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Combined bracketing methods for solving nonlinear equations

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ABSTRACT

Several methods based on combinations of bisection, regula falsi, and parabolic interpolation has been developed. An interval bracketing ensures the global convergence while the combination with the parabolic interpolation increases the speed of the convergence. The proposed methods have been tested on a series of examples published in the literature and show good results.

1. Introduction

Many bracketing algorithms have been described in literature for finding a single root of a nonlinear equation

$$f(x) = 0. \tag{1}$$

The most basic bracketing method is a dichotomy method also known as a bisection method with a rather slow convergence [1]. The method is guaranteed to converge for a continuous function f on the interval $[x_a, x_b]$ where $f(x_a)f(x_b) < 0$. Another basic bracketing root finding method is a regula falsi technique (false position) [1]. Both methods have a linear order of convergence, but the regula falsi method suffers due to the slow convergence in some cases. A variant of the regula falsi method called an Illinois algorithm improves this drawback [2]. In order to increase reliability, combinations of several methods were introduced by Dekker [3] who combined the bisection and secant method. This method was further improved by Brent [4] and later by Alefeld and Potra [5] with several new algorithms. An approximation with an exponential function was also used on the closed interval in order to calculate a root of the nonlinear function [6].

Instead of a linear interpolation, a quadratic polynomial interpolation can be also used for the root determination. This open technique with a superlinear order of convergence was introduced by Muller [7] and successfully applied on the closed interval by Suhadolnik [8]. Muller's method can also be found in the combination with the bisection [9–11] or inverse quadratic interpolation [12]. Recently, several new algorithms of enclosing methods were published [13–18].

2. Proposed zero finding methods

In the rest of the text, it will be assumed that the function $f(x)$ is continuous on a closed interval $[x_a, x_b]$. In this interval, the function has a root and the following inequality holds

$$f(x_a)f(x_b) < 0. \tag{2}$$

Without loss of generality we suppose that the function values on the border of the interval are $f(x_a) < 0$ and $f(x_b) > 0$. A new iterative value on the closed interval is calculated by fitting a parabola to the three points of the function $f(x)$. The first

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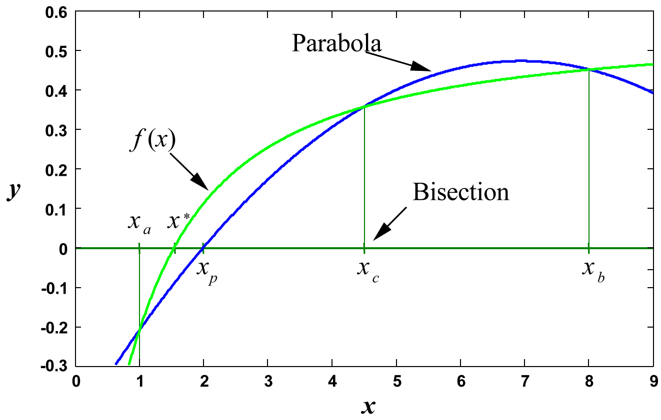

Fig. 1. Bisection-parabolic method (BP).

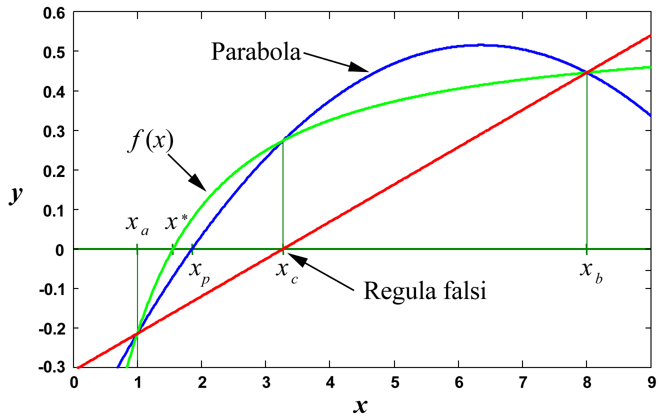

Fig. 2. Regula falsi–parabolic method (RP).

