MATH IN AI WINTER 2023

Graph Theory

Lecture Notes

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https://github.com/flaricy/notes-for-graph-theory

The author hopes to take notes while learning graph theory. Reference books are *Algebraic Graph Theory* and 离散数学基础. Starts from Dec 8th.
Not released yet

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1.1 Convexity

1.1.1 Cone

Definition 1.1.1 — Cone. A set $K \in \mathbb{R}^n$, when $x \in K$ implies $\alpha x \in K$.

A non convex cone can be hyper-plane.
For convex cone $x+y \in K, \forall x, y \in K$.
Cone don't need to be "pointed". e.g.
Direct sums of cones $C_1 + C_2 = \{x = x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$.

■ **Example 1.1** $S_1^n\{X | X = X^n, \lambda(x) \geq 0\}$
A matrix with positive eigenvalues. ■

Operations preserving convexity

Intersection $C \cap_{i \in \mathbb{I}} C_i$
Linear map Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If $C \in \mathbb{R}^n$ is convex, so is $A(C) = \{Ax \forall x \in C\}$
Inverse image $A^{-1}(D) = \{x \in \mathbb{R} | Ax \in D\}$

Operations that induce convexity

Convex hull on $S = \cap\{C | S \in C, Cisconvex\}$

■ **Example 1.2** $Co\{x_1, x_2, \cdots, x_m\} = \{\sum_{i=1}^m \alpha_i x_i | \alpha \in \delta_m\}$ ■

For a convex set $x \in C \Rightarrow x = \sum \alpha_i x_i$.

Theorem 1.1.1 — Carathéodory's theorem. If a point $x \in \mathbb{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d+1$ or fewer points such that x lies in the convex hull of P'. Equivalently, x lies in an r-simplex with vertices in P.

1.2 Convex Functions

Definition 1.2.1 — Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.2 — Strictly Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is strictly convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \langle f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.3 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules $u \geq 0$ if $f - \frac{1}{2}u||\cdot||^2$ is convex.

Interpretation: There is a convex quadratic $\frac{1}{2}u||\cdot||^2$ that lower bounds f.

■ **Example 1.3** $\min_{x \in C} f(x) \leftrightarrow \min \bar{f}(x)$ Useful to turn this into an unconstrained problem.

$$\bar{f}(x) = \begin{cases} f(x) & if x \in C \\ \infty & elsewhere \end{cases}$$

■

Definition 1.2.4 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \infty$ $\bar{\mathbb{R}}$ is convex if $x, y \in \mathbb{R}^n \times \mathbb{R}^n$, $\forall x, y, \bar{f}(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1 is equivalent to definition 2 if $f(x) = \infty$.

■ **Example 1.4** $f(x) = \sup_{j \in J} f_j(x)$ ■

1.2.1 Epigraph

Definition 1.2.5 — Epigraph. For $f : \mathbb{R}^n \to \bar{R}$, its epigraph $epi(f) \in \mathbb{R}^{n+1} is the set epi(f) \{(x, \alpha) | f(x) \in \alpha\}$

Next: a function is convex i.f.f. its epigraph is convex.

Definition 1.2.6 A function $f : C \to \mathbb{R}, C \in \mathbb{R}^n$ is convex if $\forall x, y \in C$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) \quad \forall a \in (0,1)$.
Strict convex: $x \neq y \Rightarrow f(ax + (1-a)x) \leq af(x) + (1-a)f(x)$

R f is convex $\Rightarrow -f$ is concave.

Level set: $S_\alpha f = \{x | f(x) \leq \alpha\}$.
$S_\alpha f$ is convex $\Leftrightarrow f$ is convex.

Definition 1.2.7 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules μ if $\forall x, y \in C$, $\forall \alpha \in (0,1)$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) - \frac{1}{2\mu}\alpha(1-\alpha)\|x - y\|^2$.

