Probability with Engineering Applications

Lecture 20

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Review

- **Jointly CDF**

$$F_{X,Y}(u_o, v_o) \;=\; P\{X \le u_o, Y \le v_o\}.$$

- **Jointly PMF**

$$p_{X,Y}(u,v) = P\{X = u, Y = v\}$$

- **Jointly PDF**

$$F_{X,Y}(u_o, v_o) \;=\; \int_{-\infty}^{u_o}\int_{-\infty}^{v_o} f_{X,Y}(u,v)\,dv\,du.$$

- **Conditional PDF**

$$f_{Y|X}(v|u_o) = \frac{f_{X,Y}(u_o, v)}{f_X(u_o)}$$

Independent Random Variables

A B

$P(A,B) = P(A)P(B)$

- **Definition**

Definition 4.4.1 *Random variables X and Y are defined to be independent if any pair of events of the form $\{X \in A\}$ and $\{Y \in B\}$, are independent. That is:*

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

$\forall\ x \in A,\ y \in B$

$$\Updownarrow$$

- **CDF-Form (both discrete and continuous)**

$$\boxed{F_{X,Y}(u_o, v_o)} = F_X(u_o)F_Y(v_o).$$

CDF factorization property

Joint CDF

$P\{X \le u_0, Y \le v_0\}$ $P\{X \le u_0\}$ $P\{Y \le v_0\}$

$A = \{X \le u_0\}$ $B = \{Y \le v_0\}$

$\Uparrow$

$\{u_1, u_1+1, \cdots\}$

Independent Random Variables

- **PMF-Form (Discrete)**

$P\{X=u, Y=v\}$

$\Updownarrow$

$A=\{X=u\}$

$B=\{Y=v\}$

$$\underbrace{p_{X,Y}(u,v)}_{\text{Joint PMF}} = p_X(u)p_Y(v)$$

PMF factorization property

$A=\{X \geq u_1\}$

$B=\{y \geq v_1\}$

$\{v_1, v_{1+1}, \dots\}$

$P(A,B) = P(A)P(B)$

- **PDF-Form (Continuous)**

$$\boxed{f_{X,Y}(u,v) = f_X(u)f_Y(v)}$$

PDF factorization property

$$F_{X,Y}(u_0,v_0) = \int_{-\infty}^{u_0}\int_{-\infty}^{v_0} f_{X,Y}(u,v)\,du\,dv$$

Joint PDF

X Y

$$= F_X(u_0)\,F_Y(v_0) = \int_{-\infty}^{u_0} f_X(u)\,du \cdot \int_{-\infty}^{v_0} f_Y(v)\,dv$$

marginal PDF

$\forall u_0, v_0$

(I) $$\boxed{f_{X,Y}(u,v) \equiv f_X(u)\,f_Y(v)}$$

Independent Random Variables

- **Equivalent Conditional PDF/PMF -Form (discrete and continuous)**

Proposition 4.4.2 *X and Y are independent if and only if the following condition holds: For all $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all $v \in \mathbb{R}$.*

(I) $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ ☆ (II)

How to prove? why?

Step1: (II) $\longrightarrow$ (I) sufficient condition

case 1: $f_X(u) = 0 = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$

$f_{X,Y}(u,v) = f_X(u) f_Y(u)$

0 ✓ 0

=

case 2: $f_{Y|X}(v|u) = f_Y(v)$

$f_{X,Y}(u,v) = f_X(u) \cdot \underline{f_{Y|X}(v|u)}$

$f_Y(v)$ ✓

Step2: (I) $\longrightarrow$ (II) necessary condition!

=☆

$$\boxed{f_{Y|X}(v|u)} = \frac{f_{X,Y}(u,v)}{f_X(u)} \quad \text{if } f_X(u) > 0$$

$$= \frac{\cancel{f_X(u)}\, f_Y(v)}{\cancel{f_X(u)}} = \boxed{f_Y(v)} \quad f$$

Independent Random Variables

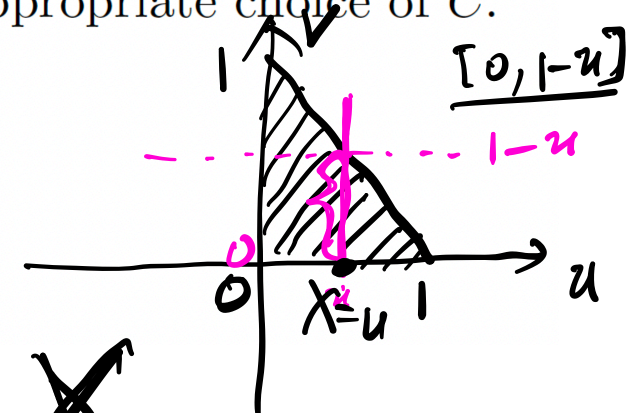
- **Example**

Decide whether X and Y are independent for each of the following three pdfs:

(a) $f_{X,Y}(u,v) = \begin{cases} Cu^2v^2 & u,v \geq 0, u+v \leq 1 \\ 0 & \text{else,} \end{cases}$ for an appropriate choice of C.

