Probability with Engineering Applications

Lecture 21

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Review

- **Independent Random Variables**

Def

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}. \quad \forall\ A, B$$

CDF factorization property $\underbrace{F_{X,Y}(u_o, v_o)}_{\text{Joint CDF}} = F_X(u_o)F_Y(v_o).$ continuous / discrete

PMF factorization property $\underbrace{p_{X,Y}(u, v)}_{\text{Joint pmf}} = p_X(u)\underbrace{p_Y(v)}_{\text{marginal}}$

PDF factorization property $f_{X,Y}(u, v) = f_X(u)f_Y(v)$ $\boxed{\text{Support}}$

- **Distribution of Sum of Random Variables**

$$S = X + Y$$

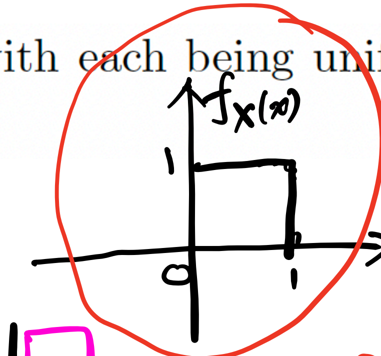
case 1: discrete case $P_S(s=k) = \sum_i \overbrace{P(X=i, Y=k-i)}^{\text{Joint pmf}} = \overbrace{\sum_i P_X(X=i)\,P_Y(Y=k-i)}^{\text{convolution}}$ $\boxed{P_X(k) * P_Y(k)}$

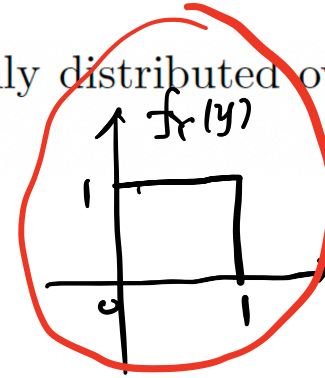
case 2: continuous case $f_S(s) \overset{X,Y\ \text{independent}}{=} \int_{-\infty}^{\infty} \overbrace{f_X(z)\, f_Y(s-z)\, dz}^{\text{convolution}}$ $\boxed{f_X(s) * f_Y(s)}$ "卷积"

Distribution of Sum of Random Variables

- **Example**

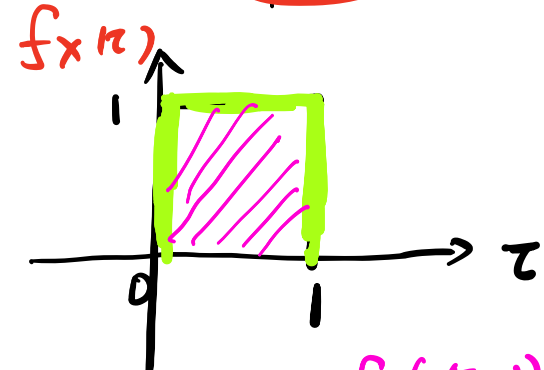
Example 4.5.4 Suppose X and Y are independent, with each being uniformly distributed over the interval $[0,1]$. Find the pdf of $S = X + Y$.





