MATH IN AI WINTER 2023

Graph Theory

Lecture Notes

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https://github.com/flaricy/notes-for-graph-theory

The author hopes to take notes while learning graph theory. Reference books are *Algebraic Graph Theory* and 离散数学基础. Starts from Dec 8th.
Not released yet

Contents

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1. Convex Sets

1.1 Convexity

1.1.1 Cone

Definition 1.1.1 — Cone. A set $K \in \mathbb{R}^n$, when $x \in K$ implies $\alpha x \in K$.

A non convex cone can be hyper-plane.
For convex cone $x + y \in K, \forall x, y \in K$.
Cone don't need to be "pointed". e.g.
Direct sums of cones $C_1 + C_2 = \{x = x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$.

■ **Example 1.1** $S_1^n\{X | X = X^n, \lambda(x) \geq 0\}$
A matrix with positive eigenvalues. ■

Operations preserving convexity

Intersection $C \cap_{i \in \mathbb{I}} C_i$
Linear map Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If $C \in \mathbb{R}^n$ is convex, so is $A(C) = \{Ax \forall x \in C\}$
Inverse image $A^{-1}(D) = \{x \in \mathbb{R} | Ax \in D\}$

Operations that induce convexity

Convex hull on $S = \cap\{C | S \in C, Cisconvex\}$

■ **Example 1.2** $Co\{x_1, x_2, \cdots, x_m\} = \{\sum_{i=1}^m \alpha_i x_i | \alpha \in \delta_m\}$ ■

For a convex set $x \in C \Rightarrow x = \sum \alpha_i x_i$.

Theorem 1.1.1 — Carathéodory's theorem. If a point $x \in \mathbb{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d+1$ or fewer points such that x lies in the convex hull of P'. Equivalently, x lies in an r-simplex with vertices in P.

1.2 Convex Functions

Definition 1.2.1 — Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.2 — Strictly Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is strictly convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \langle f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.3 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules $u \geq 0$ if $f - \frac{1}{2}u||\cdot||^2$ is convex.

Interpretation: There is a convex quadratic $\frac{1}{2}u||\cdot||^2$ that lower bounds f.

■ **Example 1.3** $\min_{x \in C} f(x) \leftrightarrow \min \bar{f}(x)$ Useful to turn this into an unconstrained problem.

$$\bar{f}(x) = \begin{cases} f(x) & if x \in C \\ \infty & elsewhere \end{cases}$$

■

Definition 1.2.4 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \infty$ $\bar{\mathbb{R}}$ is convex if $x, y \in \mathbb{R}^n \times \mathbb{R}^n$, $\forall x, y, \bar{f}(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1 is equivalent to definition 2 if $f(x) = \infty$.

■ **Example 1.4** $f(x) = \sup_{j \in J} f_j(x)$ ■

1.2.1 Epigraph

Definition 1.2.5 — Epigraph. For $f : \mathbb{R}^n \to \bar{R}$, its epigraph $epi(f) \in \mathbb{R}^{n+1} is the set epi(f) \{(x, \alpha) | f(x) \in \alpha\}$

Next: a function is convex i.f.f. its epigraph is convex.

Definition 1.2.6 A function $f : C \to \mathbb{R}, C \in \mathbb{R}^n$ is convex if $\forall x, y \in C$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) \quad \forall a \in (0,1)$.
Strict convex: $x \neq y \Rightarrow f(ax + (1-a)x) \leq af(x) + (1-a)f(x)$

R f is convex $\Rightarrow -f$ is concave.

Level set: $S_\alpha f = \{x | f(x) \leq \alpha\}$.
$S_\alpha f$ is convex $\Leftrightarrow f$ is convex.

Definition 1.2.7 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules μ if $\forall x, y \in C$, $\forall \alpha \in (0,1)$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) - \frac{1}{2\mu}\alpha(1-\alpha)\|x - y\|^2$.

