The Centeral Limit Theorem

Thanks to 子预, his article 辉煌的中心极限定理 helps me a lot.

$This\ is\ a\ brief\ but\ not\ strict\ proof\ of\ Center\ Limit\ Theorem(CLT).$

$CLT:$

$$i.i.d.\ \ r.v.\ \ X_1, X_2, X_3, ..., X_n,\quad E(X_k)=\mu,\ Var(X_k)=\sigma^2<\infty$$

$$Define\ \ Z=\frac{\Sigma_k X_k - n\mu}{\sqrt{n}\cdot\sigma},\quad \lim_{n\to+\infty} P(Z\le\tau)=\Phi(\tau)$$

$Or,\ we\ can\ also\ state\ that:$

$$\lim_{n\to+\infty} Z\ \xrightarrow{D} N(0,1)$$

$Proof:$

$$Define\ \ Y_k=\frac{X_k-\mu}{\sigma},\ \ E(Y_k)=0,\ \ Var(Y_k)=1,\ \ then\ \ Z=\frac{1}{\sqrt{n}}\Sigma_k Y_k$$

$$The\ Characteristic\ Function\ for\ Y_j:\ \varphi_{Y_j}(t)=E(e^{itY_j})\quad (see\ Characteristic\ Function\ Part)$$

$$\begin{aligned}
\varphi_{Y_j}(t) &= E(\Sigma_k\frac{(itY_j)^k}{k!}) \qquad (Taylor Expansion)\\
&= \Sigma_k\frac{i^k E(Y_j^k)\cdot t^k}{k!}\\
&= 1+iE(Y_j)t\ -\frac{E(Y_j^2)}{2!}t^2\ -\frac{iE(Y_j^3)}{3!}t^3\ +\frac{E(Y_j^4)}{4!}t^4\ ......\\
&= 1-\frac{1}{2}t^2+o(t^2)\\
\varphi_Z(t) &= \varphi_{\frac{1}{\sqrt{n}}\Sigma_j Y_j}(t)\\
&= \Pi_j\varphi_{Y_j}(\frac{t}{\sqrt{n}}) = (\varphi_{Y_j}(\frac{t}{\sqrt{n}}))^n\\
&= (1-\frac{t^2}{2n}+o(\frac{t^2}{n}))^n\ \ =\ (1-\frac{t^2}{2n}+o(\frac{1}{n}))^n
\end{aligned}$$

$$Since\ \ e^x=\ \lim_{n\to+\infty}(1+\frac{x}{n})^n,\ \ \lim_{n\to+\infty}\varphi_Z(t)=\lim_{n\to+\infty}(1-\frac{t^2}{2n}+o(\frac{1}{n}))^n=e^{-\frac{1}{2}t^2}$$

$$We\ know\ there's\ a\ one-to-one\ mapping\ relationship\ between\ the characteristic\ function$$
$$and\ pdf/pmf,(See\ below\ for\ details)\ so\ we\ need\ to\ find\ \varphi_N(t)$$

$$\begin{aligned}
\varphi_N(t) &= \int_{-\infty}^{+\infty} e^{itx}\cdot\ \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2+itx}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-it)^2+\frac{1}{2}(it)^2}dx\\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\cdot\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-it)^2}d(x-it)\\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\cdot\sqrt{2\pi}\ = e^{-\frac{1}{2}t^2}
\end{aligned}$$

$$Since\ \lim_{n\to+\infty}\varphi_Z(t)=\ e^{-\frac{1}{2}t^2}\ =\ \varphi_N(t),\ \ \lim_{n\to+\infty} Z\ \xrightarrow{D} N(0,1)\qquad Q.E.D.$$

Characteristic Function

we mainly use three properties of characteristic function.

1. $\varphi_{aX+b}(t) = e^{ibt}\varphi_X(at)$
2. $\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)\ given\ X\ and\ Y\ are\ independent$
3. for a $r.v.\ X,$ There exists a one-to-one mapping relationship between $Characteristic\ Function$ and $pdf(continuous)\ \ or\ \ pmf(discrete).$
4. Taylor expansion gives us the moments:
   $\varphi_X(t) = 1\ +\ i\mu t - \frac{i}{2}\sigma_2 t^2......$
   You can easily read/derivate the $Expectation,\ Variance\ and\ higher\ order\ Moments$

We will introduce what is Characteristic Function and brief proove the first three properties(the fourth has been shown above).
We will only cover the continuous case. For diecrete case, proove it yourself.

Bwyond previous three properties, characteristic functions have other great properties, like:

5. Good smoothness: The characteristic function is always continuous and often differentiable.
6. Convenient for convolution calculation.

