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On the local convergence of fast two-step Newton-like methods for solving nonlinear equations

I.K. Argyros [a], S. Hilout [b,*]

[a] Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA

[b] Poitiers University, Laboratoire de Mathématiques et Applications, 86962 Futuroscope Chasseneuil Cedex, France

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ABSTRACT

A local convergence analysis is presented for a fast two-step Newton-like method (TSNLM) for solving nonlinear equations in a Banach space setting. The TSNLM unifies earlier methods such as Newton's, Secant, Newton-like, Chebyshev–Secant, Chebyshev–Newton, Steffensen, Stirling's and other single or multistep methods. Numerical examples and a comparative study of these methods validating our theoretical results are also given in the concluding section of this paper.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of an equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a non-empty, open, convex subset Ω of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Many problems in computational sciences can be brought in the form of Eq. (1.1) using mathematical modeling [1–3]. The solutions of these equations can rarely be found in closed form. Therefore, iterative processes are used to find such solutions. These processes are usually variants of Newton's methods (NMs) [2,4,5].

A classic iterative process for solving nonlinear equations is Chebyshev's method (see [6,7]):

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n - \dfrac{1}{2}F'(x_n)^{-1}F''(x_n)(y_n - x_n)^2, \quad n \geq 0. \end{cases}$$

* Corresponding author.
E-mail addresses: iargyros@cameron.edu (I.K. Argyros), said.hilout@math.univ-poitiers.fr (S. Hilout).

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This one-point iterative process depends explicitly on the two first derivatives of F (namely, $x_{n+1} = \psi(x_n, F(x_n), F'(x_n), F''(x_n)))$. In [6], Ezquerro and Hernández introduced some modifications of Chebyshev's method by reducing the number of evaluations only one of the first derivative of F (i.e., $x_{n+1} = \overline{\psi}(x_n, F'(x_n))$, but with third order of convergence. This method is called the Chebyshev–Newton-type method (CNTM), and it has the following form:

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = x_n + a(y_n - x_n) \\ x_{n+1} = x_n - \dfrac{1}{a^2}F'(x_n)^{-1}((a^2 + a - 1)F(x_n) + F(z_n)), \quad n \geq 0, \end{cases}$$

where $F'(x)(x \in \Omega)$ is the Fréchet-derivative of F. Semilocal convergence results are provided by Ezquerro and Hernández in [6].

Recently, in [8], we constructed a family of iterative processes free of derivatives as the classic Secant method (SM) [9]. Then, we consider an approximation of the first derivative of F from a divided difference of first order (i.e., $F'(x_n) \approx [x_{n-1}, x_n, F]$), where $[x, y; F]$ is a divided difference of order 1 for the operator F at the points $x, y \in \Omega$. So, we introduced the Chebyshev–Secant-type method (CSTM)

$$\begin{cases} x_{-1}, \quad x_0 \in \Omega, \\ y_n = x_n - B_n^{-1}F(x_n), \quad B_n = [x_{n-1}, x_n; F], \\ z_n = x_n + a(y_n - x_n), \\ x_{n+1} = x_n - B_n^{-1}(bF(x_n) + cF(z_n)), \quad k \geq 0, \end{cases}$$

where a, b, c are non-negative parameters to be chosen so that sequence $\{x_n\}$ converges to $x^\star$. We provided semilocal convergence result for the CSTM. Note that, if $a = 0, b = c = 1/2$, and $y_n = x_{n+1}$, then the CSTM is reduced to the SM.

Relevant works on the SM using Hölder or Lipschitz-type conditions are given by Bosarge and Falb [10], Döring [11], Dennis [12], Potra [13], Argyros [1,14–16,2], Hernández et al. [17] and others [9,5,18] (see also studies in [19,20,3,10,21,12,22,23,13,24,25]).

In this study, we unify these methods by considering the TSNLM as follows:

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - A_n^{-1}F(x_n), \quad A_n = A(x_n), \\ z_n = x_n + a(y_n - x_n), \\ x_{n+1} = x_n - A_n^{-1}(bF(x_n) + cF(z_n)), \quad n \geq 0. \end{cases}$$

Here, $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x \in \Omega)$ which is the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. Clearly, the NM, SM, CNTM and CSTM are special cases of the TSNLM.

