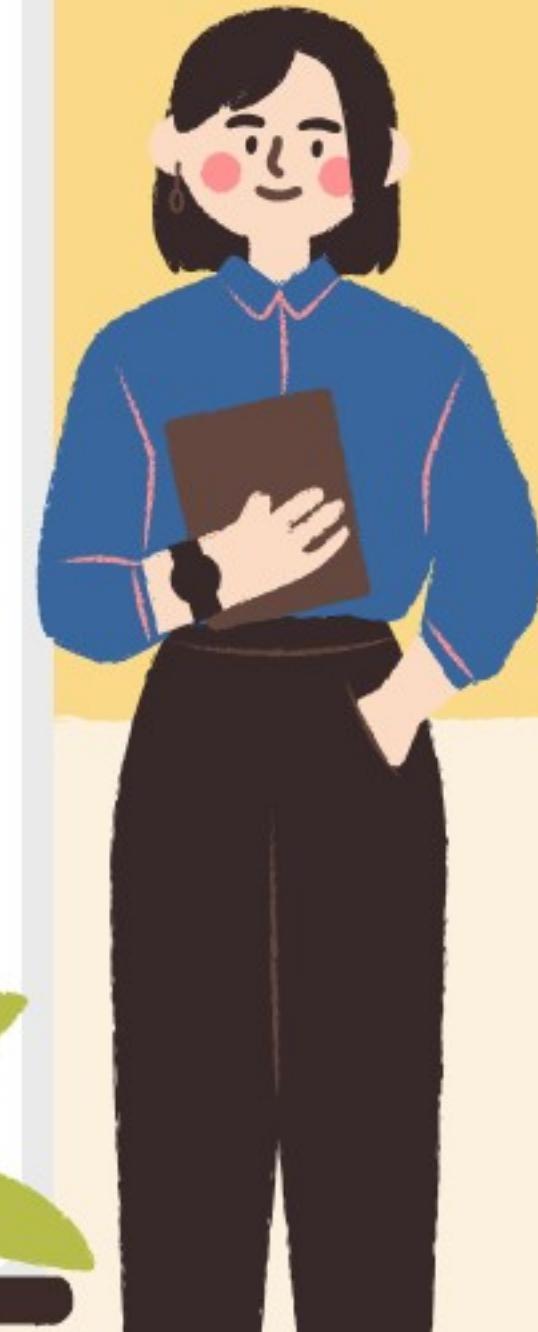
Kiri = Kanan \Rightarrow mk aksioma 2 terpenuhi

$$③ \langle k\bar{u}, \bar{w} \rangle = k \langle \bar{u}, \bar{w} \rangle$$

Kiri $\Rightarrow \langle k\bar{u}, \bar{w} \rangle = 3k\bar{u}_1\bar{w}_1 + 2k\bar{u}_2\bar{w}_2$

Kanan $\Rightarrow k \langle \bar{u}, \bar{w} \rangle = k(3\bar{u}_1\bar{w}_1 + 2\bar{u}_2\bar{w}_2)$
 $= 3k\bar{u}_1\bar{w}_1 + 2k\bar{u}_2\bar{w}_2$

Kiri = Kanan \Rightarrow mk aksioma 3 terpenuhi





→ $\text{kanan} \sim 4.22$ (b)

④ $\langle \vec{v}, \vec{v} \rangle = 3 v_1 v_1 + 2 v_2 v_2$
 $= 3 v_1^2 + 2 v_2^2$

jelas bahwa $3 v_1^2 + 2 v_2^2 \geq 0$

selanjutnya $\langle \vec{v}, \vec{v} \rangle = 3 v_1^2 + 2 v_2^2 = 0$ jika $v_1 = v_2 = 0$

contoh 99: ~ what buku

$$\vec{u} = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

rumus berikut mendefinisikan hasil kali dalam pd M_{22}

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

misal:

$$\bar{u} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

mis:

$$\begin{aligned}\langle \bar{u}, \bar{v} \rangle &= 1(-1) + 2(0) + 3(3) + 4(2) \\ &= 16\end{aligned}$$

contoh so:

$$P = a_0 + a_1 x + a_2 x^2$$

$$q = b_0 + b_1 x + b_2 x^2$$

mis hasil kali dalam pd P₂ adalah :

$$\langle P, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

$$\|P\| = \sqrt{a_0^2 + a_1^2 + a_2^2}$$



$$\begin{aligned} \textcircled{1} \quad \|\bar{u}\| &= \sqrt{\langle \bar{u}, \bar{u} \rangle}^{\frac{1}{2}} \rightarrow \text{if ada bobot} \\ &= \sqrt{\langle \bar{u}, \bar{u} \rangle}^{\frac{1}{2}} \\ &= \sqrt{\bar{u} \cdot \bar{u}} \\ &= \sqrt{u_1^2 + u_2^2} \end{aligned}$$

} \rightarrow if \neq bobot (biasa)

$\textcircled{2}$ jarak antar 2 vektor $\bar{u} \times \bar{v} \Rightarrow d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$

$\textcircled{3}$ aturan ketaksamaan Cauchy - Schwarz

$$\langle \bar{u}, \bar{v} \rangle^2 \leq \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

ada bobot $\rightarrow = \langle \bar{u} - \bar{v}, \bar{u} - \bar{v} \rangle^{\frac{1}{2}}$

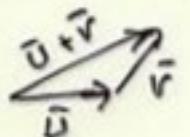
sifat dasar panjang

$$\|\bar{u}\| \geq 0$$

$$\|\bar{u}\| = 0 \text{ if } \bar{u} = 0$$

$$\|k\bar{u}\| = |k| \|\bar{u}\|$$

$$\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$$



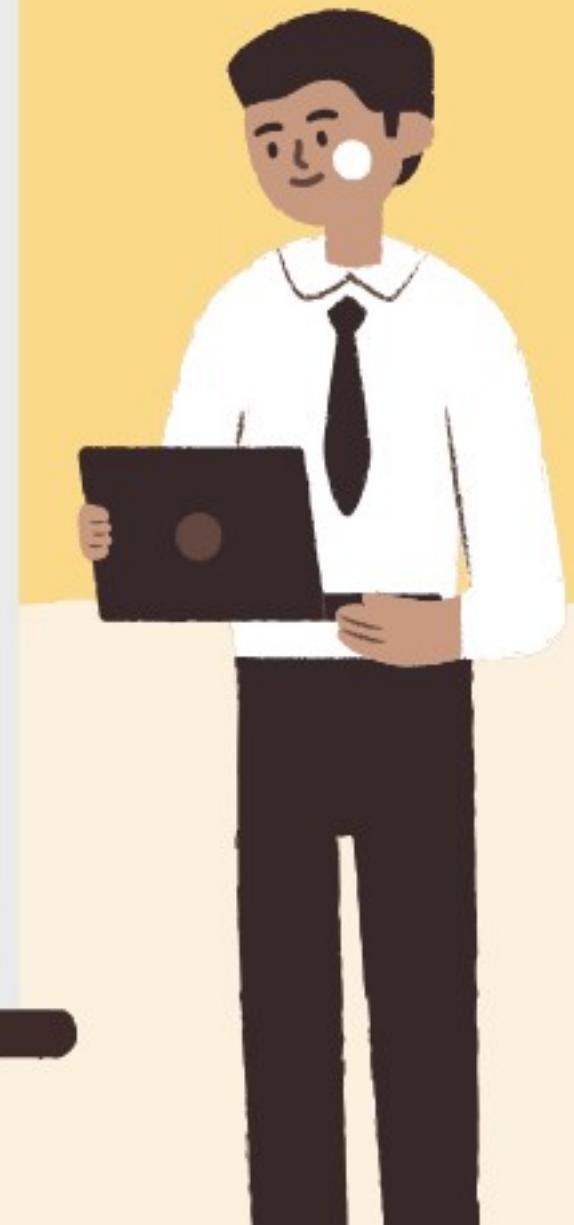
sifat dasar jarak

$$d(\bar{u}, \bar{v}) \geq 0$$

$$d(\bar{u}, \bar{v}) = 0 \text{ if } \bar{u} = \bar{v}$$

$$d(\bar{u}, \bar{v}) = d(\bar{v}, \bar{u})$$

$$d(\bar{u}, \bar{v}) \leq d(\bar{u}, \bar{w}) + d(\bar{w}, \bar{v})$$





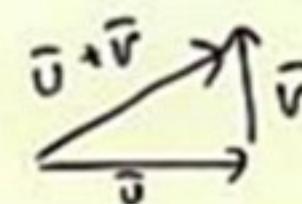
o) Sudut antara $\bar{u} \times \bar{v}$

$$\cos \theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|}$$

o) $\bar{u} \times \bar{v}$ saing^o \perp if $\Rightarrow \langle \bar{u}, \bar{v} \rangle = 0$

o) if $\bar{u} \perp \bar{v}$, then:

$$\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$$



Contoh 55 :

$$\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pola $\langle \bar{u}, \bar{v} \rangle = 3 u_1 v_1 + 2 u_2 v_2$



