Modified Electrodynamics: Fixing Relativistic Field Theories

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Abstract

This thesis investigates the problem of modifying relativistic field theories so as to remove their singularities. A particular modified relativistic theory, Born-Infeld Electrodynamics, is first considered. Some simple solutions are obtained. Comments are made on the general solution structure. For instance, a notion of multipoles is developed to organize the class of stationary solutions of the theory. Theoretical difficulties in the Born-Infeld theory motivate a general discussion of model building, first for scalar fields, and then for vector fields. Two theories are described that fix some of the problems arising in the Born-Infeld theory. Along the way, the related problem of removing essential singularities from the theory of General Relativity is discussed.

Introduction

The subject of this thesis is model building: we would like to build a mathematical model from a minimal set of assumptions that corresponds most accurately to the real world. There are two immediate challenges. The first concerns the development of a robust mathematical structure to store and convey all relevant information. The description should neither be so broad that it contains more than what actually occurs, nor should it be so restrictive that it cannot scale up as generalization is needed. The Lagrangian formalism provides an efficient method for such constructions. We will outline this formalism in Chapter 1. The second challenge concerns the development of a minimal set of assumptions, slogans like "energy is conserved", that encode what is physically acceptable. The problem is, what you take to be acceptable physics is entirely up to you. There is currently a mess of competing ideas about acceptable physics. In Loop Quantum Gravity, for instance, one claims that space is quantized. For the String Theorist, the appropriate statement is that particles are extended bodies, in particular, strings. On the other hand, there are certain principles that appear, in light of direct observation, to be ubiquitous to physical processes. In this thesis we will work frequently with two such principles: the energy associated with physical objects and processes is finite, and physical objects and processes are defined for all of space. These certainly sit high on the list.

We will not be making models from scratch. Instead we will consider the problem of fixing models that are already quite good. Einstein's theory of General Relativity (GR) is the particular model that I have in mind. Working from two basic physical principles, a natural theory of spacetime emerges. More importantly, the predictions of the theory agree very well with experiment. Yet the theory admits a number of peculiar, some would say unacceptable, solutions. For instance, Gödel found a solution to Einstein's theory that allows for time travel (provided that the universe has peculiar dynamics).As a second instance, the theory admits singular solutions. These are solutions for which the geometry of certain regions of space is undefined—precisely, some components of the Riemann curvature tensor are infinite. Unlike Gödel's solution, singular solutions are common in the description of our universe. Aesthetically, one may find singular solutions unpleasing. Shouldn't space be defined everywhere? I don't know. The problem becomes more significant when on tries to unify GR with other physical theories.

The Schwarzschild solution is one example of a singular solution. It describes the structure of spacetime away from a spherically symmetric stationary mass. In particular, it describes spacetime away from a stationary spherically symmetric black hole. The solution may be encoded in a matrix g. The rr component of the matrix reads

$$g_{rr} = \frac{1}{1 - \frac{2GM}{c^2 r}},$$

where M is the mass of the source and G is the gravitational constant. This component (hence the full solution) appears to be infinite in two regions of space, once at the origin and once at $r = 2GM/c^2$. The latter is a coordinate singularity and can be removed by an appropriate change of coordinates; the singularity at the origin is essential, it is a feature of spacetime itself.

Several attempts have been made at removing singular solutions from General Relativity altogether. Presumably, this removal of the singular solution corresponds to the advent of new physics and new governing principles near the singularity. Fixing the singular solution amounts to inserting new physics into the theory. The problem is that there is no agreement in how one should proceed, nor has any particular attempt been successful. Without considering in detail the physics near singular points, one may still proceed by insisting that spacetime singularities are not allowed. In practice one makes the slightly stronger assertion that components of the Riemann curvature tensor tensor are bounded in magnitude. The challenge is to construct a modified theory of GR that obeys not only the principles of GR, but also the principle of boundedness. In general, such a construction will not be unique; it is not intended to describe in any detail the physics near singularities. Rather, we interpret the modified theory as a first approximation of some more complete model that unifies GR with physics near singularities. It turns out that constructing an acceptable modified theory is itself a challenging project [5]. This is in part due to the fact that the modified theory is often intractable (as if GR was not hard enough) and in part due to the fact that modifications, while solving one problem, may introduce others. In this thesis we attempt to understand the difficulties of such constructions.

En route to this problem, we will study a simpler problem of fixing singularities in the classical theory of Electrodynamics (E&M). A discussion of singularities arising in this theory is given in Chapter 2. As in the case of GR, the challenge is to construct a modified theory of Electrodynamics for which the singularities are absent. And, as in the case of GR, we do this by insisting that relevant fields are bounded in magnitude. The Born-Infeld theory of Electrodynamics (BI E&M) is one modified theory designed to remove singularities. We will spend some time studying this model, its structure and its failings. In light of this discussion, we will generate a class of modified theories that fix the singular solutions while avoiding the problematic features possessed by BI E&M. It is hoped that this systematic development of acceptable modified Electrodynamic theories informs a similar development for GR.

Chapter 1

Background

This chapter serves to orient the reader within the mathematical and theoretical framework of this thesis. We begin with a brief introduction to the theory and machinery of relativistic fields. As an example, and for reference in the main body of this text, the field theoretic formulation of classical Electrodynamics is developed. Some conventions are established along the way. Sections may be skipped as the reader sees fit.

1.1 A Review of Relativistic Field Theory

We first outline the theory of relativistic fields. Emphasis is placed upon the machinery and notation of the theory rather than a rigorous development of the underlying mathematics. For an introduction to relativistic field theory, see [10] or [6]. Eric Lawerence's thesis *Conservation Laws in General Relativity* [16] provides a concise introduction to the calculus of tensors on manifolds. More can be found in [1].

1.1.1 Tensor Calculus

The setting is an $n+1$ dimensional (one temporal dimension and n spatial dimensions) **Riemannian manifold** (M, g), where M is a differentiable manifold and g is a differentiable inner product defined on the tangent space to the manifold. Actually the manifold will be pseudo-Riemannian, but more on this in a moment. Introduce to the manifold local coordinates, $(x^0, x^1, \ldots, x^n)$. In index notation they are written as (x^0, x^i) or x^μ, where latin indices run from 1 to n and greek indices run from 0 to n. In terms of local coordinates, the tangent space of the manifold is a Euclidean vector space with basis set of $n+1$ partial derivatives $\{\frac{\partial}{\partial x^\mu}\}$.

Using the local coordinates, we are free to introduce the usual coordinate dependent structures, like scalar fields $\phi : M \to \mathbb{R}$, and vector fields $f^\mu : M \to \mathbb{R}^{n+1}$, etc. Though we have introduced one set of local coordinates, x^μ, to the manifold, we could have just as easily picked a different set of local coordinates, say the $\bar{x}^\mu$ coordinates. We assume that barred coordinates can (almost) always be written in terms of original coordinates: $\bar{x}^\mu = \bar{x}^\mu(x^\nu)$, and vice versa. Notice that in the barred coordinates, there is no reason to think that scalars or vectors should look like they did in the original coordinate system. But in fact, there are certain objects that look the same in any coordinate system. These are **tensors**. They are essentially coordinate independent, and can be thought of as living on the manifold itself.

In terms of local coordinates, define a k^{th} rank **contravariant tensor** $T^{\mu_1 \cdots \mu_k}$ to be a collection of $(n+1)^k$ coordinate dependent values, in particular a k-linear form, that transform under a change of coordinates according to the transformation law:

$$\bar{T}^{\mu_1 \cdots \mu_k}(\bar{x}^\sigma) = \frac{\partial \bar{x}^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial \bar{x}^{\mu_k}}{\partial x^{\nu_k}} T^{\nu_1 \cdots \nu_k}(x^\sigma), \tag{1.1}$$

where $\bar{T}$ is the tensor T written entirely in terms of the barred coordinates. We will explain how to interpret the repeated indices of this expression in the following paragraphs. Similarly, define a k^{th} rank **covariant tensor** $T_{\mu_1 \cdots \mu_k}$ to be a collection of $(n+1)^k$ coordinate dependent values that transform under a change of coordinates according to the transformation law:

$$\bar{T}_{\mu_1 \cdots \mu_k}(\bar{x}^\sigma) = \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\mu_1}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} T_{\nu_1 \cdots \nu_k}(x^\sigma). \tag{1.2}$$

We also allow for mixed tensors with l contravariant indices and m covariant indices, $T^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \nu_m}$.[1] We will sometimes refer to 0^{th} rank tensors as scalars and 1^{st} rank contravariant tensors as contravariant vectors or just vectors.

Any symmetric second rank covariant tensor, which is invertible almost everywhere induces a differentiable (pseudo)**inner product** on the manifold according to the prescription: $(f, h) \equiv f^\mu g_{\mu\nu} h^\nu \equiv f^\mu h_\mu$, where $g_{\mu\nu}$ is the **metric tensor** and repeated indices are summed over.[2] In Einstein's theory of General Relativity, the name of the game is solving for the metric. In this thesis the metric will be left unspecified when possible; but for most calculations, we will work in flat space, described by the **Minkowski metric**:

$$\eta_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As mentioned before, the setting is not a Riemannian manifold, but a pseudo-Riemannian manifold. We are now in a position to say what that means. A Riemannian manifold is locally Euclidian. A pseudo-Riemannian manifold, on the other hand, is locally Minkowskian, which just means that locally the metric can be brought to the form of the diagonal matrix: $\mathrm{diag}(-1, \cdots, -1, 1, \cdots 1)$. The setting for GR and the work of this thesis is a pseudo-Riemannian manifold for which the local metric looks like the Minkowski metric, given above.

Define $g^{\mu\nu}$ to be the matrix inverse of $g_{\mu\nu}$. Together they provide the full range of tensorial index manipulations: contraction, raising and lowering. For example, the

[1] Precisely, $T^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \nu_m}$ is a multilinear mappings from m copies of the tangent space and l copies of its dual to $\mathbb{R}$.

[2] Explicitly, $f^\mu g_{\mu\nu} f^\nu \equiv \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} f^\mu g_{\mu\nu} f^\nu$.

contraction $f^{\mu\nu}g_{\nu\sigma} = f^{\mu}{}_{\sigma}$ lowers one of f's indices. And the contraction $f_{\mu}h^{\mu\tau} = f^{\sigma}g_{\sigma\mu}h^{\mu\tau}$ returns a singly indexed tensor. Note that these operations transform tensors to tensors.

We are in position to describe the calculus of tensors on manifolds. We are faced with the task of generalizing the directional derivative $\nabla = (\frac{\partial}{\partial x^0}, \cdots, \frac{\partial}{\partial x^n})$ of multivariable calculus to manifolds (provided that it needs generalization). For a scalar field ϕ, the directional derivative of ϕ, which we write in index notation as $\phi_{,\mu}$ is a covariant vector.[3] By the chain rule,

$$\bar{\phi}_{,\mu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}}\phi_{,\nu}.$$

Evidently, the directional derivative of a scalar is a tensor, so no need for generalization. For higher rank tensors, this is usually not the case. The issue is seen most readily for the contravariant vector f^{μ}. It satisfies the transformation law:

$$\bar{f}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} f^{\nu}.$$

Now apply the directional derivative to both sides; by the chain rule,

$$\bar{f}^{\mu}{}_{,\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}\left(f^{\nu}{}_{,\gamma}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}}\right) + \frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\nu}\partial x^{\gamma}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\sigma}} f^{\nu}.$$

Comparison with the defining transformation laws shows that the second term is the problem. The problem is fixed by introducing the **covariant derivative**; this will be our generalized derivative. For a mixed second rank tensor $T^{\mu}{}_{\nu}$, the covariant derivative, denoted with a semicolon is:

$$T^{\mu}{}_{\nu;\sigma} = T^{\mu}{}_{\nu,\sigma} + \Gamma^{\mu}_{\sigma\tau}T^{\tau}{}_{\nu} - \Gamma^{\tau}_{\nu\sigma}T^{\mu}{}_{\tau}, \tag{1.3}$$

where $\Gamma^{\mu}_{\sigma\tau}$ is the **metric connection** defined by

$$\Gamma^{\mu}_{\sigma\tau} = \frac{1}{2}g^{\mu\gamma}(g_{\gamma\tau,\sigma} + g_{\gamma\sigma,\tau} - g_{\sigma\tau,\gamma}).$$

The definition generalizes to arbitrary tensors. Note that the usual properties of differentiation hold for the covariant derivative.