For these two properties, I haven't grasped it yet. Since they are not used in previous proof, I ommit them.

$Definition:$

$$For\ a\ r.v.\ X, the\ characteristic\ function\ \varphi_X(t)\ is\ defined\ as\ E(e^{itX}):$$

$$\begin{aligned} \varphi_X(t) &= \int_D e^{itx} f_X(x)dx \qquad\qquad for\ continuous\ r.v. \\ \varphi_X(t) &= \Sigma_i e^{itx_i}\, p_X(x_i) \qquad\qquad\quad for\ discrete\ r.v. \end{aligned}$$

$Proof\ \ for\ properties$

$Property 1:$

$$\varphi_{aX+b} = E(e^{it(aX+b)}) = e^{itb} \cdot E(e^{itaX}) = e^{itb}\varphi_X(at)$$

$Property 2:$

$$\begin{aligned} \varphi_{X+Y} &= E(e^{it(X+Y)}) = E(e^{itX}e^{itY}) \\ &= E(e^{itX}) \cdot E(e^{itY}) \qquad (X, Y\ are\ independent,\ see\ below) \\ &= \varphi_X(t)\ \cdot\ \varphi_Y(t) \end{aligned}$$

$$\begin{aligned} &Given\ X\ and\ Y\ are\ independent: \\ E(\alpha(X)\beta(Y)) &= \\ &= \iint_{D_{X,Y}} f_{X,Y}(x,y)\cdot\alpha(x)\beta(y)\ dx\, dy \\ &= \iint_{D_{X,Y}} f_X(x)\, f_Y(y)\cdot\alpha(x)\beta(y)\ dx\, dy \\ &= \int_{D_X}(\int_{D_Y} f_Y(y)\,\beta(y)dy\quad)\cdot\ f_X(x)\,\alpha(x)dx \\ &= \int_{D_X} f_X(x)\,\alpha(x)\,dx\ \cdot\ \int_{D_Y} f_Y(y)\,\beta(y)\,dy \\ &= E(\alpha(X))\,E(\beta(Y)) \end{aligned}$$

$Property 3:$

$$The\ \ Characteristic\ \ Function\ \ is\ \ actually\ \ a\ \ Fourier\ \ Transform\ \ for\ \ pdf$$

$$For\ \ a\ \ definite\ \ Characteristic\ \ Function\ \ \varphi_X(t), we\ \ can\ \ find\ \ the\ \ only\ \ pdf:$$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\varphi_X(t)\,dt$$

$$This\ is\ guaranteed\ by\ Fourier\ Transform.\ For\ details,\ see\ the\ below\ about\ Fourier.$$

$$Thus\ for\ any\ \varphi_X(t) = \varphi_Y(t),\ f_X(z) = f_Y(z).$$

A little more about Characteristic Function

Fourier Transformation means separating a function into combinations of waves of different "frequency".

The characteristic function is just the Fourier Transformation of pdf, it means we use waves of different frequencies to detect the Amptitude of pdf at time t. This seems a little bit strange, but is quite interesting. The Fourier Transformation opens a new door for my original world.

Besides, CLT also holds for different but independent distributions, and even holds for different and low-relavant distributions. I briefly search some proof, but it is too complex for me. I have no time. You can search it if interested.

About Fourier Transform(FT)

For this part, thanks to tetradecane. His articles also help me a lot.
I highly recommand you read his articles, the first one is here.

I will first give a personal and intuitive understanding of Fourier:
it uses a linear combination of waves of different frequencies to approximate a wave of certain period, and finally to approximate any function.

Let's start from Fourier Series

Fourier Series

To denote a function $f(t)$ with period T, we have:

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T}\right) + b_n\sin\left(\frac{2\pi nt}{T}\right)\right],\text{where}$$

$$\begin{cases} a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos\left(\frac{2\pi nt}{T}\right)dt \\ \\ b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin\left(\frac{2\pi nt}{T}\right)dt \end{cases}$$

The reason why $f(t)$ can be denoted like this will be omitted(How to proove is too difficult for me, and I don't have enough time).
But, a naive understanding will be provided.

