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Some efficient derivative free methods with memory for solving nonlinear equations

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A B S T R A C T

Based on the family of three-point derivative free methods without memory of eighth order convergence proposed by Džunić et al. [J. Džunić, M.S. Petković, L.D. Petković, Three-point methods with and without memory for solving nonlinear equations, Appl. Math. Comput. 218 (2012) 4917–4927], we present three methods with memory by suitable variation of a free parameter in each iterative step. This free parameter is calculated using Newton's interpolatory polynomial of the third degree in two ways and Newton's interpolatory polynomial of the fourth degree. Consequently, the *R*-order of convergence is increased from 8 to $\frac{11+\sqrt{137}}{2} \approx 11.352$, $6+4\sqrt{2} \approx 11.657$ and 12. The increase in the convergence order is achieved without any additional function evaluations and therefore, the proposed methods possess a very high computational efficiency. Numerical examples are presented and the performance is compared with the existing three-point methods with and without memory of the basic family. Moreover, theoretical order of convergence is verified on the examples.

1. Introduction

Derivative free iterative methods for solving nonlinear equations $f(x) = 0$ are important in the sense that in many practical situations it is preferable to avoid calculations of derivative of f. One such scheme is

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

which is obtained from the Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots,$$

by approximating the derivative $f'(x_k)$ by the quotient $\frac{f(x_k+\gamma f(x_k))-f(x_k)}{\gamma f(x_k)}$. The scheme (1) defines a one-parameter (γ) family of methods and has same order and efficiency index [1] as that of Newton's method. The choice $\gamma = 1$ produces the well-known Steffensen method [2].

Recently, based on the scheme (1), Petković et al. [3] proposed a derivative free family of two-step methods with memory having improved order and efficiency. Džunić et al. [4] have extended the idea of this family and presented derivative free

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families of three-point methods with and without memory. In case of methods without memory the three-point family is given by

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{\varphi(x_k)}, \\ z_k = y_k - h(u_k, v_k)\frac{f(y_k)}{\varphi(x_k)}, \quad k = 0,1,2\ldots \\ x_{k+1} = z_k - \dfrac{f(z_k)}{f[z_k,y_k] + f[z_k,y_k,x_k](z_k - y_k) + f[z_k,y_k,x_k,w_k](z_k - y_k)(z_k - x_k)}, \end{cases} \tag{2}$$

where

$$u_k = \frac{f(y_k)}{f(x_k)}, \quad v_k = \frac{f(y_k)}{f(w_k)}, \quad \varphi(x_k) = \frac{f(w_k) - f(x_k)}{\gamma f(x_k)}, \quad w_k = x_k + \gamma f(x_k)$$

and h is a two-valued function that satisfies the conditions

$$h(0,0) = h_u(0,0) = h_v(0,0) = 1, \quad h_{vv}(0,0) = 2, \quad |h_{uu}(0,0)| < \infty, \quad |h_{uv}(0,0)| < \infty. \tag{3}$$

Here the subscript indices denote corresponding partial derivatives of h.

If x_0 is an initial approximation sufficiently closer to zero of f (say α), it was proved in [4] that the family of three-point methods (2) is of order eight and satisfies the error equation

$$\begin{aligned} e_{k+1} = x_{k+1} - \alpha = \frac{c_2^2}{4}(1 + \gamma f'(\alpha))^4 [2c_3 + c_2^2(-8 + 2h_{uv}(0,0) + \gamma f'(\alpha)(h_{uu}(0,0) - 2) + h_{uu}(0,0))] \\ \times [2c_2c_3 - 2c_4 + c_2^3(-8 + 2h_{uv}(0,0) + \gamma f'(\alpha)(h_{uu}(0,0) - 2) + h_{uu}(0,0))]e_k^8 + O(e_k^9), \end{aligned} \tag{4}$$

where $e_k = x_k - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2,3,\ldots$.

Since the method (2) uses four function evaluations, it has optimal eighth order convergence (see [5]). Some simple forms of the function $h(u, v)$ satisfying the conditions (3) are given below:

(1) $h(u, v) = \frac{1+u}{1-v}$,
(2) $h(u, v) = \frac{1}{(1-u)(1-v)}$,
(3) $h(u, v) = 1 + u + v + v^2$,
(4) $h(u, v) = 1 + u + v + (u + v)^2$,
(5) $h(u, v) = u + \frac{1}{1-v}$.

Džunić et al. [4] have explored the three-point methods with memory from the family (2) by using the idea which consists of calculation of the parameter $\gamma = \gamma_k$ as the iteration proceeds by the formula $\gamma_k = -1/\bar{f}'(\alpha)$ for $k = 1,2,\ldots$, where $\bar{f}'(\alpha)$ is an approximation of $f'(\alpha)$. A similar idea has been applied to the methods with memory proposed by Džunić and Petković in [6]. In [4], Džunić et al. have presented four methods through the following forms of γ_k

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad \text{(secant approach)}, \tag{5}$$

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{x_k - y_{k-1}}{f(x_k) - f(y_{k-1})} \quad \text{(better secant approach)}, \tag{6}$$

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{x_k - z_{k-1}}{f(x_k) - f(z_{k-1})} \quad \text{(best secant approach)}, \tag{7}$$

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{1}{N_2'(x_k)} \quad \text{(Newton interpolatory approach)}, \tag{8}$$

where $N_2(t) = N_2(t; x_k, z_{k-1}, y_{k-1})$ is Newton's interpolatory polynomial of second degree, set through three available approximations (x_k, z_{k-1}, y_{k-1}) and

$$\begin{aligned} N_2'(x_k) &= \left[\frac{d}{dt}N_2(t)\right]_{t=x_k} = \left[\frac{d}{dt}(f(x_k) + f[x_k, z_{k-1}](t - x_k) + f[x_k, z_{k-1}, y_{k-1}](t - x_k)(t - z_{k-1}))\right]_{t=x_k} \\ &= f[x_k, z_{k-1}] + f[x_k, z_{k-1}, y_{k-1}](x_k - z_{k-1}) = f[x_k, y_{k-1}] + f[x_k, z_{k-1}] - f[z_{k-1}, y_{k-1}]. \end{aligned} \tag{9}$$