To complete the picture, we describe integration of tensors over manifolds. In our case It will suffice to consider the integration of scalar fields over the neighborhood $\Omega \subseteq M$. Since integration returns a real number, the integral should transform like a scalar, *i.e.* the integral should be invariant under coordinate transformations. But this feature is already built into the integral of multivariable calculus. The integral of a scalar field over Ω in barred coordinates is related to the integral in unbarred coordinates by,

$$I = \bar{I} = \int_{\bar{\Omega}} \phi(\bar{x}^{\mu}) \det(J^{\sigma}_{\gamma}) d\bar{\tau}, \tag{1.4}$$

[3]Actually, when we write $\nabla\phi$ we usually mean the contravariant vector $\phi_{,\nu}g^{\mu\nu}$.

where $J^\sigma_\gamma \equiv \frac{\partial x^\sigma}{\partial \bar{x}^\gamma}$ is the Jacobian matrix and $d\bar{\tau} \equiv d\bar{x}^0 \cdots d\bar{x}^n$ is the volume element. But from the transformation law for the metric, the Jacobian matrix is found to be related to the determinant of the metric by $\sqrt{-g} \equiv \sqrt{-\det(g_{\mu\nu})} = \det(J^\mu_\nu)$.[4] Replacement in (1.4) yields our invariant integral. Stokes' theorem carries over to the setting of tensors on manifolds. In particular, we have the following useful relationship. Given vector and scalar fields A^μ and ϕ,

$$\int_\Omega d\tau \sqrt{-g} A^\mu \phi_{;\mu} = \int_{\partial\Omega} d\sigma_\mu \sqrt{-g} A^\mu \phi - \int_\Omega d\tau \sqrt{-g} A^\mu{}_{;\mu} \phi, \tag{1.5}$$

where $\partial\Omega$ is the boundary of Ω and $d\sigma_\mu$ is the surface element.

1.1.2 Lagrangian Formulation

We would like to describe the behavior of a field ϕ. In our case, ϕ will usually be a scalar, a vector, or second rank tensor field. The Lagrange formulation of field theory provides a framework for systematically generating systems of equations that govern the dynamics of ϕ and specify how it interacts with other fields; in essence the Lagrangian framework enables us to generate field theories. The central premise of the formulation is that ϕ extremizes the **action** S given by:

$$S = \int_\Omega \mathcal{L}(\phi, D\phi, D^2\phi, \cdots)\, d\tau \tag{1.6}$$

where $\mathcal{L}$ is the **Lagrangian density** or just the **Lagrangian**, a function of ϕ, $D\phi$, $D^2\phi$, $\cdots$, where $D^i\phi$ stands in for all i^{th} order derivatives of the field, and Ω is the domain of ϕ. As for the Lagrangian, it remains undefined until a particular physical system is on the table. For a classical system, the Lagrangian is determined by Newton's Laws (up to a choice of coordinates); it is just the difference of the kinetic and potential energy for the system: $\mathcal{L} = \mathcal{K} - \mathcal{U}$. Of course, the classical theory has regimes of validity. As our theories move away from the classical regime, this natural construction becomes less compelling. And in the end, one is left to construct an appropriate Lagrangian from first principles alone, by hook and crook, and by simple luck. We will refer to this process as model building. One might expect the Lagrangian framework to fall flat at this point. The surprising thing about this approach to generating field theories is that simple principles, properly encoded, generate good theories.

Special Relativistic field theories work under the premise that laws of physics are unchanged under Lorentz transformations. This pre-supposes that spacetime in vacuum is Minkowskian. General Relativistic field theories work under the premise that the laws of physics are unchanged under arbitrary coordinate transformations. These principles fit easily into the Lagrangian formulation if we model spacetime as a pseudo-Riemannian manifold, locally Minkowski, and insist that all of physics be described by tensor equations. So, we restrict our attention to theories that come from Lagrangians of the form

$$\mathcal{L} = \sqrt{-g}\bar{\mathcal{L}}(\phi, \phi_{;\mu}, \phi_{;\nu\sigma}, \cdots),$$

[4] The minus sign corrects for the fact that the manifold is locally Minkowskian.

where $\bar{\mathcal{L}}$ is a scalar formed from various contractions of ϕ[5] and its derivatives, say up to order k. We will sometimes refer to $\bar{\mathcal{L}}$ as the **scalar Lagrangian** or just the **Lagrangian**.

The general method for generating field equations goes as follows. Impose the extremal condition on S by taking the functional derivative of S, denoted δS, with respect to ϕ and its derivatives, and require that it be zero. For the relativistic field ϕ we have

$$\begin{aligned}\delta S &= \delta \int_\Omega d\tau \sqrt{-g}\, \bar{\mathcal{L}}(\phi, \phi_{;\mu}, \cdots) = \int_\Omega d\tau \sqrt{-g}\, \delta\bar{\mathcal{L}} \\ &= \int_\Omega d\tau \sqrt{-g}\, \frac{\partial \bar{\mathcal{L}}}{\partial \phi}\delta\phi + \int_\Omega d\tau \sqrt{-g}\, \frac{\partial \bar{\mathcal{L}}}{\partial \phi_{;\mu}}\delta\phi_{;\mu} + \cdots\end{aligned}$$

Now integrate by parts (see (1.5)) in all terms except the first until all volume integrals contain are weighted by the variation $\delta\phi$:

$$= \int_\Omega d\tau \sqrt{-g}\, \frac{\partial \bar{\mathcal{L}}}{\partial \phi}\delta\phi + \int_{\partial\Omega} d\sigma_\mu \sqrt{-g}\, \frac{\partial \bar{\mathcal{L}}}{\partial \phi_{;\mu}}\delta\phi - \int_\Omega d\tau \left(\sqrt{-g}\frac{\partial \bar{\mathcal{L}}}{\partial \phi_{;\nu}}\right)_{;\nu} \delta\phi + \cdots$$

Now assume that $\delta\phi,\ \delta\phi_{;\mu}, \cdots, \delta\phi_{;\mu_1 \cdots \mu_{k-1}}$ all vanish at $\partial\Omega$. Then the boundary integrals vanish, leaving

$$= \int_\Omega d\tau \sqrt{-g} \left\{ \frac{\partial \bar{\mathcal{L}}}{\partial \phi} - \left(\frac{\partial \bar{\mathcal{L}}}{\partial \phi_{;\mu}}\right)_{;\mu} + \cdots \right\} \delta\phi,$$

where we have used the fact that $(\sqrt{-g})_{;\mu} = 0$. For $\delta S = 0$ under arbitrary variation $\delta\phi$, the integrand must be identically zero. Therefore, ϕ must satisfy the **field equations**, or the **Euler-Lagrange equations**[6]:

$$\frac{\partial \bar{\mathcal{L}}}{\partial \phi} - \left(\frac{\partial \bar{\mathcal{L}}}{\partial \phi_{;\mu}}\right)_{;\mu} + \cdots = 0. \tag{1.7}$$

If we again think of ϕ as a generic tensor field, then (1.7) is a system of partial differential equations, $n+1$ for each index attached to ϕ . This is an essential, though straightforward, calculation that will be carried out in specific cases a few more times through this work.

For the generic relativistic field theory described above, we define a few useful quantities, which connect the fields back to familiar physics. The definitions are not necessarily obvious, but are natural extension of the usual physical definitions.

[5] ϕ looks like a scalar once the derivatives show up, and will usually refer to a scalar, but it is intended for the moment to stand in for an arbitrary tensor. If you like, put indices on it, the discussion proceeds just the same.

[6] By the way, I could have added or multiplied $\mathcal{L}$ through by a constant without changing the result.

Derivation and motivation of these relationships can be found in the references previously mentioned. We define the **momentum field** π^μ conjugate to ϕ by

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{;\mu}}. \tag{1.8}$$

Note that this is the fully relativistic version of momentum and not simply the time derivative of the the field (what is meant in the quantum setting). The **stress tensor** is given by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}.$$

By the chain rule, the stress tensor may be written as

$$T^{\mu\nu} = -\left(g^{\mu\nu}\bar{\mathcal{L}} + 2\frac{\partial \bar{\mathcal{L}}}{\partial g_{\mu\nu}} \right). \tag{1.9}$$

The T^{00} component of the stress tensor admits the interpretation of spatial **energy density** $\mathcal{E}$ for the field ϕ:

$$\mathcal{E} = T^{00} = -\left(g^{00}\bar{\mathcal{L}} + 2\frac{\partial \bar{\mathcal{L}}}{\partial g_{00}} \right). \tag{1.10}$$

This interpretation comes from the equivalence of $T^0{}_0$ and the Hamiltonian associated with $\mathcal{L}$. The total energy U of a field is obtained by integrating $\mathcal{E}$ over all of space.

1.2 The Classical Theory of Electrodynamics

Here we present the basic results of the classical theory of Electrodynamics, using the field theoretic formalism of the previous section. See [13] or [15] for details. For this and subsequent treatments of E&M and modifications thereof, we will work in natural units, with $\epsilon_0 = \mu_0 = c = 1$. Consider the free field action for E&M:

$$S_{EM} = \frac{1}{4}\int d\tau \sqrt{-g}\, F^{\mu\nu}F_{\mu\nu} \tag{1.11}$$

where $F_{\mu\nu}$ is the **field strength tensor** given by $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and A^μ is the **vector potential** of the theory. By construction, $F_{\mu\nu}$ satisfies the **Bianchi identity**:

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = 0. \tag{1.12}$$

The source free field equations are obtained by the usual variational procedure. Sources will be introduced at the end. Note that

$$F^{\mu\nu}F_{\mu\nu} = 2A_{\mu,\sigma}g^{\mu\nu}g^{\sigma\tau}A_{\nu,\tau} - 2A_{\mu,\sigma}g^{\mu\nu}g^{\sigma\tau}A_{\tau,\nu},$$

and

$$\frac{\partial}{\partial A_{\mu,\nu}}F^{\sigma\tau}F_{\sigma\tau} = 4g^{\mu\sigma}g^{\nu\tau}A_{\sigma,\tau} - 4g^{\mu\sigma}g^{\nu\tau}A_{\tau,\sigma} = -4F^{\mu\nu}.$$

Then, the variation of (1.11) gives:

$$\begin{aligned}\delta S_{EM} &= \frac{1}{4}\int d\tau\sqrt{-g}\,\frac{\partial}{\partial A_{\mu,\nu}}\big(F^{\sigma\tau}F_{\sigma\tau}\big)\delta A_{\mu,\nu}\\ &= -\int d\tau\sqrt{-g}\,F^{\mu\nu}\delta A_{\mu,\nu}\end{aligned}$$

then, integrating by parts and requiring that the boundary terms vanish,

$$\delta S_{EM} = \int d\tau\sqrt{-g}\,F^{\mu\nu}{}_{;\nu}\delta A_\mu. \tag{1.13}$$

Finally, requiring that $\delta S_{EM} = 0$ for arbitrary variation of δA_μ we obtain the source-free E&M field equations:

$$F^{\mu\nu}{}_{;\nu} = 0. \tag{1.14}$$

The source current $j^\mu \doteq (\rho, \mathbf{j})$, where ρ is the charge density and $\mathbf{j}$ is the current density, may be coupled to the field by introducing a coupling term to the action of the form

$$S_J = \alpha\int d\tau\sqrt{-g}\,j^\mu A_\mu. \tag{1.15}$$

where α is a coupling constant. To avoid unnecessary clutter, we set $\alpha = 1$. Variation of (1.15) yields, on the RHS of (1.14), the modification $0 \to j^\mu$. The resulting equation is the vector potential formulation of Maxwell's equations.

To recover the familiar formulation of Maxwell's equations in terms of $\mathbf{E}$ and $\mathbf{B}$, the electric and magnetic field, set

$$E^i = F^{0i} \qquad \text{and} \qquad B^i = \epsilon^{ijk}F_{jk} \tag{1.16}$$

where ϵ^{ijk} is the Levi-Civita symbol. In the standard language of vector calculus, the field equations for $\mathbf{E}$ and $\mathbf{B}$ are

$$\begin{aligned}\nabla\cdot\mathbf{E} = \rho \qquad &\text{and} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial}{\partial t}\mathbf{E} \qquad &\text{and} \qquad \nabla\cdot\mathbf{B} = 0;\end{aligned} \tag{1.17}$$

the two equations on the left come from Eqs. (1.14, 1.15) and the two on the right come from the Bianchi identity, (1.12).

For reference, we also compute the stress tensor and energy density. For E&M, the Lagrangian is

$$\bar{\mathcal{L}} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

Noting that

$$\frac{\partial}{\partial g_{\mu\nu}}F_{\sigma\tau}g^{\sigma\alpha}g^{\rho\beta}F_{\alpha\beta} = \frac{\partial}{\partial g_{\mu\nu}}\Big(F_{\nu\rho}g^{\mu\nu}g^{\beta\rho}F_{\mu\beta}\Big) = -2F^{\nu\rho}F^{\mu}{}_{\rho} \tag{1.18}$$

it follows from definition of the stress tensor (1.9) that

$$T^{\mu\nu} = -\frac{1}{4}g^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} + F^{\nu\sigma}F^{\mu}_{\ \sigma}. \tag{1.19}$$

From (1.10), and noting that $F^{0\sigma}F^{0}_{\ \sigma} = E^2$, the energy density in terms of **E** and **B** is

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2).$$

Lorentz invariance of the scalar $\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = B^2 - E^2$ enables us to classify solutions in three ways. For a given solution, if $F^{\mu\nu}F_{\mu\nu} < 0$, the solution is **timelike**. If $F^{\mu\nu}F_{\mu\nu} = 0$, the solution is **lightlike**. And if $F^{\mu\nu}F_{\mu\nu} > 0$, the solution is **spacelike**.

