UNIVERSITY OF CAPE TOWN
DEPARTMENT OF STATISTICAL SCIENCES
STA2004F – TUTORIAL 1 – 2019

Name:	....	**Student no.:**	....
Date:	....	**Tut Group:**	Wednesday

---

Show all your workings.

1. A biased coin with $\mathbb{P}(\{\text{H}\}) = p \neq 0.5$ is tossed 100 times. We want to find the probability of observing an even number of heads. Let p_n be the probability that there is an even number of heads after n tosses (zero is an even number).

   (a) Show that $p_0 = 1$ and $p_n = (1-p)p_{n-1} + (1-p_{n-1})p$ for $n \geq 1$.
   Hint: Use the Law of Total Probability and condition on the outcome of the first toss.

   (b) Define the function $G : (-1, 1) \to \mathbb{R}$ by

   $$G(x) := \sum_{n=0}^{\infty} p_n x^n.$$

   Use the recurrence derived in (a) above to show that

   $$G(x) = 1 + (1-2p)xG(x) + \frac{px}{1-x}. \tag{1}$$

   (c) The relation (1) above can further be simplified using partial fractions (do this as homework) to get

   $$G(x) = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1-(1-2p)x}\right).$$

   By expanding this expression for G using power series and comparing coefficients, show that

   $$p_n = \frac{1}{2}\left(1 + (1-2p)^n\right) \qquad n = 0, 1, 2, \ldots.$$

[10]

UNIVERSITY OF CAPE TOWN
DEPARTMENT OF STATISTICAL SCIENCES
STA2004F – TUTORIAL 1 – 2019

Name:	....	**Student no.:**	....
Date:	....	**Tut Group:**	Thursday

---

Show all your workings.

1. Let X be an absolutely continuous random variable with density function given by

$$f_X(x) = \frac{1}{2} I_{[1,2]}(x) + \frac{1}{2} e^x I_{(-\infty,0)}(x), \; x \in \mathbb{R}.$$

   (a) Find and sketch the CDF of X.

   (b) Find and sketch the CDF of $Y := X I_{\{X \leq 1\}}$.

   (c) Show that Y is neither discrete nor continuous and write F_Y as

$$F_Y = pF_1 + (1-p)F_2$$

   where $0 \leq p \leq 1$, F_1 is the CDF of a discrete random variable, and F_2 is the CDF of a continuous random variable.

[10]

UNIVERSITY OF CAPE TOWN
DEPARTMENT OF STATISTICAL SCIENCES
STA2004F – TUTORIAL 1 – 2019

Name:	....	**Student no.:**	....
Date:	....	**Tut Group:**	Friday Morning

---

Show all your workings.

1. Let X have an exponential distribution with parameter $\lambda = 1$. Define $Y := X - \lfloor X \rfloor$ to be the fractional part of X (for instance, if $X = 2.38$, then $Y = 0.38$). We want to find the distribution of Y. Clearly, $0 \leq Y < 1$, so $F_Y(y) = 0$ when $y < 0$ and $F_Y(y) = 1$ when $y \geq 1$. Now let $0 \leq y < 1$ and define the events A_n by

$$A_n = \{\lfloor X \rfloor = n\}, \quad n = 0, 1, 2, \ldots.$$

   (a) Explain (briefly) why $\{A_0, A_1, A_2, \ldots\}$ are pairwise mutually exclusive and exhaustive events.

   (b) Explain why the following is true

$$\{Y \leq y\} \cap A_n = \{n \leq X \leq n + y\}.$$

   (c) Show that

$$\mathbb{P}(\{Y \leq y\} \cap A_n) = e^{-n} - e^{-(n+y)}.$$

   (d) Hence show that

$$F_Y(y) = \frac{1 - e^{-y}}{1 - e^{-1}}, \qquad 0 \leq y < 1.$$

   Hint: Total Probability.

   (e) Conclude that Y is absolutely continuous and find its density.

[10]

UNIVERSITY OF CAPE TOWN
DEPARTMENT OF STATISTICAL SCIENCES
STA2004F – TUTORIAL 1 – 2019

Name:	....	**Student no.:**	....
Date:	....	**Tut Group:**	Friday Afternoon

---

Show all your workings.

1. Let $X \sim \mathcal{U}(-2, 1)$ and define $X^+ := XI_{\{X>0\}}$.

   (a) Find and sketch the CDF of X.

   (b) Find and sketch the CDF of X^+. Is X^+ discrete, continuous or neither?

   (c) Find a function g such that $Y := g(X) \sim \mathcal{U}(0, 1)$ and show that the distribution of Y is indeed this uniform.

[10]