(b) $f_{X,Y}(u,v) = \begin{cases} u+v & u,v \in [0,1] \\ 0, & \text{else} \end{cases}$

(c) $f_{X,Y}(u,v) = \begin{cases} \frac{9u^2v^2}{} & u,v \in [0,1] \\ 0, & \text{else} \end{cases}$



(a) $f_{X,Y}(u,v) = Cu^2v^2 = \underset{f(u)}{u^2} \cdot \underset{f(v)}{v^2}$ X

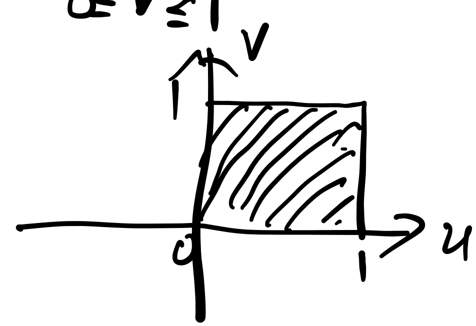
X

$f_X(u) = \boxed{\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv} = \int_0^{1-u} Cu^2v^2\, dv = \boxed{Cu^2 \frac{(1-u)^2}{3}}$

$0 \leq u \leq 1$

$f_Y(v) = \int_0^{1-v} Cu^2v^2\, du = Cv^2 \frac{(1-v)^2}{3}$

$0 \leq v \leq 1$



(b) $u+v \neq g(u)h(v)$ X

Independent Random Variables

- **Example**

(c) $f_{X,Y}(u,v) = \begin{cases} 9u^2v^2 & u,v \in [0,1] \\ 0 & \end{cases}$

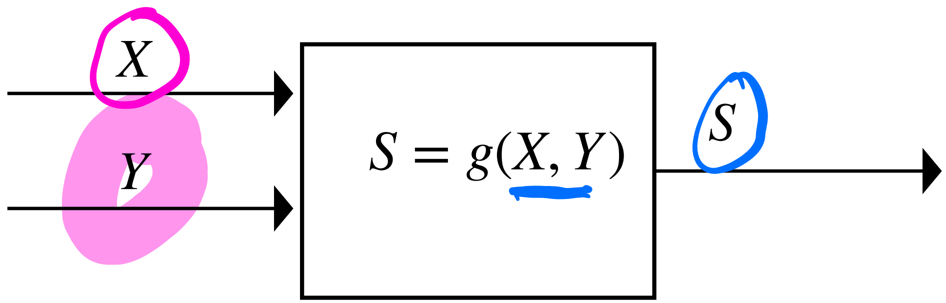
$f_X(u) = \int_0^1 9u^2v^2\,dv = \begin{cases} 9u^2\frac{1}{3} = 3u^2 & u \in [0,1] \\ 0 & \text{else} \end{cases}$

$f_Y(v) = \begin{cases} 3v^2 & v \in [0,1] \\ 0 & \text{else} \end{cases}$

$$\boxed{f_{X,Y}(u,v) = f_X(u)\, f_Y(v)}$$

✓

Distribution of Function of Random Variables



Question: ***What is the distribution of S?***

E.g. $g(X, Y) = X + Y$

Distribution of Function of Random Variables

$$S = X + Y.$$

Question: ***What is the distribution of S=X+Y?***

- **Discrete Case**

Suppose $S = X + Y$, where X and Y are integer-valued random variables.

$Y, X = -2, -1, 0, 1, 2 \ldots$

PMF

k integer

$$P(S=k) = P(X+Y=k)$$

$$\boxed{= \sum_i P_{XY}(X=i, Y=k-i)}$$

$k = 2$

X	0	1	2	-1
Y	2	1	0	3

if $X \perp Y$

X, Y are independent

$$= \boxed{\sum_i P_X(X=i)\, P_Y(Y=k-i)} = P(S=k)$$

"卷积" Convolution of $P_X(u)$ and $P_Y(v)$

Distribution of Sum of Random Variables

- **Example**

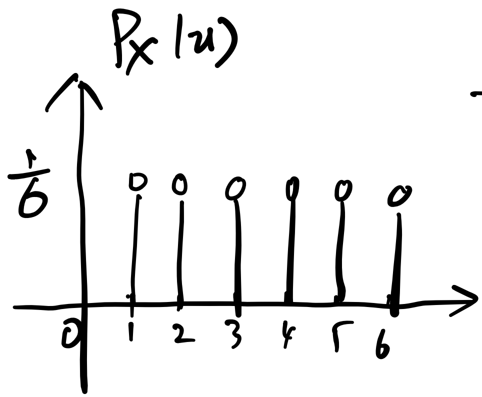
$X \sim U \qquad Y \sim U$

Example 4.5.3 Suppose X and Y represent the numbers showing for rolls of two fair dice. Thus, each is uniformly distributed over the set $\{1, 2, 3, 4, 5, 6\}$ and they are independent. Describe the distribution of $S = X + Y$.