The methods so obtained are given by

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{\widetilde{\varphi}(x_k)}, \\ z_k = y_k - h(u_k, v_k)\frac{f(y_k)}{\widetilde{\varphi}(x_k)}, \quad k = 0,1,2\ldots \\ x_{k+1} = z_k - \dfrac{f(z_k)}{f[z_k,y_k] + f[z_k,y_k,x_k](z_k - y_k) + f[z_k,y_k,x_k,w_k](z_k - y_k)(z_k - x_k)}, \end{cases} \tag{10}$$

where $\widetilde{\varphi}(x_k) = \frac{f(w_k)-f(x_k)}{\gamma_k f(x_k)}$, $w_k = x_k + \gamma_k f(x_k)$ and γ_k is calculated by one of the forms (5)–(8). It is assumed that the initial estimate γ_0 should be chosen before starting the iterative process of (10). Using an idea given by Traub [7, p. 186], Džunić et al. [4] suggested the ways for appropriate choice of the estimate of γ_0. The R-order of convergence [8] of the family of three-point methods (10) with the corresponding expressions (5)–(8) of γ_k is $2(2+\sqrt{5}) \approx 8.472$, 9, 10 and 11 (see [4]).

It is clear that in case of the methods with memory the convergence speed can be increased significantly without any addition to function evaluations and therefore, possess high computational efficiency than the methods without memory. Motivated by this fact, based on the family without memory (2), here we present three methods with memory. For the calculation of the varying parameter γ_k, we use Newton's interpolatory polynomials of third degree in two different ways and Newton's interpolatory polynomial of fourth degree. It is shown that the R-order of convergence of the presented methods is increased from 8 (which is the order of the basic family without memory) to $\frac{11+\sqrt{137}}{2} \approx 11.352$, $6+4\sqrt{2} \approx 11.657$ and 12, depending upon the form of varying parameter. The methods thus obtained still use four function evaluations and hence possess a very high computational efficiency. Numerical examples and the comparison with the existing methods of the families (2) and (10) are given to confirm the theoretical results and high computational efficiency of the proposed methods.

2. Development of the methods

We have seen that the main idea in constructing three-point methods with memory from the basic family without memory (2) consists of the calculation of the parameter $\gamma = \gamma_k$ as the iteration proceeds by an approximation $-1/\bar{f}'(\alpha)$ of $-1/f'(\alpha)$ using available data. In what follows, we consider the family without memory (2) and thereby keeping the family (10) in mind construct the methods with memory by assuming the following three forms of varying parameter γ_k (each of which is an approximation of $-1/f'(\alpha)$)

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{1}{N_3'(x_k)} \quad \text{(Newton interpolatory approach with third degree polynomial } N_3(t)), \tag{11}$$

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{1}{\overline{N}_3'(x_k)} \quad \text{(Newton interpolatory approach with third degree polynomial } \overline{N}_3(t)), \tag{12}$$

$$\gamma_k = -\frac{1}{\bar{f}'(\alpha)} = -\frac{1}{N_4'(x_k)} \quad \text{(Newton interpolatory approach with fourth degree polynomial } N_4(t)), \tag{13}$$

where $N_3(t) = N_3(t; x_k, z_{k-1}, y_{k-1}, x_{k-1})$ and $\overline{N}_3(t) = \overline{N}_3(t; x_k, z_{k-1}, y_{k-1}, w_{k-1})$ are Newton's interpolatory polynomials of third degree, set through four available approximations $(x_k, z_{k-1}, y_{k-1}, x_{k-1})$ and $(x_k, z_{k-1}, y_{k-1}, w_{k-1})$, respectively, and $N_4(t) = N_4(t; x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1})$ is Newton's interpolatory polynomial of fourth degree, set through five available approximations $(x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1})$.

The derivatives $N_3'(t)$, $\overline{N}_3'(t)$ and $N_4'(t)$ at $x = x_k$ are calculated by using the following formulas:

$$\begin{aligned} N_3'(x_k) = \left[\frac{d}{dt}N_3(t)\right]_{t=x_k} &= \left[\frac{d}{dt}(f(x_k) + f[x_k, z_{k-1}](t-x_k) + f[x_k, z_{k-1}, y_{k-1}](t-x_k)(t-z_{k-1})\right. \\ &\quad \left. + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}](t-x_k)(t-z_{k-1})(t-y_{k-1}))\right]_{t=x_k} = f[x_k, z_{k-1}] + f[x_k, z_{k-1}, y_{k-1}](x_k - z_{k-1}) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}](x_k - z_{k-1})(x_k - y_{k-1}), \end{aligned} \tag{14}$$

$$\begin{aligned} \overline{N}_3'(x_k) = \left[\frac{d}{dt}\overline{N}_3(t)\right]_{t=x_k} &= \left[\frac{d}{dt}(f(x_k) + f[x_k, z_{k-1}](t-x_k) + f[x_k, z_{k-1}, y_{k-1}](t-x_k)(t-z_{k-1})\right. \\ &\quad \left. + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}](t-x_k)(t-z_{k-1})(t-y_{k-1}))\right]_{t=x_k} = f[x_k, z_{k-1}] + f[x_k, z_{k-1}, y_{k-1}](x_k - z_{k-1}) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}](x_k - z_{k-1})(x_k - y_{k-1}) \end{aligned} \tag{15}$$

and

$$\begin{aligned} N_4'(x_k) = \left[\frac{d}{dt}N_4(t)\right]_{t=x_k} &= \left[\frac{d}{dt}(f(x_k) + f[x_k, z_{k-1}](t-x_k) + f[x_k, z_{k-1}, y_{k-1}](t-x_k)(t-z_{k-1})\right. \\ &+ f[x_k, z_{k-1}, y_{k-1}, w_{k-1}](t-x_k)(t-z_{k-1})(t-y_{k-1}) + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}] \\ &\times \left.(t-x_k)(t-z_{k-1})(t-y_{k-1})(t-w_{k-1}))\right]_{t=x_k} = f[x_k, z_{k-1}] + f[x_k, z_{k-1}, y_{k-1}](x_k - z_{k-1}) \\ &+ f[x_k, z_{k-1}, y_{k-1}, w_{k-1}](x_k - z_{k-1})(x_k - y_{k-1}) + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}](x_k - z_{k-1})(x_k - y_{k-1})(x_k - w_{k-1}). \end{aligned} \tag{16}$$