$$Remember\ the\ product-to-sum\ formulas:$$

$$\begin{cases} \sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right] \\ \\ \sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right] \\ \\ \cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right] \end{cases}$$

$$\int_0^T \sin(mx)\cos(nx)dx = \frac{1}{2}\int_0^T \frac{1}{2}\left[\sin(m+n)x - \sin(m-n)x\right]dx = \begin{cases} 0, & \text{if}\ \ m \neq n \\ \frac{2}{T}, & \text{if}\ \ m = n \end{cases}$$

$$\int_0^T \sin(mx)\sin(nx)dx = \frac{1}{2}\int_0^T \frac{1}{2}\left[\cos(m+n)x - \cos(m-n)x\right]dx = \begin{cases} 0, & \text{if}\ \ m \neq n \\ \frac{2}{T}, & \text{if}\ \ m = n \end{cases}$$

$$\int_0^T \cos(mx)\cos(nx)dx = \frac{1}{2}\int_0^T \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]dx = \begin{cases} 0, & \text{if}\ \ m \neq n \\ \frac{2}{T}, & \text{if}\ \ m = n \end{cases}$$

Thus, given that $f_T(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T}\right)+b_n\sin\left(\frac{2\pi nt}{T}\right)\right]$ holds,
we can use the above way to calculate a_n nad b_n.

Here is a more interesting understanding of previous contents. I assume that you grasp the basic knowledge of $Linear\ \ Space$.
And you should understand that $vectors\ are\ sth\ can\ be\ added\ and\ sclarly\ multiplied$.

The functions $\sin(\frac{2\pi n}{T}t)$ and $\cos(\frac{2\pi n}{T}t)$ are $orthogonal\ \ bases$ of $Function\ \ Space$.

We can then define the $inner\ product$ for 2 functions:

$$Just\ like\ the\ inner\ product\ we\ are\ familiar\ with:$$
$$\boldsymbol{a}=(a_1,a_2,a_3,.....),\quad \boldsymbol{b}=(b_1,b_2,b_3,.....),\quad <\boldsymbol{a},\boldsymbol{b}>=\Sigma_i a_i b_i,$$

$$We\ define\ the\ inner\ product\ of\ 2\ functions\ f,g\ (both\ defined\ on\ D)\ as:$$
$$<\boldsymbol{f},\boldsymbol{g}>=\int_D f(x)g(x)dx$$

The previous three integral shows that $\{1,\sin(x),\cos(x),\sin(2x),\cos(2x),.....\}$ are orthogonal to each other.
The $\{1,\sin(x),\cos(x),\sin(2x),\cos(2x),.....\}$ is a set of orthogonam basis for $Function\ \ Space$.

The process of multiplying $\sin(\frac{2\pi n}{T}t)$, $\cos(\frac{2\pi n}{T}t)$ and then integrating them is the same as $projection$.
We are just looking for $the\ position\ of\ f_T(t)$ in the trigonometric-based coordinates.

Introduce of Euler Formula

$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

Quite a beautiful formula! For the proof, just substitute the x in $Taylor\ \ Expansion$ of e^x to be $i\theta$.

We can then rewrite the Fourier Series in the exponential form.

$$\begin{cases}\sin(x)=\frac{1}{2i}(e^{ix}-e^{-ix})\\ \\ \cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})\end{cases},we\ rewrite\ \frac{2\pi n}{T}as\ \omega_n,$$

$$f_T(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T}\right)+b_n\sin\left(\frac{2\pi nt}{T}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left((e^{i\omega_n t}+e^{-i\omega_n t}\right)+\frac{b_n}{2i}\left(e^{i\omega_n t}-e^{-i\omega_n t}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\frac{ia_n+b_n}{2i}e^{i\omega_n t}+\frac{ia_n-b_n}{2i}e^{-i\omega_n t}\right]$$

$$let\ c_n\ =\begin{cases}\frac{ia_n+b_n}{2i}, & \text{if}\quad n>0\\ \\ \frac{a_0}{2}, & \text{if}\quad n=0\\ \\ \frac{ia_n-b_n}{2i} & \text{if}\quad n<0\end{cases}$$

$$We\ can\ then\ rewrite\ f_T(t)\ in\ a\ more\ compact\ form:\ f_T(t)=\sum_{n=-\infty}^{+\infty}c_n e^{i\omega_n t}$$

We introduce the imaginary unit, but $f_T(t)$ is still a real function. Interesting, right?

We've already examined the orthogonality of trigonometric basis, let's examine that of $e^{i\omega_b t}$.

$$\begin{aligned}\int_0^T e^{imx}e^{inx}dx &= \int_0^T (\cos(mx) + i\sin(mx))(\cos(nx) + i\sin(nx))\,dx \\ &= \int_0^T [\cos(mx)\cos(nx) - \sin(mx)(nx) + i\sin(mx)\cos(nx) + i\sin(nx)cos(mx)]\,dx \\ &= 0 \qquad (m \neq n)\end{aligned}$$