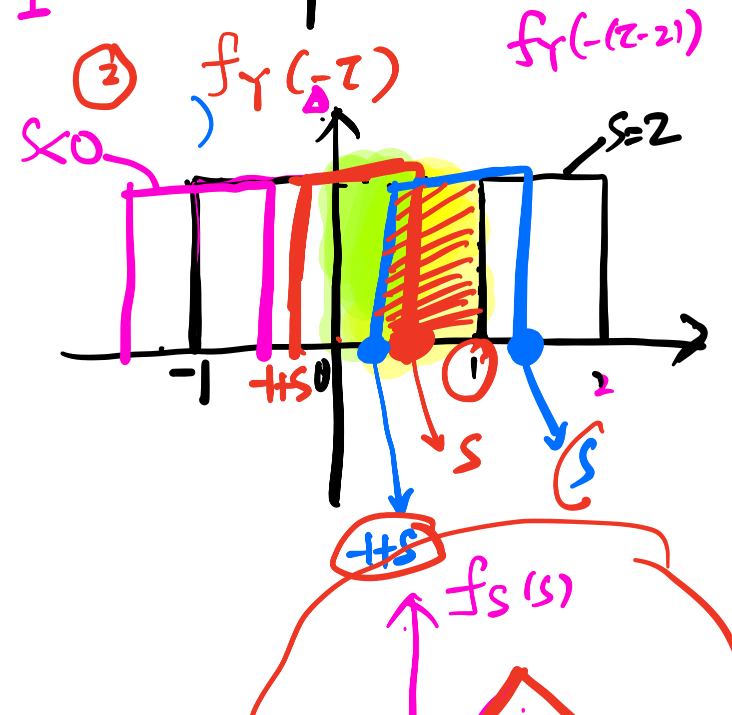
$$f_S(s) = f_X(s) * f_Y(s)$$

$$= \int_{-\infty}^{\infty} f_X(\tau)\, f_Y(s-\tau)\, d\tau$$

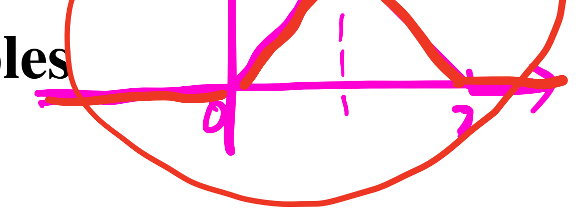


$$f_Y(s-\tau) = f_Y(-(\tau-s))$$ from $f_Y(-\tau)$ I

$$= \begin{cases} \int \boxed{0}\, d\tau = 0 & \text{if } s<0 \\ \int_0^s 1\cdot 1\, d\tau = s & \text{if } 0<s<1 \\ \int_{-1+s}^{1} 1\cdot 1\, d\tau = 2-s & 1<s<2 \\ \int 0\, d\tau = 0 & \text{if } s>2 \end{cases}$$



Distribution of Sum of Random Variables



- **Example**

Suppose X and Y are independent standard normal (Gaussian) distribution random variables.

Find the PDF of Z= X+Y

$X \sim N(0,1)$ $\quad Y \sim N(0,1)$ $\quad X\ Y$ indep.

$$f_Z(z) = f_X(z) * f_Y(z)$$

$$= \int_{-\infty}^{\infty} \underline{f_X(\tau)}\, \underline{f_Y(z-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\tau)^2}{2}}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{\tau^2}{2}} \cdot e^{-\frac{z^2-2z\tau+\tau^2}{2}}\, d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\tau^2 - z\tau)} \cdot e^{-\frac{z^2}{2}}\, d\tau$$

Distribution of Sum of Random Variables

$\left[\tau^2 - z\tau + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^2\right]$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \boxed{\int_{-\infty}^{\infty} e^{-(\tau^2 - z\tau)} \, d\tau}$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} e^{+\left(\frac{z}{2}\right)^2} \sqrt{2\pi \tfrac{1}{2}} \int_{-\infty}^{\infty} \boxed{\frac{1}{\sqrt{2\pi \frac{1}{2}}}} e^{-\frac{\left(\tau - \frac{z}{2}\right)^2}{2\cdot\frac{1}{2}}} \, d\tau \quad = 1$$

$\sqrt{\pi}$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} \qquad \frac{1}{2\pi} \cdot \sqrt{\pi}$$

$$\sim N(0, \sigma^2 = 2)$$

$X \sim N(\mu_1, \sigma_1^2)$ $\quad$ $Y \sim N(\mu_2, \sigma_2^2)$ $\quad$ $S = X + Y$, X, Y ind

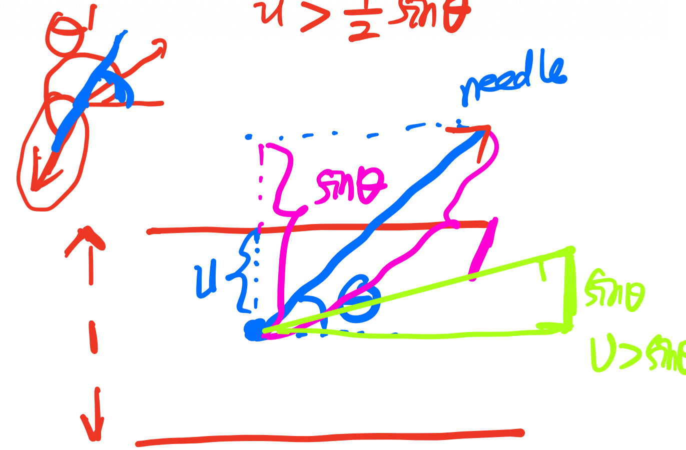
$$S \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_1^2)$$

$S' = \alpha X + \beta Y$

$N(\alpha\mu_1 + \beta\mu_2, \alpha^2\sigma_1^2 + \beta^2\sigma_2^2)$

Additional Examples using Joint Distributions

Example 4.6.3 Buffon's needle problem Suppose a needle of unit length is thrown at random onto a large grid of lines with unit spacing. Find the probability the needle, after it comes to rest, intersects a grid line.