1.2.1 Multipole Expansion for E and B

Here we restrict ourselves to consideration of static source configurations on a Minkowski background. In terms of the vector potential, the sourced field equation, (1.15), reads:

$$A^{\nu}_{\ ;\nu}{}^{\mu} - A^{\mu}_{\ ;\nu}{}^{\nu} = j^{\mu}. \tag{1.20}$$

Invoking gauge freedom for the vector potential, we are free to set $A^{\nu}_{\ ;\nu} = 0$. Under the stated restrictions, (1.20) reduces to

$$A^{\mu}_{\ ;i}{}^{i} = -j^{\mu} \tag{1.21}$$

four copies of Poisson's equation. We use roman indices here since A^{μ} is independent of time. Poisson's equation has a convenient integral form; for (1.21) the integral form of the equation reads

$$A^{\mu}(\mathbf{r}) = \frac{1}{4\pi}\int_{\text{all space}} \frac{j^{\mu}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV', \tag{1.22}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are position vectors, and dV is the spatial volume element with scale factor included. Suppose that j^{μ} is bounded, say by a ball of radius B. For $|\mathbf{r}| > B$, the denominator of the integrand has the series expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta')$$

where P_n is is the n^{th} Legendre Polynomial, θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Plugging the expansion into (1.22) gives

$$A^{\mu}(\mathbf{r}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\text{all space}} r'^{n} P_n(\cos\theta') j^{\mu}(\mathbf{r}')\, dV'. \tag{1.23}$$

This is the **multipole expansion** of the static vector potential. The first term in the sum is called the **monopole**, the second term is the **dipole**, and so on—collectively

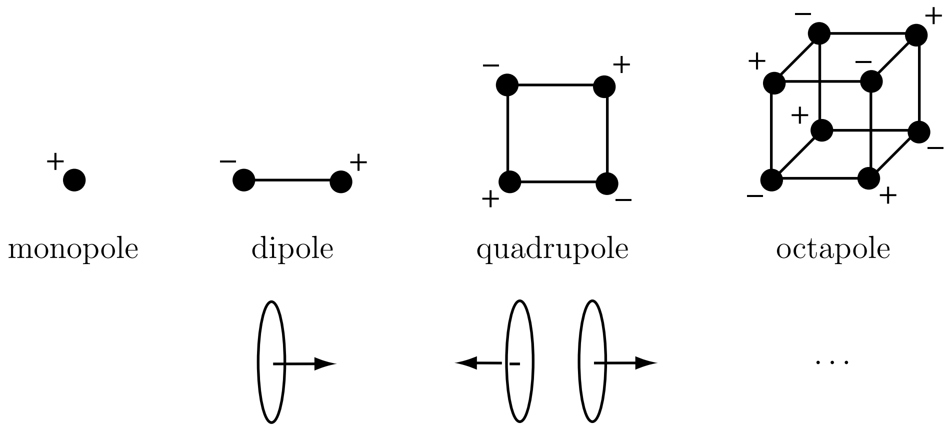

Figure 1.1: Source configurations for monopole, dipole, ... of the electric (top) and magnetic (bottom) fields. The loops are current loops flowing in the direction given by the right hand rule.

multipoles. The utility of this decomposition is that it serves as a natural basis set for real source configurations and provides a hierarchy of solution types. We may also separate multipoles into their time and space components; the A^0 component is the electric potential multipole and the A^i component is the magnetic potential multipole.

While the multipole fields provide a basis for describing the field of a generic source distribution, each (say the k^{th} multipole) is also generated by a particular source distribution. Such distributions are constructed from stationary point sources with both positive and negative charge (these source A^0) and current loops of zero net charge (these source A^i), arranged so as to exhibit the requisite symmetries, then shrunk down so that the multipole field is defined over all of space except the origin. A few of the multipole source configurations are shown schematically in Figure 1.1.

The potential formulation is convenient in this setting, but later on it will be useful to speak of multipoles for the **E** and **B** fields themselves. There are a few ways to develop these terms. The rigorous approach is to expand the integral equations for **E** and **B** in much the same way that we did for A^μ. The simpler route, and the route that we shall take, is just to construct the **E** and **B** fields from the multipoles of the vector potential according to (1.16).

Chapter 2

Born–Infeld Electrodynamics

The classical theory of Electrodynamics suffers a rather embarrassing feature. Consider a stationary point particle with charge q sitting at the origin of a coordinate system. According to Maxwell's theory, the electric field for this charge configuration is given by[1]

$$\mathbf{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, \tag{2.1}$$

and the total energy, U, stored in the electric field is

$$U = \frac{1}{2}\int_{\text{all space}} \mathbf{E}\cdot\mathbf{E}\,dV = \frac{q^2}{8\pi\epsilon_0^2}\int_{r=0}^{\infty}\frac{1}{r^2}dr = -\frac{q^2}{8\pi\epsilon_0^2}\frac{1}{r}\Big|_{r=0}^{\infty} = \infty. \tag{2.2}$$

Evidently, the electric field of a charged point particle (say an electron) stores an infinite amount of energy.[2] Notice that essentially all this energy lies within a δ-ball ($\delta > 0$) about the source:

$$\frac{q^2}{8\pi\epsilon_0^2}\int_{r=0}^{\delta}\frac{1}{r^2}dr = \frac{q^2}{8\pi\epsilon_0^2}(-\frac{1}{\delta}+\infty) = \infty.$$

We say that the electric field and energy density are **singular** when they are infinite.

Classically, this result is an unsightly blemish. But provided we never create point sources and only move them around, the issue may be ignored. Griffiths says, "Still, the infinite energy of a point particle is a recurring source of embarrassment for electromagnetic theory, afflicting the quantum version as well as the classical." [13] In the 1930s, Born and Infield tried to address this problem by developing a modified relativistic theory of electrodynamics, equivalent to Maxwell's theory in its linearized form, but designed specifically to remove the infinite energy problem described above [2, 3, 4]. With the success of Quantum Electrodynamics in the 1960s, however, Born-Infeld Electrodynamics fell out of favor.

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[1] We will work in SI units for this particular calculation.

[2] There are other problems with Maxwell's theory, from the prediction of spontaneous runaway acceleration of charged particles to their acausal preacceleration. We will not address these matters here, but it would be interesting to see whether these problems persist in the Born-Infeld theory. For further discussion of problems in the classical theory, see [13] Chapter 11.

Recently, the Born-Infeld theory has seen renewed interest. There are two reasons for this. First, it happens to be the case that the Born-Infeld action governs open strings on D-branes in the low energy limit for some versions of String Theory [22]. But I am not so concerned with this. The second reason, the one of relevance to this thesis, is that this problem of singularities arises in General Relativity. We have already gone over this story in the introduction, so I drop the discussion for now.

This chapter proceeds as follows. The field equations of BI E&M are first developed. Some simple solutions are found to highlight the basic features of the modified theory. Lacking linearity, more complicated solutions are not so easily obtained. This problem leads to an approach inspired by the classical multipole decomposition. Next we reinterpret the problem in terms of minimal surfaces. Along the way, two significant problems with this theory are uncovered. These problems will motivate the general investigation of relativistic field theories in Chapter 3.

2.1 Field Equations

The goal is to put a finite cutoff on the magnitude of the electric field. With such a cutoff in place, the energy associated with the field of a point source will be finite. To that end, take a theory that already has a cutoff, namely relativistic kinematics, and impose its structure upon E&M. The theory can also be developed in more formal terms, but for our purposes, the simpler version will suffice. A complete argument for the development of this theory may be found in [4].

The theory of relativistic kinematics works from the non–relativistic free particle action,

$$S_K = \frac{1}{2} m \int d\tau \, v^2. \tag{2.3}$$

where v is the velocity of some point particle in space and here τ is the arclength parameter. We then postulate a maximum velocity c and impose this limit on the action by way of the modification

$$S_K \longrightarrow S_{KR} = m\, c^2 \int d\tau \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right). \tag{2.4}$$

The physics lies within the square root, by which we mean, the inclusion of additive and multiplicative factors has no effect on the resulting equations of motion. We include the constant 1 and the minus sign in the action so that S_{KR} reduces exactly to S_K as $c^2 \to \infty$:

$$S_{KR} = mc^2 \int d\tau \left(1 - 1 + \frac{v^2}{c^2} + \mathcal{O}(v^4/c^4)\right) = \frac{1}{2} m \int d\tau \, v^2 + \mathcal{O}(v^4/c^4).$$

To develop BI E&M, start with the usual E&M free field action (see Section 1.2):

$$S_{EM} = \frac{1}{4} \int d\tau \sqrt{-g} F^{\mu\nu} F_{\mu\nu} \tag{2.5}$$

Cribbing off the relativistic action, (2.4), we write the Born–Infeld free field action as

$$S_{BI} = b^2 \int d\tau \sqrt{-g} \left(-1 + \sqrt{1 + \frac{1}{2b^2} F^{\mu\nu} F_{\mu\nu}} \right), \tag{2.6}$$

where b is a constant [11]. Note that as $b \to \infty$, S_{BI} reduces to S_{EM}.

Actually, the full action contains an additional term inside the square-root of order $1/b^4$. It comes rather quickly from the following observation. Any invertible symmetric second rank covariant tensor defines a metric on a manifold. However the metric is unaffected by the addition of an antisymmetric covariant second rank tensor. This holds in particular for $m_{\mu\nu} \equiv \eta_{\mu\nu} - F_{\mu\nu}/b$. Because $F_{\mu\nu}$ is antisymmetric, the modified "metric" behaves in the same way as $\eta_{\mu\nu}$. So, in a sense $F_{\mu\nu}$ is just along for the ride. It follows from (1.4) that

$$I = \int d\tau \sqrt{-m}$$

is a scalar. As far as our formalism for generating relativistic field theories is concerned, an action of this form generates a perfectly acceptible theory for $F_{\mu\nu}$. The full Born-Infeld action reads [21]:

$$S_{BIF} = b^2 \int d\tau \sqrt{-\det\left(\eta_{\mu\nu} - \frac{1}{b} F_{\mu\nu}\right)} = b^2 \int d\tau \sqrt{1 + \frac{1}{2b^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4b^4} (F_{\mu\nu} \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau})}. \tag{2.7}$$

Notice that in Minkowski spacetime, S_{BI} and S_{BIF} only differ by constants and terms under the square-root of $\mathcal{O}(1/b^4)$. Incidentally (2.7) is the action that falls out of some versions of string theory, but again, this is not my concern.

In the following, we restrict ourselves to consideration of the theory generated by the "simpler" Born-Infeld action, S_{BI}, of (2.6).[3] Incidentally, if the $F_{\mu\nu}$ of immediate interest is known to satisfy $F_{\mu\nu}\epsilon^{\mu\nu\sigma\tau}F_{\sigma\tau} = 0$, then it provides a solution to the theory generated by S_{BIF} as well as S_{BI}. In terms of the physical fields, $\mathbf{E}$ and $\mathbf{B}$ are solutions to both theories whenever $\mathbf{E} \cdot \mathbf{B} = 0$. Most of the solutions that we discuss will have this property.

The source free field equations are obtained by the usual variational procedure of Section 1.1.2. Sources will be introduced at the end. The variation of (2.6) gives, after dropping the constant term and noting that part of the compuation has already been performed in the variation of the Maxwell action back in Section 1.2,

$$\begin{aligned}
\delta S_{BI} &= b^2 \int d\tau \sqrt{-g} \, \frac{\partial}{\partial A_{\mu,\nu}} \left(1 + \frac{1}{2b^2} F^{\sigma\tau} F_{\sigma\tau}\right)^{1/2} \delta A_{\mu,\nu} \\
&= b^2 \int d\tau \sqrt{-g} \left(1 + \frac{1}{2b^2} F^{\sigma\tau} F_{\sigma\tau}\right)^{-1/2} \frac{1}{4b^2} \left(\frac{\partial}{\partial A_{\mu,\nu}} F^{\gamma\delta} F_{\gamma\delta}\right) \delta A_{\mu,\nu} \\
&= -\int d\tau \sqrt{-g} \left(1 + \frac{1}{2b^2} F^{\sigma\tau} F_{\sigma\tau}\right)^{-1/2} F^{\mu\nu} \delta A_{\mu,\nu}.
\end{aligned}$$

[3] If we really want to replace the classical theory with the Born-Infeld theory, we are morally obliged to consider the more complicated Born-Infeld action since it contains all possible players in the theory.