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- f is 2nd-differentiable, f ix convex $\iff \nabla^2 f(x) \rangle 0$.
- f is strongly convex $\iff \nabla^2 f(x) \rangle \mu I \iff x \geq \mu$

Definition 1.2.8 — 2. $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is convexif $x,y \in \mathbb{R}, \alpha \in (0,1), f(ax+(1-a)x) \le af(x)+(1-a)f(x)$.

$$\text{The effective domain of } f \text{ is } domf = \{x|f(x)\langle+\infty\}$$

■ **Example 1.5 — Iudcator function.** $\delta_c(x) = \begin{cases} 0 & x \in C \\ +\infty & elsewhere \end{cases}$.

$dom\delta_c(x) = C$ ■

Definition 1.2.9 — Epigraph. The epigraph of f is $epif = \{(x,\alpha)|f(x) \le \alpha\}$

$$\text{The graph of } epif \text{ is } \{(x,f(x)|x \in domf\}.$$

Definition 1.2.10 — III. A function $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is

Theorem 1.2.1 $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is convex $\iff \forall x,y \in \mathbb{R}^n, \alpha \in (0,1), f(ax+(1-a)x) \le af(x)+(1-a)f(x)$.

Proof. $\Rightarrow$ take $x,y \in domf$, $(x,f(x)) \in epif$,$(y,f(y)) \in epif$. ■

■ **Example 1.6 — Distance.** Distance to a convexset $d_c(x) = \inf\{\|z-x\||z \in C\}$. Take any two sequence $\{y_k\}and\{\bar{y}_k\} \subset C$ s.t. $\|y_k - x\| \to d_c(x)$, $\|\bar{y}_k - \bar{x}\| \to d_c(\bar{x})$. $z_k = \alpha y_k + (1-\alpha)\bar{y}_k$.

$$\begin{aligned} d_c(\alpha x + (1-\alpha)\bar{x}) &\le \|z_k - \alpha x - (1-\alpha)\bar{x}\| \\ &= \|\alpha(y_k - x) + (1-\alpha)(\bar{y}_k - \bar{x})\| \\ &\le \alpha\|y_k - x\| + (1-\alpha)\|\bar{y}_k - \bar{x}\| \end{aligned}$$

Take $k \to \infty$, $d_c(\alpha x + (1-\alpha)\bar{x}) \le \alpha d(x) + (1-\alpha)d(\bar{x})$ ■

■ **Example 1.7 — Eigenvalues.** Let $X \in S^n := \{n \times nsymmetricmatrix\}$. $\lambda_1(x) \ge \lambda_2(X) \ge \ldots \ge \lambda_n(x)$.
$f_k(x) = \sum_1^n \lambda_i(x)$.
Equivalent characterization

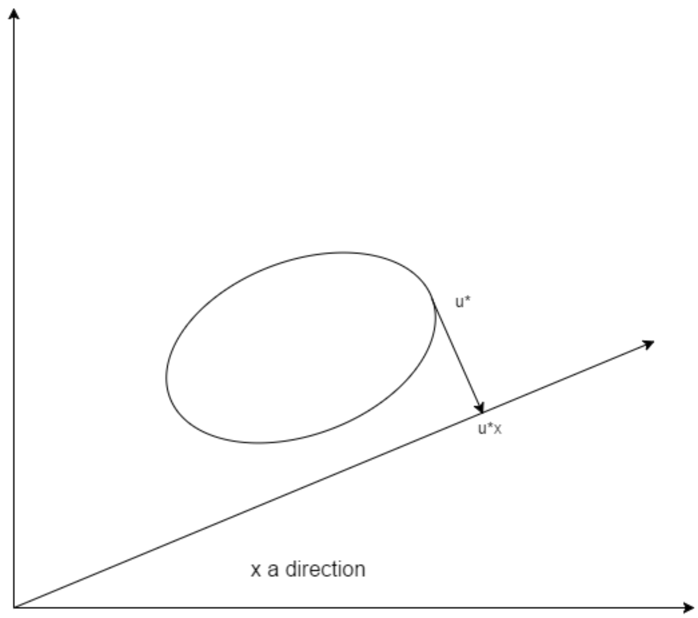
$$\begin{aligned} f_k(x) &= \max\{\sum_i v_i^T X v_i | v_i \perp v_j, i \ne j\} \\ &= \max\{tr(V^T X V | V^T V = I_k\} \end{aligned}$$

$$\max\{tr(VV^T X)\}\text{by circularity}$$

Note $\langle A,B\rangle = tr(A,B)$ is true for symmetric matrix.
$\langle A,A\rangle = |A|_F^2 = \sum_i A_{ii}^2$ ■

1.3 Support Function

Take a set $C \in \mathbb{R}^n$, not necessarily convex.The support function is $\sigma_C = \mathbb{R}^n \to \bar{\mathbb{R}}$.