The semilocal convergence issue is, based on the information around an initial point, to give criteria ensuring the convergence of the TSNLM. The semilocal convergence of the TSNLM has been given by us in [20]. In the present study, we are concerned with the local convergence analysis of the TSNLM. The local convergence analysis issue for the TSNLM is, based on the information around a solution, to find estimates of the radii of convergence balls. Local convergence results were not given for the CNTM and the CSTM.

The paper is organized as follows. Section 2 contains the local convergence analysis of the TSNLM, when the Fréchet derivatives are Lipschitz continuous around a solution. Numerical examples can be found in the concluding section, Section 3.

2. Local convergence analysis of the TSNLM

Let $U(x, r)$ and $\overline{U}(x, r)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center x and radius $r > 0$. The local convergence analysis of the TSNLM is based on the following conditions.

($\mathcal{S}_1$) $F : \Omega \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ is Fréchet-differentiable, $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x \in \Omega)$, and there exist $x^\star \in \Omega, K > 0, L > 0, M \geq 0, \mu \geq 0, \ell \geq 0$ such that, for all $x, y \in \Omega$,

$$F(x^\star) = 0, \quad A(x^\star)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.1}$$

$$\|A(x^\star)^{-1}(F'(x) - F'(y))\| \leq K\|x - y\|, \tag{2.2}$$

$$\|A(x^\star)^{-1}(F'(x) - A(x))\| \leq M\|x - x^\star\| + \mu, \tag{2.3}$$

$$\|A(x^\star)^{-1}(A(x^\star) - A(x))\| \leq L\|x - x^\star\| + \ell. \tag{2.4}$$

($\mathcal{S}_2$)

$$U(x^\star, R_1) \subseteq \Omega, \tag{2.5}$$

for some $R_1 > 0$ to be given later in Lemma 2.1 and (2.26).

($\mathcal{S}_3$)

$$\ell + \mu < 1. \tag{2.6}$$

($\mathcal{S}_4$) For $a \in [0, 1]$, $b \geq 0$ and $c > 0$ given in the TSNLM, we suppose that

$$(1 - a)c = 1 - b.$$

We need some auxiliary results on the zeros of functions. With the above notation, we have the following lemmas.

Lemma 2.1. *Let us define functions ψ_0 on $[0, \infty)$ and ψ on $\left[0, \frac{1-\ell}{L}\right)$ by*

$$\psi_0(r) = \frac{K}{2} r^2 + (M + L)r + \ell + \mu - 1 \tag{2.7}$$

and

$$\begin{aligned} \psi(r) = {} & 4\varpi^2(r)(1 - ac)\rho(r) + 2\varpi(r)(1 - b)Ka(\rho(r) + 2\varpi(r))r \\ & + Ka^2 c(\rho(r) + 2\varpi(r))\rho(r)r + 2ac\varpi(r)\rho(r)^2 - 8\varpi(r)^3, \end{aligned} \tag{2.8}$$

where ρ, ϖ are functions defined on $[0, \infty)$ by

$$\rho(r) = 2Mr + 2\mu + Kr^2 \quad \text{and} \quad \varpi(r) = 1 - (Lr + \ell). \tag{2.9}$$

Then, the following assertions hold.

(a) *Function ψ_0 has a unique positive zero R_0 given by*

$$R_0 = \frac{2(1 - (\ell + \mu))}{M + L + \sqrt{(M + L)^2 + 2K(1 - (\ell + \mu))}} \tag{2.10}$$

provided that M, L, and K are not all zero.

(b) *If*

$$(1 - \ell)|1 - ac|\mu + ac\mu^2 - (1 - \ell)^2 < 0 \tag{2.11}$$

and

$$\psi(R_0) > 0, \tag{2.12}$$

then, function ψ has a zero in $(0, R_0)$. Denote the minimal such zero by R.

Proof. (a) R_0 is well defined since M, L, K are not all zero and, by (2.6), $R_0 > 0$. Moreover, we have $\psi_0(R_0) = 0$.

(b) We have that

$$\psi(0) = 8(1 - \ell)\left((1 - \ell)|1 - ac|\mu + ac\mu^2 - (1 - \ell)^2\right) < 0, \tag{2.13}$$

by (2.6) and (2.11). Then, the existence of a zero for function ψ follows from intermediate value theorem, (2.12) and (2.13).