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- f is 2nd-differentiable, f ix convex $\iff \nabla^2 f(x) \rangle 0$.
- f is strongly convex $\iff \nabla^2 f(x) \rangle \mu I \iff x \geq \mu$

Definition 1.2.8 — 2. $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is convexif $x, y \in \mathbb{R}, \alpha \in (0,1), f(ax + (1-a)x) \leq af(x) + (1-a)f(x)$.

$$\text{The effective domain of } f \text{ is } domf = \{x | f(x) \langle +\infty\}$$

■ **Example 1.5 — Iudcator function.** $\delta_c(x) = \begin{cases} 0 & x \in C \\ +\infty & elsewhere \end{cases}$.

$dom\delta_c(x) = C$ ■

Definition 1.2.9 — Epigraph. The epigraph of f is $epif = \{(x,\alpha) | f(x) \leq \alpha\}$

$$\text{The graph of } epif \text{ is } \{(x, f(x) | x \in domf\}.$$

Definition 1.2.10 — III. A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is

Theorem 1.2.1 $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is convex $\iff \forall x, y \in \mathbb{R}^n, \alpha \in (0,1), f(ax + (1-a)x) \leq af(x) + (1-a)f(x)$.

Proof. $\Rightarrow$ take $x, y \in domf$, $(x, f(x)) \in epif, (y, f(y)) \in epif$. ■

■ **Example 1.6 — Distance.** Distance to a convexset $d_c(x) = \inf\{\|z - x\| | z \in C\}$. Take any two sequence $\{y_k\} and \{\bar{y}_k\} \subset C$ s.t. $\|y_k - x\| \to d_c(x)$, $\|\bar{y}_k - \bar{x}\| \to d_c(\bar{x})$. $z_k = \alpha y_k + (1-\alpha)\bar{y}_k$.

$$\begin{aligned} d_c(\alpha x + (1-\alpha)\bar{x}) \leq & \|z_k - \alpha x - (1-\alpha)\bar{x}\| \\ & = \|\alpha(y_k - x) + (1-\alpha)(\bar{y}_k - \bar{x})\| \\ & \leq \alpha\|y_k - x\| + (1-\alpha)\|\bar{y}_k - \bar{x}\| \end{aligned}$$

Take $k \to \infty$, $d_c(\alpha x + (1-\alpha)\bar{x}) \leq \alpha d(x) + (1-\alpha)d(\bar{x})$ ■

■ **Example 1.7 — Eigenvalues.** Let $X \in S^n := \{n \times nsymmetricmatrix\}$. $\lambda_1(x) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(x)$.
$f_k(x) = \sum_1^n \lambda_i(x)$.
Equivalent characterization

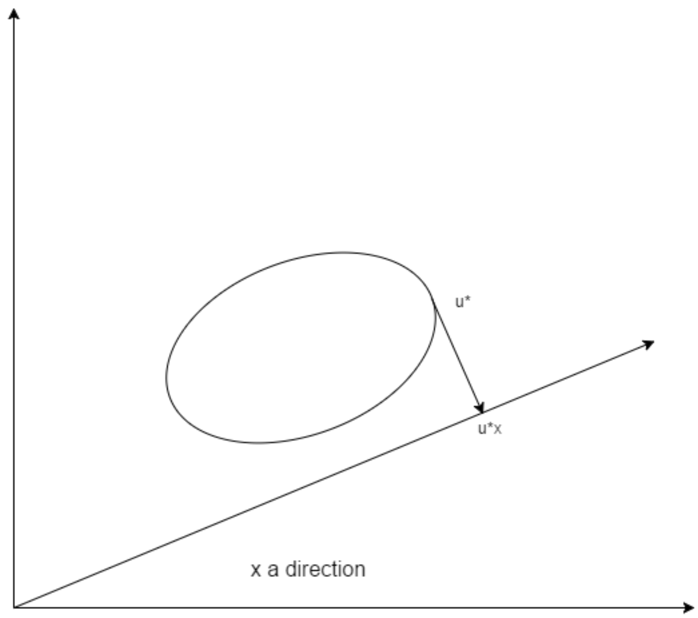
$$\begin{aligned} f_k(x) &= \max\{\sum_i v_i^T X v_i | v_i \perp v_j, i \neq j\} \\ &= \max\{tr(V^T XV | V^T V = I_k\} \end{aligned}$$

$$\max\{tr(VV^T X)\} \text{by circularity}$$

Note $\langle A, B \rangle = tr(A, B)$ is true for symmetric matrix.
$\langle A, A \rangle = |A|_F^2 = \sum_i A_{ii}^2$ ■

1.3 Support Function

Take a set $C \in \mathbb{R}^n$, not necessarily convex.The support function is $\sigma_C = \mathbb{R}^n \to \bar{\mathbb{R}}$.