Here $f[r,s]$, $f[r,s,t]$, $f[r,s,t,u]$ and $f[r,s,t,u,v]$ are the divided differences of first, second, third and fourth order, respectively. Note that the evaluation of the parameter γ_k depends on the data available from the current and the previous iterative steps. It is assumed that the initial estimate γ_0 should be chosen before starting the iterative process, for example, using the way suggested in [4].

3. Convergence theorem

In order to obtain the order of convergence of the family of three-point methods with memory (10), where γ_k is calculated using one of the formulas (11)–(13), we will use the concept of the R-order of convergence introduced by Ortega and Rheinboldt [8]. Now we state the following convergence theorem:

Theorem 1. *If an initial approximation x_0 is sufficiently close to the zero α of $f(x)$ and the parameter γ_k in the iterative scheme (10) is recursively calculated by the forms given in (11)–(13). Then, the R-order of convergence of the three-point methods (10) with the corresponding expressions (11)–(13) of γ_k is at least $\frac{11+\sqrt{137}}{2} \approx 11.352$, $6+4\sqrt{2} \approx 11.657$ and 12.*

Proof. Let $\{x_k\}$ be a sequence of approximations generated by an iterative method with memory (IM). If this sequence converges to the zero α of f with the R-order ($\geqslant r$) of IM, then we write

$$e_{k+1} \sim D_{k,r} e_k^r, \quad e_k = x_k - \alpha, \qquad (\text{see [4]}) \tag{17}$$

where $D_{k,r}$ tends to the asymptotic error constant D_r of IM when $k \to \infty$. Thus

$$e_{k+1} \sim D_{k,r}(D_{k-1,r} e_{k-1}^r)^r = D_{k,r} D_{k-1,r}^r e_{k-1}^{r^2}. \tag{18}$$

Let $\bar{e}_k = w_k - \alpha$, $\tilde{e}_k = y_k - \alpha$, $\hat{e}_k = z_k - \alpha$, then in view of the methods (2) without memory we have (see [4])

$$\bar{e}_k = (1 + \gamma f'(\alpha)) e_k + O(e_k^2), \tag{19}$$
$$\tilde{e}_k = c_2(1 + \gamma f'(\alpha)) e_k^2 + O(e_k^3), \tag{20}$$
$$\hat{e}_k = A_4(\alpha) e_k^4 + O(e_k^5), \tag{21}$$

where $A_4(\alpha) = -c_2(1+\gamma f'(\alpha))^2[c_3 + c_2^2(-4 + h_{uu}(0,0)/2 + h_{uv}(0,0) + (h_{uu}(0,0)/2 - 1)\gamma f'(\alpha))]$.

According to the error relations (19)–(21) and (4) with the self-accelerating parameter $\gamma = \gamma_k$, we can write the corresponding error relations for the methods (10) with memory

$$\bar{e}_k \sim (1 + \gamma_k f'(\alpha)) e_k, \tag{22}$$
$$\tilde{e}_k \sim c_2(1 + \gamma_k f'(\alpha)) e_k^2, \tag{23}$$
$$\hat{e}_k \sim a_{k,4}(1 + \gamma_k f'(\alpha))^2 e_k^4, \tag{24}$$
$$e_{k+1} \sim a_{k,8}(1 + \gamma_k f'(\alpha))^4 e_k^8. \tag{25}$$

where the expressions of $a_{k,4}$ and $a_{k,8}$ are evident from (21) and (4) and depend on iteration index since γ_k is recalculated in each step.

Using Taylor's expansion of $f(x)$ about the root α, we have

$$f(x) = f'(\alpha)(e + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + \cdots), \tag{26}$$

where $e = x - \alpha$.

In the sequel, we obtain the R-order of convergence of the family (10) for the approaches (11)–(13) applied to the calculation of γ_k.

Method (I): *γ_k is calculated by (11)*

Using the expansion (26) for $x = x_k$ and $x = z_{k-1}$, we obtain

$$\begin{aligned} f[x_k, z_{k-1}] &= \frac{f'(\alpha)((e_k - \hat{e}_{k-1}) + c_2(e_k^2 - \hat{e}_{k-1}^2) + c_3(e_k^3 - \hat{e}_{k-1}^3) + c_4(e_k^4 - \hat{e}_{k-1}^4) + c_5(e_k^5 - \hat{e}_{k-1}^5) + \cdots)}{e_k - \hat{e}_{k-1}} \\ &= f'(\alpha)(1 + c_2(e_k + \hat{e}_{k-1}) + c_3(e_k^2 + e_k\hat{e}_{k-1} + \hat{e}_{k-1}^2) + c_4(e_k + \hat{e}_{k-1})(e_k^2 + \hat{e}_{k-1}^2) + c_5(e_k^4 + e_k^3\hat{e}_{k-1} + e_k^2\hat{e}_{k-1}^2 \\ &\quad + e_k\hat{e}_{k-1}^3 + \hat{e}_{k-1}^4) + \cdots). \end{aligned} \tag{27}$$

