

# MATH IN AI WINTER 2023

**Graph Theory**

Lecture Notes

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PERSONAL USE

<https://github.com/flaricy/notes-for-graph-theory>

The author hopes to take notes while learning graph theory. Reference books are *Algebraic Graph Theory* and 离散数学基础. Starts from Dec 8th.

*Not released yet*



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# 1. Convex Sets

## 1.1 Convexity

### 1.1.1 Cone

**Definition 1.1.1 — Cone.** A set  $K \in \mathbb{R}^n$ , when  $x \in K$  implies  $\alpha x \in K$ .

A non convex cone can be hyper-plane.

For convex cone  $x + y \in K, \forall x, y \in K$ .

Cone don't need to be "pointed". e.g.

Direct sums of cones  $C_1 + C_2 = \{x = x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$ .

■ **Example 1.1**  $S_1^n \{X | X = X^n, \lambda(x) \geq 0\}$

A matrix with positive eigenvalues.

### Operations preserving convexity

Intersection  $C \cap_{i \in \mathbb{I}} C_i$

Linear map Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map. If  $C \in \mathbb{R}^n$  is convex, so is  $A(C) = \{Ax | x \in C\}$

Inverse image  $A^{-1}(D) = \{x \in \mathbb{R} | Ax \in D\}$

### Operations that induce convexity

Convex hull on  $S = \cap \{C | S \in C, C \text{ is convex}\}$

■ **Example 1.2**  $Co\{x_1, x_2, \dots, x_m\} = \{\sum_{i=1}^m \alpha_i x_i | \alpha \in \Delta_m\}$

For a convex set  $x \in C \Rightarrow x = \sum \alpha_i x_i$ .

**Theorem 1.1.1 — Carathéodory's theorem.** If a point  $x \in \mathbb{R}^d$  lies in the convex hull of a set  $P$ , there is a subset  $P'$  of  $P$  consisting of  $d+1$  or fewer points such that  $x$  lies in the convex hull of  $P'$ . Equivalently,  $x$  lies in an  $r$ -simplex with vertices in  $P$ .

## 1.2 Convex Functions

**Definition 1.2.1 — Convex function.** Let  $C \in \mathbb{R}^n$  be convex,  $f : C \rightarrow \mathbb{R}$  is convex on  $f$  if  $x, y \in C \times C$ .  $\forall \alpha \in (0, 1)$ ,  $f(\alpha x + (1 - \alpha)y) \leq f(\alpha x) + f((1 - \alpha)y)$

**Definition 1.2.2 — Strictly Convex function.** Let  $C \in \mathbb{R}^n$  be convex,  $f : C \rightarrow \mathbb{R}$  is strictly convex on  $f$  if  $x, y \in C \times C$ .  $\forall \alpha \in (0, 1)$ ,  $f(\alpha x + (1 - \alpha)y) < f(\alpha x) + f((1 - \alpha)y)$

**Definition 1.2.3 — Strongly convex.**  $f : C \rightarrow \mathbb{R}$  is strongly convex with modulus  $u \geq 0$  if  $f - \frac{1}{2}u\|\cdot\|^2$  is convex.

Interpretation: There is a convex quadratic  $\frac{1}{2}u\|\cdot\|^2$  that lower bounds  $f$ .

■ **Example 1.3**  $\min_{x \in C} f(x) \leftrightarrow \min \bar{f}(x)$  Useful to turn this into an unconstrained problem.

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in C \\ \infty & \text{elsewhere} \end{cases}$$

**Definition 1.2.4** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \infty \bar{\mathbb{R}}$  is convex if  $x, y \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $\forall x, y, \bar{f}(\alpha x + (1 - \alpha)y) \leq f(\alpha x) + f((1 - \alpha)y)$

Definition 1 is equivalent to definition 2 if  $f(x) = \infty$ .

■ **Example 1.4**  $f(x) = \sup_{j \in J} f_j(x)$

### 1.2.1 Epigraph

**Definition 1.2.5 — Epigraph.** For  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ , its epigraph  $epi(f) \in \mathbb{R}^{n+1}$  is the set  $epi(f) = \{(x, \alpha) | f(x) \leq \alpha\}$

Next: a function is convex i.f.f. its epigraph is convex.

**Definition 1.2.6** A function  $f : C \rightarrow \mathbb{R}$ ,  $C \in \mathbb{R}^n$  is convex if  $\forall x, y \in C$ ,  $f(ax + (1 - a)x) \leq af(x) + (1 - a)f(y) \quad \forall a \in (0, 1)$ .

Strict convex:  $x \neq y \Rightarrow f(ax + (1 - a)x) < af(x) + (1 - a)f(y)$

(R)  $f$  is convex  $\Rightarrow -f$  is concave.

Level set:  $S_\alpha f = \{x | f(x) \leq \alpha\}$ .  
 $S_\alpha f$  is convex  $\Leftrightarrow f$  is convex.

**Definition 1.2.7 — Strongly convex.**  $f : C \rightarrow \mathbb{R}$  is strongly convex with modulus  $\mu$  if  $\forall x, y \in C$ ,  $\forall \alpha \in (0, 1)$ ,  $f(\alpha x + (1 - \alpha)y) \leq af(x) + (1 - a)f(y) - \frac{1}{2\mu}\alpha(1 - \alpha)\|x - y\|^2$ .

(R)

- $f$  is 2nd-differentiable,  $f$  is convex  $\Leftrightarrow \nabla^2 f(x) \succ 0$ .
- $f$  is strongly convex  $\Leftrightarrow \nabla^2 f(x) \succ \mu I \Leftrightarrow x \geq \mu$

**Definition 1.2.8 — 2.**  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  is convex if  $x, y \in \mathbb{R}$ ,  $\alpha \in (0, 1)$ ,  $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ .

The effective domain of  $f$  is  $\text{dom } f = \{x | f(x) < +\infty\}$

■ **Example 1.5 — Indicator function.**  $\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & \text{elsewhere} \end{cases}$ .  
 $\text{dom } \delta_C(x) = C$

**Definition 1.2.9 — Epigraph.** The epigraph of  $f$  is  $\text{epif} = \{(x, \alpha) | f(x) \leq \alpha\}$

The graph of  $\text{epif}$  is  $\{(x, f(x)) | x \in \text{dom } f\}$ .

**Definition 1.2.10 — III.** A function  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  is

**Theorem 1.2.1**  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  is convex  $\iff \forall x, y \in \mathbb{R}^n, \alpha \in (0, 1), f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ .

*Proof.*  $\Rightarrow$  take  $x, y \in \text{dom } f$ ,  $(x, f(x)) \in \text{epif}, (y, f(y)) \in \text{epif}$ . ■

■ **Example 1.6 — Distance.** Distance to a convex set  $d_C(x) = \inf\{\|z - x\| | z \in C\}$ . Take any two sequences  $\{y_k\}$  and  $\{\bar{y}_k\} \subset C$  s.t.  $\|y_k - x\| \rightarrow d_C(x)$ ,  $\|\bar{y}_k - \bar{x}\| \rightarrow d_C(\bar{x})$ .  $z_k = \alpha y_k + (1 - \alpha)\bar{y}_k$ .

$$\begin{aligned} d_C(\alpha x + (1 - \alpha)\bar{x}) &\leq \|z_k - \alpha x - (1 - \alpha)\bar{x}\| \\ &= \|\alpha(y_k - x) + (1 - \alpha)(\bar{y}_k - \bar{x})\| \\ &\leq \alpha\|y_k - x\| + (1 - \alpha)\|\bar{y}_k - \bar{x}\| \end{aligned}$$

Take  $k \rightarrow \infty$ ,  $d_C(\alpha x + (1 - \alpha)\bar{x}) \leq \alpha d_C(x) + (1 - \alpha)d_C(\bar{x})$  ■

■ **Example 1.7 — Eigenvalues.** Let  $X \in S^n := \{n \times \text{nsymmetricmatrix}\}$ .  $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_n(x)$ .

$$f_k(x) = \sum_i^n \lambda_i(x).$$

Equivalent characterization

$$\begin{aligned} f_k(x) &= \max\{\sum_i v_i^T X v_i | v_i \perp v_j, i \neq j\} \\ &= \max\{\text{tr}(V^T X V) | V^T V = I_k\} \end{aligned}$$

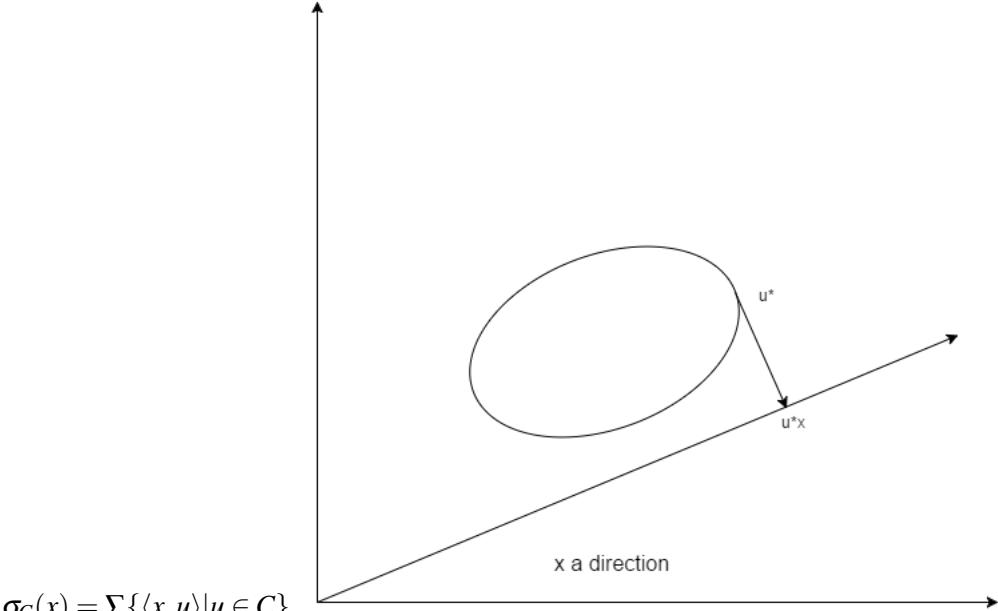
$\max\{\text{tr}(V^T X V)\}$  by circularity

Note  $\langle A, B \rangle = \text{tr}(A, B)$  is true for symmetric matrix.

$$\langle A, A \rangle = |A|_F^2 = \sum_i A_{ii}^2$$

### 1.3 Support Function

Take a set  $C \in \mathbb{R}^n$ , not necessarily convex. The support function is  $\sigma_C = \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ .



**Fact 1.3.1** The support function binds the supporting hyper-plane.

Supporting functions are

- Positively homogeneous

$$\sigma_C(\alpha x) = \alpha \sigma_C(x) \forall \alpha > 0$$

$$\sigma_C(\alpha x) = \sup_{u \in C} \langle \alpha x, u \rangle = \alpha \sup_{u \in C} \langle x, u \rangle = \alpha \sigma_C(x)$$

- Sub-linear (a special case of convex, linear combination holds  $\forall \alpha$ ).

$$\sigma_C(\alpha x + (1 - \alpha)y) = \sup_{u \in C} \langle \alpha x + (1 - \alpha)y, u \rangle \leq \alpha \sup_{u \in C} \langle x, u \rangle + (1 - \alpha) \sup_{u \in C} \langle y, u \rangle$$

■ **Example 1.8 — L2-norm.**  $\|x\| = \sup_{u \in C} \{\langle x, u \rangle, u \in \mathbb{R}^n\}$ .

$$\|x\|_p = \sup\{\langle x, u \rangle, u \in B_q\} \text{ where } \frac{1}{p} + \frac{1}{q} = 1. B_q = \{\|x\|_q \leq 1\}.$$

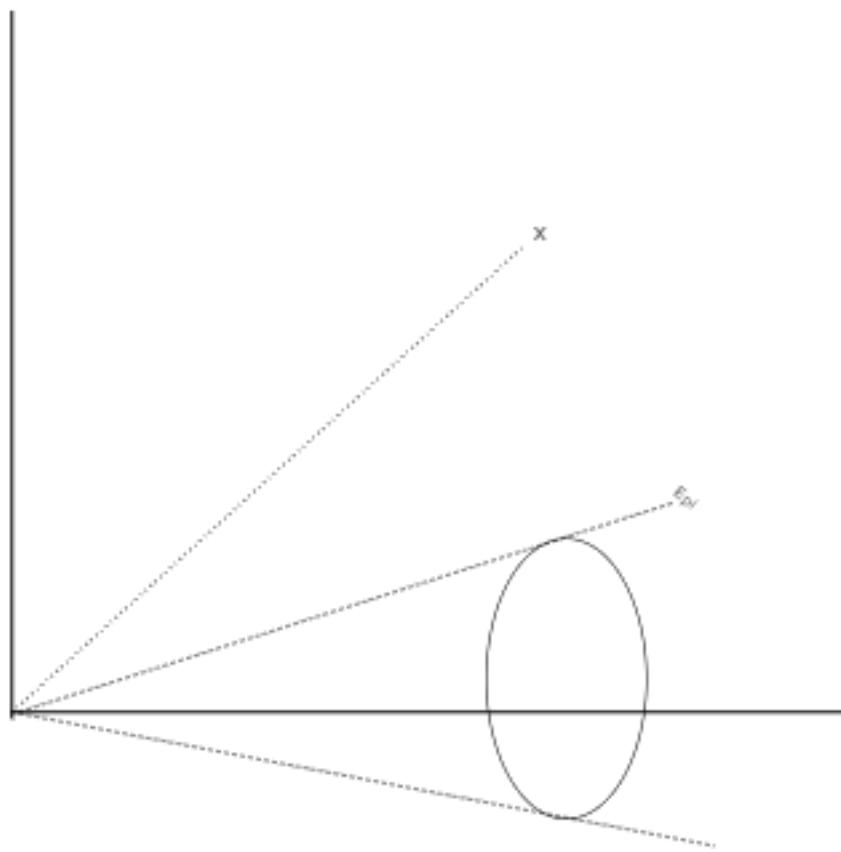
The norm is

- Positive homogeneous
- sub-linear
- If  $0 \in C$ ,  $\sigma_C$  is non-negative.
- If  $C$  is central-symmetric,  $\sigma_C(0) = 0$  and  $\sigma_C(x) = \sigma_C(-x)$

■ **Fact 1.3.2 — Epigraph of a support function.**  $epi\sigma_C = \{(x, t) | \sigma_C(x) \leq t\}$ . Suppose  $(x, t) \in epi\sigma_C$ .

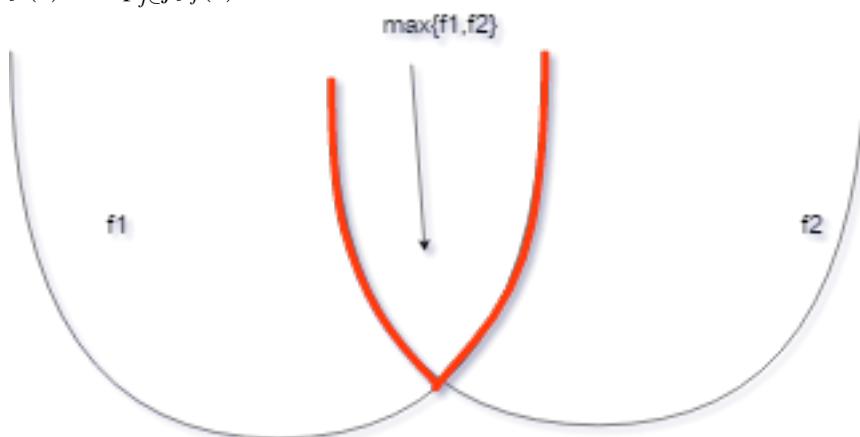
Take any  $\alpha > 0$ .  $\alpha(x, t) = (\alpha x, \alpha t)$ .

$$\alpha \sigma_C(x) = \alpha \sigma_C(x) \leq \alpha t. \alpha(x, t) \in epi\sigma_C$$



#### 1.4 Operations Preserve Convexity of Functions

- Positive affine transformation  
 $f_1, f_2, \dots, f_k \in \text{cvx} \mathbb{R}^n$   
 $f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_k f_k$
- Supremum of functions. Let  $\{f_i\}_{i \in I}$  be arbitrary family of functions. If  $\exists x \sup_{j \in J} f_j(x) < \infty \Leftrightarrow f(x) = \sup_{j \in J} f_j(x)$



- Composition with linear map.  
 $f \in \text{cvx} \mathbb{R}^n, A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear map.  $f \circ A(x) = f(Ax) \in \text{cvx} \mathbb{R}^n$

$$\begin{aligned}f \circ A(x) &= f(A(\alpha x + (1 - \alpha)y)) \\&= f(A\alpha x + (1 - \alpha)Ay) \\&\leq \alpha f(Ax) + (1 - \alpha)f(Ay)\end{aligned}$$