Contoh 55 :

$$\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

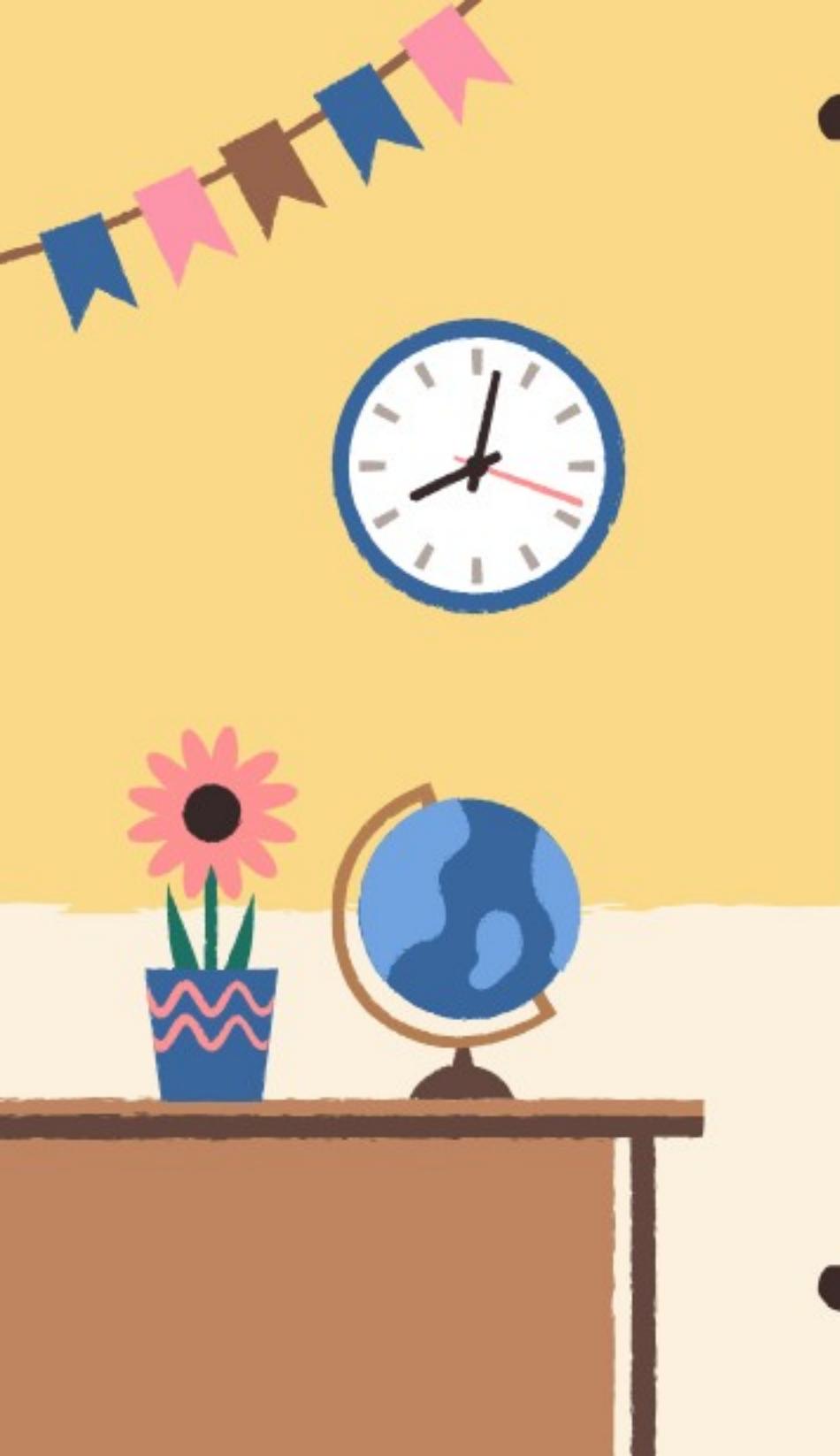
pola $\langle \bar{u}, \bar{v} \rangle = 3 u_1 v_1 + 2 u_2 v_2$

$$\begin{aligned}\|\bar{u}\| &= \langle \bar{u}, \bar{u} \rangle^{1/2} \\ &= (3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0)^{1/2} \\ &= (3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0)^{1/2} \\ &= \sqrt{3}\end{aligned}$$

$$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$$

$$\begin{aligned}&= \langle \bar{u} - \bar{v}, \bar{u} - \bar{v} \rangle^{1/2} \\ &= (3 \cdot u_1 v_1 + 2 \cdot u_2 v_2)^{1/2} \quad \begin{cases} u_1 = u_1 - v_1 \\ u_2 = u_2 - v_2 \end{cases} \\ &= (3 \cdot 1 \cdot 1 + 2 \cdot -1 \cdot -1)^{1/2} \\ &= (3 + 2)^{1/2} \\ &= \sqrt{5}\end{aligned}$$





Example 2 Let \mathbb{R}^4 have the Euclidean inner product. Find the cosine of the angle θ between the vectors $u = (4, 3, 1, -2)$ and $v = (-2, 1, 2, 3)$.

Solution. We leave it for the reader to verify that

$$\|u\| = \sqrt{30}, \quad \|v\| = \sqrt{18}, \quad \text{and} \quad \langle u, v \rangle = -9$$

so that

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\|\|v\|} = -\frac{9}{\sqrt{30}\sqrt{18}} \approx -\frac{3}{2\sqrt{15}}$$

INNER PRODUCT dikatakan ORTHOGONAL, jika:

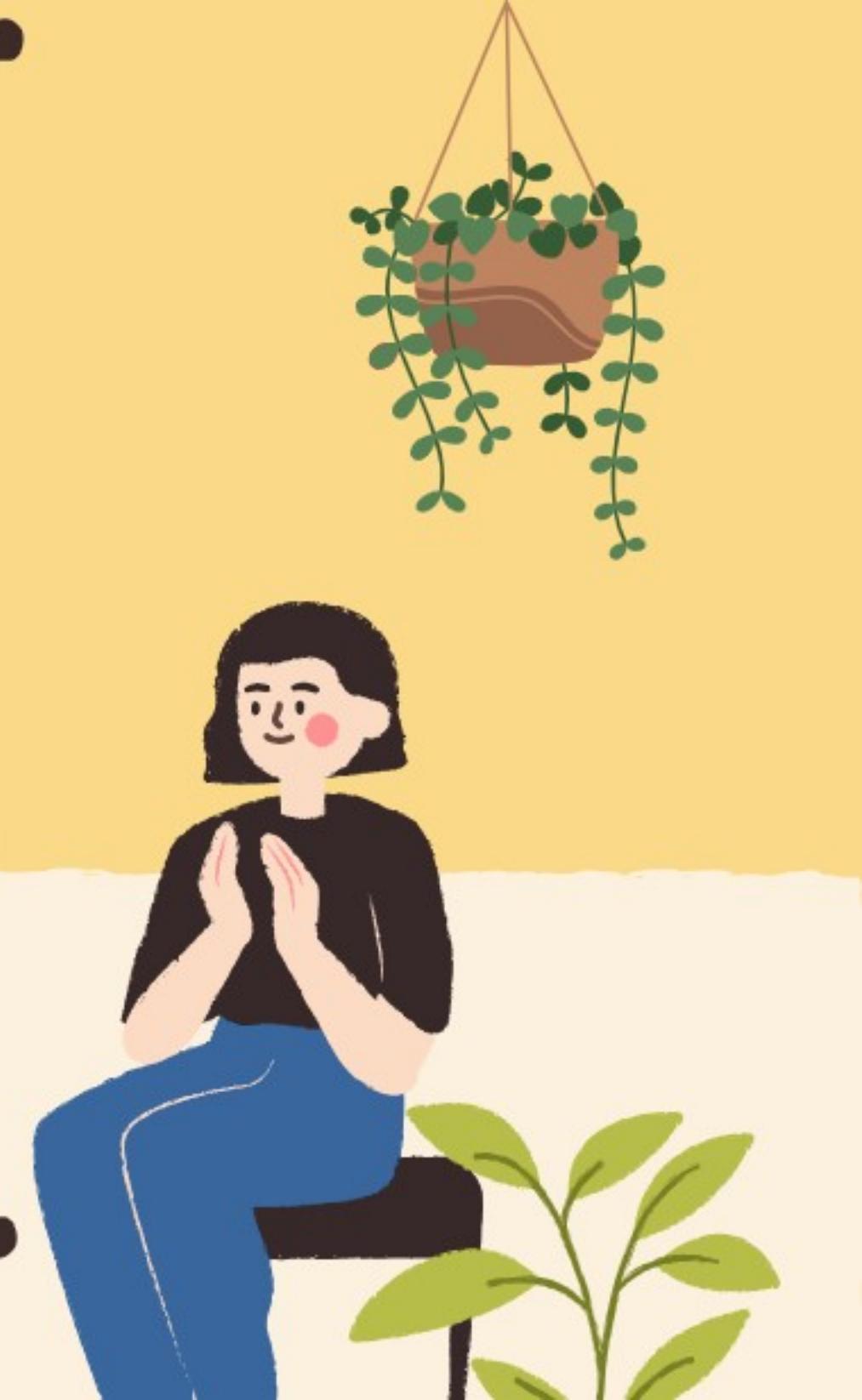
$$\langle u, v \rangle = 0$$

Example 3 If M_{22} has the inner product of Example 7 in the preceding section, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal, since

$$\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$$



Definition. Let W be a subspace of an inner product space V . A vector u in V is said to be *orthogonal to W* if it is orthogonal to every vector in W , and the set of all vectors in V that are orthogonal to W is called the *orthogonal complement of W* .