This completes the proof of Lemma 2.1. □

It is convenient for us to set

$$e_n = x_n - x^\star, \tag{2.14}$$

$$u_n = y_n - x^\star, \tag{2.15}$$

$$v_n = z_n - x^\star, \tag{2.16}$$

$$G_n = G_n(x^\star, x_n) = \int_0^1 F'(x_n + t(x^\star - x_n))dt, \tag{2.17}$$

$$Q_n = Q_n(x^\star, z_n) = \int_0^1 F'(z_n + t(x^\star - z_n))dt, \tag{2.18}$$

$$D_n = A_n^{-1}(A_n - G_n) \tag{2.19}$$

and

$$E_n = A_n^{-1}(G_n - Q_n). \tag{2.20}$$

We shall show the following Ostrowski-type representations for the TSNLM.

Lemma 2.2. *Assume that the TSNLM is well defined and that there exists $x^\star \in \Omega$, such that $F(x^\star) = 0$, $F(x^\star)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $(\mathcal{S}_4)$ holds. Then, the following assertions hold.*

$$F(x_n) = G_n e_n, \tag{2.21}$$
$$F(z_n) = Q_n v_n, \tag{2.22}$$
$$v_n = a u_n + (1-a) e_n, \tag{2.23}$$
$$u_n = D_n e_n, \tag{2.24}$$

and

$$e_{n+1} = \left((1-ac)D_n + (1-b)E_n + acE_nD_n + acD_n^2\right) e_n. \tag{2.25}$$

Proof. We have in turn that

$$\begin{aligned}
&F(x_n) = F(x_n) - F(x^\star) = G_n e_n,\\
&F(z_n) = F(z_n) - F(x^\star) = Q_n v_n,\\
&v_n - x^\star = e_n + a(y_n - x_n) = e_n + a(u_n - e_n) = a u_n + (1-a)e_n,\\
&u_n = e_n - A_n^{-1}F(x_n) = A_n^{-1}(A_n - G_n)e_n = D_n e_n,
\end{aligned}$$

and, by the TSNLM, $(\mathcal{S}_4)$, (2.14) and (2.21)–(2.24)

$$\begin{aligned}
e_{n+1} &= e_n - A_n^{-1}(bF(x_n) + cF(z_n))\\
&= A_n^{-1}(A_n e_n - (bG_n e_n + cQ_n v_n))\\
&= A_n^{-1}(A_n e_n - bG_n e_n - cQ_n(a u_n + (1-a)e_n))\\
&= A_n^{-1}(A_n e_n - bG_n e_n - (1-a)cQ_n e_n - acQ_n u_n)\\
&= A_n^{-1}(A_n e_n - bG_n e_n - (1-b)Q_n e_n - acQ_n u_n)\\
&= A_n^{-1}((A_n - G_n)e_n + (1-b)Q_n e_n + G_n e_n - bG_n e_n - acQ_n u_n)\\
&= A_n^{-1}((A_n - G_n)e_n + (1-b)(Q_n - G_n)e_n + ac(G_n - Q_n + A_n - G_n) - acA_n u_n)\\
&= A_n^{-1}((A_n - G_n)e_n + (1-b)(Q_n - G_n)e_n + ac(G_n - Q_n) + ac(A_n - G_n)) - acu_n\\
&= D_n e_n + (1-b)E_n e_n + acE_n u_n + acD_n u_n - acu_n\\
&= D_n e_n + (1-b)E_n e_n + acD_n e_n + acD_n^2 e_n - acD_n e_n\\
&= (1-ac)D_n e_n + (1-b)E_n e_n + acE_nD_n e_n + acD_n^2 e_n.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

It is also convenient for us to define sequences on $[0, R_1)$, where

$$R_1 = \min\left\{R, \frac{1-\ell}{L}\right\}, \tag{2.26}$$

by

$$h_n = \frac{1}{2(1-(L\|e_n\| + \ell))}, \tag{2.27}$$

$$\begin{aligned}
\xi_n \;=\; &|1 - ac|h_n\varrho(\|e_n\|) + (1-b)Kah_n^2(\varrho(\|e_n\|) + 2\varpi(\|e_n\|))\|e_n\|\\
&+ Ka^2ch_n^2(\varrho(\|e_n\|) + 2\varpi(\|e_n\|))\varrho(\|e_n\|)\|e_n\| + ach_n^2\varrho(\|e_n\|)^2,
\end{aligned} \tag{2.28}$$

and

$$\xi(r) = \frac{\psi(r) + 8\varpi(r)^3}{8\varpi(r)^3}, \tag{2.29}$$

where ρ and ϖ are given in (2.8). Note that, by the definition of R and R_1, we have that

$$\xi(r) = 1 \tag{2.30}$$

and

$$\xi(R_1) \le 1. \tag{2.31}$$

We can now show the main local convergence result for the TSNLM.