## 2. Basic Concepts

### 2.1 Simple graph, complete graph, tournament graph

Definition 2.1.1 — simple graph. 有向图或者无向图，如果无平行边（重边）和自环。

### 2.2 Operations on a graph

点、边的删除；收缩；两个图之间的运算

### 2.3 Havel-Hakimi algorithm & Erdos-Gallai theorem

给定一个度数序列  $\{d_i\}$ ，判断是否可以根据这个度数序列构造出简单无向图。

Theorem 2.3.1 — Havel-Hakimi. Let  $d = (d_1, d_2, \dots, d_n)$ ,  $\sum_{i=1}^n d_i = 0(mod2)$  且  $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ , then  $d$  简单可图化  $\iff d' = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, \dots)$  简单可图化。

Proof. to be written

Theorem 2.3.2 — Erdos-Gallai. 设  $d = (d_1, d_2, \dots, d_n)$  满足  $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ , 则  $d$  简单可图化  $\iff \forall 1 \leq r \leq n - 1$ ,

$$(1) \sum_{i=1}^r d_i \leq r(r-1) + \sum_{i=r+1}^n \min\{r, d_i\}$$

$$(2) \sum_{i=1}^n d_i = 0(mod2)$$

### 2.4 通路与回路

Definition 2.4.1 — 通路. 设  $G$  为无向标定图,  $G$  中顶点与边的交替序列  $\Gamma = v_{i_0} e_{j_1} v_{i_1} e_{j_2} \dots e_{j_l} v_{i_l}$  称为 顶点  $v_{i_0}$  到  $v_{i_l}$  的 通路。

■ **Definition 2.4.2 — walk** 简单通路。若  $\Gamma$  中所有边各异。

■ **Definition 2.4.3 — closed walk** 简单回路。若  $\Gamma$  是简单通路且  $v_{i_0} = v_{i_l}$

■ **Definition 2.4.4 — path** 初级通路，路径。如果  $\Gamma$  的所有顶点各异，所有边各异。

■ **Definition 2.4.5 —初级回路，圈。** 若  $\Gamma$  是初级通路且起始点 = 终点。

(R)

- 初级通路是简单通路。
- 上述定义针对的是无向图。有向图类似。

■ **Definition 2.4.6 — Girth.** 围长。简单无向图中最短圈的长度。

■ **Definition 2.4.7 — Perimeter.** 周长。简单无向图中最长圈的长度。

### 3. Eulerian Graph and Hamiltonian Graph

#### 3.1 Eulerian Graph

**Definition 3.1.1 — Eulerian trail/path** 欧拉通路. an Eulerian trail (or Eulerian path) is a trail in a finite graph that visits every edge exactly once (allowing for revisiting vertices).

**Definition 3.1.2 — Eulerian circuit/cycle.** an Eulerian trail that starts and ends on the same vertex

**Definition 3.1.3 — Eulerian Graph.** 具有欧拉回路的图。规定平凡图为欧拉图。

**Definition 3.1.4 — 半欧拉图.** 具有欧拉通路但没有欧拉回路的图。

**Lemma 3.1.1** 如果G是欧拉图，那么从G的任何一个顶点出发都可以找到一条简单通路构成欧拉回路。

*Proof.* 由定义验证即可。 ■

**Theorem 3.1.2** Suppose G is a indirected graph. The following 3 propositions are equivalent:

- (1) G is Eulerian
- (2) The degree of every vertex in G are even. And G is connected.
- (3) G is the union of several cycles with no intersected edges. And G is connected.

*Proof.* (1)  $\implies$  (2)

如果 G 平凡，结论显然。否则结论也显然。

(2)  $\implies$  (3)

我们证明: G的任意顶点度数为偶数  $\iff$  G是若干个边不交的权的并。对边数归纳：  
如果边数 = 2， 结论显然。假设2( $m - 1$ )条边时成立,当 $2m$ 条边时。找一个连通分支 $G' = <V', E'>$ , 设其中的一条极大路径为 $v_1, v_2, \dots, v_n$ , 由于 $d(v_1) \geq 2$ , 从而存在 $v_i (i \geq 2)$ , 使得 $v_1$  和 $v_i$  相邻, 从而找到一个简单回路L。从 $G'$ 中删去L, 即可归纳。

(3)  $\implies$  (1)

对圈的个数归纳。

一个圈时显然成立。假设 $m-1$ 个圈时成立，则 $m \geq 2$ 个圈时：先任意选一个圈 $L_1$ ，由于图连通，从而 $\exists v_1, v_2 \in L_1, L_2, v_1, v_2$ 可以是同一个点。由归纳假设和Lemma 1，可以从 $v_1$ 出发在剩下的圈中走到 $v_2$ ，最后绕 $L_1$ 这个圈。 ■

**Theorem 3.1.3** — 半欧拉图的判定。设 $G$ 为连通的无向图，则 $G$ 是半欧拉图当且仅当 $G$ 中恰好有两个奇度顶点。

*Proof.* 在这两个顶点之间连一条边。由Theorem 3.1.2知存在欧拉回路，然后在回路中去掉那条添加的边即可。

假设由欧拉回路，那么由Theorem 3.1.2知没有奇数度的顶点。矛盾。 ■

**Corollary 3.1.4** 设 $G$ 为连通的无向图， $G$ 中有 $2k$ 个奇度顶点，则 $G$ 中存在 $k$ 条边不交的简单通路 $P_1, P_2, \dots, P_k$ ，使得 $E(G) = \bigcup_{i=1}^n E(P_i)$ 。

*Proof.* 将 $2k$ 个点两两配对，每对连一条边。在新图中取一个欧拉回路，然后删去这 $k$ 条边，一定能得到 $k$ 段不交的简单通路。 ■

**Theorem 3.1.5** — 有向欧拉图的判定。设 $D$ 为有向图，则下面三个命题等价：

- (1)  $D$  is Eulerian.
- (2)  $D$  is connected and  $\forall v \in V(D), d^+(v) = d^-(v)$ .
- (3)  $D$  is connected and is the union of several indirected cycles with no intersected edges.