Integrating by parts, taking the boundary terms to vanish

$$\delta S_{BI} = \int d\tau \sqrt{-g} \left\{ \left(-1 + \frac{1}{2b^2} F^{\sigma\tau} F_{\sigma\tau} \right)^{-1/2} F^{\mu\nu} \right\}_{;\nu} \delta A_\mu .$$

Requiring that $\delta S = 0$ under arbitrary variation, it must be the case that

$$\left\{ \left(1 + \frac{1}{2b^2} F^{\sigma\tau} F_{\sigma\tau} \right)^{-1/2} F^{\mu\nu} \right\}_{;\nu} = 0, \tag{2.8}$$

these are the **source free Born–Infeld field equations**. Again, notice that when $b \to \infty$, (2.8) reduces to Maxwell's equations: ${F^{\mu\nu}}_{;\nu} = 0$, as expected. The source current j^μ may be coupled to the fields just as in E&M (see (1.15)).

In terms of $\mathbf{E}$ and $\mathbf{B}$, the field equations read:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho & \qquad \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} & \qquad \nabla \cdot \mathbf{B} &= 0, \end{aligned} \tag{2.9}$$

where $\mathbf{D}$ and $\mathbf{H}$ are the momenta conjugate to $\mathbf{E}$ and $\mathbf{D}$ respectively, defined by

$$\mathbf{D} \equiv \frac{\mathbf{E}}{\sqrt{1 + \frac{1}{b^2}(B^2 - E^2)}} \quad \text{and} \quad \mathbf{H} \equiv \frac{\mathbf{B}}{\sqrt{1 + \frac{1}{b^2}(B^2 - E^2)}}. \tag{2.10}$$

2.2 Exact Solutions

The Born-Infeld field equations are a system of coupled nonlinear partial differential equations, so there is no reason to think that we can solve them in general. Nonetheless, certain classes of solutions are readily obtained. Since few systematic treatments of the Born-Infeld theory exist in the literature, we begin by cataloguing some of the trivial exact solutions to the theory. Calculations are omitted except when informative.

2.2.1 Lightlike Solutions

Take any lightlike solution to Maxwell's equations. For example, the plane wave solution with

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)\hat{\mathbf{n}} \quad \text{and} \quad \mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{k}} \times \mathbf{E},$$

where $\hat{\mathbf{k}}$ points in the direction of propagation and $\hat{\mathbf{n}}$ indicates $\mathbf{E}$'s plane of oscillation. By definition, $B^2 - E^2 = 0$; this is certainly the case above. From the definition of the conjugate momenta, (2.10), $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$. It follows that the nonlinear field equations reduce to the classical field equations, which the solutions already satisfy. So, any lightlike solution from the classical theory also satisfies the Born-Infeld theory. Note that superposition of solution holds.

2.2.2 Timelike Solutions

Suppose that $\mathbf{B} = 0$. Then the relevant field equations are:

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{E} = 0, \quad \text{and} \quad \mathbf{D} = \frac{\mathbf{E}}{\sqrt{1 - \frac{1}{b^2}E^2}}. \tag{2.11}$$

For sources with rectangular, cylindrical, or spherical symmetry, $\mathbf{E}$ may be expressed as a function of one variable. The field equations then reduce to a single ordinary differential equation of one variable, which can be solved by standard methods. As we run through the basic results, notice that each solution is bounded in magnitude by b. This feature is a direct result of the model building of the previous section. Let's start with the electric field of a stationary point source with charge q, since this is the solution that motivates the Born-Infeld theory. The source is spherically symmetric, so we may assume that $\mathbf{E} = E(r)\hat{\mathbf{r}}$. In spherical coordinates, and for $r > 0$, (2.11) reads:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{E(r)}{\sqrt{1 - \frac{1}{b^2}E(r)^2}}\right) = 0 \implies r^2 \frac{E(r)}{\sqrt{1 - \frac{1}{b^2}E(r)^2}} = A,$$

where A is a constant. Solve for $E(r)$ and conclude:

$$\mathbf{E} = \frac{A}{\sqrt{r^4 + \frac{A^2}{b^2}}}\hat{\mathbf{r}} \quad (r > 0). \tag{2.12}$$

Comparison with the classical result as $b \to \infty$ fixes $A = q/4\pi$.

A similar story holds for the other obvious symmetric charge configurations—the reader is spared the details. In Cartesian coordinates, the electric field sourced by an infinite plane of uniform charge density $\sigma\,\delta(z)$, is

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\sqrt{1+\frac{\sigma^2}{4b^2}}}\,\hat{\mathbf{z}}, & z > 0 \\ -\frac{\sigma}{2\sqrt{1+\frac{\sigma^2}{4b^2}}}\,\hat{\mathbf{z}}, & z < 0 \end{cases}$$

In cylindrical coordinates, an infinite line of uniform charge density $\lambda\,\delta(s)$ has electric field

$$\mathbf{E} = \frac{\lambda}{2\pi\sqrt{s^2 + \frac{1}{b^2}\frac{\lambda^2}{4\pi^2}}}\hat{\mathbf{s}}, \quad s > 0.$$

Similar solutions are easily found for continuous charge distributions having appropriate symmetry, but let's move on.

2.2.3 Spacelike Solutions

Now suppose that $\mathbf{E} = 0$. Then the field equations reduce to the relevant set

$$\nabla \times \mathbf{H} = \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \quad \text{and} \quad \mathbf{H} = \frac{\mathbf{B}}{\sqrt{1 + \frac{1}{b^2}B^2}}. \tag{2.13}$$

As before, look for symmetric solutions that can be parameterized by a single variable. Lacking a spherically symmetric current source, we will resign ourselves to solutions with the two remaining symmetries.

Working in Cartesian coordinates, consider the magnetic field sourced by an infinite neutral plane carrying a current $\mathbf{K} = K\,\hat{\mathbf{y}}$. Away from the plane, we may assume $\mathbf{B} = B(z)\,\hat{\mathbf{x}}$. Then (2.13) reads

$$\frac{d}{dz}\left(\frac{B(z)}{\sqrt{1+\frac{1}{b^2}B(z)^2}}\right)\hat{\mathbf{y}} = 0 \implies \frac{B(z)}{\sqrt{1+\frac{1}{b^2}B(z)^2}} = A,$$

where A is a constant. Solve for $B(z)$ and conclude:

$$\mathbf{B} = \begin{cases} \frac{K}{2\sqrt{1-\frac{K^2}{4b^2}}}\,\hat{\mathbf{x}}, & z > 0 \\ -\frac{K}{2\sqrt{1-\frac{K^2}{4b^2}}}\,\hat{\mathbf{x}}, & z < 0 \end{cases}$$

In cylindrical coordinates, an infinite neutral wire carrying a uniform current $\mathbf{I}$ in the $\hat{\mathbf{z}}$ direction has magnetic field:

$$\mathbf{B} = \frac{I}{2\pi\sqrt{s^2 - \frac{1}{b^2}\frac{I^2}{4\pi^2}}}\hat{\boldsymbol{\phi}}, \quad s > 0. \tag{2.14}$$

Of course we could continue compiling this list, but let's pause for a moment to see what has happened.

The important thing to observe is that the spacelike solutions found above are undefined in certain cases.[4] For instance, the magnetic field due to the plane of current is undefined when $k > 2b$. For the current carrying wire, the magnetic field is undefined on a region about the source, $s \leq I/2\pi\,b$, for all nonzero currents. Evidently, the problem is not with our source but with the field equations themselves.

For pure electric solutions, the field equations are undefined when $E^2 \geq b^2$ and thereby limit the magnitude of $\mathbf{E}$. For pure magnetic solutions, however, the field equations are defined for all values B^2. To see why B^2 is not well behaved, we look to the field equation relating $\mathbf{H}$ and $\mathbf{j}$. Integrating both sides over a surface Ω and applying Stoke's theorem,

$$\int (\nabla \times \mathbf{H}) \cdot d\mathbf{a} = \oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{j} \cdot d\mathbf{a} \stackrel{\text{call}}{=} I_{enc}. \tag{2.15}$$

In terms of $\mathbf{B}$, this gives

$$I_{enc} = \oint \frac{\mathbf{B}}{\sqrt{1+\frac{1}{b^2}B^2}} \cdot d\mathbf{l} \leq \oint \sqrt{\frac{B^2}{1+\frac{1}{b^2}B^2}}\,|d\mathbf{l}|, \tag{2.16}$$

[4]Technically, the fields would be imaginary in the bad regions of space. But because $\mathbf{E}$ and $\mathbf{B}$ are physical fields, they must take on real values. Solutions are understood to be undefined when imaginary.

where the inequality follows from the fact that $\mathbf{B}\cdot d\mathbf{l} \leq |\mathbf{B}||d\mathbf{l}|$. To draw a contradiction, suppose that $\mathbf{B}$ is real valued (possibly infinite). Then the integrand on the right of (2.16) is bounded by b. Hence $I_{enc} \leq b\,l$, where l is the length of $\partial\Omega$. As far as I_{enc} is concerned, this bound is nonsense—the bound is only reasonable if there is a constraint on current density imposed by the theory, but in this theory there is no such constraint.[5] Hence, $\mathbf{B}$ is undefined whenever $I_{enc} > b\,l$. For point, line, and surface currents, there will always be a region on which $\mathbf{B}$ is undefined. For certain spatial current densities this will also be the case.

2.3 Multipoles

So far we have found a few simple solutions of the Born-Infeld theory, simple because the partial differential equations reduced to ordinary differential equations. In the development of the classical theory, the next step is to solve the field equations in the presence of more complicated source configurations. This step proceeds along two routes: use superposition to solve the field equations, or, develop the hierarchy of multipole solutions and use them to approximate/characterize the full solutions. Lacking linearity in the Born-Infeld theory, it is not clear how to solve for the fields in the presence of multiple separated sources. There is, however, a natural notion of multipoles for this nonlinear theory. Keep in mind, multipoles of a nonlinear theory are not well defined in the way that they are for linear theories; what we are looking for is a hierarchy of solutions related to the linear multipoles in some natural way, having either computational or conceptual utility. We will proceed axiomatically, first defining multipoles of the nonlinear theory, then trying to justify their definition.

Definition (Version 1)**.** The i^{th} **multipole** of the nonlinear theory is the solution that reduces to the i^{th} multipole of the associated linear theory as $1/b$, the parameter of the nonlinear theory, tends to zero.

We would certainly like the nonlinear multipole solutions to agree with the multipoles of the linear theory. So the definition is at least reasonable. But it is not the only reasonable definition for the multipoles of a nonlinear theory. Here are two more:

Definition (Version 2)**.** The i^{th} **multipole** of the nonlinear theory is the solution due to the appropriate multipole source distribution of Maxwellian E&M, described in Section 1.2.1.

Definition (Version 3)**.** The i^{th} **multipole** of the nonlinear theory is the solution that shares the symmetries and domain of the i^{th} multipole of the associated linear theory.

While the second and third definitions are at least as reasonable as the first, the first connects directly to the computation of the multipole solutions. For this reason we

[5] Such a constraint would amount to the statement: $\mathrm{avg}_\Omega\,\mathbf{j} \leq b\,l/A$, where l is the length of $\partial\Omega$ and A is the area of Ω.

adopt it as our primary definition.[6] For the remainder of this section, we will focus on the computation of electric multipole solutions. The magnetic multipoles can be found by a similar procedure.

By definition, the classical multipoles provide a good approximation to the Born-Infeld multipoles far away from the origin. The idea is to use the classical multipole as a first approximation in a series expansion of the full Born-Infeld multipole. Then employ the perturbative structure of the Born-Infeld field equations to generate higher order corrections to the first approximation. The corrections are the higher order terms in the series. The following construction actually provides a way to generate a Born-Infeld solution from any classical solution, with the property that the Born-Infeld solution reduces to the classical solution in the appropriate limit. But we will only apply it to the multipoles.