Theorem 2.3. *Assume that conditions* $(\mathcal{S}_1)$–$(\mathcal{S}_4)$ *hold. Then, sequence* $\{x_n\}$ *generated by the TSNLM started at* $x_0 \in U(x^\star, R_1)$ *is well defined, remains in* $U(x^\star, R_1)$ *for all* $n \geq 0$, *and converges to* $x^\star$. *Moreover, the following error estimates hold:*

$$\|e_{n+1}\| \leq \xi_n \|e_n\| < \xi(R_1)\|e_n\| \leq \|e_n\|. \tag{2.32}$$

Proof. We shall use induction to show (2.32). We shall first show that (2.32) holds for $n = 0$ and $y_0, z_0, x_1 \in U(x^\star, R_1)$. By hypothesis, $x_0 \in U(x^\star, R_1)$. Using (2.4) and the definition of R_1, we get

$$\|A(x^\star)^{-1}(A(x^\star) - A(x_0))\| \leq L\|x_0 - x^\star\| + \ell < LR_1 + \ell < 1. \tag{2.33}$$

It follows from (2.33) and the Banach Lemma on invertible operators [2,4] that $A_0^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\|A_0^{-1}A(x^\star)\| \leq \frac{1}{1 - (L\|e_0\| + \ell)} < \frac{1}{1 - (LR_1 + \ell)}. \tag{2.34}$$

Thus, y_0 is well defined. It follows from (2.24), for $n = 0$, (2.2), (2.3), (2.34), and the definition of R_0 that

$$\begin{aligned}
\|u_0\| &\leq \|A_0^{-1}A(x^\star)\| \|A(x^\star)^{-1}(A(x_0) - G_0)\| \|e_0\| \\
&\leq \|A_0^{-1}A(x^\star)\| \left(\|A(x^\star)^{-1}(A(x_0) - F'(x_0))\| + \|A(x^\star)^{-1}(F'(x_0) - G_0)\|\right) \|e_0\| \\
&\leq \frac{1}{1 - (L\|e_0\| + \ell)} \left(M\|e_0\| + \mu + \frac{K}{2}\|e_0\|^2\right) \|e_0\| \\
&= \frac{\rho(\|e_0\|)}{2\varpi(\|e_0\|)}\|e_0\| \leq \frac{\rho(R_0)}{2\varpi(R_0)}\|e_0\| \leq \|e_0\| < R_1.
\end{aligned} \tag{2.35}$$

That is, $y_0 \in U(x^\star, R_1)$. The vector z_0 is a linear combination of x_0, y_0, and $U(x^\star, R_1) \subseteq \Omega$. Then, using (2.23), for $n = 0$, we get

$$\|v_0\| \leq a\|u_0\| + (1 - a)\|e_0\| \leq \|e_0\| < R_1. \tag{2.36}$$

That is, $z_0 \in U(x^\star, R_1)$. We need to find upper bounds on the norms $\|D_0\|$ and $\|E_0\|$. We have in turn using the TSNLM, (2.2), (2.3), (2.34), (2.19) and (2.20) (for $n = 0$)

$$\|D_0\| \leq \|A_0^{-1}A(x^\star)\| \|A(x^\star)^{-1}(A_0 - G_0)\| \leq \frac{\rho(\|e_0\|)}{2\varpi(\|e_0\|)} \tag{2.37}$$