and second points are the interval border points $(x_a, f(x_a))$ and $(x_b, f(x_b))$ while the third point $(x_c, f(x_c))$; $x_c \in (x_a, x_b)$ is calculated by using the bisection (Fig. 1)

$$x_c = \frac{x_a + x_b}{2}, \tag{3}$$

or regula falsi algorithm (Fig. 2)

$$x_c = \frac{f(x_a)x_b - f(x_b)x_a}{f(x_a) - f(x_b)}. \tag{4}$$

We call the first method a bisection–parabolic (BP) method and the second a regula falsi–parabolic (RP) method. Unfortunately, in some cases the regula falsi method has a rather slow convergence (see examples in Table 1) which can remain even in the combined regula falsi–parabolic method. In order to prevent slow convergence in such cases a switching between the bisection and regula falsi is used to calculate the third point x_c. This method is called a regula falsi–bisection–parabolic (RBP) method.

The switching mechanism between the regula falsi and bisection bases on the observation of an interpolating line slope. If the slope is too high or too low the bisection is engaged instead of the regula falsi. This procedure prevents slow convergence in some cases when only the RP method is used (see examples in Table 1 for comparison) and exploits cases where the RP method is faster than the BP method.

After calculating x_c, three points $(x_a, f(x_a))$, $(x_c, f(x_c))$ and $(x_b, f(x_b))$ are finally available and through these points, a second order polynomial can be constructed

$$p(x) = A(x - x_c)^2 + B(x - x_c) + C. \tag{5}$$

The complete polynomial equations for all three points are then

$$\begin{cases} f(x_a) = A(x_a - x_c)^2 + B(x_a - x_c) + C, \\ f(x_b) = A(x_b - x_c)^2 + B(x_b - x_c) + C, \\ f(x_c) = C. \end{cases} \tag{6}$$

From these equations the parameters A, B and C can be determined

$$\begin{cases} A = \dfrac{f(x_a) - f(x_c)}{(x_a - x_c)(x_a - x_b)} + \dfrac{f(x_c) - f(x_b)}{(x_b - x_c)(x_a - x_b)}, \\ B = \dfrac{[f(x_c) - f(x_a)](x_b - x_c)}{(x_a - x_c)(x_a - x_b)} - \dfrac{[f(x_c) - f(x_b)](x_a - x_c)}{(x_b - x_c)(x_a - x_b)}, \\ C = f(x_c). \end{cases} \tag{7}$$

Considering A, B and C as parameters of the parabola defined in Eq. (5) the zeros $x_{1,2}^{*p}$ can be calculated

$$x_{1,2}^{*p} = x_c - \frac{2C}{B \pm \sqrt{B^2 - 4AC}}. \tag{8}$$

In this equation, the sign before the square root is selected in the following way. First, both roots $x_{1,2}^{*p}$ are calculated and then the root which is closer to x_c is selected. More precisely, in further calculations only the root with $x_p = \min_{j=1,2} |x_c - x_j^{*p}|$ is carried. After this procedure, the initial interval can be narrowed. The first endpoint of a shrinking interval becomes root of the parabola x_p and the second one x_c if the zero of the function f lies between them. If not, then one endpoint remains unchanged while the other one is x_p or x_c. In particular, if $f(x_a)f(x_p) < 0$ then $x_b = x_p$ and if $f(x_a)f(x_c) > 0$ then $x_a = x_c$. In the opposite case, if $f(x_a)f(x_p) > 0$ then $x_a = x_p$ and if $f(x_b)f(x_c) > 0$ then $x_b = x_c$ (see Section 4 for further details).