Away from the origin, the relevant field equations read:

$$\nabla \cdot \frac{\mathbf{E}}{\sqrt{1-\frac{1}{b^2}E^2}} = 0, \qquad \nabla \times \mathbf{E} = 0. \tag{2.17}$$

Now expand (2.17) as a sum of the static Maxwell's equation and a nonlinear part. By the usual rules of vector calculus, (2.17) gives:

$$\frac{\nabla \cdot \mathbf{E}}{\sqrt{1-\frac{1}{b^2}E^2}} + \mathbf{E}\cdot\nabla\left(\frac{1}{\sqrt{1-\frac{1}{b^2}E^2}}\right) = \frac{\nabla \cdot \mathbf{E}}{\sqrt{1-\frac{1}{b^2}E^2}} + \mathbf{E}\cdot\left(\frac{\nabla(\mathbf{E}\cdot\mathbf{E})}{2b^2(1-\frac{1}{b^2}E^2)^{3/2}}\right) = 0.$$

Then multiplying through by $(1-\frac{1}{b^2}E^2)^{3/2}$:

$$\nabla\cdot\mathbf{E} + \frac{1}{2b^2}\mathbf{E}\cdot\nabla(E^2) - \frac{1}{b^2}E^2\nabla\cdot\mathbf{E} = 0$$

And so,

$$\nabla\cdot\mathbf{E} = \frac{1}{2b^2}\big(3E^2\nabla\cdot\mathbf{E} - \nabla\cdot(E^2\mathbf{E})\big). \tag{2.18}$$

Assume that the solution has a series expansion in $1/b^2$:

$$\mathbf{E} = \sum_{i=0}^{\infty}\frac{1}{b^{2i}}\mathbf{E}_{(i)},$$

where $\mathbf{E}_{(i)}$ denotes the i^{th} term in the expansion. Insertion of the expansion into (2.18) yields the expression:

$$\sum_{i=0}^{\infty}\frac{1}{b^{2i}}\nabla\cdot\mathbf{E}_{(i)} = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j+k+\ell=i-1}\left(3\mathbf{E}_{(j)}\cdot\mathbf{E}_{(k)}\nabla\cdot\mathbf{E}_{(\ell)} - \nabla\cdot\big(\mathbf{E}_{(j)}\cdot\mathbf{E}_{(k)}\mathbf{E}_{(\ell)}\big)\right)\frac{1}{b^{2i}}. \tag{2.19}$$

[6]It is natural to wonder how these three definitions are related. Unfortunately, such question will not be addressed in this work. Incidentally, a similar issue arises in General Relativity. See [14] for details.

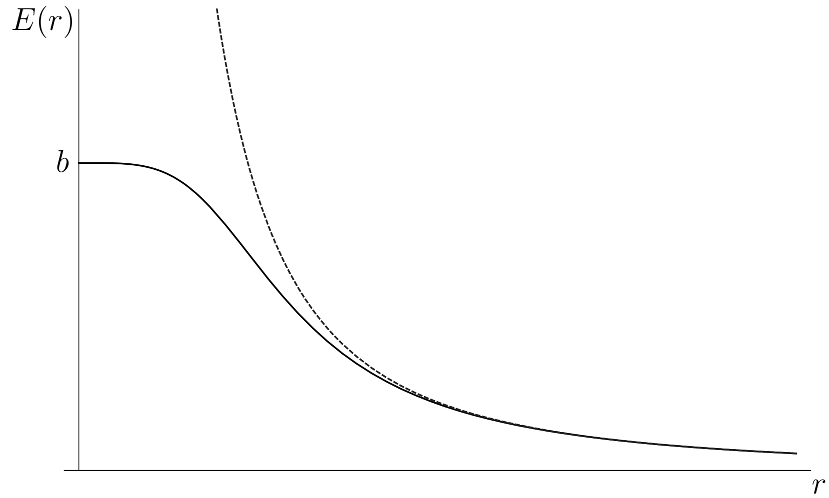

Figure 2.1: Magnitude of the Born-Infeld monopole as a function of radius. Magnitude of the electric Maxwell monopole (dashed) is given for comparison.

We require that equality hold at each order of $1/b^2$, so

$$\nabla\cdot\mathbf{E}_{(i)} = \frac{1}{2}\sum_{j+k+l=i-1}\left(3\mathbf{E}_{(j)}\cdot\mathbf{E}_{(k)}\nabla\cdot\mathbf{E}_{(\ell)} - \nabla\cdot\left(\mathbf{E}_{(j)}\cdot\mathbf{E}_{(k)}\mathbf{E}_{(\ell)}\right)\right), \quad i\in\{1,2,\dots\} \tag{2.20}$$

and $\nabla\cdot\mathbf{E}_{(0)} = 0$. This is a recursive system of linear partial differential equations. Solving the system amounts to solving Poisson's equation. The full solution is found by summing the series. It turns out that the potential formulation of (2.20), obtained by the replacement $\mathbf{E}_{(i)} \to \nabla\phi_{(i)}$, is a little easier to work with.

Let's start with the monopole. Take $\phi_{(0)} = q/r$. The method described above yields a series expansion; in this case the series is even summable, so that an exact solution is found.[7] The electric monopole actually comes most easily from the second definition of multipoles: it is just the field sourced by a stationary point charge. The solution is given by (2.12). Expanding the solution in powers of $1/b^2$ gives

$$\mathbf{E}_{mon} = \frac{q}{\sqrt{r^4+\frac{1}{b^2}q^2}}\,\hat{\mathbf{r}} = \left(\frac{q}{r^2} - \frac{q^3}{2b^2r^6} + \mathcal{O}(1/b^4)\right)\hat{\mathbf{r}}. \tag{2.21}$$

Since the solution reduces to the classical monopole when $b\to\infty$, the solution is the Born-Infeld electric monopole. See Figure 2.1 for a visual comparison of the Maxwell and Born-Infeld monopoles. Also note that the third multipole definition holds. Evidently all three multipole definitions are consistent in this case.

[7] The summation is not easy and will in general rely upon numerical recursion relations of the form

$$c_i = \frac{1}{-2}\sum_{j+k+\ell=i-1} c_j c_k c_\ell\left(1-\frac{3\ell}{i}\right), \quad \text{where} \quad c_i = \frac{1}{i!}\prod_{j=1}^{i}\frac{(2j-1)}{(-4)}.$$

I have not been able to prove this identity directly, but it is correct out to 100 terms, and consistency requires it.

The electric dipole is not so easily obtained. It is not clear from looking at the relevant field equations that a closed form solution can be found, nor is it clear that a solution will be defined over all of space. After struggling with the exact equation and several inspired separation assumptions, we are left to proceed perturbatively. Set $V_{(0)} = p\cos\theta/r^2$, which is just the Maxwell dipole in potential form with dipole strength p. By (2.20), the equation for $\phi_{(1)}$ reads

$$\begin{aligned}\nabla^2\phi_{(1)} &= -\frac{1}{2}\nabla\cdot\big(|\nabla\phi_{(0)}|^2\nabla\phi_{(0)}\big) = -\frac{1}{2}\nabla\phi_0\cdot\nabla\big(|\nabla\phi_{(0)}|^2\big)\\ &= -\frac{3p^3}{4r^{10}}(27\cos\theta+5\cos3\theta) = -\frac{6\,p^3}{r^{10}}\big((3P_1(\cos\theta)+P_3(\cos\theta)\big).\end{aligned} \tag{2.22}$$

To solve Poisson's equation, we employ the following result.

Proposition. *The general solution to Poisson's equation with the particular form:*

$$\nabla^2\phi = r^m P_n(\cos\theta) \quad (m\neq n-2,\ \ m\neq -n-3)$$

is

$$\phi = \frac{r^{m+2}P_n(\cos\theta)}{(m+2)(m+3)-n(n+1)}. \tag{2.23}$$

The proof is by direct computation, and makes use of the relation for Legendre polynomials:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}P_n(\cos\theta)\right) = -n(n+1)P_n(\cos\theta).$$

By the proposition, (2.22) has the solution:

$$\phi_{(1)} = -\frac{p^3}{r^8}\left(\frac{1}{3}P_1(\cos\theta)+\frac{3}{22}P_3(\cos\theta)\right).$$

One could continue indefinitely, computing higher and higher terms. But life is short. It would be more informative to show that the series converged on some region of space, or even to sum the series. Both projects turn out be nontrivial and in spite of my efforts, results are still pending.

I would like to mention in passing **Picard iteration** (or the method of iterated integrals) as an alternative approach to questions concerning the multipole solutions. See Appendix A for a review of this topic and its role in the simpler setting of ordinary differential equations. As far as calculating actual approximate solutions is concerned, you will have better luck using the perturbative method described above. Picard iteration on the other hand lends itself to the pressing questions of existence and uniqueness. Define the mapping $\mathbf{\Lambda}$ by

$$\mathbf{\Lambda}(\mathbf{E}(\mathbf{r})) \equiv \mathbf{E}_0(\mathbf{r}) + \frac{1}{8\pi b^2}\int_{\text{all space}}\left(3E^2(\mathbf{r}')\nabla\cdot\mathbf{E}(\mathbf{r}') - \nabla\cdot\big(E^2(\mathbf{r}')\mathbf{E}(\mathbf{r}')\big)\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dV', \tag{2.24}$$

where $\mathbf{E}_{(0)}$ satisfies $\nabla\cdot\mathbf{E}_{(0)} = 0$, $\mathscr{r} \equiv \mathbf{r}-\mathbf{r}'$, and the full combination of $\mathscr{r}$ on the far right hand side of (2.17) is the Green's function for $\nabla\cdot$. Comparison with (2.18)

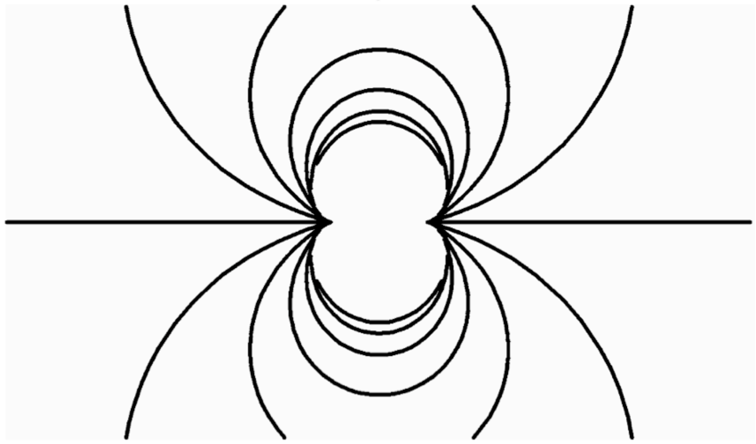

Figure 2.2: Field lines of Born-Infeld 2D electric dipole. Courtesy of [9].

shows that every fixed point of $\mathbf{\Lambda}$ is a solution to the Born-Infeld field equations, (2.17). As in the case of ordinary differential equations, the important question is whether or not there exist subsets in the domain of $\mathbf{\Lambda}$ on which $\mathbf{\Lambda}$ acts as a contraction mapping. Supposing we could find such a subspace, we could pick any function in that subspace and iterate $\mathbf{\Lambda}$ until the fixed point was reached. But before we can discuss contractions, we require a suitable norm. The choice of norm is complicated in our case (and generally for PDEs) by the fact that $\mathbf{\Lambda}$ depends upon $\nabla \cdot \mathbf{F}$ as well as $\mathbf{F}$. So L_p spaces, for instance, will not do since they do not discriminate between functions with bounded and unbounded derivatives. It turns out that the hard work of analyzing $\mathbf{\Lambda}$ is finding the appropriate function space in which to work. Unfortunately the relevant spaces are complicated, so it is still not clear how to proceed. See Chapter 5 in [8] for further discussion.

While the three dimensional multipoles resist obvious analysis, some progress has been made in the case of the two dimensional multipoles. Solutions to Laplace's Equation (electro-static solutions away from sources) are readily obtained using the methods of Complex Analysis [17]. R. Ferraro has recently extended this procedure using a non-holomorphic complex mapping to capture the structure of Born-Infeld electro-statics [9]. In particular, he determines the two dimensional electric multipoles. It would be interesting to explore the implications of this approach and his results. Unfortunately, time constraints limit our discussion to a passing mention of a few relevant features of the electric mutipoles. Multipoles in 2D do not extend to all of space, but are defined outside epicycloids of appropriate symmetry. The fields achieve their maximum value at the cusps. The field lines for the dipole are shown in Figure 2.2. Incidentally, these results might suggest that one look for solutions with electric and magnetic components, since either one by itself seems physically problematic (recall the discussion of magnetic fields at the end of Section 2.2; the problem carries over to the two dimensional case). S.O. Vellozo et. al. recently describe such a solution in the three dimensional case, with electric field going like the monopole and magnetic field going like the dipole, and both fields finite everywhere [20].

2.4 Maximal Surfaces

Having come to an impasse with the multipole solutions, we return to the original action. Suppose that $\mathbf{B} = 0$, then away from sources the action may be written (here in terms of the electric potential) as

$$S_{BI} = b^2 \int d\tau \sqrt{-g} \left(-1 + \sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2} \right). \tag{2.25}$$

In close analogy with its relativistic precursor, we can interpret (2.25) as the volume of the spacelike hypersurface $(\phi(x^i), x^i)$ of Minkowski spacetime in which b plays the role of c [12]. By varying the action we seek surfaces of maximum volume.[8] Similarly, we say that the hypersurface defined by ϕ is **maximal** if its mean curvature vanishes identically, that is, ϕ satisfies the field equation

$$\nabla \cdot \left(\frac{\nabla\phi}{\sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2}} \right) = 0, \tag{2.26}$$

sometimes called the **maximal surface equation** [7]. From our perspective, the two interpretations of ϕ are equivalent.

The problem of finding maximal surfaces has received considerable attention, both classically and from more modern perspectives. And much analysis is focused upon the particular problems of existence and uniqueness. In light of the consideration of the previous section, it would be interesting and valuable to incorporate associated methods and results into our analysis of the Born-Infeld theory. That being said, Y. Yang has pursued this line of thought; the interested reader should see [21].