Theorem 6.2.6. If A is an $m \times n$ matrix, then:

- (a) The nullspace of A and the row space of A are orthogonal complements in \mathbb{R}^n with respect to the Euclidean inner product.
- (b) The nullspace of A^T and the column space of A are orthogonal complements in \mathbb{R}^m with respect to the Euclidean inner product.





Example 6 Let W be the subspace of \mathbb{R}^5 spanned by the vectors

$$w_1 = (2, 2, -1, 0, 1), \quad w_2 = (-1, -1, 2, -3, 1),$$

$$w_3 = (1, 1, -2, 0, -1), \quad w_4 = (0, 0, 1, 1, 1)$$

Find a basis for the orthogonal complement of W .



Example 6 Let W be the subspace of \mathbb{R}^5 spanned by the vectors

$$\begin{aligned}w_1 &= (2, 2, -1, 0, 1), & w_2 &= (-1, -1, 2, -3, 1), \\w_3 &= (1, 1, -2, 0, -1), & w_4 &= (0, 0, 1, 1, 1)\end{aligned}$$

Find a basis for the orthogonal complement of W .

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and by part (a) of Theorem 6.2.6 the nullspace of A is the orthogonal complement of W . In Example 4 of Section 5.5 we showed that

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for this nullspace. Expressing these vectors in the same notation as w_1 , w_2 , w_3 , and w_4 , we conclude that the vectors

$$v_1 = (-1, 1, 0, 0, 0) \quad \text{and} \quad v_2 = (-1, 0, -1, 0, 1)$$

form a basis for the orthogonal complement of W . As a check, the reader may want to verify that v_1 and v_2 are orthogonal to w_1 , w_2 , w_3 , and w_4 by calculating the necessary dot products.

BASIS
ORTHO
NORMAL



BASIS ORTHONORMAL

- Cara menentukan basis
 - Cari basis sendiri → basis baru
 - Dari vektor-vektor yang ada → basis dari vektor-vektor lama
- Ortonormal
 - Ortogonal
 - Tiap vektor normanya (panjangnya) 1
- Normalisasi → proses membagi V dengan panjangnya $\|V\|$ agar normanya 1



Example I Let

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = (1, 0, 1), \quad \mathbf{u}_3 = (1, 0, -1)$$

and assume that \mathbb{R}^3 has the Euclidean inner product. It follows that the set of vector $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$.

Example I Let

$$u_1 = (0, 1, 0), \quad u_2 = (1, 0, 1), \quad u_3 = (1, 0, -1)$$

and assume that \mathbb{R}^3 has the Euclidean inner product. It follows that the set of vectors $S = \{u_1, u_2, u_3\}$ is orthogonal since $\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0$.

If v is a nonzero vector in an inner product space, then by part (c) of Theorem 6.2.2 the vector

$$\frac{1}{\|v\|} v$$

has norm 1, since

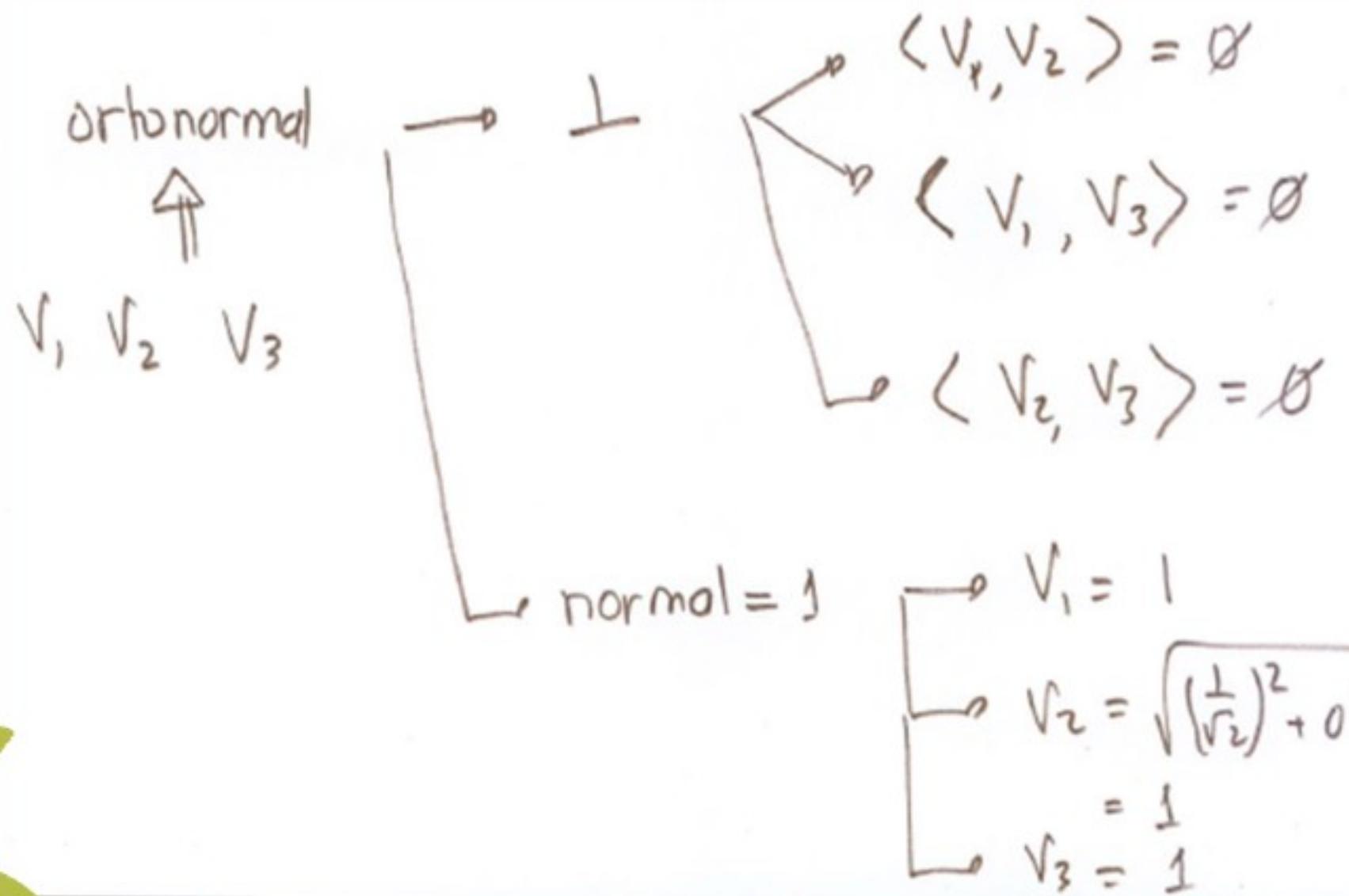
$$\left\| \frac{1}{\|v\|} v \right\| = \left\| \frac{1}{\|v\|} \right\| \|v\| = \frac{1}{\|v\|} \|v\| = 1$$

The process of multiplying a nonzero vector v by the reciprocal of its length to obtain a vector of norm 1 is called *normalizing* v . An orthogonal set of nonzero vectors can always be converted to an orthonormal set by normalizing each of its vectors.

$$V_1 = \frac{U_1}{|U_1|} = \frac{(0, 1, 0)}{1} = (0, 1, 0)$$

$$V_2 = \frac{U_2}{|U_2|} = \frac{(1, 0, 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$V_3 = \frac{U_3}{|U_3|} = \frac{(1, 0, -1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$



Example 2 The Euclidean norms of the vectors in Example 1 are

$$\|u_1\| = 1, \quad \|u_2\| = \sqrt{2}, \quad \|u_3\| = \sqrt{2}$$

Consequently, normalizing u_1 , u_2 , and u_3 yields

$$v_1 = \frac{u_1}{\|u_1\|} = (0, 1, 0), \quad v_2 = \frac{u_2}{\|u_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right),$$

$$v_3 = \frac{u_3}{\|u_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

We leave it for you to verify that the set $S = \{v_1, v_2, v_3\}$ is orthonormal by showing

That $\langle v_1, v_2 \rangle = \langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$

$$\|v_1\| = \|v_2\| = \|v_3\| = 1$$

Keuntungan Orthonormal

Theorem 6.3.1. If $S = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for an inner product space V , and u is any vector in V , then