$\sigma_C(x) = \sum\{\langle x,u\rangle | u \in C\}$.

Fact 1.3.1 The support function binds the supporting hyper-plane.

Supporting functions are

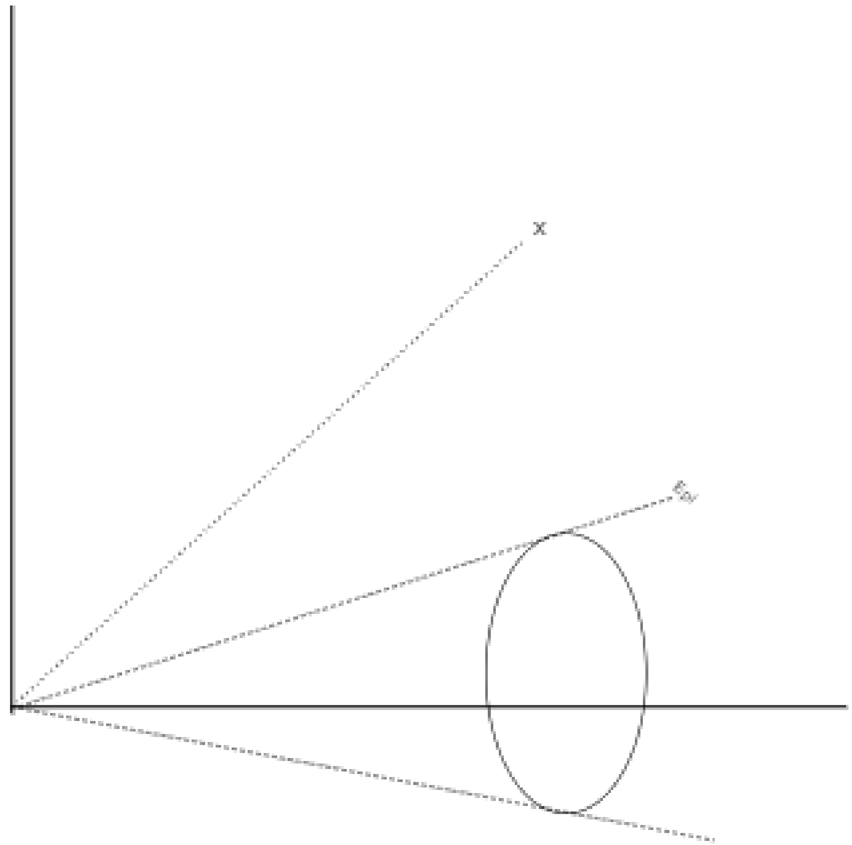
- Positively homogeneous
  $\sigma_C(\alpha x) = \alpha\sigma_C(x)\forall\alpha\rangle 0$
  $\sigma_C(\alpha x) = \sup_{u\in C}\langle \alpha x,u\rangle = \alpha \sup_{u\in C}\langle x,u\rangle = \alpha\sigma_C(x)$
- Sub-linear(a special case of convex, linear combination holds $\forall\alpha$.
  $\sigma_C(\alpha x + (1-\alpha)y) = \sup_{u\in C}\langle \alpha x + (1-\alpha)y,u\rangle \leq \alpha \sup_{u\in C}\langle x,u\rangle + (1-\alpha)\sup_{u\in C}\langle y,u\rangle$

■ **Example 1.8 — L2-norm.** $\|x\| = \sup_{u\in C}\{\langle x,u\rangle, u \in \mathbb{R}^n\}$.
$\|x\|_p = \sup\{\langle x,u\rangle, u \in B_q\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. $B_q = \{\|x\|_q \leq 1\}$.
The norm is

- Positive homogeneous
- sub-linear
- If $0 \in C$, σ_C is non-negative.
- If C is central-symmetric, $\sigma_C(0) = 0$ and $\sigma_C(x) = \sigma_C(-x)$