$\sigma_C(x) = \sum\{\langle x, u\rangle | u \in C\}$.

Fact 1.3.1 The support function binds the supporting hyper-plane.

Supporting functions are

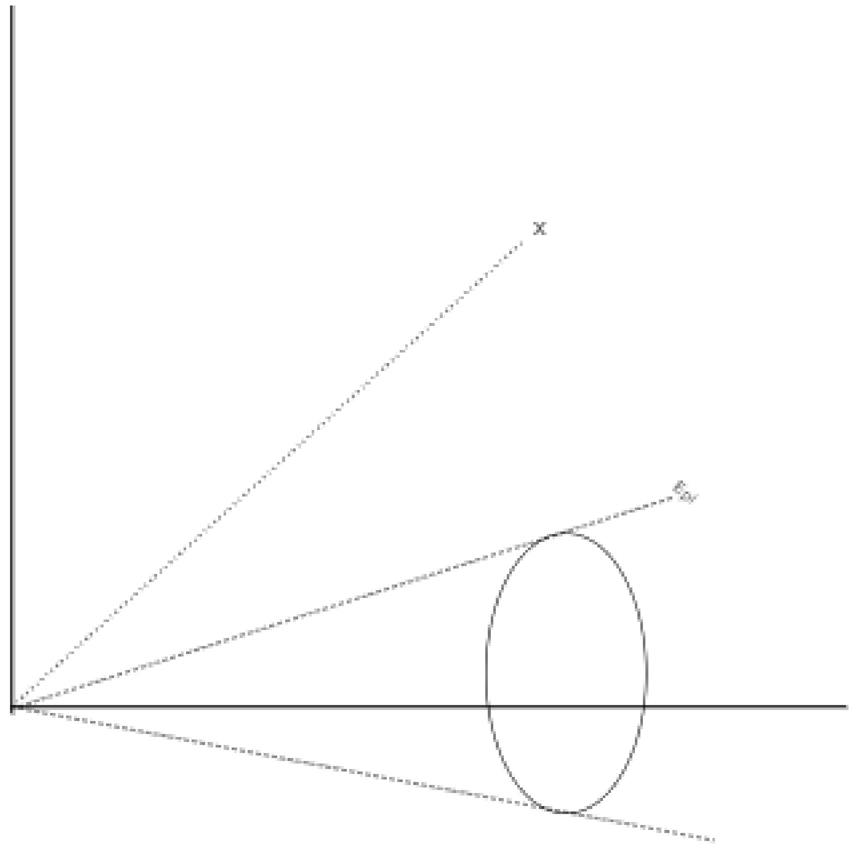
- Positively homogeneous
  $\sigma_C(\alpha x) = \alpha\sigma_C(x)\forall\alpha\rangle 0$
  $\sigma_C(\alpha x) = \sup_{u\in C}\langle \alpha x, u\rangle = \alpha \sup_{u\in C}\langle x, u\rangle = \alpha\sigma_C(x)$
- Sub-linear(a special case of convex, linear combination holds $\forall\alpha$.
  $\sigma_C(\alpha x + (1-\alpha)y) = \sup_{u\in C}\langle \alpha x + (1-\alpha)y, u\rangle \leq \alpha \sup_{u\in C}\langle x, u\rangle + (1-\alpha)\sup_{u\in C}\langle y, u\rangle$

■ **Example 1.8 — L2-norm.** $\|x\| = \sup_{u\in C}\{\langle x, u\rangle, u \in \mathbb{R}^n\}$.
$\|x\|_p = \sup\{\langle x, u\rangle, u \in B_q\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. $B_q = \{\|x\|_q \leq 1\}$.
The norm is