Amazing! $\{e^{i\omega_n t}\}\Big|_{n=1}^{\infty}$ can also be an orthogonal basis set! We can then use this property: $using\ \ e^{imt}\ to\ elimiate\ e^{int}(m \neq n).$

$$c_n = \frac{1}{T}\int_0^T e^{-i\omega_n t} f_T(t)dt$$

From Fourier Series to Fourier Transform(FT)

Let's go beyond functions with a period. Just imagine we take period T to be the $+\infty$.
Then, its discrete frequencies ω_n becomes the continuous. See how we do this.

$$For\ a\ function\ with\ period\ T\ f_T(t), we\ have$$

$$\omega_n = \frac{2n\pi}{T},\quad \Delta\omega = \omega_{n+1} - \omega_n = \frac{2\pi}{T}$$

$$You\ can\ see\ this\ direct\ propotional\ relationship\ between\ n\ and\ \omega_n, so\ we\ define:$$
$$c(\omega_n) = c_n.$$

$$What\ we\ do\ is\ just\ change\ a\ denotion,\ using\ function\ c(\omega_n)\ instead\ of\ sequence\ c_n.$$
$$c(\omega_n) = \frac{1}{T}\int_0^T e^{-i\omega_n t} f_T(t)dt$$

$$As\ for\ its\ Fourier\ Series,\quad f_T(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i\omega_n t} = \sum_{n=-\infty}^{+\infty} c(\omega_n)e^{i\omega_n t}$$

$$\begin{aligned}f_T(t) &= \sum_{n=-\infty}^{+\infty} c(\omega_n)e^{i\omega_n t} \\ &= \sum_{n=-\infty}^{+\infty} c(\omega_n)e^{i\omega_n t}\cdot\frac{1}{\Delta\omega}\cdot\Delta\omega \\ &= \sum_{n=-\infty}^{+\infty}\left[c(\omega_n)\cdot\frac{1}{\Delta\omega}\right]e^{i\omega_n t}\cdot\Delta\omega\end{aligned}$$

$$We\ define\quad \tilde{c}(\omega_n) = c(\omega_n)\cdot\frac{1}{\Delta\omega},\ and\ then\ we\ get\ :$$
$$f_T(t) = \sum_{n=-\infty}^{+\infty} \tilde{c}(\omega_n)\ e^{i\omega_n t}\cdot\Delta\omega$$

$$Then\ \ came\ \ the\ \ most\ \ important\ \ part:$$
$$As\ T \to +\infty, f_T(t) \to f(t),\ \ \Delta\omega \to 0\ \ and\ \ \omega_n\ \ becomes\ \ a\ \ continuous\ \ variable\ \omega!$$

$$\begin{aligned}f(t) &= \lim_{T\to+\infty} f_T(t) \qquad\qquad (f(t): a\ non-periodic\ function) \\ &= \lim_{T\to+\infty}\sum_{n=-\infty}^{+\infty}\tilde{c}(\omega_n)\ e^{i\omega_n t}\cdot\Delta\omega \\ &= \int_{-\infty}^{+\infty}\tilde{c}(\omega)\ e^{i\omega t}\ d\omega \qquad\qquad (*)\end{aligned}$$

Recall our aim: to use a combination of trigonometric functions of different frequencies to denote a gneral function.

Since the trigonometric functions of different frequencies are seen as a function of $frequency\ \ \omega$, and the general function is a function of t. See $(*)$ above, we are almost done.

$$\text{Recall that we define: } \tilde{c}(\omega_n) = c(\omega_n) \cdot \frac{1}{\Delta\omega}$$
$$= \frac{1}{T}\int_0^T e^{-i\omega_n t} f_T(t)dt \cdot \frac{T}{2\pi}$$
$$= \frac{1}{2\pi}\int_0^T e^{-i\omega_n t} f_T(t)dt$$
$$\text{As we take } T \to +\infty, f_T(t) \to f(t) \quad \text{and} \quad \omega_n \to \omega.$$
$$\Rightarrow \tilde{c}(\omega) = \frac{1}{2\pi}\int_0^{+\infty} e^{-i\omega t} f(t)dt$$

$$\text{We finally define } F(\omega) = \int_0^{+\infty} e^{-i\omega t} f(t)dt$$

$$\text{We call :}$$
$$F(\omega) = \int f(t)e^{-iwt}dt \qquad \text{the Fourier Transform}$$
$$f(t) = \int F(\omega)e^{iwt}d\omega \qquad \text{the Inverse Fourier Transform}$$

The One-To-One Mapping Relationship

If you have the patience to read until here, I appreciate your effort. Thanks for reading it. However, I will apologetically inform you that I cannot offer a strict proof for this relationship. However, intutively, they have the one-to-one relationship, right?

I am burnt out. If I have time in the future, I will make up the last part.