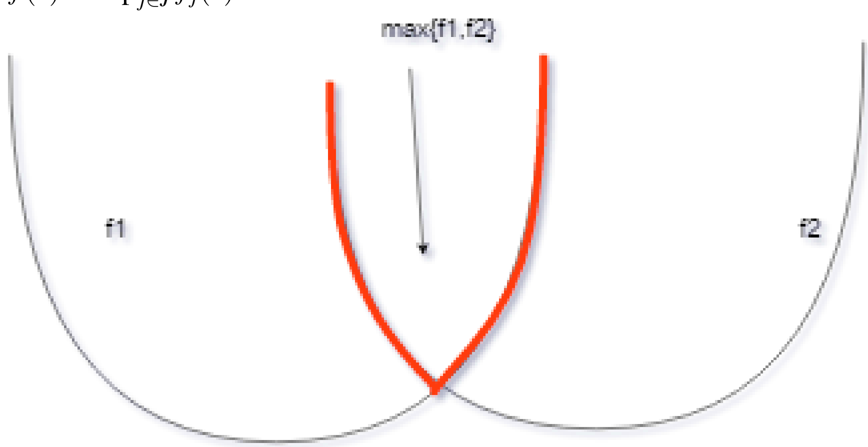
■

Fact 1.3.2 — Epigraph of a support function. $epi\sigma_C = \{(x,t)|\sigma_C(x) \leq t\}$. Suppose $(x,t) \in epi\sigma_C$.
Take any $\alpha > 0$. $\alpha(x,t) = (\alpha x, \alpha t)$.
$\alpha\sigma_C(x) = \alpha\sigma_C(x) \leq \alpha t$. $\alpha(x,c) \in epi\sigma_C$



1.4 Operations Preserve Convexity of Functions

- Positive affine transformation
  $f_1, f_2, \dots, f_k \in cvx\mathbb{R}^n$.
  $f = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k$
- Supremum of functions. Let $\{f_i\}_{i \in I}$ be arbitrary family of functions. If $\exists x \sup_{j \in J} f_j(x) < \infty \Leftrightarrow$ $f(x) = \sup_{j \in J} f_j(x)$



- Composition with linear map.
  $f \in cvx\mathbb{R}^n, A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. $f \circ A(x) = f(Ax) \in cvx\mathbb{R}^n$

$$
\begin{aligned}
f \circ A(x) &= f(A(\alpha x + (1-\alpha)y)) \\
&= f(A\alpha x + (1-\alpha)Ay) \\
&\leq \alpha f(Ax) + (a-\alpha) f(Ay)
\end{aligned}
$$

2. Basic Concepts

2.1 简单图

Definition 2.1.1 — simple graph. 有向图或者无向图，如果无平行边（重边）和自环。

2.2 Havel-Hakimi algorithm & Erdos-Gallai theorem

给定一个度数序列 $\{d_i\}$，判断是否可以根据这个度数序列构造出简单无向图。

Theorem 2.2.1 — Havel-Hakimi. Let $d = (d_1, d_2, ..., d_n)$, $\sum_{i=1}^{n} d_i = 0(mod2)$ 且 $n-1 \geq d_1 \geq d_2 \geq ... \geq d_n \geq 0$, then d 简单可图化 $\iff$ $d' = (d_2-1, d_3-1, ...d_{d_1+1}-1, ...)$ 简单可图化。

Proof. to be written ■

Theorem 2.2.2 — Erdos-Gallai. 设 $d = (d_1, d_2, ...d_n)$ 满足 $n-1 \geq d_1 \geq d_2 \geq ... \geq d_n \geq 0$, 则 d 简单可图化 $\iff$ $\forall 1 \leq r \leq n-1$,

$$(1) \sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} \qquad (2) \sum_{i=1}^{n} d_i = 0(mod2)$$

2.3 通路与回路

Definition 2.3.1 — 通路. 设G为无向标定图,G中顶点与边的交替序列 $\Gamma = v_{i_0} e_{j_1} v_{i_1} e_{j_2} ... e_{j_l} v_{i_l}$ 称为 顶点 v_{i_0} 到 v_{i_l} 的 通路。