- Positive homogeneous
- sub-linear
- If $0 \in C$, σ_C is non-negative.
- If C is central-symmetric, $\sigma_C(0) = 0$ and $\sigma_C(x) = \sigma_C(-x)$

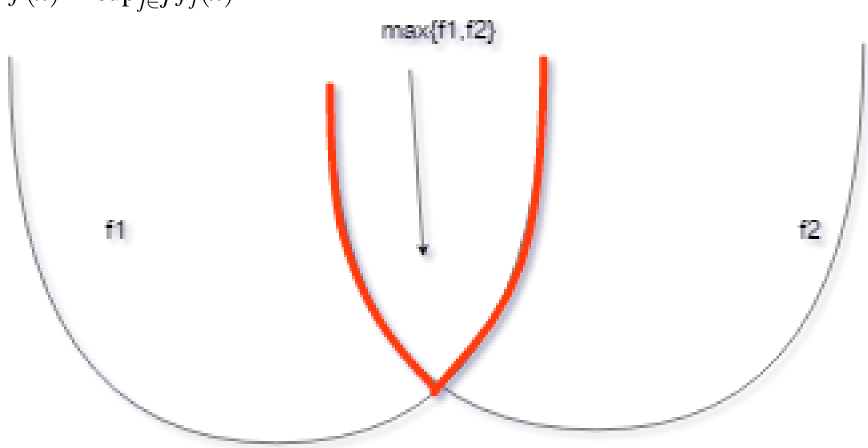
■

Fact 1.3.2 — Epigraph of a support function. $epi\sigma_C = \{(x,t)|\sigma_C(x) \leq t\}$. Suppose $(x,t) \in epi\sigma_C$.
Take any $\alpha > 0$. $\alpha(x,t) = (\alpha x, \alpha t)$.
$\alpha\sigma_C(x) = \alpha\sigma_C(x) \leq \alpha t$. $\alpha(x,c) \in epi\sigma_C$



1.4 Operations Preserve Convexity of Functions

- Positive affine transformation
  $f_1, f_2, \ldots, f_k \in cvx\mathbb{R}^n$.
  $f = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k$
- Supremum of functions. Let $\{f_i\}_{i\in I}$ be arbitrary family of functions. If $\exists x \sup_{j\in J} f_j(x) < \infty \Leftrightarrow$ $f(x) = \sup_{j\in J} f_j(x)$



- Composition with linear map.
  $f \in cvx\mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. $f \circ A(x) = f(Ax) \in cvx\mathbb{R}^n$

$$
\begin{aligned}
f \circ A(x) &= f(A(\alpha x + (1-\alpha)y)) \\
&= f(A\alpha x + (1-\alpha)Ay) \\
&\leq \alpha f(Ax) + (a-\alpha) f(Ay)
\end{aligned}
$$

2.1 Simple graph, complete graph, tournament graph

Definition 2.1.1 — simple graph. 有向图或者无向图，如果无平行边（重边）和自环。

2.2 Operations on a graph

点、边的删除；收缩；两个图之间的运算

2.3 Havel-Hakimi algorithm & Erdos-Gallai theorem

给定一个度数序列 $\{d_i\}$，判断是否可以根据这个度数序列构造出简单无向图。

Theorem 2.3.1 — Havel-Hakimi. Let $d = (d_1, d_2, ..., d_n)$, $\sum_{i=1}^{n} d_i = 0(mod2)$ 且 $n-1 \geq d_1 \geq d_2 \geq ... \geq d_n \geq 0$, then d 简单可图化 $\iff$ $d' = (d_2-1, d_3-1, ...d_{d_1+1}-1, ...)$ 简单可图化。

Proof. *to be written* ■

Theorem 2.3.2 — Erdos-Gallai. 设 $d = (d_1, d_2, ...d_n)$ 满足 $n-1 \geq d_1 \geq d_2 \geq ... \geq d_n \geq 0$, 则 d 简单可图化 $\iff \forall 1 \leq r \leq n-1$,

$$(1)\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} \qquad (2)\sum_{i=1}^{n} d_i = 0(mod2)$$