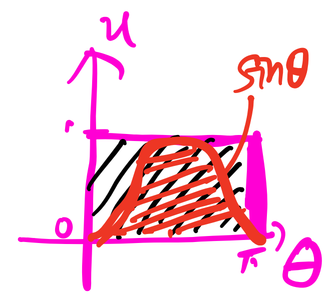
$\theta \in [0, \pi]$ uniform

$u \in [0, 1]$ uniform

Joint PDF of (θ, u) is

$$f_{\Theta,U}(\theta, u) = \begin{cases} \frac{1}{\pi} & \theta \in [0,\pi],\ u \in [0,1] \\ 0 & \text{else} \end{cases}$$

A: intersection



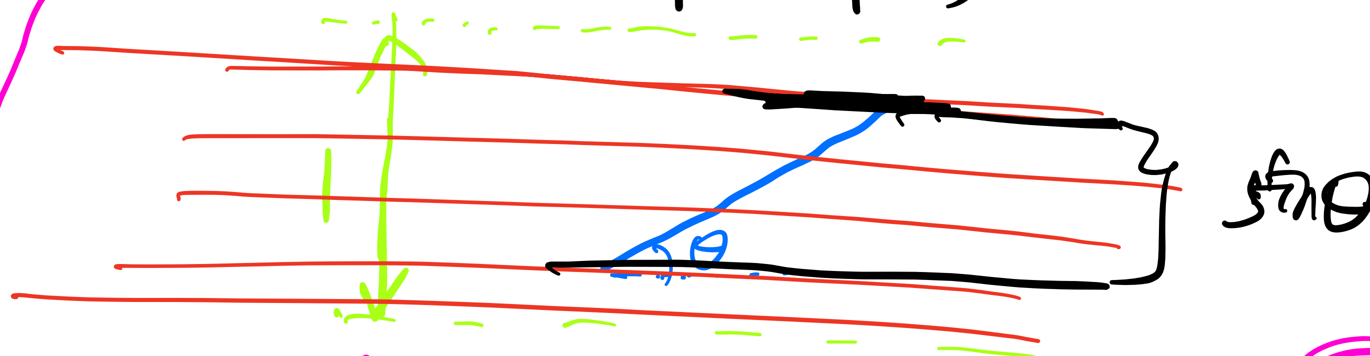
$$P(A) = P(u \le \sin\theta) = \int_0^{\pi}\int_0^{\sin\theta} \frac{1}{\pi}\, du\, d\theta = \boxed{\frac{2}{\pi}}$$

Additional Examples using Joint Distributions

$$\boxed{P(A) = \frac{2}{\pi}} \Rightarrow \pi = \frac{2}{P(A)}$$

$$P(A) = \int_0^{\pi} P(A|\theta)\, f_\Theta(\theta)\, d\theta$$

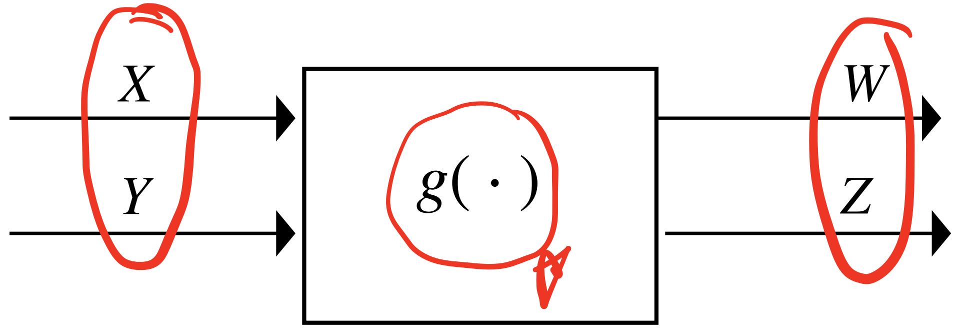
$$P(A|\theta) = \sin\theta$$



$$= \int_0^{\pi} \sin\theta \, \frac{1}{\pi}\, d\theta = \frac{2}{\pi}$$

Joint pdfs of functions of random variables

$S = X + Y$



Question: ***What is the joint distribution of*** $\binom{W}{Z} = g(\binom{X}{Y})$***?***