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots + \langle u, v_n \rangle v_n$$



Kita bisa langsung mencari $k_1, k_2, k_3, \dots, k_n$ untuk kombinasi linier



teorema 2.3 :

$\rightarrow \overline{S} = \{\overline{v}_1, \overline{v}_2, \dots, \overline{v}_n\} \rightsquigarrow$ basis ortonormal
 nyatakan bahwa vektor \overline{u} adalah (kombinasi) linear
 dr vektor $\in S$, mt:

$$\Rightarrow \bar{u} = k_1 \bar{v}_1 + k_2 \bar{v}_2 + \dots + k_n \bar{v}_n$$

$\downarrow \quad \downarrow \quad \downarrow$

$\langle \bar{u}, \bar{v}_1 \rangle \quad \langle \bar{u}, \bar{v}_2 \rangle \quad \langle \bar{u}, \bar{v}_n \rangle$

k_1, k_2, \dots, k_n
 bisq dicari
 spt ini, k_n
 v_1, v_2, \dots, v_n \nwarrow saling \perp
 \circ norma = 1

contoh 62:

$$\vec{v}_1 = (0, 1, 0) \quad \vec{v}_2 = (-\frac{4}{5}, 0, \frac{3}{5}) \quad \vec{v}_3 = (\frac{3}{5}, 0, \frac{4}{5})$$

$s = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ y es una base orthonormal \rightarrow orthogonal

nyatakan $\vec{u} = (1, 1, 1)$ sebagai kombinasi linier dari himp. S

$$\vec{v}_1 = (0, 1, 0) \quad \vec{v}_2 = (-\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}) \quad \vec{v}_3 = (\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$$

$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ y ~ basis orthonormal

→ orthogonal

normal = 1

nyatakan $\vec{u} = (1, 1, 1)$ sbg kombinasi linier dr himp. S

$$\vec{u} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3$$

$$k_1 = \langle \vec{u}, \vec{v}_1 \rangle = 1 ; \quad k_2 = \langle \vec{u}, \vec{v}_2 \rangle = -\frac{1}{\sqrt{5}} ; \quad k_3 = \langle \vec{u}, \vec{v}_3 \rangle = \frac{2}{\sqrt{5}}$$

maka: $\vec{u} = \vec{v}_1 - \frac{1}{\sqrt{5}} \vec{v}_2 + \frac{2}{\sqrt{5}} \vec{v}_3$

o) basis orthonormal

contoh 63 :

$$\bar{v}_1 = (0, 1, 0)$$

o merentang

o bebas linier

- o saling \perp
- o norma = 1

$$\bar{v}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \quad \bar{v}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$$

o merentang $\rightarrow \bar{u} = k_1 \bar{v}_1 + k_2 \bar{v}_2 + k_3 \bar{v}_3$

o bebas linier $\rightarrow 0 = k_1 \bar{v}_1 + k_2 \bar{v}_2 + k_3 \bar{v}_3$

siswa:

o merentang

o mis : $\bar{u} = (1, 2, 3)$, mt $\begin{cases} k_1 = \langle \bar{u}, \bar{v}_1 \rangle = 2 \\ k_2 = \langle \bar{u}, \bar{v}_2 \rangle = \frac{4}{\sqrt{2}} \\ k_3 = \langle \bar{u}, \bar{v}_3 \rangle = -\frac{2}{\sqrt{2}} \end{cases}$

$$mt = (1, 2, 3) = 2 \bar{v}_1 + \frac{4}{\sqrt{2}} \bar{v}_2 - \frac{2}{\sqrt{2}} \bar{v}_3$$

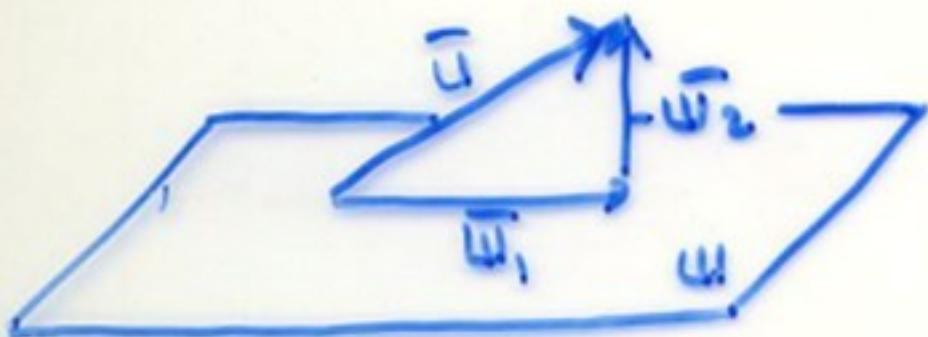
o mis : $\bar{u} = (2, 2, 2)$, mt $\begin{cases} k_1 = 2 \\ k_2 = \frac{4}{\sqrt{2}} \\ k_3 = 0 \end{cases}$

$$mt = (2, 2, 2) = 2 \bar{v}_1 + \frac{4}{\sqrt{2}} \bar{v}_2$$

bebas linier $\rightarrow 0 = (0, 0, 0)$, mt $\begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$

$$mt = (0, 0, 0) = 0 \bar{v}_1 + 0 \bar{v}_2 + 0 \bar{v}_3$$

c) projekcií



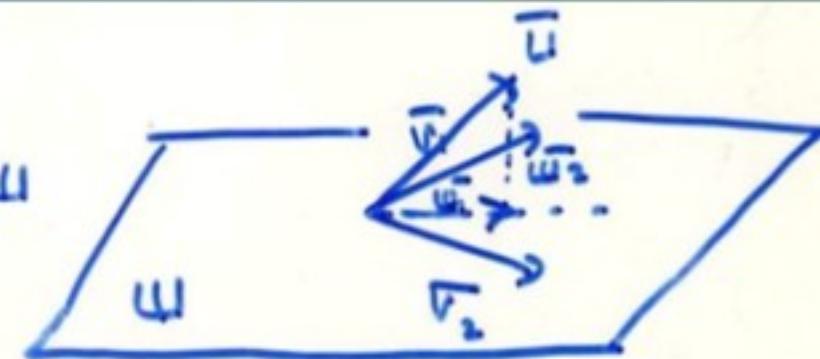
$$u = \bar{w}_1 + \bar{w}_2$$

\bar{w}_1 je projekcií ortogonalní \bar{u} pd \mathbb{W}
Proy $_{\mathbb{W}}$ $\bar{u} = \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 + \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2 + \dots + \langle \bar{u}, \bar{v}_n \rangle \bar{v}_n$

\bar{w}_2 komp. \bar{u} yo ortogonal thp \mathbb{W}
 $\bar{u} - \text{Proy}_{\mathbb{W}} \bar{u} = \bar{u} - \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 - \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2 - \dots - \langle \bar{u}, \bar{v}_n \rangle \bar{v}_n$

contoh 69:

$$\begin{aligned}\bar{v}_1 &= \langle 0, 1, 0 \rangle \\ \bar{v}_2 &= \left\langle -\frac{4}{5}, 0, \frac{3}{5} \right\rangle\end{aligned}$$



① Proyeksi ortogonal $\bar{u} = \langle 1, 1, 1 \rangle$ pd \mathbb{W} adalah :

$$\begin{aligned}\bar{v}_1 &= \text{Proy}_{\mathbb{W}} \bar{u} = \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 + \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2 \\ &= \bar{v}_1 + -\frac{1}{5} \bar{v}_2 \\ &= \langle 0, 1, 0 \rangle + \left\langle \frac{4}{25}, 0, -\frac{3}{25} \right\rangle \\ &= \left\langle \frac{4}{25}, 1, -\frac{3}{25} \right\rangle\end{aligned}$$

4.27

① komp \bar{u} yg ortogonal thp W adalah :

$$\begin{aligned}\bar{u}_2 &= \bar{u} - \text{proy}_W \bar{u} = (1, 1, 1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right) \\ &= \left(\frac{21}{25}, 0, \frac{28}{25}\right)\end{aligned}$$

\bar{u}_2 \rightarrow orthogonal thp \bar{v}_1 & \bar{v}_2

$\bar{u}_2 \perp$ thp setiap vektor pd W yg direntang \bar{v}_1 & \bar{v}_2

Teorema 2.6 :

~ setiap ruang hasil kali dalam berdimensi berhingga tak nol mempunyai sebuah basis ortonormal.