2.4 通路与回路

Definition 2.4.1 — 通路. 设G为无向标定图,G中顶点与边的交替序列 $\Gamma = v_{i_0} e_{j_1} v_{i_1} e_{j_2} ... e_{j_l} v_{i_l}$ 称为 顶点 v_{i_0} 到 v_{i_l} 的 通路。

Definition 2.4.2 — 简单通路. 若Γ中所有边各异。

Definition 2.4.3 — 简单回路. 若Γ是简单通路且 $v_{i_0} = v_{i_l}$

Definition 2.4.4 — 初级通路，路径. 如果Γ的所有顶点各异,所有边各异。

Definition 2.4.5 — 初级回路，圈. 若Γ是初级通路且起始点 = 终点。

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- 初级通路是简单通路。
- 上述定义针对的是无向图。有向图类似。

Definition 2.4.6 — Girth. 围长。简单无向图中最短圈的长度。

Definition 2.4.7 — Perimeter. 周长。简单无向图中最长圈的长度。

3.1 Eulerian Graph

Definition 3.1.1 — Eulerian trail/path 欧拉通路. an Eulerian trail (or Eulerian path) is a trail in a finite graph that visits every edge exactly once (allowing for revisiting vertices).

Definition 3.1.2 — Eulerian circuit/cycle. an Eulerian trail that starts and ends on the same vertex

Definition 3.1.3 — Eulerian Graph. 具有欧拉回路的图。规定平凡图为欧拉图。

Definition 3.1.4 — 半欧拉图. 具有欧拉通路但没有欧拉回路的图。

Lemma 3.1.1 如果G是欧拉图，那么从G的任何一个顶点出发都可以找到一条简单通路构成欧拉回路。

Proof. 由定义验证即可。 ■

Theorem 3.1.2 Suppose G is a indirected graph. The following 3 propositions are equivalent:
(1) G is Eulerian
(2) The degree of every vertex in G are even. And G is connected.
(3) G is the union of several cycles with no intersected edges. And G is connected.

Proof. (1) $\implies$ (2)
如果 G 平凡，结论显然。否则结论也显然。
(2) $\implies$ (3)
我们证明: G的任意顶点度数为偶数 $\iff$ G是若干个边不交的权的并。 对边数归纳：
如果边数 $= 2$， 结论显然。 假设$2(m-1)$条边时成立,当$2m$条边时。找一个连通分支$G' =< V', E' >$, 设其中的一条极大路径为$v_1, v_2, ..., v_n$,由于$d(v_1) \geq 2$， 从而存在 $v_i(i \geq 2)$，使得v_1 和 v_i 相邻，从而找到一个简单回路L。从G'中删去L，即可归纳。
(3) $\implies$ (1)

对圈的个数归纳。
一个圈时显然成立。假设m-1个圈时成立，则$m \geq 2$个圈时：先任意选一个圈L_1，由于图连通，从而$\exists v_1, v_2 \in L_1, L_2$, v_1, v_2可以是同一个点。由归纳假设和Lemma 1，可以从$v1$出发在剩下的圈中走到$v2$,最后绕L_1这个圈。

■

Theorem 3.1.3 — 半欧拉图的判定. 设G为连通的无向图，则G是半欧拉图当且仅当G中恰好有两个奇度顶点。

Proof. 在这两个顶点之间连一条边。由Theorem 3.1.2知存在欧拉回路，然后在回路中去掉那条添加的边即可。
假设由欧拉回路，那么由Theorem 3.1.2知没有奇数度的顶点。矛盾。 ■

Corollary 3.1.4 设G为连通的无向图，G中有$2k$个奇度顶点，则G中存在k条边不交的简单通路$P_1, P_2, ...P_k$,使得$E(G) = \cup_{i=1}^{n} E(P_i)$.