*Proof.* (1)  $\implies$  (2) : trivial

(2)  $\implies$  (3) : similar to Theorem 8.1.1

(3)  $\implies$  (1) : similar to Theorem 8.1.1 ■

**Theorem 3.1.6** — 有向半欧拉图的判定。 $D$ 中恰有两个奇数度的顶点，且一个点出度比入度大1，一个点入度比出度大1. 其他点如度都等于出度。

**Definition 3.1.5 — Fleury's algorithm.** 求无向图中的欧拉回路。

**Definition 3.1.6 — 逐步插入回路算法.**

## 3.2 Hamiltonian Graph

**Proposition 3.2.1 — motive.** (1859 Willian Hamilton) **Traverse the World Problem**  
正十二面体图上是否存在一条初级回路（圈）遍历所有顶点？

**(R)** [Traveling Salesman Problem, TSP] 在一个赋权的无向图中，去找一个哈密尔顿回路，  
并且使得该回路的总权值最小。  
这是一个NP-complete问题。

**Definition 3.2.1** Hamiltonian path

- 经过所有点恰好1次的通路称为哈密顿路。
- 如果上面的通路是回路，那么称为哈密顿回路。
- 具有哈密顿回路的图称为哈密顿图。
- 具有哈密顿通路但没有哈密顿回路的图称为半哈密顿图。
- 规定：平凡图是哈密顿图。

**Theorem 3.2.2**  $G = \langle V, E \rangle$  is Hamiltonian. For every  $\emptyset \neq V_1 \subsetneq V$  we have

$$p(G - V_1) \leq |V_1|$$

, where  $p(G)$  denotes the number of connected components of  $G$ .

*Sketch.*

**Fact 3.2.3** 往一个图中添加边，那么连通分支数不会减少。

所以只需要考虑  $G$  中哈密顿回路所包含的边，如果删去一些节点，对连通分支数的影响。 ■

**Corollary 3.2.4 — case of semi-Hamiltonian graph.** 设  $G = \langle V, E \rangle$  is semi-Hamiltonian, For every  $\emptyset \neq V_1 \subsetneq V$  we have

$$p(G - V_1) \leq |V_1| + 1$$

, where  $p(G)$  denotes the number of connected components of  $G$ .

*Proof.* 同样考虑在哈密顿通路（不是回路）中删除一些点能产生多少段即可。 ■

■ **Example 3.1 — Peterson Graph.** Show that Peterson Graph is semi-Hamiltonian. ■

(R) 彼得森图满足Theorem 3.2.2，但不是哈密顿图。

**Theorem 3.2.5 — a sufficient condition.** 设  $G$  为  $n \geq 1$  阶简单无向图，若对于  $G$  中不相邻的任意两点  $v_1, v_2$ ，均有

$$d(v_1) + d(v_2) \geq n - 1$$

则  $G$  中存在哈密顿通路。

*Proof.* First, show that  $G$  is connected.

if  $G$  is not connected, then consider two connected components  $G_1, G_2$ , pick  $v_1 \in G_1, v_2 \in G_2$ . We have  $d(v_1) \leq |V(G_1)| - 1, d(v_2) \leq |V(G_2)| - 1$ , which implies  $d(v_1) + d(v_2) \leq |V| - 2$ . Contradiction.

考虑极大路径法。

1. 任选一条极大路径  $\Gamma = v_1v_2\dots v_l$ 。如果  $l = n$ , 那么这就是哈密顿通路。如果不是：
2. 证明存在一个圈经过  $\Gamma$  上所有顶点。为此，只要证明：存在顶点  $v_i \in \Gamma$  使得  $v_{i-1}$  与  $v_1$  相邻， $v_i$  与  $v_n$  相邻。这一点可以通过条件证明。

因为图G中有 $\Gamma$ 中未出现的点，所以可以找到这样一个点和上面构造的圈中某个点相邻，我们就可以找到一条更长的哈密顿通路。

3. 重复此过程，我们可以在有限步之内得到最长的极大通路，此即哈密顿通路。 ■

**Corollary 3.2.6 — Øystein Ore, Norwegian.** 设 $G$ 为 $n \geq 1$ 阶简单无向图，若对于 $G$ 中不相邻的任意两点 $v_1, v_2$ ，均有

$$d(v_1) + d(v_2) \geq n$$

则 $G$ 中存在哈密顿通路。

*Proof.* 由Theorem3.2.5知， $G$ 中存在哈密顿通路 $\Gamma = v_1v_2\dots v_n$ ，如果 $v_1v_n$ 相邻，那么找到了回路。如果不相邻，用类似的方法可以找到一个圈。 ■

**Corollary 3.2.7** 设 $G$ 为 $n \geq 1$ 阶简单无向图，若对于 $G$ 中任意 $v$ ，均有

$$d(v) \geq \frac{n}{2}$$

则 $G$ 中存在哈密顿通路。

**Theorem 3.2.8** 设 $u, v$ 为无向 $n$ 阶简单图 $G$ 中的任意两个不相邻的顶点，且 $d(u) + d(v) \geq n$ ，则

$G$ 为哈密顿图  $\iff G \cup e = (u, v)$  为哈密顿图。

*Proof.*  $\implies$  trivial.