~ $S = \{\bar{u}_1, \bar{u}_2, \bar{u}_3, \dots, \bar{u}_n\}$ ~ basis binaan ~ di ket

$R = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n\}$ ~ basis ortonormal

~ yg dicari :

GRAM
SCHMIT



GRAM-SCHMIT

contoh 65:

terapkan proses Gram Schmit u mentransfor -
masikan basis

$$U_1 = (1, 1, 1)$$

$$U_2 = (0, 1, 1)$$

$$U_3 = (0, 0, 1)$$

} ke alm basis orthonormal
 $\begin{matrix} \uparrow \sqrt{3} \\ U_1 = \sqrt{3} \end{matrix}$

Metode ini
digunakan
untuk :
Mencari basis
orthonormal





contoh 65:

terapkan proses Gram Schmidt untuk transformasi

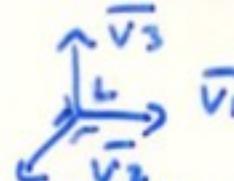
masukan basis

$$\bar{U}_1 = (1, 1, 1)$$

$$\bar{U}_2 = (0, 1, 1)$$

$$\bar{U}_3 = (0, 0, 1)$$

ke dalam basis orthonormal



jwb:

$S = \{\bar{U}_1, \bar{U}_2, \bar{U}_3\}$ } \rightarrow basis biasa
 $B = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ } \rightarrow basis orthonormal yg akan dicari

$$\text{langkah 1} \rightarrow \bar{V}_1 = \frac{\bar{U}_1}{\|\bar{U}_1\|} = \frac{(1, 1, 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\text{langkah 2} \rightarrow \bar{U}_2 - \text{Proy}_{\bar{V}_1} \bar{U}_2 = \bar{U}_2 - \langle \bar{U}_2, \bar{V}_1 \rangle \bar{V}_1 \\ = (0, 1, 1) - \frac{3}{\sqrt{3}} (1, 1, 1) \\ = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

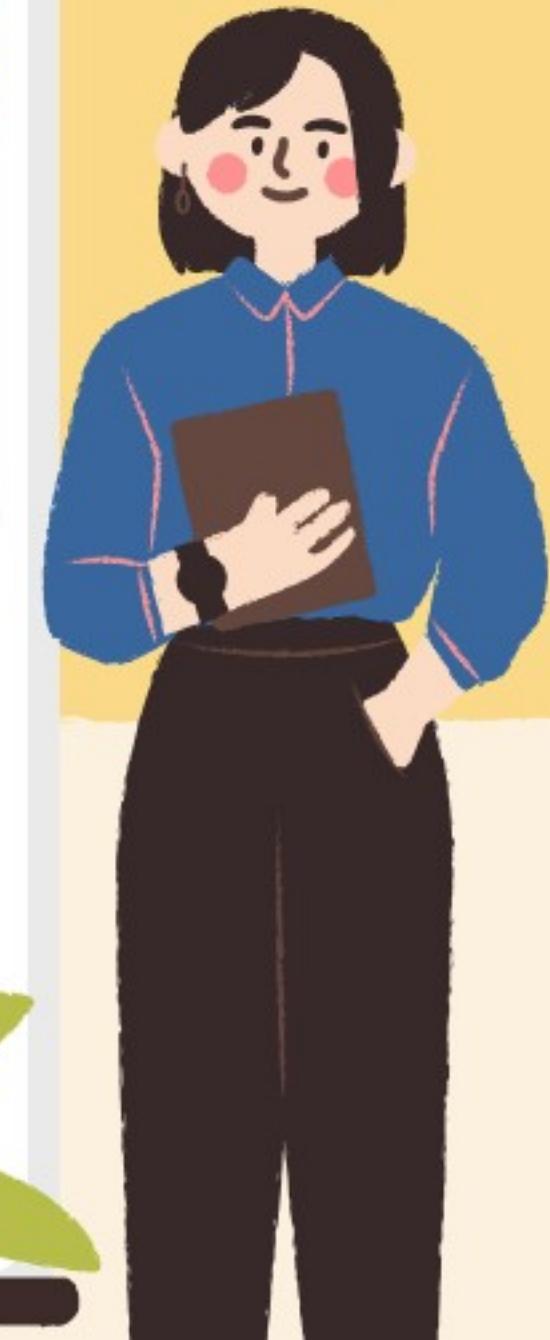
$$\text{mt} \rightarrow \bar{V}_2 = \frac{\bar{U}_2 - \text{Proy}_{\bar{V}_1} \bar{U}_2}{\|\bar{U}_2 - \text{Proy}_{\bar{V}_1} \bar{U}_2\|} = \frac{3}{\sqrt{6}} (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\ = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

langkah 3 $\rightarrow \bar{U}_3 - \text{Proy}_{\bar{V}_2} \bar{U}_3 = \bar{U}_3 - \langle \bar{U}_3, \bar{V}_1 \rangle \bar{V}_1 - \langle \bar{U}_3, \bar{V}_2 \rangle \bar{V}_2$
 $= (0, 0, 1) - Y_{\bar{V}_3} (Y_{\bar{V}_3}, Y_{\bar{V}_2}, Y_{\bar{V}_1}) - Y_{\bar{V}_2} (-Y_{\bar{V}_3}, Y_{\bar{V}_2}, Y_{\bar{V}_1})$
 $= (0, -Y_2, Y_2)$

Rk:
 $v_3 = \frac{\bar{U}_3 - \text{Proy}_{\bar{V}_2} \bar{U}_3}{\|\bar{U}_3 - \text{Proy}_{\bar{V}_2} \bar{U}_3\|} = \sqrt{2} (0, -Y_2, Y_2)$
 $= (0, -Y\sqrt{2}, Y\sqrt{2})$

$\therefore \bar{V}_1, \bar{V}_2, \bar{V}_3$ membentuk basis orthonormal $\subseteq \mathbb{R}^3$

$$V_1 = \frac{U_1}{\|U_1\|} = \frac{(1, 1, 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$



$$V_2 = \frac{U_2 - \langle U_2, V_1 \rangle V_1}{\|U_2 - \langle U_2, V_1 \rangle V_1\|}$$

$$\begin{aligned} U_2 - \langle U_2, V_1 \rangle \cdot V_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_2 &= \frac{(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})}{\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}} (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\ &= (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \end{aligned}$$



$$V_3 = \frac{U_3 - \langle U_3, V_1 \rangle V_1 - \langle U_3, V_2 \rangle V_2}{\| \dots \dots \dots \dots \dots \|}$$

$$V_3 = (0, 0, 1) - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \frac{1}{\sqrt{6}} \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right)$$

$$V_3 = \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\| \dots \dots \dots \|}$$

$$V_3 = \sqrt{2} (0, -\frac{1}{2}, \frac{1}{2})$$

$$V_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$



Terapkan proses Gram Schimt untuk mentransformasikan basis u_1, u_2, u_3 ke dalam basis ortonormal.

u_1	=	-4	8	2
u_2	=	7	-3	6
u_3	=	6	-3	7

CONTOH SOAL



Terapkan proses Gram Schimt untuk mentransformasikan basis u_1, u_2, u_3 ke dalam basis ortonormal.

u_1	=	-4	8	2
u_2	=	7	-3	6
u_3	=	6	-3	7

$$v_1 = \frac{u_1}{|u_1|}$$

$$|u_1| = 9.17$$

$$v_1 = \boxed{-0.44 \quad 0.87 \quad 0.22}$$

CONTOH SOAL



CONTOH SOAL



$$v_2 = \frac{u_2 - \langle u_2, v_1 \rangle \cdot v_1}{|u_2 - \langle u_2, v_1 \rangle \cdot v_1|}$$

$$\langle u_2, v_1 \rangle = -4.37$$

$$\frac{u_2 - \langle u_2, v_1 \rangle \cdot v_1}{|u_2 - \langle u_2, v_1 \rangle \cdot v_1|} = \begin{array}{ccc} 5.08 & 0.80 & 6.96 \end{array}$$

$$|u_2 - \langle u_2, v_1 \rangle \cdot v_1| = \begin{array}{c} 8.65 \end{array}$$

$$V_2 = \begin{array}{c} 0.59 \\ 0.09 \\ 0.80 \end{array} \quad |v_2| = 0.998098$$