The Eq. (8) can be rewritten in an iterative form by introducing $x_i = x_c$, $i \in \mathbb{N}$ as an approximate value of the function zero in the current iteration. A new calculated value is then $x_{i+1} = x_p$. In order to keep $|x_{i+1} - x_i|$ minimal, a sign before the square root in Eq. (8) must be equal to the sign of B

$$x_{i+1} = x_i - \frac{2f(x_i)}{B + \operatorname{sgn}(B)\sqrt{B^2 - 4Af(x_i)}}. \tag{9}$$

3. Convergence of the proposed methods

In the first theorem the convergence order of the BP method is determined.

Theorem 1. *Let $f(x)$ contain only one zero x^* in the closed interval $[x_a, x_b]$ with $f(x_a)f(x_b) < 0$ and $f'(x^*) \neq 0$. Suppose $f \in C^3$, at least in a neighborhood of x^*. If the midpoint of the interval is defined in the iterative process with Eq. (3) and the second order polynomial is defined by Eqs. (5)–(8), then the sequence of polynomial zero points $\{x_i\}$ defined in Eq. (9) converges to x^* with the convergence order of 2.*

Proof. If only parabola is used in the iterative process, the following error equation can be assumed [7,8]

$$e_{i+1} \approx -e_i e_{i-1} e_{i-2} \frac{f'''(x^*)}{6f'(x^*)} \tag{10}$$

where

$$\begin{cases} e_{i+1} = x_{i+1} - x^*, \\ e_i = x_i - x^*, \\ e_{i-1} = x_{i-1} - x^*, \\ e_{i-2} = x_{i-2} - x^*. \end{cases} \tag{11}$$

The interval at each iteration using the bisection is first halved. The bisection converges linearly and an error ratio between two consecutive iterations can be estimated as

$$\frac{e_{i-1}}{e_{i-2}} \approx \frac{1}{2}. \tag{12}$$

This ratio can be introduced in Eq. (10)

$$e_{i+1} \approx -e_i e_{i-1}^2 \frac{f'''(x^*)}{3f'(x^*)}. \tag{13}$$

Suppose that the sequence converges with order $r > 0$ to the zero point x^*

$$\lim_{i \to \infty} \frac{|e_{i+1}|}{|e_i|^r} = K, \tag{14}$$

where K is an asymptotic error constant ($K \neq 0$). This equation can be written in an alternate form

$$\begin{cases} |e_{i+1}| \approx K|e_i|^r, \\ |e_i| \approx K|e_{i-1}|^r. \end{cases} \tag{15}$$

By inserting 15 into 13, one can calculate

$$K^r|e_{i-1}|^{r^2} \approx |e_{i-1}|^{r+2}\left|\frac{f'''(x^*)}{3f'(x^*)}\right|. \tag{16}$$

If the exponents are equalized, r can be calculated as a positive root of equation

$$r^2 - r - 2 = 0 \tag{17}$$

which gives finally the convergence of the order $r = 2$ [12]. This is the BP method. □

Theorem 2. *Let $f(x)$ contain only one zero x^* in the closed interval $[x_a, x_b]$ with $f(x_a)f(x_b) < 0$ and $f'(x^*) \neq 0$. Suppose $f \in C^3$, at least in a neighborhood of x^*. If a division point of the interval is defined in the iterative process with Eq. (4) and the second order polynomial is defined by Eqs. (5)–(8), then the sequence of polynomial zero points $\{x_i\}$ defined in Eq. (9) converges to x^* with the convergence order of 2.*

Proof. The error ratio between two consecutive iterations in case of the regula falsi method can be estimated as [1]

$$\frac{e_{i-1}}{e_{i-2}} \approx K_0, \tag{18}$$

where K_0 is the constant. By inserting Eq. (18) into (10) we get

$$e_{i+1} \approx -e_i e_{i-1}^2 \frac{f'''(x^*)}{6K_0 f'(x^*)}. \tag{19}$$

Obtaining similar procedure as in the proof of the Theorem 1, the final polynomial equation is $r^2 - r - 2 = 0$ with a positive root $r = 2$. This is the RP method. □