and

$$\begin{aligned}
\|E_0\| &\leq \|A_0^{-1}A(x^\star)\| \|A(x^\star)^{-1}(G_0 - Q_0)\| \\
&\leq \frac{K}{\varpi(\|e_0\|)} \int_0^1 (1 - t)\|z_0 - x_0\| dt \\
&\leq \frac{K\|z_0 - x_0\|}{2\varpi(\|e_0\|)} \leq \frac{Ka\|y_0 - x^\star + x^\star - x_0\|}{2\varpi(\|e_0\|)} \leq \frac{Ka(\|u_0\| + \|e_0\|)}{2\varpi(\|e_0\|)} \\
&\leq Ka\left(\frac{\rho(\|e_0\|)}{2\varpi(\|e_0\|)} + 1\right)\|e_0\| \\
&\leq \frac{KaM\|e_0\|^2}{2\varpi(\|e_0\|)} + \frac{Ka\mu\|e_0\|}{2\varpi(\|e_0\|)^2} + \frac{K^2a\|e_0\|^3}{4\varpi(\|e_0\|)^2} + \frac{Ka\|e_0\|}{2\varpi(\|e_0\|)} \\
&\leq \frac{Ka}{4\varpi(\|e_0\|)^2}(\rho(\|e_0\|) + 2\varpi(\|e_0\|))\|e_0\|.
\end{aligned} \tag{2.38}$$

Using (2.37), (2.38), (2.25), (2.27) and (2.28) (for $n = 0$), we obtain in turn that

$$\begin{aligned}
\|e_1\| &\leq |1 - ac|\|D_0\|\|e_0\| + (1 - b)\|E_0\|\|e_0\| + ac\|E_0\|\|D_0\|\|e_0\| + ac\|D_0\|^2\|e_0\| \\
&\leq \frac{|1 - ac|}{2\varpi(\|e_0\|)}\rho(\|e_0\|)\|e_0\| + \frac{(1 - b)Ka}{4\varpi(\|e_0\|)^2}(\rho(\|e_0\|) + 2\varpi(\|e_0\|))\|e_0\|^2 \\
&\quad + \frac{acKa}{4\varpi(\|e_0\|)^2}(\rho(\|e_0\|) + 2\varpi(\|e_0\|))\frac{\rho(\|e_0\|)\|e_0\|^2}{2\varpi(\|e_0\|)} + \frac{ac}{4\varpi(\|e_0\|)^2}\rho(\|e_0\|)\|e_0\| \\
&\leq |1 - ac|h_0\rho(\|e_0\|)\|e_0\| + (1 - b)Kah_0^2(\rho(\|e_0\|) + 2\varpi(\|e_0\|))\|e_0\|^2 \\
&\quad + Ka^2ch_0^3(\rho(\|e_0\|) + 2\varpi(\|e_0\|))\rho(\|e_0\|)\|e_0\|^2 + ach_0^2\rho(\|e_0\|)^2\|e_0\| \\
&= \xi_0\|e_0\|
\end{aligned}$$

$$\begin{aligned}&< \frac{1}{8\varpi(R_1)^3}(4\varpi^2(R_1)|1-ac|\rho(R_1)+2\varpi(R_1)(1-b)Ka(\rho(R_1)+2\varpi(R_1))R_1\\&\quad +Ka^2c(\rho(R_1)+2\varpi(R_1))\rho(R_1)R_1+2ac\varpi(R_1)\rho(R_1)^2)\|e_0\|\\&\le \xi(R_1)\|e_0\|\le\|e_0\|<R_1.\end{aligned} \tag{2.39}$$

Hence, we deduce that $x_1 \in U(x^\star, R_1)$. Suppose that $\{x_k\}$ $(0 \le n \le k)$ is well defined and $x_n \in U(x^\star, R_1)$, where k is a fixed natural number. By simply exchanging $x_0, y_0, z_0, A_0, G_0, Q_0, h_0, H_0, \xi_0$ by $x_k, y_k, z_k, A_k, G_k, Q_k, h_k, H_k, \xi_k$, respectively, we deduce the following results.

(i_k) $A_k \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\|A_k^{-1}A(x^\star)\| \le \frac{1}{\varpi(\|e_k\|)} < \frac{1}{\varpi(R_1)}. \tag{2.40}$$

(ii_k) y_k is well defined, $y_k \in U(x^\star, R_1)$, and

$$\|u_k\| \le \frac{\rho(\|e_k\|)\|e_k\|}{2\varpi(\|e_k\|)} < \frac{\rho(R_1)}{2\varpi(R_1)}\|e_k\| \le \|e_k\| < R_1. \tag{2.41}$$

(iii_k) z_k is well defined, $z_k \in U(x^\star, R_1)$;
(iv_k) x_{k+1} is well defined and

$$\|e_{k+1}\| \le \xi_k\|e_k\| < \xi(R_1)\|e_k\| \le \|e_k\| < R_1. \tag{2.42}$$

Hence, (2.32) holds for all n and $x_{k+1} \in U(x^\star, R_1)$. The induction is completed. By letting $k \longrightarrow \infty$ in (2.42), we obtain $\lim_{k\to\infty} x_k = x^\star$. This completes the proof of Theorem 2.3. □