CONTOH SOAL



$$v_3 = \frac{u_3 - \langle u_3, v_2 \rangle v_2 - \langle u_3, v_1 \rangle v_1}{|u_3 - \langle u_3, v_2 \rangle v_2 - \langle u_3, v_1 \rangle v_1|}$$

$$\langle u_3, v_2 \rangle = 8.87$$

$$\langle u_3, v_2 \rangle V_2 =$$

5.23	0.80	7.10
------	------	------

$$\langle u_3, v_1 \rangle = -3.71$$

$$\langle u_3, v_1 \rangle V_1 =$$

1.63	-3.23	-0.82
------	-------	-------

$$u_3 - \langle u_3, v_2 \rangle v_2 - \langle u_3, v_1 \rangle v_1 =$$

-0.86	-0.57	0.72
-------	-------	------

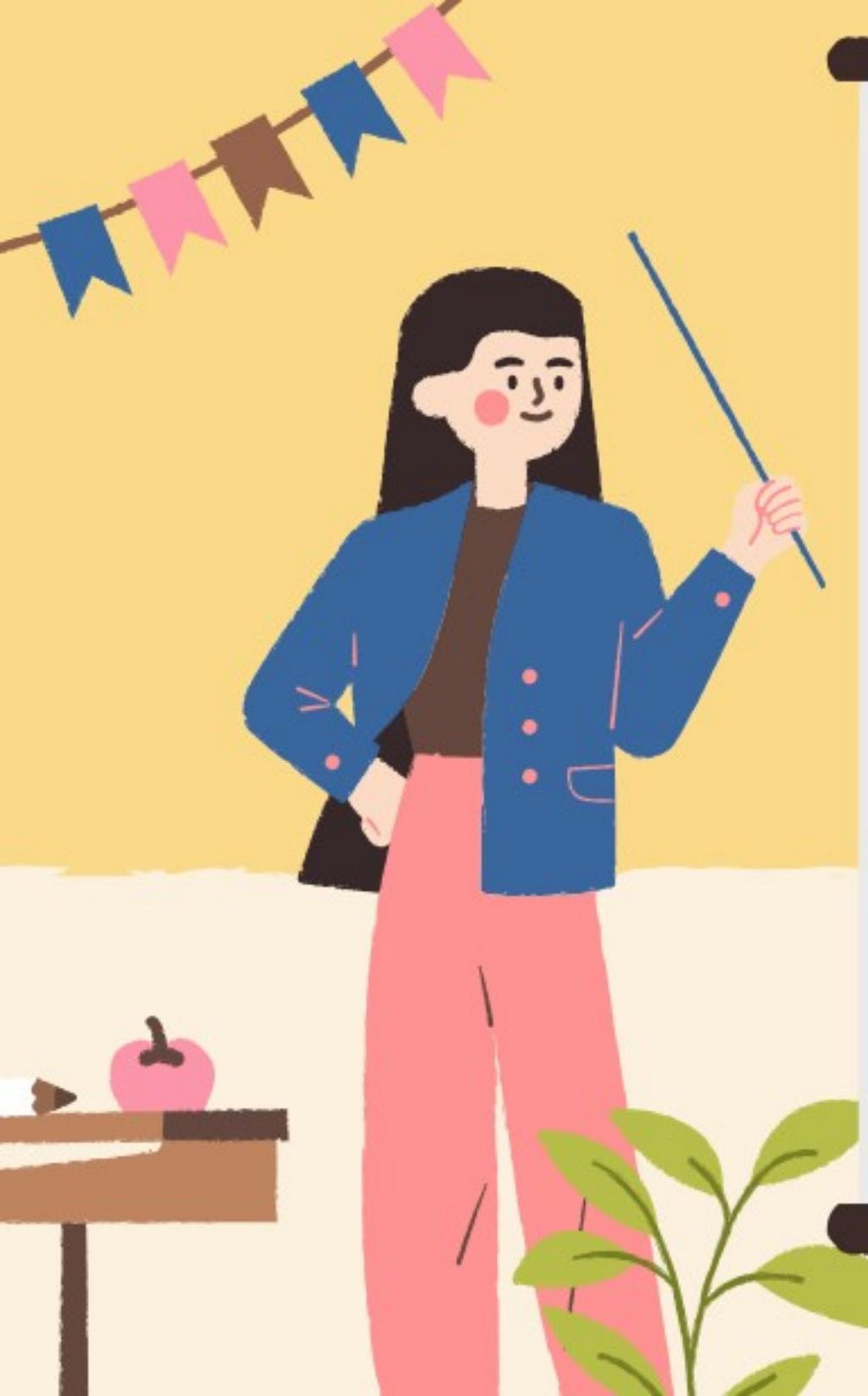
$$|u_3 - \langle u_3, v_2 \rangle v_2 - \langle u_3, v_1 \rangle v_1|$$

1.26

v3

-0.68	-0.45	0.57
-------	-------	------

$$|v_3| = 0.994887$$



6.5 Orthogonal matrix ; change of Basis

Orthogonal matrix :

$$\begin{array}{l} \text{• } A \cdot A^T = A^T \cdot A = I \\ \text{• } A^{-1} = A^T \end{array}$$

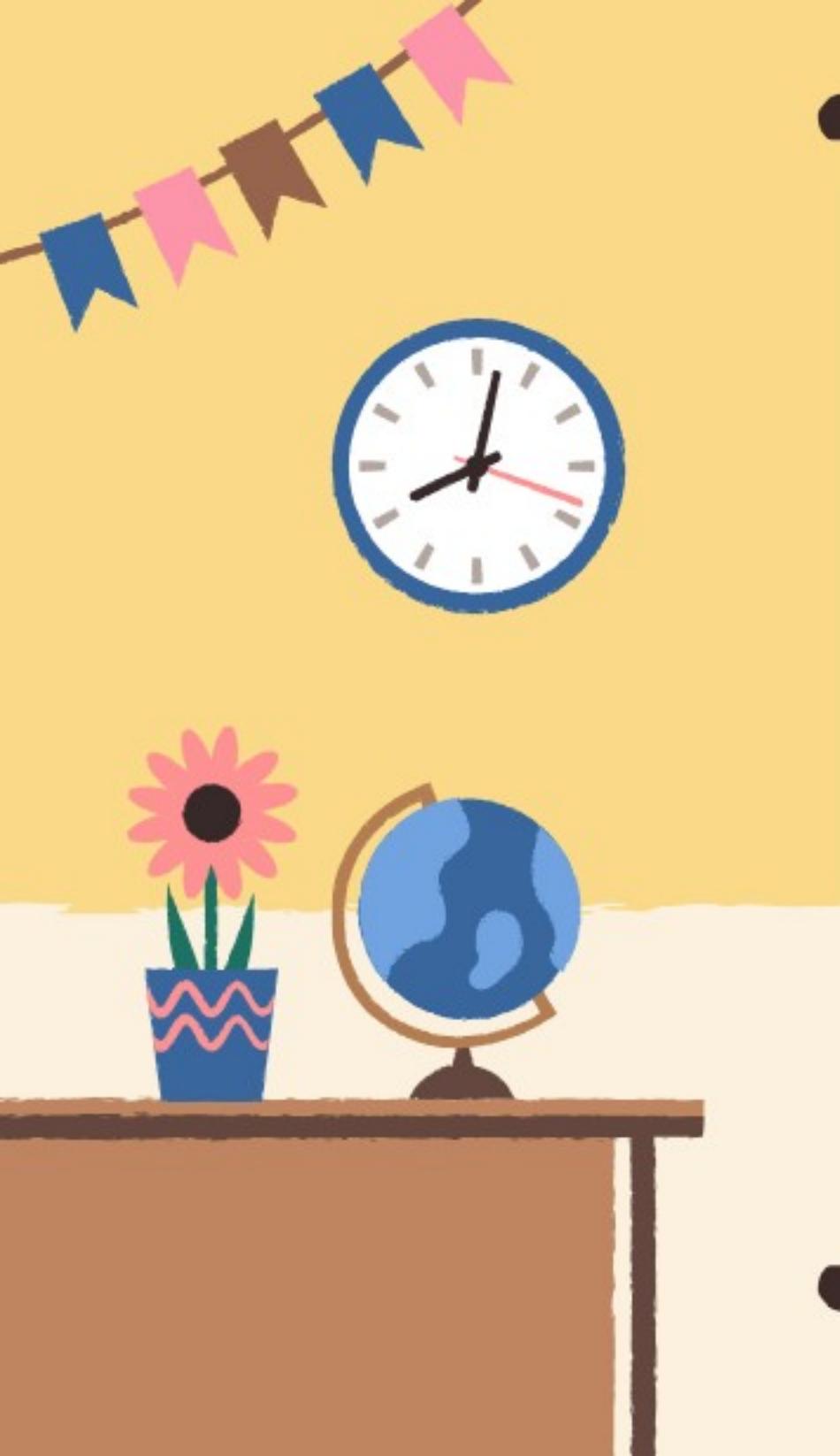
Example 1 The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal, since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





TEOREMA

Theorem 6.5.1. *The following are equivalent for an $n \times n$ matrix A .*

- (a) *A is orthogonal.*
- (b) *The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.*
- (c) *The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