Proof. 将2k个点两两配对，每对连一条边。在新图中取一个欧拉回路，然后删去这k条边，一定能得到k段不交的简单通路。 ■

Theorem 3.1.5 — 有向欧拉图的判定. 设D为有向图，则下面三个命题等价：
(1) D is Eulerian.
(2) D is connected and $\forall v \in V(D), d^+(v) = d^-(v)$.
(3) D is connected and is the union of several indirected cycles with no intersected edges.

Proof. (1) $\Longrightarrow$ (2) : trival
(2) $\Longrightarrow$ (3) : similar to Theorem 8.1.1
(3) $\Longrightarrow$ (1) : similar to Theorem 8.1.1

■

Theorem 3.1.6 — 有向半欧拉图的判定. D中恰有两个奇数度的顶点，且一个点出度比如度大1，一个点入度比出度大1.其他点如度都等于出度。

Definition 3.1.5 — Fleury's algorithm. 求无向图中的欧拉回路。

Definition 3.1.6 — 逐步插入回路算法.

3.2 Hamiltonian Graph

Proposition 3.2.1 — motive. (1859 Willian Hamilton) **Traverse the World Problem**
正十二面体图上是否存在一条初级回路（圈）遍历所有顶点?

R [Traveling Salesman Problem, TSP] 在一个赋权的无向图中，去找一个哈密尔顿回路，并且使得该回路的总权值最小。
这是一个NP-complete问题。

Definition 3.2.1 Hamiltonian path

- 经过所有点恰好1次的通路称为哈密顿路。
- 如果上面的通路是回路，那么称为哈密顿回路。
- 具有哈密顿回路的图称为哈密顿图。
- 具有哈密顿通路但没有哈密顿回路的图称为半哈密顿图。
- 规定：平凡图是哈密顿图。

Theorem 3.2.2 $G=<V,E>$ is Hamiltonian. For every $\emptyset \neq V_1 \subsetneq V$ we have

$$p(G-V_1) \leq |V_1|$$

, where $p(G)$ denotes the number of connected components of G.

Sketch.
Fact 3.2.3 往一个图中添加边，那么连通分支数不会减少。
所以只需要考虑G中哈密顿回路所包含的边，如果删去一些节点，对连通分支数的影响。 ■

Corollary 3.2.4 — case of semi-Hamiltonian graph. 设$G=<V,E>$ is semi-Hamiltonian, For every $\emptyset \neq V_1 \subsetneq V$ we have

$$p(G-V_1) \leq |V_1|+1$$

, where $p(G)$ denotes the number of connected components of G.

Proof. 同样考虑在哈密顿通路（不是回路）中删除一些点能产生多少段即可。 ■

■ **Example 3.1 — Peterson Graph.** Show that Peterson Graph is semi-Hamiltonian. ■

R 彼得森图满足Theorem 3.2.2，但不是哈密顿图。

Theorem 3.2.5 — a sufficent condition. 设G为$n \geq 1$阶简单无向图，若对于G中不相邻的任意两点v_1, v_2，均有

$$d(v_1)+d(v_2) \geq n-1$$

则G中存在哈密顿通路。

Proof. First, show that G is connected.
if G is not connected, then consider two connected components G_1, G_2, pick $v_1 \in G_1, v_2 \in G_2$. We have $d(v_1) \leq |V(G_1)|-1, d(v_2) \leq |V(G_2)|-1$, which implies $d(v_1)+d(v_2) \leq |V|-2$. Contradiction.
考虑极大路径法。
1. 任选一条极大路径 $\Gamma = v_1 v_2 ... v_l$。 如果 $l=n$, 那么这就是哈密顿通路。如果不是：
2. 证明存在一个圈经过Γ 上所有顶点。为此，只要证明：存在顶点 $v_i \in \Gamma$ 使得 v_{i-1} 与 v_1 相邻， v_i 与 v_n 相邻。这一点可以通过条件证明。

因为图G中有Γ中未出现的点，所以可以找到这样一个点和上面构造的圈中某个点相邻，我们就可以找到一条更长的哈密顿通路。
3. 重复此过程，我们可以在有限步之内得到最长的极大通路，此即哈密顿通路。 ■