$\impliedby$  如果 $G \cup e = (u, v)$  的一条哈密顿回路中有边 $e$ ，那么删去这条边，用同样的方法构造一个圈。 ■

■ **Example 3.2** 对于 $n \geq 4$ 阶简单无向图 $G$ ，只要 $\delta(G) \geq \frac{n}{2} + 1$ ， $G$ 中至少存在2条不同的哈密顿回路。 ■

**Theorem 3.2.9** 设 $D$ 为 $n \geq 2$ 阶竞赛图，则 $D$ 具有哈密顿通路。

*Proof.* induction by  $n$ .

$n = 2$  : trivial

Suppose  $n = k$  OK. When  $n = k + 1$ . WLOG, let  $\Gamma = v_1v_2\dots v_k$  be a Hamiltonian path of  $G - v_{k+1}$ , If  $\forall 1 \leq i \leq k$ , directed edge  $(v_i, v_{k+1}) \in E(D)$ , then  $\Gamma' = v_1\dots v_kv_{k+1}$  is the desired Hamiltonian path. Otherwise, there  $\exists r \in \{1, 2, \dots, k\}$  such that  $(v_i, v_{k+1}) \in E(D), \forall i < r$ , but  $(v_{k+1}, v_r) \in E(D)$ . Then  $\Gamma' = v_1\dots v_{r-1}v_{k+1}v_r\dots v_k$  is the desired path. ■

下面探讨竞赛图中何时有哈密顿回路。我们假定竞赛图是强连通的，那么有如下的两个引理。

**Lemma 3.2.10** 强连通的竞赛图( $n \geq 3$ )中存在长度为3的圈。

*Proof.* Take any vertex  $v_0 \in D$ . Let  $\Gamma_D^+(v_0) = \{v | < v_0, v > \in E(D)\}$ ,  $\Gamma_D^-(v_0) = \{v | <, v_0 > \in E(D)\}$ . We claim that both  $\Gamma^+$  and  $\Gamma^-$  are non-empty. Moreover, there  $\exists u \in \Gamma^+, v \in \Gamma^-$  such that  $(u, v) \in E(D)$  (otherwise  $D$  is not strongly connected). Thus,  $v_0 \rightarrow u \rightarrow v \rightarrow v_0$  forms a 3-loop. ■

**Lemma 3.2.11** 强连通的竞赛图( $n \geq 3$ )中，如果存在长为 $k < n$ 的圈，则存在长为 $k+1$ 的圈。

*Proof.* 1. 先考虑是否有点 $u$ 同时是长为 $k$ 的圈有关的边的from和to。如果是，那么必然存在 $(v_i, u) \in E \wedge (u, v_{i+1}) \in E$ ，这一点用之前的套路即可说明。

2. 如果不存在这样的点，那么圈以外的点可以分成两类，一类是圈上的点到该点都是入边，一类是圈上的点到该点都是出边。

由强连通性可知，这两个集合都非空，且任何一个都有指向对方集合的边。那么任意选出圈上的点 $v_1 \rightarrow v_2 \rightarrow v_3$ ，将其换成 $v_1 \rightarrow v' \rightarrow v'' \rightarrow v_3$ . ■

**Theorem 3.2.12** 强连通的竞赛图是哈密顿图。

*Proof.* Easy by the previous 2 lemmas.

Note that  $K_2$  cannot be strongly connected but other tournament graphs can. ■

**Theorem 3.2.13**  $K_{2n}$  中有  $n-1$  条边不重的哈密顿回路,  $K_{2n+1}$  中有  $n$  条边不重的哈密顿回路。



## 4. Matrix Theory for Graphs

以下提到的**graph**指的是无向图，简单起见可以认为没有自环。允许有重边。

### 4.1 Then Adjacency Matrix

**Definition 4.1.1 — adjacency matrix of a directed graph** 邻接矩阵. The adjacency matrix  $A(X)$  of a directed graph  $X$  is the integer matrix with rows and columns indexed by the vertices of  $X$ , such that the  $uv$ -entry of  $A(X)$  is equal to the number of arcs from  $u$  to  $v$ . If  $X$  is simple, then the elements are 0 or 1.

**Definition 4.1.2 — adjacency matrix of an undirected graph.** We view each edge as a pair of arcs in opposite directions, and  $A(X)$  is a symmetric 01-matrix. If the graph has no loops (自环), the diagonal entries of  $A(X)$  are 0.

(R) 注意是这里的矩阵应该是标定的(顶点指定编号)。同一个顶点集如果采用不同的方式，那么得到的邻接矩阵不同。但是，这些矩阵之间存在关联。具体而言：

**Lemma 4.1.1** Let  $X$  and  $Y$  be directed graphs on the same vertex set. Then they are isomorphic if and only if there is a permutation matrix  $P$  such that  $P^T A(X) P = A(Y)$ .

*Proof.* 将邻接矩阵视为双线性函数的度量矩阵。在不同的基下双线性函数的度量矩阵是合同的。在此处基的变换矩阵是permutation matrix. ■

(R) Since permutation matrices are orthogonal, i.e.  $P^T = P^{-1}$ , and so if  $X$  and  $Y$  are isomorphic, then  $A(X)$  and  $A(Y)$  are similar matrices.

**Definition 4.1.3 — the spectrum of a matrix** 矩阵的谱. The *spectrum* of a matrix is the list of its eigenvalues together with their multiplicities.

**Definition 4.1.4 — the spectrum of a graph.** The spectrum of a graph  $X$  is the spectrum of  $A(X)$ . 我们也称  $A(X)$  的特征值和特征向量是图  $X$  的特征值和特征向量。

- (R) Lemma 4.1.1 shows that the spectrum (or equivalently, the characteristic polynomial of  $A(X)$ ) is an invariant of the isomorphism class of a graph.  
但是两个不同构的图也可以有相同的特征多项式，

下面探讨邻接矩阵提供的更多信息。

**Definition 4.1.5 — walk** 简单通路。A walk of length  $r$  in a directed graph  $X$  is a sequence of vertices

$$v_0 \sim v_1 \sim \dots \sim v_r$$

A walk is *closed* if  $v_0 = v_r$ .