$$X \longrightarrow \boxed{g(\cdot)} \longrightarrow Y = g(X)$$

$$f_X(x)\,dx = f_Y(y)\,dy$$

$$f_Y(y) = f_X(x)\left|\frac{dy}{dx}\right|$$

Joint pdfs of functions of random variables

Proposition 4.7.4 *Suppose* $\binom{W}{Z} = g(\binom{X}{Y})$, *where* $\binom{X}{Y}$ *has pdf* $f_{X,Y}$, *and* g *is a one-to-one mapping from the support of* $f_{X,Y}$ *to* $\mathbb{R}^2$. *Suppose the Jacobian* J *of* g *exists, is continuous, and has nonzero determinant everywhere. Then* $\binom{W}{Z}$ *has joint pdf given by*

$$\underline{f_{W,Z}(\alpha,\beta)} = \frac{1}{|\det J|} f_{X,Y}\left(g^{-1}\left(\underline{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}\right)\right).$$

$$\begin{pmatrix}X\\ Y\end{pmatrix} = g^{-1}\begin{pmatrix}W\\ Z\end{pmatrix} = g^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

for (α,β) *in the support of* $f_{W,Z}$.

$$\begin{cases} g_1(u,v) = \alpha \\ g_2(u,v) = \beta \end{cases}$$

$$\boxed{J = J(u,v)} = \begin{pmatrix} \frac{\partial g_1(u,v)}{\partial u} & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2(u,v)}{\partial u} & \frac{\partial g_2(u,v)}{\partial v} \end{pmatrix}$$

$$= \frac{\partial(\alpha,\beta)}{\partial u \partial v} = \begin{pmatrix} \frac{\partial\alpha}{\partial u}, & \frac{\partial\alpha}{\partial v} \\ \frac{\partial\beta}{\partial u}, & \frac{\partial\beta}{\partial v} \end{pmatrix}$$

$$f_{X,Y}(u,v)\,du\,dv = \underline{f_{W,Z}(\alpha,\beta)}\,d\alpha\,d\beta$$

$$\underline{f_{W,Z}(\alpha,\beta)} = f_{X,Y}(u,v)\boxed{\frac{du\,dv}{d\alpha\,d\beta}}$$

Joint pdfs of functions of random variables

$(\alpha,\beta) = g(u,v)$ ☆

$$du\,dv = \left|\frac{\partial(u,v)}{\partial\alpha\,\partial\beta}\right| d\alpha\,d\beta = \left|\frac{\partial(\alpha,\beta)}{\partial u\,\partial v}\right|^{-1} d\alpha\,d\beta$$

J

$$f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v)\left|\frac{\partial(u,v)}{\partial\alpha\,\partial\beta}\right| = f_{X,Y}(u,v)\frac{1}{\left|\frac{\partial(\alpha,\beta)}{\partial u\,\partial v}\right|}$$

J

$$= f_{X,Y}(g^{-1}(\alpha,\beta))\frac{1}{|\det(J)|}$$ ☆

Joint pdfs of functions of random variables

- **Example: Linear Transform**

$$\binom{W}{Z} = A\binom{X}{Y} = g\binom{X}{Y}$$

$\binom{u}{v} = g^{-1}(\alpha,\beta) = A^{-1}\binom{\alpha}{\beta}$

$$f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v)\frac{1}{|\det(J)|} = f_{X,Y}(u,v)\frac{1}{|\det(A)|}$$

$$J = \frac{\partial(\alpha,\beta)}{\partial u \partial v} = A$$

Example 4.7.2 Suppose X and Y have joint pdf $f_{X,Y}$, and $W = X - Y$ and $Z = X + Y$. Express the joint pdf of W and Z in terms of $f_{X,Y}$. ☆

Joint pdfs of functions of random variables

- **Example: Linear Transform**

Joint pdfs of functions of random variables

- **Example: Non-Linear Transform**

Example 4.7.6 Suppose X and Y are independent $N(0, \sigma^2)$ random variables. View $\binom{X}{Y}$ as a random point in the $u - v$ plane, and let (R, Θ) denote the polar coordinates of that point. Find the joint pdf of R and Θ.

Joint pdfs of functions of random variables

- **Example: Non-Linear Transform**