In the same way as the above for $(x = z_{k-1}, x = y_{k-1})$ and $(x = y_{k-1}, x = x_{k-1})$, we calculate

$$\begin{aligned} f[z_{k-1}, y_{k-1}] &= f'(\alpha)(1 + c_2(\hat{e}_{k-1} + \tilde{e}_{k-1}) + c_3(\hat{e}_{k-1}^2 + \hat{e}_{k-1}\tilde{e}_{k-1} + \tilde{e}_{k-1}^2) + c_4(\hat{e}_{k-1} + \tilde{e}_{k-1})(\hat{e}_{k-1}^2 + \tilde{e}_{k-1}^2) + c_5(\tilde{e}_{k-1}^4 \\ &\quad + \tilde{e}_{k-1}^3\hat{e}_{k-1} + \tilde{e}_{k-1}^2\hat{e}_{k-1}^2 + \tilde{e}_{k-1}\hat{e}_{k-1}^3 + \hat{e}_{k-1}^4) + \cdots), \end{aligned} \tag{28}$$

$$\begin{aligned} f[y_{k-1}, x_{k-1}] &= f'(\alpha)(1 + c_2(e_{k-1} + \tilde{e}_{k-1}) + c_3(e_{k-1}^2 + e_{k-1}\tilde{e}_{k-1} + \tilde{e}_{k-1}^2) + c_4(e_{k-1} + \tilde{e}_{k-1})(e_{k-1}^2 + \tilde{e}_{k-1}^2) + c_5(\tilde{e}_{k-1}^4 \\ &\quad + \tilde{e}_{k-1}^3 e_{k-1} + \tilde{e}_{k-1}^2 e_{k-1}^2 + \tilde{e}_{k-1} e_{k-1}^3 + e_{k-1}^4) + \cdots). \end{aligned} \tag{29}$$

Using (27)–(29), we get

$$\begin{aligned} f[x_k, z_{k-1}, y_{k-1}] &= f'(\alpha)(c_2 + c_3(\hat{e}_{k-1} + \tilde{e}_{k-1} + e_k) + c_4(\tilde{e}_{k-1}^2 + \hat{e}_{k-1}\tilde{e}_{k-1} + \hat{e}_{k-1}^2 + \hat{e}_{k-1}e_k + \tilde{e}_{k-1}e_k + e_k^2) + c_5(e_k^3 \\ &\quad + e_k^2\tilde{e}_{k-1} + e_k\tilde{e}_k^2 + \tilde{e}_{k-1}^3 + e_k^2\hat{e}_{k-1} + e_k\tilde{e}_{k-1}\hat{e}_{k-1} + \tilde{e}_{k-1}^2\hat{e}_{k-1} + e_k\hat{e}_{k-1}^2 + \tilde{e}_{k-1}\hat{e}_{k-1}^2 + \hat{e}_{k-1}^3) + \cdots) \end{aligned} \tag{30}$$

and

$$f[z_{k-1},y_{k-1},x_{k-1}] = f'(\alpha)(c_2 + c_3(\hat{e}_{k-1} + \tilde{e}_{k-1} + e_{k-1}) + c_4(\tilde{e}_{k-1}^2 + \hat{e}_{k-1}\tilde{e}_{k-1} + \hat{e}_{k-1}^2 + \hat{e}_{k-1}e_{k-1} + \tilde{e}_{k-1}e_{k-1} + e_{k-1}^2) \\ + c_5(e_{k-1}^3 + e_{k-1}^2\tilde{e}_{k-1} + e_{k-1}\tilde{e}_{k-1}^2 + \tilde{e}_{k-1}^3 + e_{k-1}^2\hat{e}_{k-1} + e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + \tilde{e}_{k-1}^2\hat{e}_{k-1} + e_{k-1}\hat{e}_{k-1}^2 \\ + \tilde{e}_{k-1}\hat{e}_{k-1}^2 + \hat{e}_{k-1}^3) + \cdots). \tag{31}$$

Eqs. (30) and (31) yield

$$f[x_k,z_{k-1},y_{k-1},x_{k-1}] = f'(\alpha)(c_3 + c_4(e_k + e_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1}) + c_5(e_k^2 + e_{k-1}^2 + \tilde{e}_{k-1}^2 + \tilde{e}_{k-1}\hat{e}_{k-1} + \hat{e}_{k-1}^2 + e_{k-1}(\tilde{e}_{k-1} \\ + \hat{e}_{k-1}) + e_k(e_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1})) + \cdots). \tag{32}$$

Substituting (27), (30) and (32) in (14) yields

$$N_3'(x_k) = f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + c_4(e_{k-1}e_k^2 + \tilde{e}_{k-1}e_k^2 + \hat{e}_{k-1}e_k^2 - e_{k-1}\tilde{e}_{k-1}e_k - \tilde{e}_{k-1}\hat{e}_{k-1}e_k - e_{k-1}\hat{e}_{k-1}e_k + e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} \\ + 3e_k^3) + \cdots) \\ = f'(\alpha)(1 + c_4e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + O(e_k)).$$

Using above result and (11), we find

$$1 + \gamma_k f'(\alpha) \sim c_4e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}. \tag{33}$$

Assume that the iterative sequences $\{y_k\}$ and $\{z_k\}$ have the R-orders p and s, respectively, then, bearing in mind (17) we obtain

$$\tilde{e}_k \sim D_{k,p}e_k^p \sim D_{k,p}(D_{k-1,r}e_{k-1}^r)^p \sim D_{k,p}D_{k-1,r}^p e_{k-1}^{rp} \tag{34}$$

and

$$\hat{e}_k \sim D_{k,s}e_k^s \sim D_{k,s}(D_{k-1,r}e_{k-1}^r)^s \sim D_{k,s}D_{k-1,r}^s e_{k-1}^{rs}. \tag{35}$$