- (R) 注意与 path 区分。path 不允许顶点重复。

**Lemma 4.1.2** Let  $X$  be a directed graph with adjacency matrix  $A$ . The number of walks from  $u$  to  $v$  in  $X$  with length  $r$  is  $(A^r)_{uv}$ .

*Proof.* Induction by  $n$ . Consider the meaning of matrix multiplication. ■

- (R) Lemma 4.1.2 shows that the number of closed walks of length  $r$  in  $X$  is  $\text{tr}(A^r)$ .

**Corollary 4.1.3** Let  $X$  be a graph (无自环的无向图) with  $e$  edges and  $t$  triangles. A : adjacency matrix, then

- (1)  $\text{tr } A = 0$
- (2)  $\text{tr } A^2 = 2e$
- (3)  $\text{tr } A^r = 6t$

## 4.2 The Incidence Matrix

**Definition 4.2.1 — incidence matrix of an undirected graph** 无向图的关联矩阵。Let  $G = \langle V, E \rangle$ ,  $V = \{v_1, \dots, v_n\}$ ,  $E = \{e_1, \dots, e_m\}$ . The incidence matrix  $B(X) \in M_{n \times m}(\mathbb{Z})$ , such that

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j \\ 0, & \text{otherwise} \end{cases}$$

**Theorem 4.2.1** Let  $X$  be a graph with  $n$  vertices and  $c_0$  bipartite connected components 连通的二部图分量. Then  $\text{rk } B = n - c_0$ .

*Proof.* We shall show that the null space of  $B$  has dimension  $c_0$ . ■

**Lemma 4.2.2** Let  $B$  be the incidence matrix of the graph  $X$ . Then  $BB^T = \Delta(X) + A(X)$ , where

$$\Delta(X) = \text{diag}\{\deg(v_1), \dots, \deg(v_n)\}$$

*Proof.* 比较平凡。 ■

**Definition 4.2.2 — fundamental incidence matrix** 基本关联矩阵. 设  $v_i$  为参考点, 在  $B(X)$  中删去第  $i$  行得到的矩阵。

### 4.3 The Incidence Matrix of an Oriented Graph

可以理解为有向图。

An *orientation* of a (无向图)graph  $X$  is the assignment of a direction to each edge. Recall that an **arc** of a graph is an ordered pair of adjacent vertices  $(u, v)$ . To put the definition of orientation formally,

**Definition 4.3.1 — orientation of a graph.** a function  $\sigma : E \rightarrow \{-1, 1\}$ , where  $E$  is the arcs of  $X$ . The function satisfies that if  $(u, v)$  is an arc, then

$$\sigma(u, v) = -\sigma(v, u).$$

If  $\sigma(u, v) = 1$ , then we will regard the edge  $uv$  as  $u \rightarrow v$ .

Now an *Oriented Graph* is a graph together with a particular orientation. We may use  $X^\sigma$  to denote it.

**Definition 4.3.2 — incidence matrix of a directed graph.**  $D(X^\sigma)$  is the  $\{0, 1, -1\}$ -matrix with rows indexed by vertices, columns indexed by edges of  $X$ .

$$d_{uf} = \begin{cases} -1, & u \rightarrow \_ \\ 1, & \_ \rightarrow u \\ 0, & \text{else} \end{cases}$$

(R) 注意这个和离散数学书上的定义是反的。

**Theorem 4.3.1** If  $X$  has  $c$  connected components. Let  $\sigma$  be any orientation of  $X$  and  $D$  is the incidence matrix of  $X^\sigma$ , then  $\text{rk } D = n - c$ .

*Proof.* Show that the null space of  $D$  has dimension  $c$ . Suppose  $z \in \mathbb{R}^n$  such that  $z^T D = 0$ . 根据定义验证每一个连通分支中的  $z_i$  应该取相同值。 ■

**Theorem 4.3.2**  $DD^T = \Delta(X) - A(X)$ .

*Proof.* to be written. ■

#### 4.4 The Laplacian of a Graph

**Definition 4.4.1 — The Laplacian Matrix** 无向图的拉普拉斯矩阵。Let  $\sigma$  be an arbitrary orientation of a graph  $X$ , and let  $D$  be the incidence matrix of  $X^\sigma$ . Then the *Laplacian* of  $X$  is the matrix  $Q(X) = DD^T$ .

- (R) From Theorem 4.3.2 we know that  $Q$  does not depend on the orientation  $\sigma$ , and hence it is well defined.

**Lemma 4.4.1** Let  $X$  be a graph with  $n$  vertices and  $c$  connected components. Then  $\text{rk } Q = n - c$ .

*Proof.* Reduced to show that for any matrix  $D$ , we have

$$\text{rk } D = \text{rk } DD^T$$

In fact, let  $z \in \mathbb{R}^n$  such that  $DD^T z = 0$ , then  $z^T DD^T z = 0$ , which implies  $D^T z = 0$ . Thus any vector in the null space of  $DD^T$  is in the null space of  $D^T$ . ■

下面探讨拉普拉斯矩阵的特征值。

Since  $Q$  is symmetric, its eigenvalues are real.  $\mathbb{R}^n$  has an orthogonal basis consisting of eigenvectors of  $Q$ . Since  $Q = DD^T$ , it is positive semi-definite, and therefore its eigenvalues are all nonnegative.