2.5 Energy

The previous two sections begin to indicate the difficulty involved in solving the Born-Infeld field equations. So in general it will not be possible to analyze a system from closed form solutions to the theory. Nonetheless, the action and field equations carry important information about the behavior of solutions and directly inform physical quantities of the theory. We have already seen this in a few places, for instance at the end of Section 2.2 with the discussion of $\mathbf{B}$ field domains of existence, and to greater extent through the interpretation of solutions as minimal surfaces. Here we restrict our discussion to the energy of solutions, as this is the issue that motives the Born-Infeld theory in the first place.

We will construct the energy density for the Born-Infeld theory. Sticking the

[8] The problem is more commonly phrased in the setting of Euclidean space where one seeks minimal rather than maximal surfaces. The extension to Minkowski space is non-trivial and many open questions still remain. See [19] for an introduction to the general problem of minimal surfaces.

Born-Infeld Lagrangian (2.6) into (1.9) gives the stress tensor

$$\begin{aligned} T^{\mu\nu} &= -\left(g^{\mu\nu}\bar{\mathcal{L}} + 2\frac{\partial\bar{\mathcal{L}}}{\partial g_{\mu\nu}}\right) \\ &= b^2\left(g^{\mu\nu} - g^{\mu\nu}\sqrt{1+\frac{1}{2b^2}F^{\sigma\tau}F_{\sigma\tau}} - \frac{1}{2b^2}\frac{1}{\sqrt{1+\frac{1}{2b^2}F^{\sigma\tau}F_{\sigma\tau}}}\frac{\partial}{\partial g_{\mu\nu}}F^{\pi\rho}F_{\pi\rho}\right). \\ &= b^2 g^{\mu\nu}\left(1-\sqrt{1-\frac{1}{2b^2}F^{\sigma\tau}F_{\sigma\tau}}\right) + \frac{F^{\mu\rho}F^{\nu}{}_{\rho}}{\sqrt{1+\frac{1}{2b^2}F^{\gamma\delta}F_{\gamma\delta}}} \end{aligned} \tag{2.27}$$

The energy density is the zero component. In terms of **E** and **B**, and making the associated shift to Minkowski spacetime, the energy density is:

$$\mathcal{E} = b^2\left(-1+\sqrt{1+\frac{1}{b^2}(B^2-E^2)}\right) + \mathbf{D}\cdot\mathbf{E}. \tag{2.28}$$

The original motivation for the Born-Infeld theory was to put a limit on the energy stored in the electric field of a stationary charged point particle. Let's see how we have done. Actually the integration is not all that nice, so the task is left to *Mathematica*. When the job is done, we find that the energy of the Born-Infeld monopole, (2.21), to be:

$$U_{mon} = \frac{3\sqrt{q\,b}}{64\pi}\,\Gamma(-3/4)^2 \approx 0.35\,\sqrt{q\,b}. \tag{2.29}$$

So the energy of an electric field due to a point charge is finite. Great.

For the Born-Infeld theory, finite energy comes at a price. Consider the energy of a pure magnetic field blowing up on some surface boundary $\partial\Omega \subset \mathbb{R}^3$, and undefined in Ω (see Section 2.2 for an example of when this occurs). From (2.28), the energy of the solution is given by

$$U = \int_{\text{all space}} (-b^2 + b\sqrt{b^2+B^2})\,dV$$

We will restrict our attention to the small region, Ω_ϵ of space just beyond $\partial\Omega$, where $B^2 \gg b^2$. Then the energy contained in this region is approximately

$$U_{\Omega_\epsilon} \approx \int_{\Omega_\epsilon} b|\mathbf{B}|\,dV.$$

Since Ω_ϵ is a spatially extended region, the volume element plays no role in the question of convergence. Then, if **B** diverges like $1/d^p$, $p \geq 1$, where d is the minimal separation distance between the point of evaluation and $\partial\Omega$, then U_{Ω_ϵ} is infinite. It is not clear that anything should be said about the energy associated with the interior of Ω since the field is undefined there. Then we are done: the energy is infinite unless the field blows up slower than $1/d$. The Born-Infeld magnetic field due to a toroidal current distribution (a bounded distribution) is one example of a magnetic field that diverges too quickly near $\partial\Omega$ in certain cases.[9]

I have not been as thorough or precise in my discussion of the Born-Infeld theory as one could be, nor have I carried many of the interesting details of this theory to their logical conclusions. (It is my hope that someone *will* return to these interesting details.) Nonetheless, I hope my point has been made. While Born-Infeld electrodynamics resolves one problem in the classical theory, it also exhibits certain undesirable features, especially in relation to its magnetic solutions. These are due in particular to the asymmetric, in fact antisymmetric, appearance of **E** and **B** in the action: the structure that fixes **E** causes problems for **B**. This asymmetric treatment is not spurious, but due specifically to the requisite Lorentz invariance of $\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = B^2 - E^2$. So it is not so clear how to solve the problem. Problematic features include solutions that are undefined on open regions of space, and solutions with infinite energy. Of course, the classical theory also has problem with infinite energy solutions. But the modified theory was supposed, indeed developed, to solve such problems. Why should we tolerate such a defective theory? Why not build a theory that avoids these problematic features? In the next chapter we shall explore this possibility.

[9] I did not compute this solution back in Section 2.2 but it should be clear how the computation goes. For a torus of N loops of wire each carrying a current I, the magnetic field inside the torus is

$$\mathbf{B} = \frac{IN}{\sqrt{4\pi s^2 - \frac{I^2N^2}{b^2}}}\hat{\boldsymbol{\phi}}.$$

Chapter 3

General Scalar and Vector Field Theories

Our question is simple. How do we construct modified theories of electrodynamics that avoid the unphysical features of the Born-Infeld theory? This question is a particular instance of the general problem of model building and we will approach it from this perspective. Working from the Lagrangian formulation of model building, the problem amounts to determining a suitable Lagrangian. To make our Lagrangian acceptable, certain minimal requirements are imposed. We insist that theories reduce to their corresponding classical counterparts under appropriate limits. In direct response to the problems of the previous chapter, it will further be required that free solutions of the theory have finite positive energy, exist over all of space, and that their associated derivative fields are bounded in magnitude. (Remember, the physical fields $\mathbf{E}$ and $\mathbf{B}$ are the derivatives of the vector potential.) The goal of this chapter is to characterize the class of Lagrangians that ensure that such conditions are met. While we are not in the end able to fully characterize acceptable Lagrangians, a class of Lagrangians is described for which most of its members yield physically acceptable behavior. A few amusing examples of modified theories are presented at the end of the chapter.[1]

In a sense, this chapter is only a warm-up for the problem of constructing modified theories of General Relativity. Yet the problem for E&M is sufficiently complicated that we shall begin with an even simpler problem of constructing modified theories of massless scalar fields (massless to draw a closer analogy with the case of E&M). Additionally, comparison between the scalar and vector theories should inform the

[1] Incidentally, one might wonder why we cannot play similar games with the relativistic kinematic action, (2.3), from which the Born-Infeld theory came. The reason is that the relativistic action comes directly from the geometric notion that real trajectories are paths of extremal length in spacetime; and there is exactly one way to impose this condition in the action. For E&M, there is no obvious reason to adopt the minimal surface interpretation, analogous to the length minimization, as correct. So the form of the action is much less constrained. Actually, some would like to interpret $F_{\mu\nu}$ as the antisymmetric part of the modified metric tensor, $m_{\mu\nu} \equiv g_{\mu\nu} + F_{\mu\nu}$, where $g_{\mu\nu}$ is the usual symmetric metric. They would then claim that the real electromagnetic field minimizes volume in spacetime: $S = \int d\tau \sqrt{-m}$. This is basically the full BI action, (2.7). But there is no particular reason to think that $F_{\mu\nu}$ really admits this geometric interpretation.

difficulties in transitioning between the vector and second rank tensor theories.

3.1 Scalar Fields

Consider a generic theory of the real valued scalar field ϕ. This theory will be a modification of the standard linear massless scalar field theory, derivable from the action:

$$S_0 = \int d\tau \sqrt{-g}\, \bar{\mathcal{L}}_0 \quad \text{where} \quad \bar{\mathcal{L}}_0 = \frac{\alpha^2}{2}\phi_{,\mu}\phi_{,}{}^{\mu}. \tag{3.1}$$

For reference we compute the objects of interest for the linear theory. From the method of variation, the field equation for ϕ determined by S_0 is

$$\alpha^2\, \phi_{;}{}^{\mu}{}_{\mu} = 0. \tag{3.2}$$

The stress tensor for ϕ is

$$T^{\mu\nu} = -\frac{\alpha^2}{2}\left(g^{\mu\nu}\phi_{,\sigma}\phi_{,}{}^{\sigma} - 2\phi_{,}{}^{\mu}\phi_{,}{}^{\nu}\right). \tag{3.3}$$

In Minkowski spacetime, the energy associated with ϕ is

$$U = \frac{\alpha^2}{2}\int_{\text{all space}} (\phi_{,i}\phi_{,}{}^{i} + \phi_{,t}\phi_{,}{}^{t} + 2\phi_{,}{}^{t}\phi_{,}{}^{t})\, dV = \frac{\alpha^2}{2}\int_{\text{all space}} (\phi_{,i}\phi_{,}{}^{i} + \phi_{,}{}^{t}\phi_{,}{}^{t})\, dV. \tag{3.4}$$

We will call a field ϕ **localized** if it tends to zero faster than $1/r$ as $r \to \infty$. From (3.4), the classical energy of a localized field lies almost entirely within a compact region of space.

Now the generalization. Assume that ϕ is governed by an action of the form[2]

$$S = \frac{1}{\epsilon^2}\int d\tau \sqrt{-g}\, \bar{\mathcal{L}}(\epsilon^2\phi_{,\mu}\phi_{,}{}^{\mu}/2) \tag{3.5}$$

where ϵ is a small parameter of the theory with units such that $\epsilon^2\phi_{,\mu}\phi_{,}{}^{\mu}$ is dimensionless, and $\bar{\mathcal{L}}$ is analytic. This construction ensures that $\bar{\mathcal{L}}$ is a scalar and dimensionally consistent. The factor of $1/2$ shows up to simplify certain expressions down the road. The condition that $\bar{\mathcal{L}}$ be analytic is, strictly speaking, not required, but it will enable us to speak more efficiently about the problems of modeling.

It is required that the general theory reduce to the linear theory specified above as $\epsilon \to 0$. So, expand $\bar{\mathcal{L}}$ in ϵ^2 about $\epsilon = 0$,

$$\bar{\mathcal{L}} = \frac{1}{\epsilon^2}\bar{\mathcal{L}}(0) + \frac{1}{2}\bar{\mathcal{L}}'(0)\phi_{,\mu}\phi_{,}{}^{\mu} + \mathcal{O}(\epsilon^2).$$

Comparison with the linear theory yields the requirement

$$\bar{\mathcal{L}}(0) = 0 \quad \text{and} \quad \bar{\mathcal{L}}'(0) = \alpha^2, \tag{3.6}$$

[2]This is hardly the most general Lagrangain that we could write down. What we are really looking for is a *minimal fix* to problems in the precursor. Sticking the linear action inside some scalar function is the simplest thing that we could do.

where $\bar{\mathcal{L}}'$ denotes the derivative of $\bar{\mathcal{L}}$ with respect to its single argument.[3]

Field equations for ϕ are obtained in the usual way:

$$\delta S = \frac{1}{\epsilon^2}\int d\tau\sqrt{-g}\frac{\partial}{\partial\phi_{,\mu}}\left(\bar{\mathcal{L}}(\epsilon^2\phi_{,\nu}\phi_{,}^{\ \nu}/2)\right)\delta\phi_{,\mu} = \int d\tau\sqrt{-g}\ \bar{\mathcal{L}}'(\epsilon^2\phi_{,\nu}\phi_{,}^{\ \nu}/2)\phi_{,}^{\ \mu}\delta\phi_{,\mu}.$$

Then, integrating by parts and requiring that the boundary terms vanishes,

$$= -\int d\tau\left(\sqrt{-g}\ \bar{\mathcal{L}}'(\epsilon^2\phi_{,\nu}\phi_{,}^{\ \nu}/2)\phi_{,}^{\ \mu}\right)_{,\mu}\delta\phi. \tag{3.7}$$

Insisting that $\delta S = 0$ for arbitrary $\delta\phi$, it must be the case that

$$\left(\bar{\mathcal{L}}'(\epsilon^2\phi_{,\nu}\phi_{,}^{\ \nu}/2)\phi_{,}^{\ \mu}\right)_{;\mu} = 0. \tag{3.8}$$

This is the resulting field equation for ϕ. A scalar source ρ may be coupled to the scalar theory by way of the coupling action $\int d\tau\sqrt{-g}\,\rho\,\phi$, where we have absorbed the requisite coupling constant into ρ. In terms of the momenta conjugate to ϕ:

$$\pi^\mu = \bar{\mathcal{L}}'(\epsilon^2\phi_{,\nu}\phi_{,}^{\ \nu}/2)\phi_{,}^{\ \mu}. \tag{3.9}$$

Then (3.8) may be written, this time with source term included, as

$$\pi^\mu_{\ ;\mu} = \rho. \tag{3.10}$$

With that out of the way, we turn to the problem of finding a class of $\bar{\mathcal{L}}$ for which the resulting fields behave in the desired manner. To simplify the problem, take relevant fields to be static and localized. As usual, assume a Minkowski spacetime. It is required that $\phi_{,\mu}$ are bounded in magnitude. One way to impose this condition is to insist that the domain X of $\bar{\mathcal{L}}$ is bounded, say by 1. Actually, since $\phi_{,\mu}\phi_{,}^{\ \mu}$ is positive for all nontrivial stationary fields, and assuming that $\phi_{,\mu}$ is continuous, we have $X \cap \mathbb{R}_{\geq 0} = [0, 1]$.