According to (17), (23), (33) and (34), we obtain

$$\tilde{e}_k \sim c_2(1 + \gamma_k f'(\alpha))e_k^2 \sim c_2c_4e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}e_k^2 \sim c_2c_4e_{k-1}(D_{k-1,p}e_{k-1}^p)(D_{k-1,s}e_{k-1}^s)(D_{k-1,r}e_{k-1}^r)^2 \sim c_2c_4D_{k-1,p}D_{k-1,s}D_{k-1,r}^2 e_{k-1}^{2r+s+p+1}. \tag{36}$$

Combining (17), (24), (33) and (35), we have

$$\hat{e}_k \sim a_{k,4}(1 + \gamma_k f'(\alpha))^2e_k^4 \sim a_{k,4}(c_4e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1})^2e_k^4 \sim a_{k,4}c_4^2e_{k-1}^2(D_{k-1,p}e_{k-1}^p)^2(D_{k-1,s}e_{k-1}^s)^2(D_{k-1,r}e_{k-1}^r)^4 \\ \sim a_{k,4}c_4^2D_{k-1,p}^2D_{k-1,s}^2D_{k-1,r}^4 e_{k-1}^{4r+2s+2p+2}. \tag{37}$$

From (17), (25) and (33)–(35), we find the error relation

$$e_{k+1} \sim a_{k,8}(1 + \gamma_k f'(\alpha))^4e_k^8 \sim a_{k,8}(c_4e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1})^4e_k^8 \sim a_{k,8}c_4^4D_{k-1,p}^4D_{k-1,s}^4D_{k-1,r}^8 e_{k-1}^{8r+4s+4p+4}. \tag{38}$$

Comparing the exponents of e_{k-1} on the right hand sides of (34) and (36), (35) and (37), and then (18) and (38), we form the system of three equations in p, s and r

$$rp - 2r - s - p - 1 = 0, \\ rs - 4r - 2s - 2p - 2 = 0, \\ r^2 - 8r - 4s - 4p - 4 = 0.$$

Non-trivial solution of this system is $p = \frac{11+\sqrt{137}}{8}$, $s = \frac{11+\sqrt{137}}{4}$ and $r = \frac{11+\sqrt{137}}{2} \approx 11.352$. Therefore, the R-order of the methods with memory (10), when γ_k is calculated by (11), is at least 11.352.

Method (II): *γ_k is calculated by* (12)

Using (26) for $x = y_{k-1}$, $x = w_{k-1}$ and simplifying, we get

$$f[y_{k-1},w_{k-1}] = f'(\alpha)(1 + c_2(\bar{e}_{k-1} + \tilde{e}_{k-1}) + c_3(\bar{e}_{k-1}^2 + \bar{e}_{k-1}\tilde{e}_{k-1} + \tilde{e}_{k-1}^2) + c_4(\bar{e}_{k-1} + \tilde{e}_{k-1})(\bar{e}_{k-1}^2 + \tilde{e}_{k-1}^2) + c_5(\tilde{e}_{k-1}^4 \\ + \tilde{e}_{k-1}^3\bar{e}_{k-1} + \tilde{e}_{k-1}^2\bar{e}_{k-1}^2 + \tilde{e}_{k-1}\bar{e}_{k-1}^3 + \bar{e}_{k-1}^4) + \cdots). \tag{39}$$

From (28) and (39) we calculate

$$f[z_{k-1},y_{k-1},w_{k-1}] = f'(\alpha)(c_2 + c_3(\hat{e}_{k-1} + \tilde{e}_{k-1} + \bar{e}_{k-1}) + c_4(\tilde{e}_{k-1}^2 + \hat{e}_{k-1}\tilde{e}_{k-1} + \hat{e}_{k-1}^2 + \hat{e}_{k-1}\bar{e}_{k-1} + \tilde{e}_{k-1}\bar{e}_{k-1} + \bar{e}_{k-1}^2) \\ + c_5(\bar{e}_{k-1}^3 + \bar{e}_{k-1}^2\tilde{e}_{k-1} + \bar{e}_{k-1}\tilde{e}_{k-1}^2 + \tilde{e}_{k-1}^3 + \bar{e}_{k-1}^2\hat{e}_{k-1} + \bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + \tilde{e}_{k-1}^2\hat{e}_{k-1} + \bar{e}_{k-1}\hat{e}_{k-1}^2 \\ + \tilde{e}_{k-1}\hat{e}_{k-1}^2 + \hat{e}_{k-1}^3) + \cdots). \tag{40}$$

Taking (30) and (40) into account, we obtain

$$f[x_k, z_{k-1}, y_{k-1}, w_{k-1}] = f'(\alpha)(c_3 + c_4(e_k + \bar{e}_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1}) + c_5(e_k^2 + \bar{e}_{k-1}^2 + \tilde{e}_{k-1}^2 + \tilde{e}_{k-1}\hat{e}_{k-1} + \hat{e}_{k-1}^2 + \bar{e}_{k-1}(\tilde{e}_{k-1} + \hat{e}_{k-1}) + e_k(\bar{e}_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1})) + \cdots). \tag{41}$$

Using (27), (30) and (41) in (15), it follows that

$$\begin{aligned}\overline{N}_3'(x_k) &= f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + c_4(\bar{e}_{k-1}e_k^2 + \tilde{e}_{k-1}e_k^2 + \hat{e}_{k-1}e_k^2 - \bar{e}_{k-1}\tilde{e}_{k-1}e_k - \tilde{e}_{k-1}\hat{e}_{k-1}e_k - \bar{e}_{k-1}\hat{e}_{k-1}e_k + \bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} \\ &\quad + 3e_k^3) + \cdots) \\ &= f'(\alpha)(1 + c_4\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + O(e_k)).\end{aligned}$$

According to this result and (12) we find

$$1 + \gamma_k f'(\alpha) \sim c_4\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}. \tag{42}$$