$S = X + Y$

$S \in \{2, 3, 4, \cdots, 12\} \triangleq F$

$\forall k \in F \quad p(S = k) = \sum_{i=1}^{k-1} P_X(X=i)\, P_Y(Y=k-i)$



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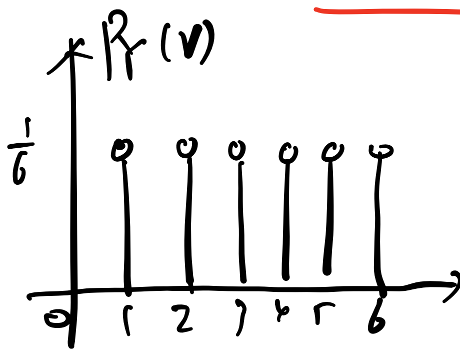
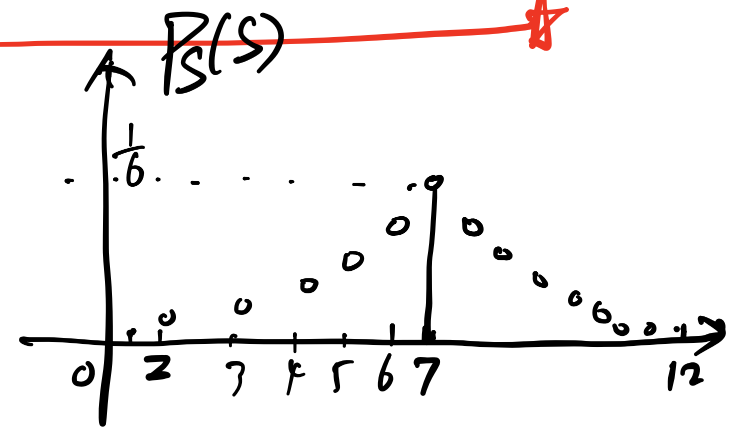




$P_X(u) \circledast P_Y(v) = P_S(s)$

$*$

Distribution of Sum of Random Variables

- **Example**

Example 4.5.2 Suppose X and Y are independent random variables such that X has the Poisson distribution with parameter λ_1 and Y has the Poisson distribution with parameter λ_2. Describe the distribution of $S = X + Y$.

$$X \sim \text{Pois}(\lambda_1) \qquad Y \sim \text{Pois}(\lambda_2)$$

$$S = X + Y \qquad P_S(s) = P_X(x) \circledast P_Y(y)$$

$$\boxed{S \sim \text{Pois}(\lambda_1 + \lambda_2)}$$

Distribution of Sum of Random Variables

Question: ***What is the distribution of S=X+Y?***

- **Continuous Case**

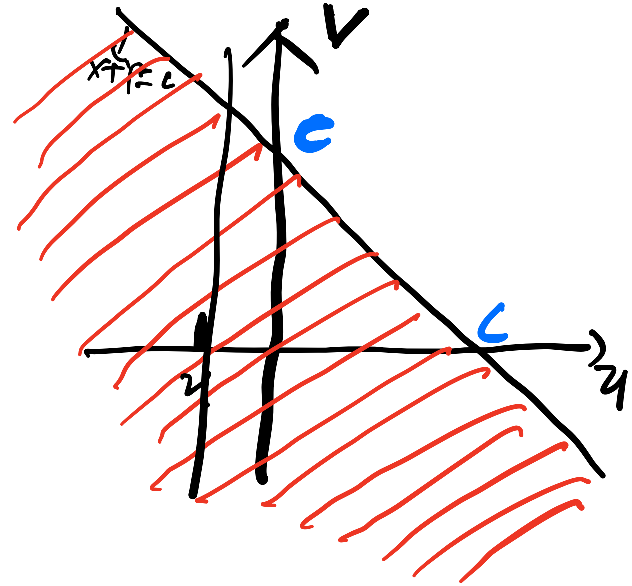
Suppose $S = X + Y$ where X and Y are jointly continuous-type.

$f_{X,Y}(u,v)$

Goal: $f_S(s)$

start with $F_S(c) = P(S \le c)$

$= P(X+Y \le c)$



$$= \int_{u=-\infty}^{\infty} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, du\, dv$$

$$f_S(c) = \frac{d F_S(c)}{dc} = \frac{d}{dc} \int_{-\infty}^{\infty} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, du\, dv$$

$$= \int_{-\infty}^{\infty} \left[\frac{d}{dc} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, dv \right] du = \int_{-\infty}^{\infty} f_{X,Y}(u, c-u) \cdot 1\, du$$

Distribution of Sum of Random Variables

- **Example**

Example 4.5.4 Suppose X and Y are independent, with each being uniformly distributed over the interval $[0,1]$. Find the pdf of $S = X + Y$.

if X, Y independent

$$\boxed{\int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\, du = f_S(c)}$$

"convolution" for continuous case.