Definition 2.3.2 — 简单通路. 若Γ中所有边各异。

Definition 2.3.3 — 简单回路. 若Γ是简单通路且 $v_{i_0} = v_{i_l}$

Definition 2.3.4 — 初级通路，路径. 如果Γ的所有顶点各异,所有边各异。

Definition 2.3.5 — 初级回路，圈. 若Γ是初级通路且起始点 = 终点。

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- 初级通路是简单通路。
- 上述定义针对的是无向图。有向图类似。

3. Eulerian Graph and Hamiltonian Graph

3.1 Eulerian Graph

Definition 3.1.1 — Eulerian trail/path 欧拉通路. an Eulerian trail (or Eulerian path) is a trail in a finite graph that visits every edge exactly once (allowing for revisiting vertices).

Definition 3.1.2 — Eulerian circuit/cycle. an Eulerian trail that starts and ends on the same vertex

Definition 3.1.3 — Eulerian Graph. 具有欧拉回路的图。规定平凡图为欧拉图。

Definition 3.1.4 — 半欧拉图. 具有欧拉通路但没有欧拉回路的图。

Lemma 3.1.1 如果G是欧拉图，那么从G的任何一个顶点出发都可以找到一条简单通路构成欧拉回路。

Proof. 由定义验证即可。 ■

Theorem 3.1.2 Suppose G is a indirected graph. The following 3 propositions are equivalent:
(1) G is Eulerian
(2) The degree of every vertex in G are even. And G is connected.
(3) G is the union of several cycles with no intersected edges. And G is connected.

Proof. (1) $\Longrightarrow$ (2)
如果 G 平凡，结论显然。否则结论也显然。
(2) $\Longrightarrow$ (3)
我们证明: G的任意顶点度数为偶数 $\Longleftrightarrow$ G是若干个边不交的权的并。 对边数归纳：
如果边数 = 2， 结论显然。 假设$2(m-1)$条边时成立,当$2m$条边时。找一个连通分支$G' =< V', E' >$,设其中的一条极大路径为$v_1, v_2, ..., v_n$,由于$d(v_1) \geq 2$， 从而存在 $v_i (i \geq 2)$，使得v_1 和 v_i 相邻，从而找到一个简单回路L。从G'中删去L，即可归纳。
(3) $\Longrightarrow$ (1)

对圈的个数归纳。
一个圈时显然成立。假设m-1个圈时成立，则$m \geq 2$个圈时：先任意选一个圈L_1，由于图连通，从而$\exists v_1, v_2 \in L_1, L_2$, v_1, v_2可以是同一个点。由归纳假设和Lemma 1，可以从$v1$出发在剩下的圈中走到$v2$,最后绕L_1这个圈。

■

Theorem 3.1.3 — 半欧拉图的判定. 设G为连通的无向图，则G是半欧拉图当且仅当G中恰好有两个奇度顶点。

Proof. 在这两个顶点之间连一条边。由Theorem 3.1.2知存在欧拉回路，然后在回路中去掉那条添加的边即可。
假设由欧拉回路，那么由Theorem 3.1.2知没有奇数度的顶点。矛盾。 ■

Corollary 3.1.4 设G为连通的无向图，G中有$2k$个奇度顶点，则G中存在k条边不交的简单通路$P_1, P_2, ...P_k$,使得$E(G) = \cup_{i=1}^{n} E(P_i)$.

