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The general Jacobi matrix method for solving some nonlinear ordinary differential equations

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ABSTRACT

In this paper, we obtain the approximate solutions for some nonlinear ordinary differential equations by using the general Jacobi matrix method. Explicit formulae which express the Jacobi expansion coefficients for the powers of derivatives and moments of any differentiable function in terms of the original expansion coefficients of the function itself are given in the matrix form. Three test problems are discussed to illustrate the efficiency of the proposed method.

1. Introduction

Approximations by orthogonal polynomials have played a main role in the development of physical sciences, engineering and mathematical analysis. Techniques for finding approximate solutions for differential equations, based on the classical orthogonal polynomials, are popularly known as spectral methods [1–5]. Approximating functions in spectral methods are related to polynomial solutions of eigenvalue problems in ordinary differential equations, known as Sturm–Liouville problems. In the past few decades, there has been growing interest in this subject. As a matter of fact, the spectral methods provide a competitive alternative to other standard approximation techniques, for a large variety of problems. The first applications of this method were concerned with the investigation of periodic solutions of boundary value problems using trigonometric polynomials. Expansions in orthogonal basis functions were performed, due to their high accuracy and flexibility in computations, see, for instance, the intensive seminal work of Doha [6–9] and Doha and Ahmed [10]. It is worth to be noted that one of the important and interesting implementations of this technique is a matrix presentation. For example Sezer and Kaynak presented a Chebyshev matrix method to solve linear differential equations [11]. Also in [12] Sezer and Dogan used this method for obtaining approximate solutions of linear and non-linear Fredholm integral equations. Furthermore Köroğlu [13] proposed this method for solving Fredholm integro-differential equations. Also we refer the interested reader to [14–18] for some semi-analytical approaches for the numerical solution of nonlinear differential equations.

In this paper, using the general Jacobi matrix method, we obtain the approximate solutions of some nonlinear ordinary differential equations. In Section 2 we introduce the Jacobi polynomials and their properties. Section 3 is allocated to obtain

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the matrix relations for the powers of derivatives and moments of the expansion of Jacobi polynomials in terms of original Jacobi polynomials. In Section 4 we report our numerical finding to demonstrate the accuracy and efficiency of the proposed method by considering three test problems. Suppose $R_{i,j}(x)$ are functions that have Taylor series.

In this paper we are interested to find the Jacobi approximations for solutions of the class of some nonlinear ordinary differential equations in the form

$$\begin{aligned}&\sum_{k=1}^{s_j} R_{k,j}(y^{(j)})^k + \sum_{k=1}^{s_{j-1}} R_{k,j-1}(y^{(j-1)})^k + \cdots + \sum_{k=1}^{s_0} R_{k,0}(y^{(0)})^k = f(x),\\ &-1 \leqslant x \leqslant 1, \quad j \geqslant 0,\\ &s_p > 0, \quad p = 0,\ldots,j,\end{aligned} \tag{1}$$

subject to

$$\sum_{i=0}^{j} b_{i,l} y^{(l)}(a_i) = \theta_l, \quad a_i \in [-1,1], \quad l = 0,\ldots,j-1, \quad i = 0,\ldots,j. \tag{2}$$

Firstly, we write Taylor series of $R_{k,j}(x)$ in the form:

$$R_{k,j}(x) \simeq \sum_{i=0}^{m_j} r_{k,i}^{(j)} x^i, \tag{3}$$

where $m_p > 0$, $(p = 0,1,\ldots,j)$ are integer numbers.