It is required that fields be defined over all of space. We will use the trick employed in Chapter 2 with magnetic fields to generate a necessary condition on $\bar{\mathcal{L}}$. Integrate both sides of (3.10) over a region Ω and apply Stoke's theorem:

$$\int d\tau\sqrt{-g}\,\pi^\mu_{\ ;\mu} = \oint d\sigma_\mu\sqrt{-g}\,\pi^\mu = \int d\tau\sqrt{-g}\,\rho \stackrel{\text{call}}{=} Q_{enc}. \tag{3.11}$$

Then,

$$|Q_{enc}| \leq \oint |d\sigma_\mu|\,|\sqrt{-g}|\,|\pi^\mu|. \tag{3.12}$$

But the magnitude of Q is unconstrained by the theory. Then, for (3.12) to hold in general, π^μ must be free to blow up. It follows from the definition of π^μ that either

[3]As previously mentioned, the action is arbitrary up to a constant, so all that is really required of $\bar{\mathcal{L}}(0)$ is that it is constant. It will be convenient for comparison with the linear theory to set the free constant to zero.

$\phi_{,\mu}$ or $\bar{\mathcal{L}}'$ is free to blow up. By assumption, $\phi_{,\mu}$ is bounded in magnitude. Hence $\bar{\mathcal{L}}'$ must be unbounded in range. However, we would like the magnitude of $\phi_{,\mu}$ to correspond with that of Q_{enc} when possible, so we will only allow $\bar{\mathcal{L}}'$ to blow up as $\phi_{,\mu}$ approaches its upper bound, so in particular at 1.

It is required that the total energy of all localized stationary fields be finite. To see what constraints this places on $\bar{\mathcal{L}}$, first construct the stress tensor. From (1.9),

$$T^{\mu\nu} = -\frac{1}{\epsilon^2}\left(g^{\mu\nu}\bar{\mathcal{L}} + 2\frac{\partial\bar{\mathcal{L}}}{\partial g_{\mu\nu}}\right) = -\frac{g^{\mu\nu}}{\epsilon^2}\bar{\mathcal{L}}(\epsilon^2\phi_{,\sigma}\phi_{,}^{\ \sigma}/2) + \bar{\mathcal{L}}'(\epsilon^2\phi_{,\sigma}\phi_{,}^{\ \sigma}/2)\phi_{,}^{\ \mu}\phi_{,}^{\ \nu} \tag{3.13}$$

and the energy density is T^{00}. By assumption, $g^{00} = -1$ and the second term drops out since $\phi_{,0} = 0$. What is left is

$$T^{00} = \frac{1}{\epsilon^2}\bar{\mathcal{L}}(\epsilon^2\phi_{,\mu}\phi_{,}^{\ \mu}/2), \tag{3.14}$$

which is just our scalar Lagrangian. The total energy of a particular stationary solution is

$$U = \frac{1}{\epsilon^2}\int_{\text{all space}}\bar{\mathcal{L}}(\epsilon^2\phi_{,\mu}\phi_{,}^{\ \mu}/2)\,dV. \tag{3.15}$$

When will U be finite? From the asymptotic structure of $\bar{\mathcal{L}}$ (see (3.6)) and the fact that ϕ is localized, the energy of the field lies almost entirely within a compact region of space. Thus, we need only worry about convergence of (3.15) on this compact region. A sufficient condition for U to be finite is that $\bar{\mathcal{L}}$ is bounded in its range. Even when $\bar{\mathcal{L}}$ is not bounded it could be the case that finite energy solutions do exist, but further conditions are required on $\bar{\mathcal{L}}$ to ensure that every localized solution has finite energy.

Putting the pieces together, we obtain the following Lagrangian: $\bar{\mathcal{L}} : X \to Y$ is analytic with $X \cap \mathbb{R}_{\geq 0} = [0, 1]$, Y is bounded, and $\bar{\mathcal{L}}$ takes the particular value $\bar{\mathcal{L}}(0) = 0$. Further, $\bar{\mathcal{L}'}$ is bounded except near 1 and takes the particular value $\bar{\mathcal{L}}'(0) = \alpha^2$. Fig. 3.1 depicts a few functions that lie in this class of Lagrangians (for $\alpha^2 = 1$). Although we will stop here, the argument may be carried one step further. One might suspect that the local convexity of the Lagrangian affects the existence and uniqueness of resulting fields. The role of this and other functional features are fleshed out in [8]. The upshot is that convex Lagrangians lead to well behaved solutions.[4] So we additionally take $\bar{\mathcal{L}}$ to be convex. What we are left with is an infinite class of functions, that are identical in form; they can all be represented schematically by the leftmost function in Figure 3.1.

3.2 Vector Fields

This section closely follows the discussion of the previous section. In fact, most of the work is already done; details will be omitted except when the vector nature of this theory significantly affects the prior calculations. We will work with the small

[4] It may be the case that strictly increasing functions are good enough.

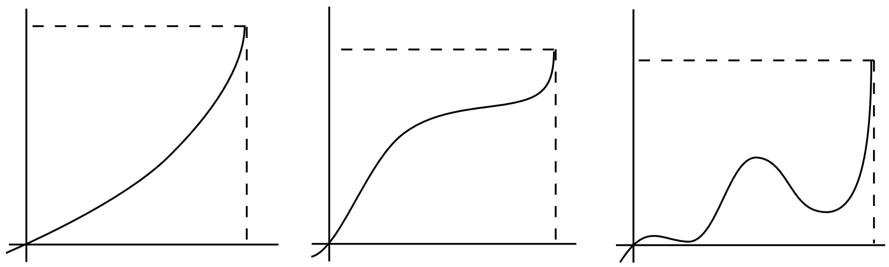

Figure 3.1: A few acceptable forms for $\bar{\mathcal{L}}$. In this case we need not worry about behavior for negative inputs.

parameter $1/b^2$ instead of ϵ^2 to make things look more like the Born-Infeld theory. The generic vector theory governing A^μ will be a modification of classical electrodynamics.

As before, assume that A^μ is governed by an action of the form

$$S = b^2 \int d\tau \sqrt{-g}\bar{\mathcal{L}}(F^{\mu\nu}F_{\mu\nu}/2b^2), \tag{3.16}$$

where $1/b$ is a small parameter as in Chapter 2. It is required that the general theory reduce to the classical Lagrangian as $b \to \infty$. So, expand $\bar{\mathcal{L}}$ in $1/b^2$ about $b = \infty$,

$$\bar{\mathcal{L}} = b^2\bar{\mathcal{L}}(0) + \frac{1}{2}\bar{\mathcal{L}}'(0)F^{\mu\nu}F_{\mu\nu} + \mathcal{O}(1/b^2).$$

Comparison with the classical Lagrangian, (1.11) yields the requirement

$$\bar{\mathcal{L}}(0) = 0 \quad \text{and} \quad \bar{\mathcal{L}}'(0) = 1/2. \tag{3.17}$$

Field equations for A^μ are obtained in the usual way. The computation is almost identical to the computation of the Born-Infeld field equations, so the details are omitted. The resulting field equations are

$$\frac{1}{2}\left(\bar{\mathcal{L}}'(F^{\sigma\tau}F_{\sigma\tau}/2b^2)F^{\mu\nu}\right)_{;\mu} = 0. \tag{3.18}$$

As in the Born-Infeld theory, the field equations may also be written in terms of the physical fields $\mathbf{E}$ and $\mathbf{B}$. They are precisely the Born-Infeld field equations, (2.9), with modified conjugate momenta:

$$\mathbf{D} \equiv \bar{\mathcal{L}}'((B^2 - E^2)/b^2)\mathbf{E} \quad \text{and} \quad \mathbf{H} \equiv \bar{\mathcal{L}}'((B^2 - E^2)/b^2)\mathbf{B}. \tag{3.19}$$

Finally, the stress tensor of the general theory is

$$T^{\mu\nu} = -(b^2\, g^{\mu\nu}\bar{\mathcal{L}} - 2\bar{\mathcal{L}}'F^{\nu\sigma}F^{\mu}{}_{\sigma})$$

For stationary solutions in Minkowski spacetime, the total energy is then

$$U = \int_{\text{all space}} (b^2\bar{\mathcal{L}} + 2\bar{\mathcal{L}}'E^2)\, dV. \tag{3.20}$$

We now turn to the problem of finding a class of $\bar{\mathcal{L}}$ for which the resulting fields behave in the desired manner. As before, take relevant fields to be static and localized. And assume a Minkowski spacetime. It is required that $\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = B^2 - E^2$ is bounded. One way to impose this condition is to insist that the domain X of $\bar{\mathcal{L}}$ is bounded. Unlike the scalar theory, $F^{\mu\nu}F_{\mu\nu}$ can be positive or negative. So we insist that X of $\bar{\mathcal{L}}$ is bounded above and below. Without loss of generality, say $X = [-1, 1]$.

It is required that fields be defined over all of space. We can again use the trick of integrating the field equations. The story is the same as before; $\bar{\mathcal{L}}'$ must blow up when the fields reach their maximum values, so, when $F^{\mu\nu}F_{\mu\nu} = \pm b^2$.

Finally, it is necessary that the total energy of all localized stationary fields $\mathbf{E}$ and $\mathbf{B}$ be real valued and finite. From the asymptotic structure of $\bar{\mathcal{L}}$ (see (3.17)) and the fact that both $\mathbf{E}$ and $\mathbf{B}$ are localized, the energy of the field lies almost entirely within a compact region of space. Thus, we need only worry about convergence of (3.20) on this compact region. To obtain finite energy solutions in the scalar theory, it was sufficient to assume that $\bar{\mathcal{L}}$ was bounded. For the vector theory this only ensures that the first term of the integrand is finite. For pure magnetic solutions this is actually enough since the second term goes to zero anyway. For the electric solutions to have finite energy it is additionally required that $\bar{\mathcal{L}}'(-E^2/b^2)$ is integrable over the relevant compact region of space. So, placing the origin at the point where $\bar{\mathcal{L}}'$ blows up, it must be the case that $\bar{\mathcal{L}}'(-E^2/b^2) \sim 1/r^p$, where $p < 3$. The condition that integrability places on $\bar{\mathcal{L}}$ is somewhat subtle and requires further analysis. The problem is that $\bar{\mathcal{L}}'$ determines $\mathbf{E}$, and in particular how fast $\mathbf{E}$ approaches its maximum. But the rate at which $\mathbf{E}$ approaches its maximum in turn determines whether or not $\bar{\mathcal{L}}'$ is integrable.

Putting everything back together, the story is much like it was for the scalar theory. We obtain the following Lagrangian: $\bar{\mathcal{L}} : X \to Y$ is analytic with $X = [-1, 1]$, Y is bounded, and $\bar{\mathcal{L}}$ takes the particular value $\bar{\mathcal{L}}(0) = 0$. Further, $\bar{\mathcal{L}}'$ is bounded except at ± 1 and takes the particular value $\bar{\mathcal{L}}'(0) = -1/2$. The new feature of acceptable Lagrangians in the vector theory is that $\bar{\mathcal{L}}'(-E^2/b^2)$ is integrable when $E^2 \to b^2$. Interestingly, there is no such constraint when $B^2 \to b^2$. Unfortunately, we are not yet able to impose this condition in a constructive way. We could continue in this vein—but let's instead turn to some examples.

3.3 Examples of Modified Electrodynamic Theories

The difficulty of constructing examples lies in finding suitable analytic functions for $\bar{\mathcal{L}}$. Because we would actually like to do physics with the resulting modified theory, it is extremely advantageous to search for functions of minimal complexity, that lend themselves to the relevant computation. To impose the required bound on the domain of $\bar{\mathcal{L}}$, we will search for (or construct) functions that admit the branch cut $(-\infty, -1) \cup (1, \infty)$, and are finite valued at ± 1.[5] To that end, look for functions that take branch cuts of the form $(-\infty, a)$ or (a, ∞) and are finite at branch points: $\sqrt[n]{z}$ and certain

[5] See [17] for a review of analytic function theory.