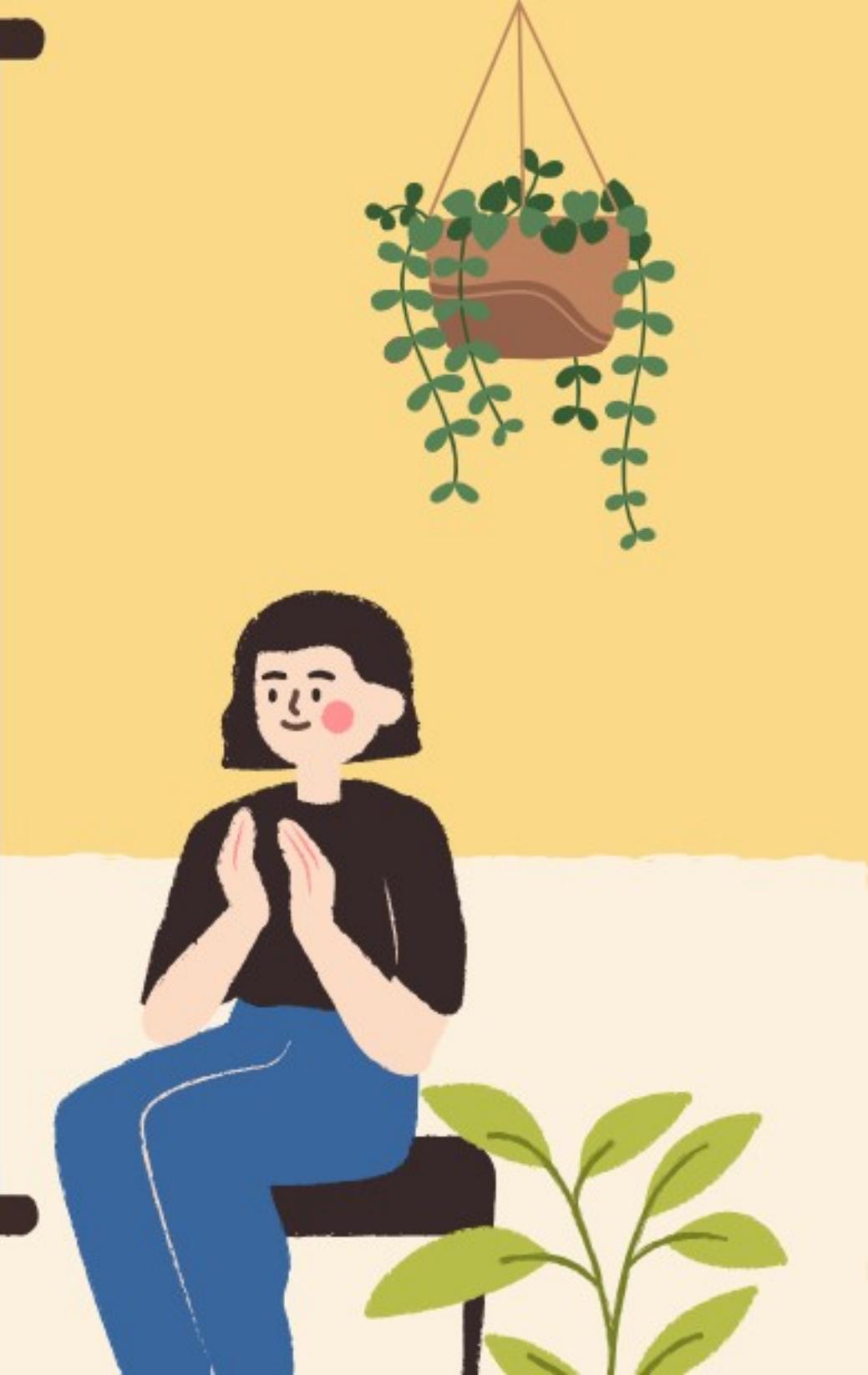
Contoh:

Show that

$$A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

is an orthogonal matrix by

- (a) calculating $A^T A$
- (b) using part (b) of theorem 6.5.1
- (c) using part (c) of theorem 6.5.1



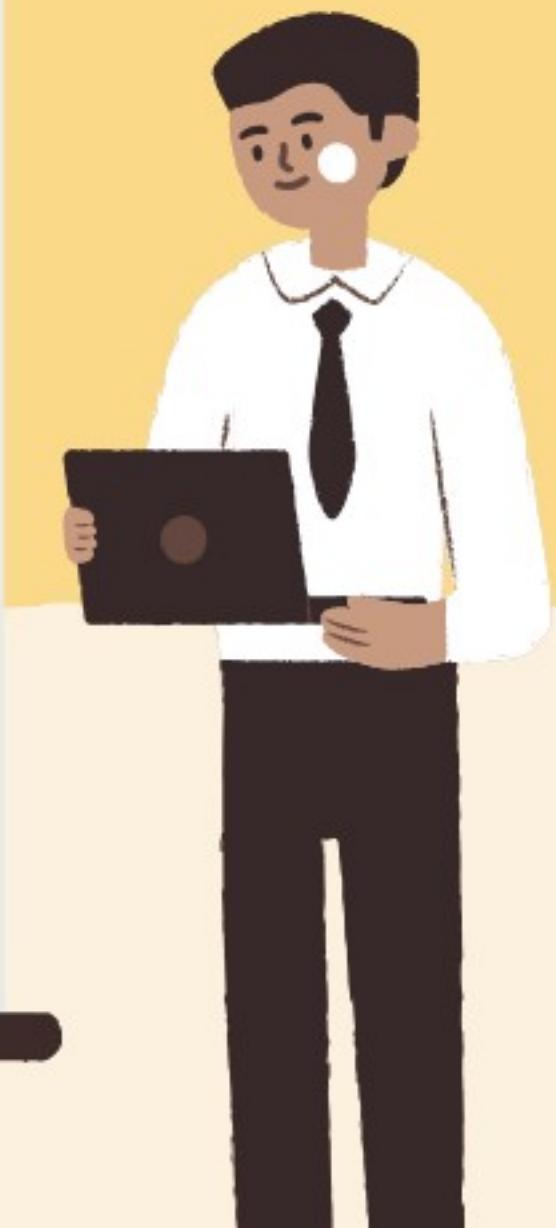
Jawab :

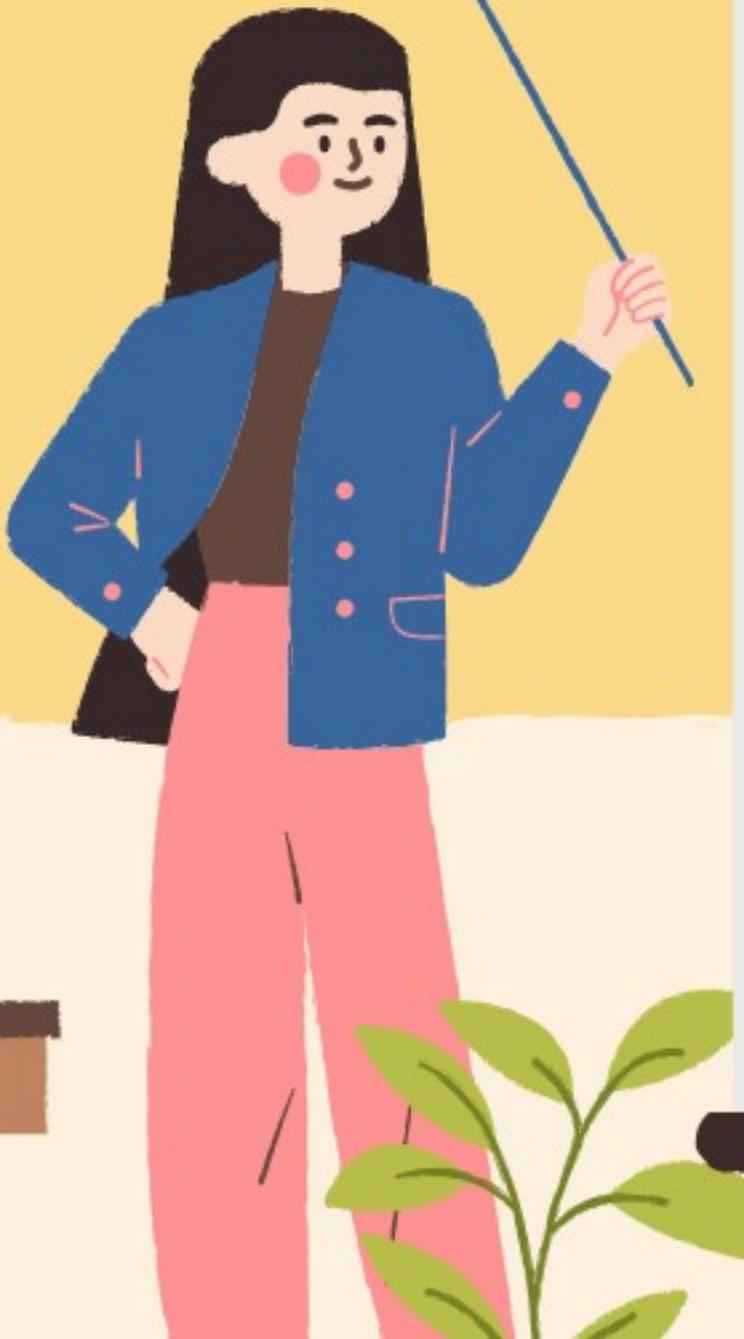
(a) $A^T \cdot A$

$$= \begin{bmatrix} \frac{4}{5} & -\frac{3}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \cdot \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{3}{25} & \frac{3}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