Putting (3) in (1) yields:

$$\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} r_{k,i}^{(j)} x^i (y^{(j)}(x))^k + \sum_{k=1}^{s_{j-1}}\sum_{i=0}^{m_{j-1}} r_{k,i}^{(j-1)} x^i (y^{(j-1)}(x))^k + \cdots + \sum_{k=1}^{s_0}\sum_{i=0}^{m_0} r_{k,i}^{(0)} x^i (y(x))^k \simeq f(x). \tag{4}$$

Now suppose we consider the Jacobi approximation of the exact solution of (1) in the form:

$$y(x) \simeq \sum_{i=0}^{n} \delta_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(\Delta_n^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(x), \tag{5}$$

then $x^i(y^{(j)}(x))^k (i,j \geqslant 0, k > 0)$ could be written as:

$$x^i (y^{(j)}(x))^k \simeq \sum_{t=0}^{n} \delta_t^{(i,j,k)} P_t^{(\alpha,\beta)}(x) = \left(\Delta_{0:n}^{(i,j,k)}\right)^T P^{(\alpha,\beta)}(x), \tag{6}$$

where

$$\Delta_{p:q}^{(i,j,k)} = [\delta_p^{(i,j,k)}, \delta_{p+1}^{(i,j,k)}, \ldots, \delta_q^{(i,j,k)}]^T; \quad (q > p); \;\; \Delta_{0:q}^{(i,j,k)} = \Delta_q^{(i,j,k)}, \tag{7}$$

and

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), P_2^{(\alpha,\beta)}(x), \ldots, P_n^{(\alpha,\beta)}(x)\right]^T. \tag{8}$$

The main goal of this paper is obtaining the coefficients $\delta_t^{(0,0,1)}; t = 0, 1 \ldots n$ (the coefficients of Jacobi approximation (5)). In the next sections we study the method of obtaining unknowns $\delta_t^{(0,0,1)}$. However before that we introduce some properties of Jacobi polynomials.

2. Some properties of Jacobi polynomials

The Jacobi polynomials associated with the real parameters $(\alpha > -1, \beta > -1)$, are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$, $(n = 0,1,2,\ldots)$, satisfying the orthogonality relation [19];

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{9}$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. \tag{10}$$

These polynomials are eigenfunctions of the following singular Sturm–Liouville equation [19]:

$$(1-x^2)\Phi''(x) + [\beta - \alpha - (\beta+\alpha+2)x]\Phi'(x) + n(n+\beta+\alpha+1)\Phi(x) = 0. \tag{11}$$

Therefore the explicit solution of (11) is of the form [20]:

$$p_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k. \tag{12}$$

Two well-known recurrence formulae that play a main role in this paper are [20]:

a. Three-term recursion formula:

$$xP_n^{(\alpha,\beta)}(x) = D_n^{(\alpha,\beta)}P_{n+1}^{(\alpha,\beta)} + E_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x) + F_n^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x), \tag{13}$$

where

$$D_n^{(\alpha,\beta)} = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}, \quad E_n^{(\alpha,\beta)} = \frac{(\beta^2-\alpha^2)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$F_n^{(\alpha,\beta)} = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

b. First derivative recursion formula:

$$P_n^{(\alpha,\beta)}(x) = A_n^{(\alpha,\beta)}(P_{n+1}^{(\alpha,\beta)}(x))' + B_n^{(\alpha,\beta)}(P_n^{(\alpha,\beta)}(x))' - C_n^{(\alpha,\beta)}(P_{n-1}^{(\alpha,\beta)}(x))', \quad n \geqslant 1, \tag{14}$$

where

$$A_n^{(\alpha,\beta)} = \frac{1}{n+1}D_n^{(\alpha,\beta)},$$
$$B_n^{(\alpha,\beta)} = \frac{-2}{\alpha+\beta}E_n^{(\alpha,\beta)},$$
$$C_n^{(\alpha,\beta)} = \frac{-1}{n+\alpha+\beta}F_n^{(\alpha,\beta)}.$$

3. The new matrix method for obtaining the Jacobi approximate solution of (1) and (2)

In this section we want to find coefficients $\delta_t^{(i,j,k)}$ (or $\Delta_n^{(i,j,k)}$) in (6) respect to $\delta_t^{(0,0,1)}$, (or $\Delta_n^{(0,0,1)}$) in (5). For this purpose we solve this problem step by step as given in Table 1 ($k > 0$ and $i, j \geqslant 0$ are integer numbers).

To study Step 1, we consider the following theorem.