Remark 2.4. In the special case of the TSNLM, i.e., if $A(x) = F'(x)$ $(x \in \Omega)$, $M = \mu = \ell = 0$, we have, by (2.32),

$$\begin{aligned}\|e_{n+1}\| &\le |1-ac|h_nK\|e_n\|^3 + (1-b)Kah_n^2(K\|e_n\|^2+2(1-L\|e_n\|))\|e_n\|^2\\&\quad + K^2a^2ch_n^3(K\|e_n\|^2+2(1-L\|e_n\|))\|e_n\|^4 + ach_n^2K^2\|e_n\|^5.\end{aligned} \tag{2.43}$$

Moreover, if $a = b = 1$, (2.43) reduces to

$$\|e_{n+1}\| \le \xi_n^1\|e_n\|^3 \le \xi^1(R_1)\|e_n\|^3, \tag{2.44}$$

where

$$\begin{aligned}\xi_n^1 &= |1-c|h_nK + K^2ch_n^3(K\|e_n\|^2+2(1-L\|e_n\|))\|e_n\| + ch_n^2K^2\|e_n\|^2\\&\le \xi^1(R_1) = |1-c|\frac{K}{2\varpi(R_1)} + \frac{K^2c}{8\varpi(R_3)^3}(KR_1^2+2(1-LR_1))R_1 + \frac{cK^2R_1^2}{4\varpi(R_1)^2}.\end{aligned} \tag{2.45}$$

Hence, we deduce that the NTM

$$\begin{aligned}y_n &= x_n - F'(x_n)^{-1}F(x_n)\\x_{n+1} &= y_n - cF'(x_n)^{-1}F(y_n), \quad c > 0\end{aligned}$$

is of convergence order 3, provided that (2.12) holds. Several other choices for A, a, b, c are also possible [2,26,3,6].

3. Numerical examples

We present in this section some numerical examples for validating the theoretical results of Section 2. We show some situations where the results provided in the paper can be applied.

Example 3.1 ([2,26]). Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$. Define function F on $\Omega = [-1, 1]$, given by

$$F(x) = e^x - 1. \tag{3.1}$$

Then, for $x^\star = 0$, using (3.1), we have $F(x^\star) = 0$ and $F'(x^\star) = e^0 = 1$. Moreover, hypotheses $(\mathcal{S}_1)$–$(\mathcal{S}_4)$ hold for $A(x) = F'(x)$ $(x \in \Omega)$ and

$$K = e = 2.718281828, \qquad L = e - 1 = 1.718281828, \qquad M = \mu = \ell = 0.$$

Function ψ_0 is given by

$$\psi_0(r) = 1.359140914r^2 + 1.718281828r - 1.$$

Table 1
Values of $\psi(R_0)$, R, and R_1.

a	b	c	$\psi(R_0)$	R	R_1
1	1	0.8333333333	0.5118934862	0.3612167951	0.3612167951
1	1	0.1	0.0614272183	0.4090368017	0.4090368017
0.1	0.1	1	0.1337025358	0.3695899743	0.3695899743
0.1	0.5	0.5555555556	0.0742791866	0.3941526077	0.3941526077
0.2	0.6	0.5	0.1256719449	0.3743666097	0.3743666097
0.4	0.7	0.5	0.2192215268	0.3493942691	0.3493942691
0.45	0.868	0.24	0.1140431053	0.3830908277	0.3830908277
0.8689	0.7859	1.633104500	1.021049270	0.3030576608	0.3030576608
0.9999	0.9123	877.00	538.7332546	0.2708407173	0.2708407173

By Lemma 2.1 (for example for $a = b = 1$ and $c = 5/6$), function ψ is given by

$$\psi(r) = -69.04763068r^2 + 46.67310733r^3 - 3.495236830r^4 - 4.422853941r^5 - 8 + 41.23876387r$$
$$R_0 = 0.4334005735, \qquad \psi(R_0) = 0.5118934862 \quad \text{and} \quad R = R_1 = 0.3612167951.$$

We give in Table 1 the values of $\psi(R_0)$, R, and R_1 for various values of parameters a, b, and c satisfying assumption $(\mathcal{S}_4)$, where $R_0 = 0.4334005735$.