Assume that the iterative sequence $\{w_k\}$ has the R-order q, then from (17)

$$\bar{e}_k \sim D_{k,q}e_k^q \sim D_{k,q}(D_{k-1,r}e_{k-1}^r)^q \sim D_{k,q}D_{k-1,r}^q e_{k-1}^{rq}. \tag{43}$$

Combining (17), (22), (42) and (43), we obtain

$$\begin{aligned}\bar{e}_k &\sim (1 + \gamma_k f'(\alpha))e_k \sim c_4\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}e_k \sim c_4(D_{k-1,q}e_{k-1}^q)(D_{k-1,p}e_{k-1}^p)(D_{k-1,s}e_{k-1}^s)(D_{k-1,r}e_{k-1}^r) \\ &\sim c_4D_{k-1,q}D_{k-1,p}D_{k-1,s}D_{k-1,r}e_{k-1}^{r+s+p+q}.\end{aligned} \tag{44}$$

In the similar way we find the error relations

$$\tilde{e}_k \sim c_2(1 + \gamma_k f'(\alpha))e_k^2 \sim c_2c_4D_{k-1,q}D_{k-1,p}D_{k-1,s}D_{k-1,r}^2 e_{k-1}^{2r+s+p+q}, \tag{45}$$

$$\hat{e}_k \sim a_{k,4}(1 + \gamma_k f'(\alpha))^2 e_k^4 \sim a_{k,4}c_4^2D_{k-1,q}^2D_{k-1,p}^2D_{k-1,s}^2D_{k-1,r}^4 e_{k-1}^{4r+2s+2p+2q} \tag{46}$$

and

$$e_{k+1} \sim a_{k,8}(1 + \gamma_k f'(\alpha))^4 e_k^8 \sim a_{k,8}c_4^4D_{k-1,q}^4D_{k-1,p}^4D_{k-1,s}^4D_{k-1,r}^8 e_{k-1}^{8r+4s+4p+4q}. \tag{47}$$

Comparing the error exponents of e_{k-1} from four pairs of relations (43) and (44), (34) and (45), (35) and (46), and (18) and (47), we form the system of four equations in q, p, s and r

$$\begin{aligned}&rq - r - s - p - q = 0, \\ &rp - 2r - s - p - q = 0, \\ &rs - 4r - 2s - 2p - 2q = 0, \\ &r^2 - 8r - 4s - 4p - 4q = 0.\end{aligned}$$

Non-trivial solution of this system is $q = \frac{1+2\sqrt{2}}{2}$, $p = \frac{3+2\sqrt{2}}{2}$, $s = 3 + 2\sqrt{2}$ and $r = 6 + 4\sqrt{2} \approx 11.657$ and we conclude that the R-order of the methods with memory (10), when γ_k is given by (12), is at least 11.657.

Method (III): *γ_k is calculated by (13)*

Using (32)and (41)

$$f[x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}] = f'(\alpha)(c_4 + c_5(e_k + e_{k-1} + \bar{e}_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1}) + \cdots). \tag{48}$$

Substituting (27), (30), (32) and (48) in (16), yields

$$\begin{aligned}N_4'(x_k) &= f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + 4c_4e_k^3 + c_5(e_k^3(e_{k-1} + \bar{e}_{k-1} + \tilde{e}_{k-1} + \hat{e}_{k-1}) - e_k^2(e_{k-1}\bar{e}_{k-1} + e_{k-1}\tilde{e}_{k-1} + \bar{e}_{k-1}\tilde{e}_{k-1} \\ &\quad + e_{k-1}\hat{e}_{k-1} + \bar{e}_{k-1}\hat{e}_{k-1} + \tilde{e}_{k-1}\hat{e}_{k-1}) + e_k(e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1} + e_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + \bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + e_{k-1}\bar{e}_{k-1}\hat{e}_{k-1}) \\ &\quad - e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + 4e_k^4) + \cdots) \\ &= f'(\alpha)(1 - c_5e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1} + O(e_k)).\end{aligned} \tag{49}$$

From this result and (13), we find

$$1 + \gamma_k f'(\alpha) \sim c_5e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}. \tag{50}$$

Using (50) and previously derived relations, we obtain the following error relations for the intermediate approximations

$$\bar{e}_k \sim (1 + \gamma_k f'(\alpha))e_k \sim c_5e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}e_k \sim c_5D_{k-1,q}D_{k-1,p}D_{k-1,s}D_{k-1,r}e_{k-1}^{r+s+p+q+1}, \tag{51}$$

$$\tilde{e}_k \sim c_2(1 + \gamma_k f'(\alpha))e_k^2 \sim c_2c_5e_{k-1}\bar{e}_{k-1}\tilde{e}_{k-1}\hat{e}_{k-1}e_k^2 \sim c_2c_5D_{k-1,q}D_{k-1,p}D_{k-1,s}D_{k-1,r}^2 e_{k-1}^{2r+s+p+q+1} \tag{52}$$

and

$$\hat{e}_k \sim a_{k,4}(1 + \gamma_k f'(\alpha))^2 e_k^4 \sim a_{k,4}c_5^2e_{k-1}^2\bar{e}_{k-1}^2\tilde{e}_{k-1}^2\hat{e}_{k-1}^2e_k^4 \sim a_{k,4}c_5^2D_{k-1,q}^2D_{k-1,p}^2D_{k-1,s}^2D_{k-1,r}^4 e_{k-1}^{4r+2s+2p+2q+2}. \tag{53}$$

In the similar fashion we find the final error relation (25) which is given by

$$e_{k+1} \sim a_{k,8}(1+\gamma_k f'(\alpha))^4 e_k^8 \sim a_{k,8} c_5^4 e_{k-1}^4 \bar{e}_{k-1}^4 \tilde{e}_{k-1}^4 \hat{e}_{k-1}^4 e_k^8 \sim a_{k,8} c_5^4 D_{k-1,q}^4 D_{k-1,p}^4 D_{k-1,s}^4 D_{k-1,r}^8 e_{k-1}^{8r+4s+4p+4q+4}. \quad (54)$$