By Lemma 4.4.1, the multiplicity of 0 as eigenvalue of  $Q$  is equal to the number of components of  $X$ .

**Lemma 4.4.2 — 拉普拉斯二次型.** Let  $X$  be a graph on  $n$  vertices with Laplacian  $Q$ . Then for any vector  $x$ ,

$$x^T Qx = \sum_{uv \in E(X)} (x_u - x_v)^2$$

*Proof.* This follows from the obeservation that

$$x^T Qx = x^T DD^T x = |D^T x|^2$$

If  $uv \in E(X)$ , then the entry of  $D^T x$  corresponding to  $uv$  is  $\pm(x_u - x_v)$ . ■

##### 4.4.1 Trees

这一节介绍矩阵树定理。一个图的生成树的个数可以被拉普拉斯矩阵决定。首先是一个准备工作。

**Definition 4.4.2 — delete an edge.** Let  $X$  be a graph, and let  $e = uv$  be an edge of  $X$ . The graph  $X - e$  with vertex set  $V(X)$  and edge set  $E(X) - e$  is said to be obtained by **deleting** the edge  $e$ .

**Definition 4.4.3 — contract an edge.** The graph  $X/e$  constructed by identifying the vertices  $u$  and  $v$  and the deleting  $e$  is said to be obtained by **contracting**  $e$ .

**Definition 4.4.4** If  $M$  is a symmetric matrix with rows and columns indexed by the set  $V$  and if  $S \subseteq V$ , then let  $M[S]$  denote the submatrix obtained by deleting rows and columns indexed by elements of  $S$ .

**Theorem 4.4.3 — Matrix Tree Theorem.** Let  $X$  be a graph with Laplacian matrix  $Q$ . If  $u$  is an arbitrary vertex of  $X$ , then  $\det Q[\{u\}]$  is equal to the number of spanning trees of  $X$ .

*Proof.* Induction on the number of edges of  $X$ .

First, Let  $\tau(X)$  be the number of spanning trees of  $X$ . we observe that

$$\tau(X) = \tau(X - e) + \tau(X/e)$$

more ...

**Corollary 4.4.4** The number of spanning trees of  $K_n$  is  $n^{n-2}$ .

*Proof.* This follows directly from the fact that  $Q[u] = nI_{n-1} - J$  for any vertex  $u$ .

下面两个定理的证明都需要一些矩阵的技巧。矩阵树定理表明，对角元的余子式(cofactor)都相同，事实上拉普拉斯矩阵的伴随(adjoint)中所有元素都相同。

**Theorem 4.4.5** Let  $\tau(X)$  denote the number of spanning trees in the graph  $X$  and let  $Q$  be its Laplacian. Then  $\text{adj}(Q) = \tau(X) \cdot J$ .

**Theorem 4.4.6** Let  $X$  be a graph on  $n$  vertices, and let  $\lambda_1 \leq \dots \leq \lambda_n$  be the  $n$  eigenvalues of Laplacian of  $X$ . Then the number of spanning trees in  $X$  is  $\frac{1}{n} \prod_{i=2}^n \lambda_i$ .

*Proof.* The result clearly holds if  $X$  is not connected, so we may assume without loss that  $X$  is connected.

Let  $\phi(t)$  denote the characteristic polynomial  $\det(tI - Q)$ . Since  $\lambda_1 = 0$ , the coefficient of  $t$  is

$$(-1)^{n-1} \prod_{i=2}^n$$

On the other hand, the coefficient of the linear term in  $\phi(t)$  is

$$(-1)^{n-1} \sum_{u \in V(X)} \det Q[u]$$

This yields the theorem.



## 5. Matrix Foundations

### 5.1 线性映射的伴随

### 5.2 Singular Value Decomposition

设  $V, W$  为  $\mathbb{R}$  上的有限维内积空间，其内积记作  $(\cdot | \cdot)_V$  和  $(\cdot | \cdot)_W$ . 设  $m = \dim V, n = \dim W$ .

**Theorem 5.2.1 — SVD.** 对于任意线性映射  $T : V \rightarrow W$ , 记  $p = \min\{m, n\}$ , 则存在

$V$  的单位正交基  $v_1, \dots, v_m$ ,

- $W$  的单位正交基  $w_1, \dots, w_m$ ,
  - 非负实数  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p$ ,
- 使得

$$Tv_i = \begin{cases} \sigma_i w_i, & 1 \leq i \leq p \\ 0, & i > p \end{cases}$$

此处的  $\sigma_1 \geq \dots \geq \sigma_p$  由  $T$  唯一确定, 称为  $T$  的奇异值。

*Proof.* to be written

**Theorem 5.2.2 — 矩阵版本.** 设  $V = \mathbb{R}^m, W = \mathbb{R}^n$ , 各自配备标准内积, 并且将  $T$  等同于矩阵  $A \in M_{n \times m}(\mathbb{R})$ . 对于定理 5.1.1 中的单位正交基, 以列向量定义正交矩阵:

$$P := (v_1 | \dots | v_m) \in M_{m \times m}(\mathbb{R}), Q := (w_1 | \dots | w_n) \in M_{n \times n}(\mathbb{R}),$$

再用奇异值定义

$$\Sigma := \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_r, 0, 0, \dots\} \in M_{n \times m}(\mathbb{R})$$

其中  $r = \text{rank}(T)$ . 则奇异值分解化为矩阵等式

$$AP = Q\Sigma$$

亦即

$$A = Q\Sigma P^T$$

### 5.3 Kronecker Product

**Definition 5.3.1** 设  $A \in M_{n \times m}(\mathbb{R}), B \in M_{p \times q}(\mathbb{R})$ , 则定义

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & & & \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

**Property 5.3.1 — 混合乘积.**

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

*Proof.* 利用分块矩阵乘法即可验证。 ■