Corollary 3.2.6 — Øystein Ore, Norwegian. 设G为$n \geq 1$阶简单无向图，若对于G中不相邻的任意两点v_1, v_2，均有

$$d(v_1)+d(v_2) \geq n$$

则G中存在哈密顿通路。

Proof. 由Theorem3.2.5知，G中存在哈密顿通路$\Gamma = v_1 v_2 ... v_n$，如果$v_1 v_n$相邻，那么找到了回路。如果不相邻，用类似的方法可以找到一个圈。 ■

Corollary 3.2.7 设G为$n \geq 1$阶简单无向图，若对于G中任意v，均有

$$d(v) \geq \frac{n}{2}$$

则G中存在哈密顿通路。

Theorem 3.2.8 设u, v为无向n阶简单图G中的任意两个不相邻的顶点，且$d(u)+d(v) \geq n$，则
G为哈密顿图 $\iff$ $G \cup e = (u,v)$ 为哈密顿图。

Proof. $\implies$ trivial.
$\impliedby$ 如果$G \cup e = (u,v)$ 的一条哈密顿回路中有边 e，那么删去这条边，用同样的方法构造一个圈。 ■

■ **Example 3.2** 对于 $n \geq 4$阶简单无向图G，只要$\delta(G) \geq \frac{n}{2}+1$，G中至少存在2条不同的哈密顿回路。 ■

Theorem 3.2.9 设D为$n \geq 2$阶竞赛图,则D具有哈密顿通路。

Proof. induction by n.
n = 2 : trivial
Suppose n = k OK. When n = k + 1. WLOG, let $\Gamma = v_1 v_2 ... v_k$ be a Hamiltonian path of $G - v_{k+1}$, If $\forall 1 \leq i \leq k$, directed edge $(v_i, v_{k+1}) \in E(G)$, then $\Gamma' = v_1 ... v_k v_{k+1}$ is the desired Hamiltonian path. Otherwise, there $\exists r \in \{1,2,...,k\}$ such that $(v_i, v_{k+1}) \in E(G), \forall i < r$, but $(v_{k+1}, v_r) \in E(G)$. Then $\Gamma' = v_1 ... v_{r-1} v_{k+1} v_r ... v_k$ is the desired path. ■

下面探讨竞赛图中何时有哈密顿回路。我们假定竞赛图是强连通的，那么有如下的两个引理。

Lemma 3.2.10 强连通的竞赛图$(n \geq 3)$中存在长度为3的圈。

Proof. Take any vertex $v_0 \in D$. Let $\Gamma^+_D(v_0) = \{v| < v_0, v > \in E(D)\}, \Gamma^-_D(v_0) = \{v| <, v_0 > \in E(D)\}$. We claim that both Γ^+ and Γ^- are non-empty. Moreover, there $\exists u \in \Gamma^+, v \in \Gamma^-$ such that $(u,v) \in E(D)$ (otherwise D is not strongly connected). Thus, $v_0 \to u \to v \to v_0$ forms a 3-loop. ■

Lemma 3.2.11 强连通的竞赛图$(n \geq 3)$中，如果存在长为$k < n$的圈，则存在长为$k+1$的圈。

Proof. 1. 先考虑是否有点u同时是长为k的圈有关的边的from和to。如果是，那么必然存在$(v_i, u) \in E \wedge (u, v_{i+1}) \in E$, 这一点用之前的套路即可说明。
2. 如果不存在这样的点，那么圈以外的点可以分成两类，一类是圈上的点到该点都是入边，一类是圈上的点到该点都是出边。
由强连通性可知，这两个集合都非空，且任何一个都有指向对方集合的边。那么任意选出圈上的点$v_1 \to v_2 \to v_3$，将其换成$v_1 \to v' \to v'' \to v_3$. ■

Theorem 3.2.12 强连通的竞赛图是哈密顿图。

Proof. Easy by the previous 2 lemmas.
Note that K_2 cannot be strongly connected but other tournament graphs can. ■

Theorem 3.2.13 K_{2n}中有n-1条边不重的哈密顿回路,K_{2n+1}中有n条边不重的哈密顿回路。

4.1 Adjacency Matrix