Theorem 3. *If the Theorems 1 and 2 hold, then at the same conditions the regula falsi–bisection–parabolic (RBP) method converges to x^* with the convergence order of 2.*

Proof. During the iterative process the RBP algorithm switches between BP and RP calculation. In one iterative step only the BP or RP procedure is used. If both procedures have the convergence order of 2 then the complete RBP method also has the convergence order of 2. □

4. Combined algorithm

In the proposed algorithm, the inputs are the left side x_a and right side x_b of the initial interval, function $f(x)$, calculated precision ε and maximum number of iterations $N_{\max}$. The initial interval $[x_a, x_b]$ must contain zero with condition $f(x_a)f(x_b) < 0$.

Start of the RBP algorithm.

Step 1. **Input:** $x_a, x_b, f, \varepsilon, N_{\max}$.

Step 2. **Initial bisection:** $x_c = \frac{x_a + x_b}{2}$, $n = 1$, $xo_p = x_c$.

Step 3. **Function values:** $f_a = f(x_a)$; $f_b = f(x_b)$; $f_c = f(x_c)$.

Step 4.
$$A = \frac{f_a - f_c}{(x_a - x_c)\,(x_a - x_b)} + \frac{f_c - f_b}{(x_b - x_c)\,(x_a - x_b)},$$
$$B = \frac{[f_c - f_a](x_b - x_c)}{(x_a - x_c)(x_a - x_b)} - \frac{[f_c - f_b](x_a - x_c)}{(x_b - x_c)(x_a - x_b)},$$
$$C = f_c.$$

Step 5. **Root of parabola:** $x_p = x_c - \frac{2C}{B + \mathrm{sgn}(B)\sqrt{B^2 - 4AC}}$.

Step 6. $f_p = f(x_p)$.

Step 7. **New endpoints of the interval:**

If $f_a f_p < 0$ then

$x_b = x_p, f_b = f_p$

If $f_a f_c > 0$ then $x_a = x_c, f_a = f_c$

else

$x_a = x_p, f_a = f_p$

If $f_b f_c > 0$ then $x_b = x_c, f_b = f_c$.

Step 8. **Switching mechanism between bisection and regula falsi**:

If $|f_a - f_b| > 10|x_a - x_b|$ or $|f_a - f_b| < 0.1|x_a - x_b|$ then

Bisection: $x_c = \dfrac{x_a + x_b}{2}$

else

Regula falsi: $x_c = \dfrac{x_a f_b - x_b f_a}{f_b - f_a}$.

Step 9. $f_c = f(x_c)$.

Step 10. If $n > N_{\max}$ then stop.

Step 11. If $n > 1$ and $\left|xo_p - x_p\right| < \varepsilon$ then print x_p, f_p and stop.

Step 12. If $f_p = 0$ then print x_p, f_p and stop.

Step 13. $n = n + 1, \quad xo_p = x_p$.

Step 14. Go to Step 4.

End of the RBP algorithm.

Step 3 of the algorithm contains three function evaluations before the main loop begins in Step 4. The algorithm Steps 6 and 9 which are part of the loop also contain function evaluation. If the number of iterations is n, then the total number of function evaluations is $2n + 3$ (see last row in Table 1).

A switching mechanism between the bisection and regula falsi prevents the slow convergence of the proposed algorithm. If the slope of the regula falsi line (see Fig. 2) is too high $|f(x_a) - f(x_b)| > 10|x_a - x_b|$ or too low $|f(x_a) - f(x_b)| < 0.1|x_a - x_b|$ the bisection is engaged.

The RP algorithm or BP algorithm is easily developed from the RBP algorithm if in Step 8 the regula falsi or bisection including if-then condition is eliminated.

The convergence of all methods are proposed with the Theorems 1–3 where $f'(x^*) \neq 0$. If $f'(x^*) = 0$ the methods may suffer due to the slow convergence (see Table 1, Examples 17 and 18).