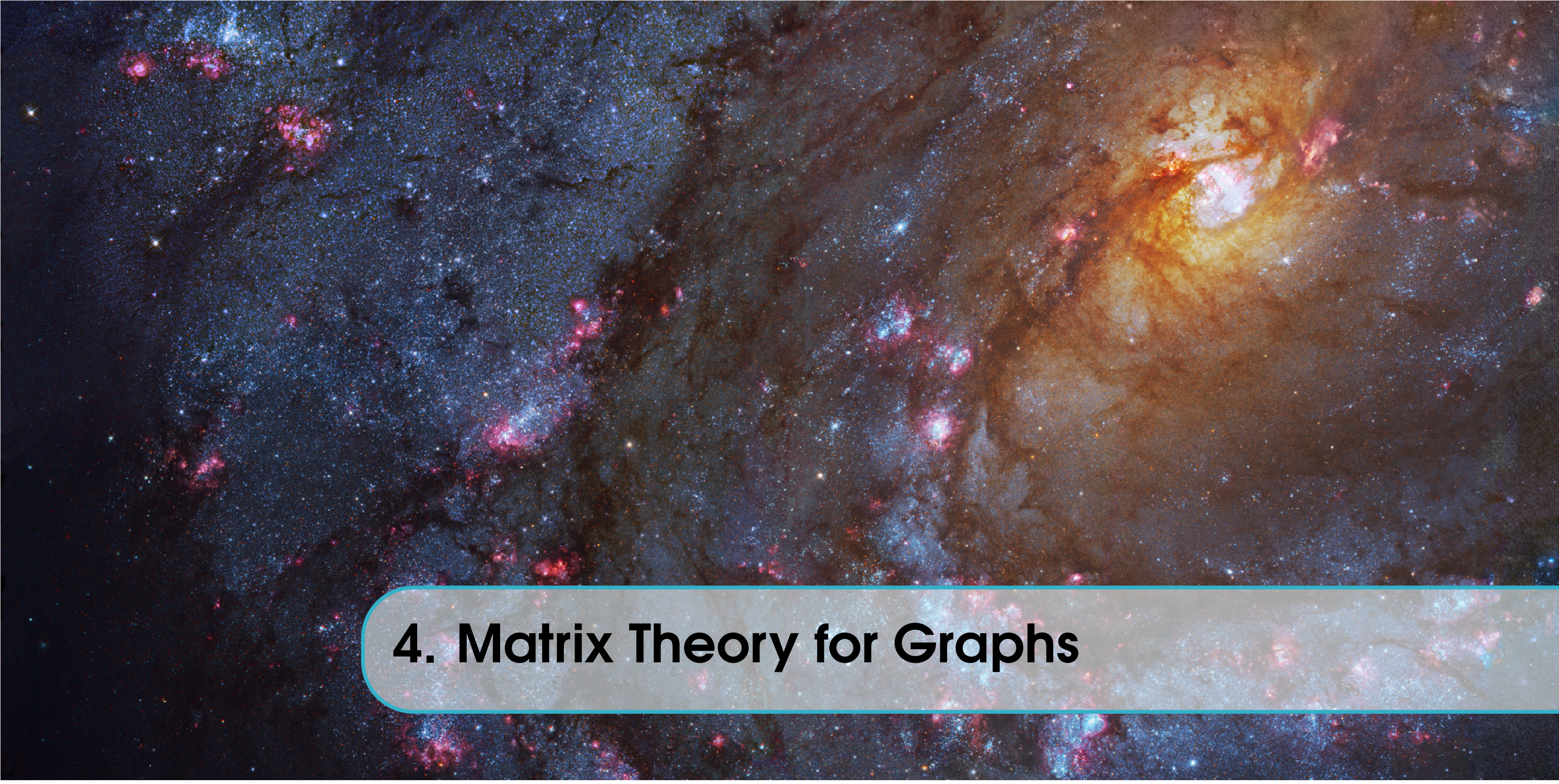
Proof. 将2k个点两两配对，每对连一条边。在新图中取一个欧拉回路，然后删去这k条边，一定能得到k段不交的简单通路。 ■

Theorem 3.1.5 — 有向欧拉图的判定. 设D为有向图，则下面三个命题等价：
(1) D is Eulerian.
(2) D is connected and $\forall v \in V(D), d^+(v) = d^-(v)$.
(3) D is connected and is the union of several indirected cycles with no intersected edges.

Proof. (1) $\Longrightarrow$ (2) : trival
(2) $\Longrightarrow$ (3) : similar to Theorem 8.1.1
(3) $\Longrightarrow$ (1) : similar to Theorem 8.1.1

■

Theorem 3.1.6 — 有向半欧拉图的判定. D中恰有两个奇数度的顶点，且一个点出度比如度大1，一个点入度比出度大1.其他点如度都等于出度。



4. Matrix Theory for Graphs

4.1 Adjacency Matrix