CONTOH SOAL





(b) row vektor membentuk orthonormal

$$r_1 = \left\{ \frac{4}{5}, 0, -\frac{3}{5} \right)$$

$$r_2 = \left(-\frac{9}{5}, \frac{4}{5}, -\frac{12}{25} \right)$$

$$r_3 = \left(\frac{12}{25}, \frac{3}{5}, \frac{16}{25} \right)$$

(c) column vektor membentuk orthonormal

$$C_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{9}{25} \\ \frac{12}{25} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

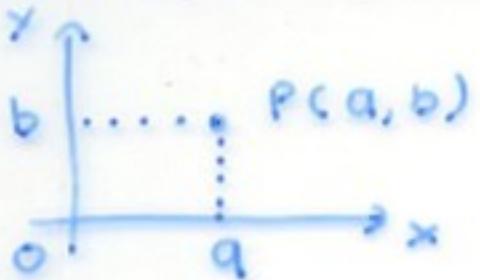
$$C_3 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}$$

KOORDINAT BASIS BARU

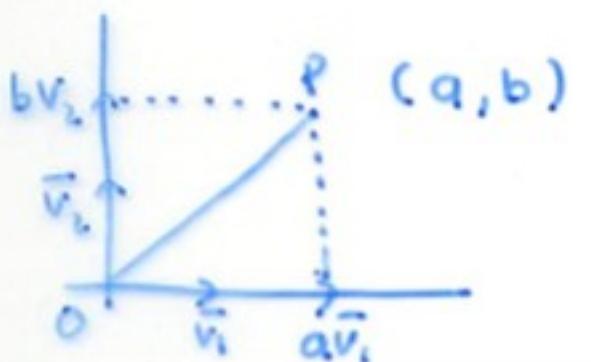


4.10 Koordinat, perubahan basis

- basis berhubungan erat dg koordinat
- koordinat biasa $\rightarrow P(a,b)$ dlm koordinat basis (x,y)

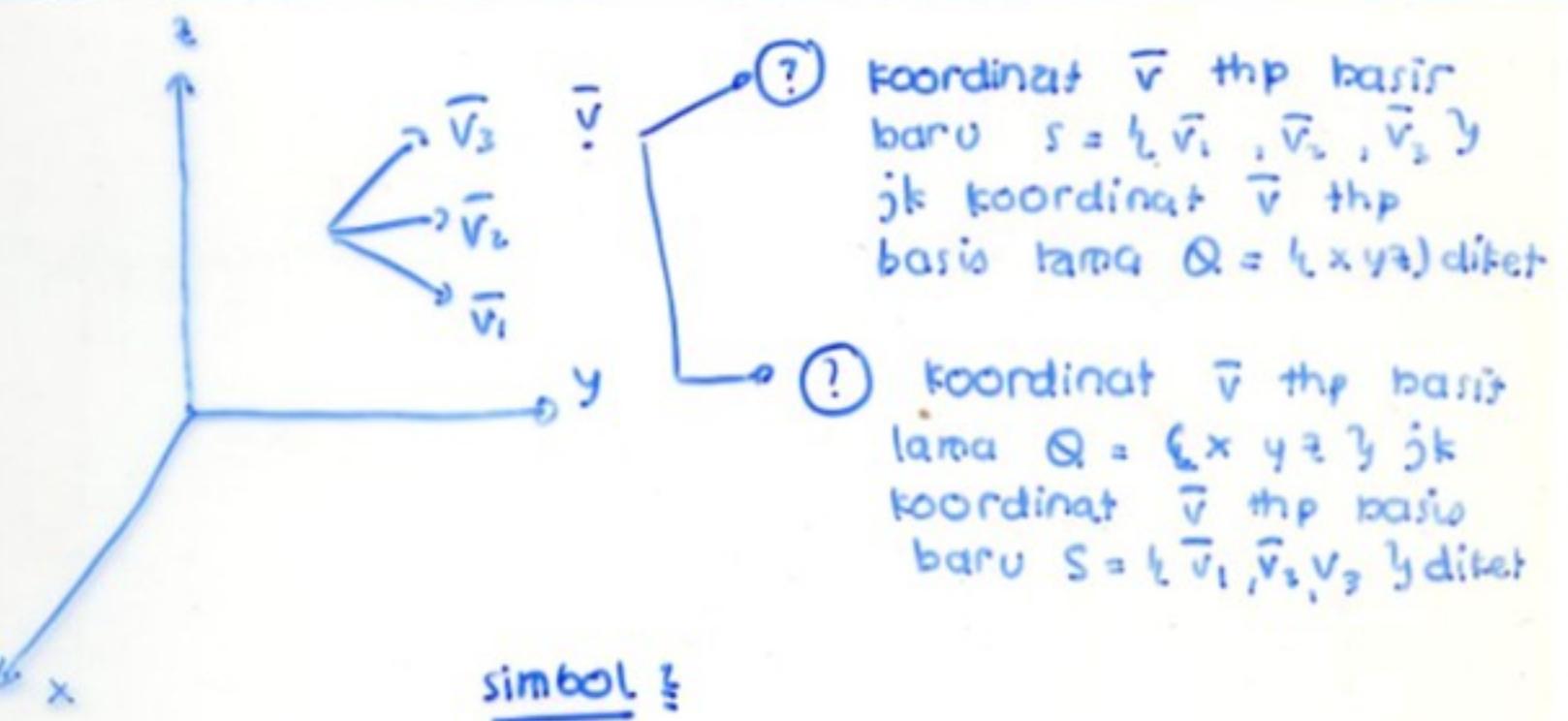


- koordinat vektor $\rightarrow P(a,b)$ dlm koordinat vektor basis $s = \{\vec{v}_1, \vec{v}_2\}$



$$\overline{OP} = a\vec{v}_1 + b\vec{v}_2$$

titik $(a,b) \Rightarrow$ koordinat P
yo relatif thp basis $s = \{\vec{v}_1, \vec{v}_2\}$



$[B]_S$ = matrix transisi dr
basis baru (S) ke
basis lama (Q)
 $= [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

$[B]_Q$ = matrix transisi dr
basis lama (Q) ke
basis baru (S)
 $= ([B]_S)^{-1}$

$[\vec{v}]_S$ = koordinat \vec{v} thp
basis baru $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$[\vec{v}]_Q$ = koordinat \vec{v} thp
basis lama $Q = \{x, y, z\}$

ELIMINAR:

$$[B]_s \cdot [\bar{v}]_s = [\bar{v}]_Q$$

$$[\bar{v}]_s = [B]_Q \cdot [\bar{v}]_Q$$

$$= ([\alpha]_s)^{-1} \cdot [\bar{v}]_Q$$

contoh:

basis baru $s = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$[B]_s = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 9 \end{bmatrix}$$

Pertanyaan:

a) Cari koordinat $[\bar{v}]_s = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$ thp basis ^{baru} $s = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$
⇒ cari $[\bar{v}]_s$

b) Cari koordinat $[\bar{v}]_s = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ thp basis ^{lama} $\alpha = \{x, y, z\}$
⇒ cari $[\bar{v}]_\alpha$

Đub:

a) $[B]_S [\vec{v}]_S = [\vec{v}]_Q$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_S = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_S = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore [\vec{v}]_S = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$



$$b) [\beta]_s [\bar{v}]_s = [\bar{v}]_Q$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 31 \\ 7 \end{bmatrix}$$

$$\therefore [\bar{v}]_Q = \begin{bmatrix} 11 \\ 31 \\ 7 \end{bmatrix}$$

contoh:

$$\text{basis baru } S = \{\bar{v}_1, \bar{v}_2\}$$

$$\bar{v}_1 = [1] \quad \bar{v}_2 = [2]$$

$$[\beta]_s = [1 \ 2]$$

Pertanyaan:

- a) Cari koordinat $[\vec{v}]_Q = \begin{bmatrix} ? \\ ? \end{bmatrix}$ terhadap
basis baru $s = \{\vec{v}_1, \vec{v}_2\}$
 \Rightarrow cari $[\vec{v}]_s$

Jwb:

$$[S]_s [\vec{v}]_s = [\vec{v}]_Q$$

$$[\vec{v}]_s = ([S]_s)^{-1} \cdot [\vec{v}]_Q$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\therefore [\vec{v}]_s = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

D)

Cari koordinat $[\vec{v}]_s = \begin{bmatrix} -3 \\ s \end{bmatrix}$ thp basis lama

\Rightarrow cari $[\vec{v}]_Q$

jawab:

$$[B]_s [\vec{v}]_s = [\vec{v}]_Q$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ s \end{bmatrix} = \begin{bmatrix} ? \\ 2 \end{bmatrix}$$

$$\therefore [\vec{v}]_Q = \begin{bmatrix} ? \\ 2 \end{bmatrix}$$

PR 4.10

2.b

8

10

TUGAS!!!!

- Tugas Kelompok →
 - cari 2 soal dan jawaban di internet yang berhubungan dengan materi ppt ini
 - Tulis alamat internetnya
 - Di kirim ke elearning, terakhir →
 - Minggu depan
- Format → subject →
 - Alin-B-melati
 - Bentuk → ppt → informasi nama kelompok + anggota



TERIMA KASIH!!!