5. Numerical examples

In this section, the numerical examples are presented and collected in Table 1. Some elementary functions $f(x)$ are listed and zeros are calculated with different methods on the intervals denoted by $[x_a, x_b]$. The number of iterations n is calculated for the bisection (Bis) [1], regula falsi (Reg) [1], Illinois' (Ill) [2], Brent's (Bre) [4], Suhadolnik's (Suh) [8] and the presented BP,

Table 1
Different bracketing methods by presenting the number of iterations n.

No.	$\varepsilon = 10^{-15},\ N_{\max} = 10^5$		Bis	Reg	Ill	Bre	Suh	BP	RP	RBP
	$f(x)$	$[x_a, x_b]$	n							
1	$\ln x$	[.5, 5]	52	29	10	8	8	6	6	6
2	$(10 - x)e^{-10x} - x^{10} + 1$	[.5, 8]	53	$>10^5$	37	15	25	10	257	10
3	$e^{\sin x} - x - 1$	[1, 4]	52	33	10	12	7	7	5	5
4	$11x^{11} - 1$	[.5, 1]	49	108	13	10	9	7	7	7
5	$2\sin x - 1$	$[.1, \frac{\pi}{3}]$	50	15	8	6	5	5	4	4
6	$x^2 + \sin\frac{x}{10} - .25$	[0,1]	50	34	11	8	4	4	3	3
7	$(x - 1)e^{-x}$	[0, 1.5]	51	74	12	9	7	6	5	5
8	$\cos x - x$	[0, 1.7]	51	18	9	6	5	6	4	4
9	$(x - 1)^3 - 1$	[1.5, 3]	51	61	12	9	7	6	5	5
10	$e^{x^2+7x-30} - 1$	[2.6, 3.5]	50	4020	20	10	9	7	7	7
11	$\arctan x - 1$	[1, 8]	53	27	11	8	8	6	6	6
12	$e^x - 2x - 1$	[.2, 3]	52	157	15	12	8	6	6	6
13	$e^{-x} - x - \sin x$	[0,.5]	49	13	8	5	5	5	4	4
14	$x^3 - 1$	[.1, 1.5]	51	36	11	7	6	6	5	5
15	$x^2 - \sin^2 x - 1$	$[-1, 2]$	52	34	12	9	7	7	5	5
16	$\sin x - \frac{x}{2}$	$[\frac{\pi}{2}, \pi]$	51	33	9	7	6	7	4	4
17	x^3	$[-0.5, \frac{1}{3}]$	48	$>10^5$	95	144	$>10^5$	44	$>10^5$	44
18	x^5	$[-0.5, \frac{1}{3}]$	48	$>10^5$	185	122	$>10^5$	49	$>10^5$	49
Number of function evaluations			$n + 1$	$n + 2$			$n + 3$	$2n + 3$		

RP and RBP algorithms. The calculation based on Brent's method is actually the standard routine ''fzero'' for finding zeros of the nonlinear functions in Matlab. In Table 1, the bottom row presents terms for calculating the number of the function evaluations in each particular column based on the number of iterations n.

In most cases the RP algorithm outperforms the BP algorithm, but in some cases the BP algorithm slowly converges to the root of the function (see Examples 2, 17 and 18). In these cases all algorithms have a large number of iterations except the most basic bisection and consequently the algorithm BP. Some algorithms even exceed the maximal number of iterations which is limited to 10^5. In these cases the combined algorithm RBP keeps a low number of iterations and eliminates the weakness of the RP algorithm.

6. Conclusions

The paper presents a bracketing algorithm for the iterative zero finding of the nonlinear equations. The algorithm is based on the combination of the bisection and regula falsi with the second order polynomial interpolation technique. The convergence of the algorithm is superlinear. The proposed algorithm can be used as a good substitute to the well-known bracketing methods. The weakness and strength of the algorithm are presented in some typical examples and a comparison with other methods is given.

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