By comparing the exponents of e_{k-1} from four pairs of relations (43) and (51), (34) and (52), (35) and (53), and (18) and (54), we arrive at the system of four equations in q, p, s and r

$$\begin{aligned} &rq - r - s - p - q - 1 = 0, \\ &rp - 2r - s - p - q - 1 = 0, \\ &rs - 4r - 2s - 2p - 2q - 2 = 0, \\ &r^2 - 8r - 4s - 4p - 4q - 4 = 0. \end{aligned}$$

Since non-trivial solution of this system is $q = 2, p = 3, s = 6$ and $r = 12$ and therefore, we conclude that the R-order of the methods with memory (10), using the formula (13) of γ_k, is at least 12. □

Remark 1. It is evident that the improvement of convergence order in all cases is attained without any additional function evaluations, which points to a very high computational efficiency of the proposed methods. To verify this we calculate Ostrowski–Traub coefficient of efficiency [1], which is given by $E = r^{1/\theta}$, where r is the R-order of the considered iterative

Table 1
Performance of the methods for example $f_1(x)$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	r_c
$h(u,v)=(1+u)/(1-v)$				
(2)	6.49(−5)	4.97(−34)	5.86(−267)	8.000
(10), (5)	6.49(−5)	2.64(−36)	1.61(−302)	8.481
(10), (6)	6.49(−5)	1.17(−40)	4.60(−360)	8.936
(10), (7)	6.49(−5)	1.77(−42)	2.22(−417)	9.980
(10), (8)	6.49(−5)	1.50(−48)	4.33(−526)	10.944
(10), (11)	6.49(−5)	5.56(−48)	4.66(−536)	11.333
(10), (12)	6.49(−5)	2.34(−48)	6.46(−554)	11.637
(10), (13)	6.49(−5)	4.83(−51)	1.17(−601)	11.936
$h(u,v)=1/((1-u)(1-v))$				
(2)	6.53(−5)	7.81(−35)	3.26(−274)	8.000
(10), (5)	6.53(−5)	1.11(−36)	1.57(−305)	8.462
(10), (6)	6.53(−5)	1.40(−40)	2.08(−359)	8.939
(10), (7)	6.53(−5)	1.92(−42)	4.68(−417)	9.981
(10), (8)	6.53(−5)	1.57(−48)	6.80(−526)	10.944
(10), (11)	6.53(−5)	5.79(−48)	7.39(−536)	11.333
(10), (12)	6.53(−5)	2.44(−48)	1.03(−553)	11.637
(10), (13)	6.53(−5)	4.86(−51)	1.27(−601)	11.936
$h(u,v)=1+u+v+v^2$				
(2)	6.45(−5)	1.27(−33)	2.90(−263)	8.000
(10), (5)	6.45(−5)	1.08(−35)	9.65(−297)	8.482
(10), (6)	6.45(−5)	9.43(−40)	6.15(−352)	8.962
(10), (7)	6.45(−5)	1.36(−41)	1.99(−408)	10.002
(10), (8)	6.45(−5)	1.38(−47)	1.98(−516)	10.987
(10), (11)	6.45(−5)	4.81(−47)	1.22(−526)	11.384
(10), (12)	6.45(−5)	2.09(−47)	2.37(−544)	11.696
(10), (13)	6.45(−5)	9.07(−51)	4.46(−598)	11.936
$h(u,v)=1+u+v+(u+v)^2$				
(2)	6.58(−5)	4.21(−35)	1.17(−276)	8.000
(10), (5)	6.58(−5)	5.96(−37)	5.85(−308)	8.458
(10), (6)	6.58(−5)	7.59(−41)	8.33(−362)	8.931
(10), (7)	6.58(−5)	1.03(−42)	4.55(−421)	10.035
(10), (8)	6.58(−5)	1.03(−48)	2.75(−529)	10.971
(10), (11)	6.58(−5)	3.73(−48)	1.59(−539)	11.362
(10), (12)	6.58(−5)	1.59(−48)	3.43(−557)	11.662
(10), (13)	6.58(−5)	1.12(−50)	3.50(−602)	12.050
$h(u,v)=u+1/(1-v)$				
(2)	6.45(−5)	1.27(−33)	2.84(−263)	8.000
(10), (5)	6.45(−5)	1.08(−35)	9.44(−297)	8.482
(10), (6)	6.45(−5)	9.39(−40)	5.88(−352)	8.962
(10), (7)	6.45(−5)	1.35(−41)	1.82(−408)	10.002
(10), (8)	6.45(−5)	1.10(−47)	2.40(−517)	10.982
(10), (11)	6.45(−5)	4.08(−47)	2.57(−527)	11.380
(10), (12)	6.45(−5)	1.71(−47)	3.52(−545)	11.690
(10), (13)	6.45(−5)	3.67(−50)	5.02(−592)	11.974

method (IM) and θ is the number of function evaluations per iteration. Thus, the coefficients of efficiency of the proposed methods are; $E = 11.352^{1/4} \approx 1.836$, $E = 11.657^{1/4} \approx 1.848$ and $E = 12^{1/4} \approx 1.861$. These E-values are much improved over the E-value ($E = 8^{1/4} \approx 1.682$) of the basic family without memory (2).

Remark 2. The families of n-point methods with memory, developed in [6], have R-order of convergence at least $23 \times 2^{n-4}$ for $n \geqslant 4$, which shows that for four and higher-step methods of families the efficiency is higher than the present methods. For instance, in case of four-step methods the coefficient of efficiency (E) is $23^{1/5} \approx 1.872$. However, the present methods are different in structure in the sense that these are three-step methods. Moreover, present developments are based on the basic family (2) which is different from the basic families considered in [6].