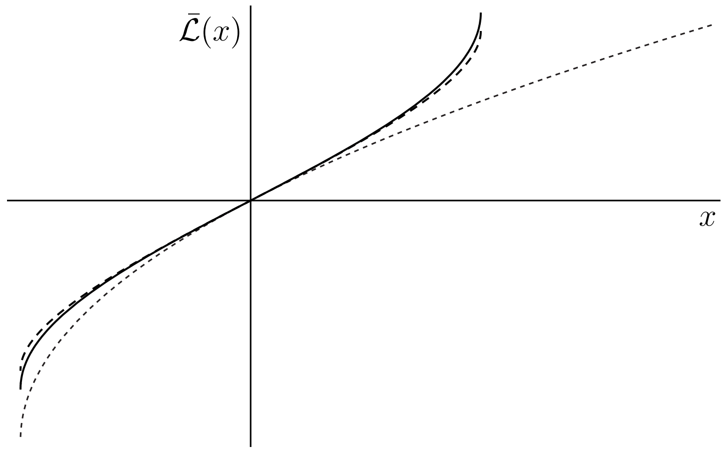

Figure 3.2: Graph of Arcsine, Root Difference (dashed), and Born-Infeld (thin dashed) Lagrangians.

trigonometric functions, for example. Next we piece them together. There are many many ways to do this, and presumably better ways than what we consider. Check that they satisfy the remaining conditions described above and we are ready to do modified Electrodynamics.

We present two modified theories of E&M that fix the problems of Born-Infeld E&M discussed in Chapter 2. Morally, we should consider these improved theories with the same enthusiasm and completeness that we paid the Born-Infeld theory. Instead we will just provide a quick overview of the resulting structures, and make a few comparisons.

Arcsine

Consider the modified theory of E&M derived from the Lagrangian:

$$\bar{\mathcal{L}}(F^{\mu\nu}F_{\mu\nu}/2b^2) = \frac{1}{2}\text{Arcsin}(F^{\mu\nu}F_{\mu\nu}/2b^2), \tag{3.21}$$

where the principle branch of Arcsine is in use (see Figure 3.2). As far as I can tell, this is the simplest acceptable Lagrangian. Note that:

$$\bar{\mathcal{L}}' = \frac{1}{2\sqrt{1-\left(\frac{1}{2b^2}F^{\mu\nu}F_{\mu\nu}\right)^2}}$$

and from (3.20)

$$U = b^2\int_{\text{all space}}\left(\frac{1}{2}\text{Arcsin}\big((B^2-E^2)/b^2\big) + \frac{1}{b^2}\frac{E^2}{\sqrt{1-\left(\frac{1}{b^2}(B^2-E^2)\right)^2}}\right)dV.$$

It follows that the field equations are defined for $B^2 - E^2 \in [-b^2, b^2]$. From the field equations, the electric monopole is computed in the usual way:

$$\mathbf{E}_{mon} = \frac{b}{\sqrt{2}q}\sqrt{-b^2r^4 + \sqrt{4q^4 + b^4r^8}}\,\hat{\mathbf{r}}. \tag{3.22}$$

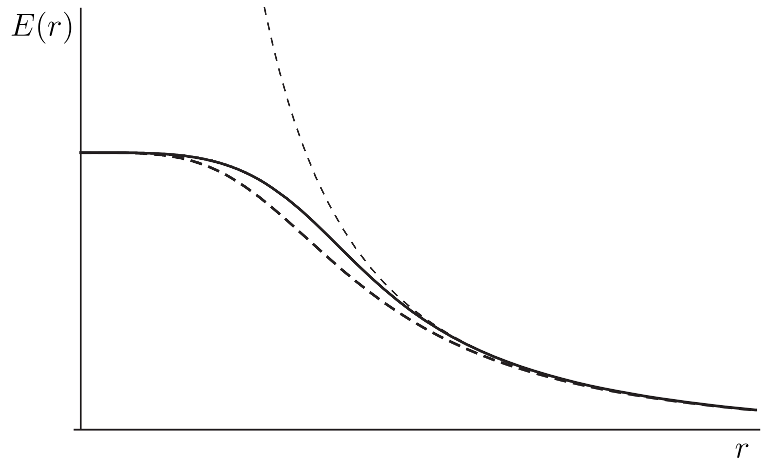

Figure 3.3: Radial dependence of the Arcsine monopole with Born-Infeld monopole (dashed) and classical monopole (thin dashed) for comparison.

This is a funny looking monopole, but does behave in the expected manner (see Figure 3.3). It is nice to see that the solution is algebraically no worse than that for the Born-Infeld theory, which is all that we can hope for. As a second example, consider the field produced by a current carrying wire. In cylindrical coordinates, an infinite neutral wire carrying a uniform current $\mathbf{I}$ in the $\hat{\mathbf{z}}$ direction has magnetic field

$$\mathbf{B} = \frac{b}{\sqrt{2}I}\sqrt{-b^2s^2 + \sqrt{4I^4 + b^4s^4}}\,\hat{\boldsymbol{\phi}}.$$

Difference of Roots

Consider the modified theory of E&M derived from the Lagrangian:

$$\bar{\mathcal{L}}(F^{\mu\nu}F_{\mu\nu}/2b^2) = \frac{1}{2}\left(\sqrt{1 + F^{\mu\nu}F_{\mu\nu}/2b^2} - \sqrt{1 - F^{\mu\nu}F_{\mu\nu}/2b^2}\right), \tag{3.23}$$

This Lagrangian is a natural extension of the Born-Infeld Lagrangian. Note that this Lagrangian is almost identical to the Arcsine Lagrangian (see Figure 3.2). The only real difference is that this Lagrangian blows up more quickly than the previous. Note that:

$$\bar{\mathcal{L}}' = \frac{1}{2}\left(\frac{1}{2\sqrt{1 + F^{\mu\nu}F_{\mu\nu}/2b^2}} + \frac{1}{2\sqrt{1 - F^{\mu\nu}F_{\mu\nu}/2b^2}}\right).$$

Now we are ready to do physics. Well, we coud try to do physics with this theory, and it would be pretty much like that of the Arcsine theory, except that the relevant equations become algebraically intractable. The monopole solution, for instance, fills a page. It is unclear whether there are theories other than the Arcsine theory for which the solutions are nice. It seems unlikely.

Conclusion

Let's see where we stand. We began with the problem of singular solution in General Relativity. The goal was to construct a modified theory for which no solutions were singular. But this problem is quite difficult. To better understand the problem, we considered its analogue in the classical theory of Electrodynamics. To that end we took a long hard look at the Born-Infeld Electrodynamics, a theory that supposedly fixed field singularities. Independent of its motivation, the Born-Infeld theory was itself an intriguing object of study. As a coupled system of nonlinear partial differential equations, it raised several interesting questions about the solution structure for the theory. For instance, we were able to develop a sensible definition of multipoles for the nonlinear theory. In the end, the Born-Infeld theory was unacceptable with regard to the problem of singularities. While the theory placed a bound on the magnitude of the electric field, it did so at the cost of the magnetic field—the magnetic field was sometimes infinite and sometimes undefined over extended regions of space.

Problems in the Born-Infeld theory led us to perform a careful study of model building for relativistic scalar and vector theories. By imposing certain minimal conditions that solutions be bounded and have finite energy, we were able to determine two classes of admissible theories, one for the scalar fields and one for the vector fields. Interestingly, the class of acceptable theories for scalar fields was very constrained, allowing for theories of basically one form. Because of complications arising for vector fields, the class of acceptable vector theories could only be described by an implicit relationship between the theory and its solutions. But it was not clear what the relationship actually said about admissible theories. We then considered two particular modified electrodynamic theories. While finite energy of all localized fields was not established, both theories did have bounded solutions, which is already an improvement on the Born-Infeld theory. The second was analytically very cumbersome. The first was quite reasonable and seems to be the most natural modified electrodynamic theory of this form with the specified properties.

There is much left to be done. Notably, an analysis of the kind carried out in Chapter 3 should be performed for GR. Barring that, there is still plenty to be said about the characterization of acceptable scalar and vector field theories. It would be particularly interesting to bring into the discussion some of the results of functional analysis concerning existence and uniqueness. Several other questions, tangential to the main project of this work, have also been posed. These concern the further analysis of the Born-Infeld and Arcsine theories, using the methods of Chapter 2.

Appendix A

Contraction Mappings

The theory of ordinary differential equations can be thought of in two parts: existence and uniqueness of solutions, and methods for obtaining solutions (or approximations thereof). Here we will only discuss a few of the fundamental results and ideas, with a brief sketch of proofs where appropriate.[1]

The fundamental tool for proving the existence and uniqueness theorem for ordinary differential equations is the contraction principle for contraction mappings on a Banach space. Let the pair $(X, ||\cdot||)$ be a **normed vector space**. In this space, a **Cauchy sequence** is a sequence of vectors for which all vectors in the sequence beyond a certain index are within $\epsilon > 0$ of each other w.r.t (with respect to) the norm. The normed vector space is called **complete** if every Cauchy sequence converges within the space. For example, $\mathbb{R}$ is Cauchy, whereas $\mathbb{Q}$ (the set of all rational numbers) is not.[2]A complete normed vector space is called a **Banach space**.

In the context of ODEs, we are interested in the normed vector space of continuous functions $C(I)$ on I, a compact interval, where the norm is defined by

$$||x|| = \sup_{t \in I} |x(t)|. \tag{A.1}$$

From real analysis, the $C(I)$ is complete, hence a Banach space.

In addition, let $K : C \subseteq X \to C$. A **fixed point** of K is an element $x \in C$ such that $K(x) = x$. We say that K is a **contraction** if there is a contraction constant $\theta \in [0, 1)$ such that

$$||K(x) - K(y)|| \leq \theta ||x - y||, \qquad x, y \in C. \tag{A.2}$$

Let $x_n = K^n(x) = K(K^{n-1}(x)), K^0(x) = x$. We are now ready to state the central result.

Theorem 1 (Contraction principle, Weissinger)**.** *Suppose $K : C \subseteq X \to C$ is a contraction. Then K has a unique fixed point $\bar{x} \in C$ such that*

$$||K^n(x) - \bar{x}|| \leq \frac{\theta^n}{1 - \theta} ||K(x) - x||, \qquad x \in C. \tag{A.3}$$

[1] This section follows G. Teschl's presentation [18].

[2] π may be defined in terms of Cauchy sequence in $\mathbb{Q}$, $\{3, 3.1, 3.14, .3, 141, \cdots\}$, but $\pi \notin \mathbb{Q}$.

Sketch of proof. Uniqueness: If x and $\tilde{x}$ are fixed points, then

$$||x - \tilde{x}|| = ||K(x) - K(\tilde{x})|| \leq \theta ||x - \tilde{x}||,$$

which can not be true.

Existence: Fix $x_0 \in C$ and consider the sequence $\{K^i(x_0)\} = \{x_i\}$. From the fact that K is a contraction, one obtains the bound (for $n > m$)

$$||x_n - x_m|| \leq \frac{\theta^m}{1-\theta} ||x_1 - x_0||. \tag{A.4}$$

Thus $\{x_i\}$ converges to $\bar{x} \in C$. Moreover, examination of the normed difference of $K(\bar{x})$ and $\bar{x}$ shows it to be the fixed point. The estimate follows from (A.4). □

With that, consider the **initial value problem** (IVP)

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{A.5}$$

where $f \in C(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R} \times \mathbb{R}^n$, and $(t_0, x_0) \in U$. Integration of both sides of (A.5) yields the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{A.6}$$

Notice that any solution to the IVP is a fixed point of (A.6). The idea is to show that for sufficiently nice functions f and suitable initial values, (A.6) is in fact a contraction mapping on some open subset of $\mathbb{R} \times \mathbb{R}^n$. Hence, the fixed points (solutions) will exist and be unique on the open subset.

We say that f is **Lipschitz continuous** in the second argument if for every set $V \in U$ the following number

$$L = \sup_{(t,x) \neq (t,y) \in V} \frac{|f(t,x) - f(t,y)|}{|x - y|} \tag{A.7}$$

(which depends on V) is finite. It turns out that such functions are precisely the ones that make (A.6) a contraction mapping

Theorem 2 (Picard–Lindelof)**.** *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, then there is a unique solution $\bar{x}(t)$ of the IVP (A.5) .*

Sketch of proof. Establish that (A.6) is a contraction mapping given the nature of f. The proof follows from Theorem 1. See [18] for details. □

In addition to proving existence and uniqueness of solutions to (A.5), Theorem 2 also provides an algorithm for computing the solution. The algorithm is known as **Picard Iteration** or the method of iterated integrals. The idea is to use the contraction mapping of (A.6) to generate successive approximations of the actual solution. Theorem 2 ensures (with some restrictions) that the approximation converges to the actual solution. For ODEs, the method is not actually suitable for finding solutions since it is generally not possible to compute the integral at each iteration.

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