Example 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, the space of continuous functions defined on $[0, 1]$ be equipped with the max norm, and $\Omega = \overline{U}(0, 1)$. Define function F on Ω by

$$F(h)(x) = h(x) - 5\int_0^1 x\theta h(\theta)^3 d\theta. \tag{3.2}$$

Note that $x^\star(t) = 0$ is zero of operator F. We also have

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\theta h(\theta)^2 u(\theta) d\theta \quad \text{for all } u \in \Omega. \tag{3.3}$$

Using (3.3), we have $F'(x^\star) = I_{\mathcal{X}}$ and

$$[(F'(x) - F'(y))w](s) = -15\int_0^1 st(x^2(t) - y^2(t))w(t)dt \quad \forall x, y, w \in \Omega. \tag{3.4}$$

In view of (3.4), we obtain that

$$\begin{aligned} \|F'(x) - F'(y)\| &\le 15\int_0^1 t\|x^2(t) - y^2(t)\|dt \\ &\le 15\int_0^1 t(\|x\| + \|y\|)\|x - y\|dt \le 15\|x - y\|, \quad \forall x, y \in \Omega \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|F'(x) - F'(x^\star)\| &\le 15\int_0^1 t\|x^2(t)\|dt \\ &\le 15\int_0^1 t\|x\|\|x - x^\star\|dt \le \frac{15}{2}\|x - x^\star\|, \quad \forall x \in \Omega. \end{aligned} \tag{3.6}$$

All hypotheses of Theorem 2.3 hold for $A(x) = F'(x)$ $(x \in \Omega)$, $x^\star = 0$, with $K = 15$, $L = 7.5$, and $M = \mu = \ell = 0$. Function ψ_0 is given by

$$\psi_0(r) = 7.5r^2 + 7.5r - 1.$$

By Lemma 2.1 (for example for $a = 0.2$, $b = 0.6$ and $c = 0.5$), function ψ is given by

$$\psi(r) = -990r^2 + 2304r^3 + 3015r^4 - 3.2 + 108r - 270r^5$$
$$R_0 = 0.1191391874, \qquad \psi(R_0) = 0.1120363027 \quad \text{and} \quad R = R_1 = 0.04888375116.$$

We give in Table 2 the values of $\psi(R_0)$, R, and R_1 for various values of parameters a, b, and c satisfying assumption $(\mathcal{S}_4)$, where $R_0 = 0.1191391874$.

Table 2
Values of $\psi(R_0)$, R, and R_1.

a	b	c	$\psi(R_0)$	R	R_1
0.1	1	0	2.9×10^{-11}	0.1333333333	0.1333333333
0.1	0.1	1	0.1240155518	0.04029188762	0.04029188762
0.1	0.5	0.5555555556	0.06889752882	0.07361139239	0.07361139239
0.2	0.6	0.5	0.1120363027	0.04888375116	0.04888375116
0.4	0.7	0.5	0.1761556092	0.01302679070	0.01302679070
0.45	0.868	0.24	0.08865523446	0.06596454142	0.06596454142
0.8	0.7	1.5	0.4819296992	0.1433430882	0.1333333333
0.99	0.98	2	0.3445246534	0.08564610417e	0.08564610417

In the last example, we are not interested in checking if the hypotheses of Theorem 2.3 are satisfied or not, but in comparing the numerical behavior of a special case of the TSNLM with the earlier CSTM.

Example 3.3. In this example, we present an application of the previous analysis to the significant Chandrasekhar integral equation [20]:

$$x(s) = 1 + \frac{s}{4}x(s)\int_0^1 \frac{x(t)}{s+t}dt, \quad s \in [0, 1]. \tag{3.7}$$

Integral equations of the form (3.7) are very important and appear in the areas of neutron transport, radiative transfer, and the kinetic theory of gasses. We refer the interested reader to [2,26,21], where a detailed description of the physical phenomenon described by (3.7) can be found. Note that solving (3.7) is equivalent to solving $F(x) = 0$, where $F : C[0, 1] \to C[0, 1]$ and

$$[F(x)](s) = x(s) - 1 - \frac{s}{4}x(s)\int_0^1 \frac{x(t)}{s+t}dt, \quad s \in [0, 1]. \tag{3.8}$$

To obtain a numerical solution of (3.7), we first discretize the problem, and we find it convenient by testing several number of nodes to approach the integral by a Gauss–Legendre numerical quadrature with eight nodes (see [26,6,17,9]):