4. Numerical examples

In this section we demonstrate the convergence behavior of the methods with memory (10), where γ_k is calculated by one of the formulas (11)–(13). For comparison, in numerical experiments we choose the methods of the families (2) and (10) proposed by Džunić et al. [4]. All computations are performed using the programming package *Mathematica* with multi-

Table 2
Performance of the methods for example $f_2(x)$.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	r_c
$h(u, v) = (1 + u)/(1 - v)$				
(2)	2.88(−6)	1.56(−42)	1.17(−332)	8.000
(10), (5)	2.88(−6)	4.81(−45)	4.33(−374)	8.486
(10), (6)	2.88(−6)	2.40(−48)	6.21(−427)	8.997
(10), (7)	2.88(−6)	1.35(−50)	1.81(−497)	10.081
(10), (8)	2.88(−6)	1.50(−55)	4.89(−601)	11.069
(10), (11)	2.88(−6)	2.34(−57)	1.76(−642)	11.453
(10), (12)	2.88(−6)	1.88(−57)	4.39(−658)	11.734
(10), (13)	2.88(−6)	8.54(−60)	8.39(−707)	12.087
$h(u, v) = 1/((1 - u)(1 - v))$				
(2)	9.22(−7)	4.70(−47)	2.15(−369)	8.000
(10), (5)	9.22(−7)	1.72(−48)	1.19(−403)	8.511
(10), (6)	9.22(−7)	2.43(−52)	7.44(−463)	9.006
(10), (7)	9.22(−7)	1.75(−54)	2.55(−536)	10.097
(10), (8)	9.22(−7)	1.94(−59)	8.36(−644)	11.094
(10), (11)	9.22(−7)	3.17(−61)	5.06(−687)	11.490
(10), (12)	9.22(−7)	2.54(−61)	1.62(−702)	11.752
(10), (13)	9.22(−7)	1.40(−63)	3.11(−752)	12.120
$h(u, v) = 1 + u + v + v^2$				
(2)	4.79(−6)	2.08(−40)	2.62(−315)	8.000
(10), (5)	4.79(−6)	2.37(−42)	4.69(−351)	8.503
(10), (6)	4.79(−6)	5.39(−46)	9.44(−406)	9.006
(10), (7)	4.79(−6)	2.77(−48)	2.42(−473)	10.064
(10), (8)	4.79(−6)	2.93(−53)	1.80(−575)	11.061
(10), (11)	4.79(−6)	4.25(−55)	2.28(−616)	11.442
(10), (12)	4.79(−6)	3.38(−55)	4.25(−632)	11.737
(10), (13)	4.79(−6)	1.08(−57)	2.48(−680)	12.056
$h(u, v) = 1 + u + v + (u + v)^2$				
(2)	2.72(−6)	5.04(−44)	7.01(−346)	8.000
(10), (5)	2.72(−6)	1.84(−45)	2.94(−378)	8.496
(10), (6)	2.72(−6)	2.60(−49)	1.38(−435)	8.979
(10), (7)	2.72(−6)	2.34(−51)	1.57(−505)	10.078
(10), (8)	2.72(−6)	2.68(−56)	4.73(−609)	11.054
(10), (11)	2.72(−6)	4.59(−58)	3.41(−650)	11.437
(10), (12)	2.72(−6)	3.70(−58)	1.58(−665)	11.710
(10), (13)	2.72(−6)	2.46(−60)	1.04(−713)	12.090
$h(u, v) = u + 1/(1 - v)$				
(2)	4.99(−6)	2.91(−40)	3.85(−314)	8.000
(10), (5)	4.99(−6)	3.32(−42)	8.15(−350)	8.503
(10), (6)	4.99(−6)	7.54(−46)	1.94(−404)	9.005
(10), (7)	4.99(−6)	3.81(−48)	5.80(−472)	10.063
(10), (8)	4.99(−6)	4.07(−53)	6.73(−574)	11.060
(10), (11)	4.99(−6)	6.04(−55)	1.26(−614)	11.441
(10), (12)	4.99(−6)	4.82(−55)	2.34(−630)	11.737
(10), (13)	4.99(−6)	1.73(−57)	2.82(−678)	12.063

ple-precision arithmetic. To check the theoretical order of convergence, we calculate the computational order of convergence (r_c) using the formula [9]

$$r_c = \frac{\log|f(x_k)/f(x_{k-1})|}{\log|f(x_{k-1})/f(x_{k-2})|}, \tag{55}$$

taking into consideration the last three approximations in the iterative process.

For demonstration, we use the following two examples (selected from [4]):

$$f_1(x) = e^{x^2+x\cos x-1}\sin\pi x + x\log(x\sin x+1), \quad \alpha = 0, \quad x_0 = 0.6, \quad \gamma = \gamma_0 = -0.1, \tag{56}$$

$$f_2(x) = \log(x^2-2x+2) + e^{x^2-5x+4}\sin(x-1), \quad \alpha = 1, \quad x_0 = 1.35, \quad \gamma = \gamma_0 = -0.1. \tag{57}$$

The errors $|x_k - \alpha|$ of approximations to the corresponding zeros of $f_1(x)$ and $f_2(x)$, and computational order of convergence (r_c) are given in Tables 1 and 2, where $A(-h)$ denotes $A \times 10^{-h}$. The results, for both categories of methods with and without memory, are produced with the same initial values of γ and γ_0. Comparing results displayed in Tables 1 and 2, we observe the higher accuracy in the successive approximations produced by the methods with memory. This is quite understood because of the higher order of such methods. Also, like the existing methods with memory the present methods show consistent convergence behavior. Though the computational order of convergence exhibited in the last column of tables is not so close to the theoretical value of order as compared to the methods without memory, still it is quite acceptable as a measure of convergence speed keeping in mind that methods with memory have somewhat complex structure dealing with information from two successive iterations. We conclude with the remark that in case of the methods with memory the higher order of convergence is attained without any additional function evaluations, which shows that such methods possess a very high computational efficiency.

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