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{8} w_j f(t_j),$$

where

$$\begin{array}{llll} t_1 = 0.019855072, & t_2 = 0.101666761, & t_3 = 0.237233795, & t_4 = 0.408282679, \\ t_5 = 0.591717321, & t_6 = 0.762766205, & t_7 = 0.898333239, & t_8 = 0.980144928, \\ w_1 = 0.050614268, & w_2 = 0.111190517, & w_3 = 0.156853323, & w_4 = 0.181341892, \\ w_5 = 0.181341892, & w_6 = 0.156853323, & w_7 = 0.111190517, & w_8 = 0.050614268. \end{array}$$

If we denote $x_i = x(t_i)$, $i = 1, 2, \ldots, 8$, Eq. (3.7) is transformed into the following nonlinear system:

$$x_i = 1 + \frac{x_i}{4}\sum_{j=1}^{8} a_{ij}x_j, \quad i = 1, 2, \ldots, 8,$$

where $a_{ij} = \frac{t_i w_j}{t_i + t_j}$.

Denote now $\overline{x} = (x_1, x_2, \ldots, x_8)^T$, $\overline{1} = (1, 1, \ldots, 1)^T$, $A = (a_{ij})$, and write the last nonlinear system in matrix form:

$$\overline{x} = \overline{1} + \frac{1}{4}\overline{x} \odot (A\overline{x}), \tag{3.9}$$

where $\odot$ represents the inner product. We shall compare the CSTM with the Steffensen-type method (STTM) defined by

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - [x_n, G(x_n); F]^{-1}F(x_k) \\ z_n = x_n + a(y_n - x_n) \\ x_{n+1} = x_n - [x_n, G(x_n); F]^{-1}(bF(x_n) + cF(z_n)), \quad n \geq 0, \end{cases}$$

where $G : \mathcal{X} \to \mathcal{X}$. Clearly, the STTM reduces to the TSNLM if $A(x) = [x, G(x); F]$ $(x \in \Omega)$. Note that the CSTM and the STTM are useful alternatives of the CNTM, especially in cases where the computation of $F''(x_n)$ of $F'(x_n)^{-1}$ is expensive

Table 3
The comparison results of $\|\bar{x}_{n+1} - \bar{x}_n\|$ for Example 3.3.

n	STTM ($a = b = c = 1$)	CSTM ($a = b = c = 1$)
1	2.49e−01	2.49e−01
2	5.69e−04	6.14e−04
3	3.40e−12	5.76e−07
4	4.34e−37	1.91e−15
5	6.36e−112	4.34e−30
6	1.54e−336	8.04e−62
ρ	2.813536145	2.16697778

or impossible, or their analytic representation is unavailable. The computational order of convergence (COC) is shown in Table 3 for various methods. Here, the COC is defined by [7]

$$\rho \approx \ln\left(\frac{\|\bar{x}_{n+1} - \bar{x}^\star\|_\infty}{\|\bar{x}_n - \bar{x}^\star\|_\infty}\right) \Big/ \ln\left(\frac{\|\bar{x}_n - \bar{x}^\star\|_\infty}{\|\bar{x}_{n-1} - \bar{x}^\star\|_\infty}\right), \quad n \in \mathbb{N}.$$

The last line in Table 3 shows that the COC of the STTM and the CSTM is, respectively, close to 2.813536145 and 2.16697778. Hence, the STTM is faster than the CSTM in this case. Set $G(x) = x$. If we choose $\bar{x}_0 = (1, 1, \dots, 1)^T$ and $\bar{x}_{-1} = (0.99, 0.99, \dots, 0.99)^T$. Assume that sequence $\{\bar{x}_n\}$ is generated by the STTM (or the CSTM) with different choices of parameters a, b, and c. Table 3 gives the comparison results for $\|\bar{x}_{n+1} - \bar{x}_n\|$ equipped with the max-norm for this example, which show that the STTM is faster than the CSTM. Here, we perform the computations by using Maple 11 on a computer equipped with Intel® Core™ i3-2310M CPU.

4. Conclusion

A local convergence analysis of the TSNLM is provided for approximating a locally unique solution of a nonlinear equation in a Banach space setting. We have used a combination of Lipschitz and center-Lipschitz conditions and a more general linear operator A than before [26,8,20].

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