Theorem 1. *If we consider the Jacobi approximation of $y^{(j)}(x)$ given in (6) in the following form*

$$y^{(j)}(x) \simeq \sum_{i=0}^{n} \delta_i^{(0,j,1)}P_i^{(\alpha,\beta)}(x) = \left(\Delta_n^{(0,j,1)}\right)^T P^{(\alpha,\beta)}(x); \quad j = 0,1,2,\ldots, \tag{15}$$

then for obtaining $\delta_i^{(0,j,1)}$(or $\Delta_n^{(0,j,1)}$) respect to $\delta_t^{(0,0,1)}$, (or $\Delta_n^{(0,0,1)}$) we have:

$$\Delta_n^{(0,j,1)} = D^j\Delta_n^{(0,0,1)}, \tag{16}$$

where

$$D = \begin{bmatrix} 0 & E^{-1} \\ 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \tag{17}$$

and

$$E_{pq} = \begin{cases} \frac{-2(p+\alpha+1)(p+\beta+1)}{(\alpha+\beta+p+1)(\alpha+\beta+2p+2)(\alpha+\beta+2p+3)}, & p = q-2, \\ \frac{2(\alpha+\beta+p)}{(\alpha+\beta+2p-1)(\alpha+\beta+2p)}, & p = q, \\ \frac{2(\alpha-\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+2)}, & p = q-1, \\ 0, & o.w. \end{cases} \quad (1 \leqslant p,q \leqslant n),$$

Table 1
Steps of computing $\delta_t^{(i,j,k)}$ respect to $\delta_t^{(0,0,1)}$.

Step	i	j	k	Proof
1	0	Free	1	Theorem 1
2	Free	Free	1	Theorem 2
3	Free	Free	Free	Theorem 3

Proof. Let us prove (16) for $j = 1$, then by repeating this process we can obtain matrix form (16) for any $j \geqslant 2$. For this purpose taking derivative both sides of (5), there exist $\delta_i^{(0,1,1)}, i = 0, 1, \ldots n$ such that:

$$y'(x) \simeq \frac{d}{dx}\left(\sum_{i=0}^{n} \delta_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x)\right) = \sum_{i=0}^{n} \delta_i^{(0,1,1)} P_i^{(\alpha,\beta)}(x). \tag{18}$$

But from (18) we obtain

$$\sum_{i=0}^{n} \delta_i^{(0,0,1)} \frac{d}{dx} P_i^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} \delta_i^{(0,1,1)} P_i^{(\alpha,\beta)}(x). \tag{19}$$

Combining (14) and (19) concludes:

$$\sum_{i=0}^{n} \delta_i^{(0,0,1)} \left(P_i^{(\alpha,\beta)}(x)\right)' = \sum_{i=1}^{n} \delta_i^{(0,1,1)} \left(A_i^{(\alpha,\beta)} \left(P_{i+1}^{(\alpha,\beta)}(x)\right)' + B_i^{(\alpha,\beta)} \left(P_i^{(\alpha,\beta)}(x)\right)' - C_i^{(\alpha,\beta)} \left(P_{i-1}^{(\alpha,\beta)}(x)\right)'\right). \tag{20}$$

Now simplifying the right summation of (20) and then equating the coefficients of the same derivatives of Jacobi polynomials of both sides of (20), we can obtain the following matrix relation:

$$E\Delta_{n-1}^{(0,1,1)} = \Delta_{1:n}^{(0,0,1)}, \tag{21}$$

where

$$E_{pq} = \begin{cases} \frac{-2(p+\alpha+1)(p+\beta+1)}{(\alpha+\beta+p+1)(\alpha+\beta+2p+2)(\alpha+\beta+2p+3)}, & p = q-2, \\ \frac{2(\alpha+\beta+p)}{(\alpha+\beta+2p-1)(\alpha+\beta+2p)}, & p = q, \\ \frac{2(\alpha-\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+2)}, & p = q-1, \\ 0, & o.w. \end{cases} \quad (1 \leqslant p, q \leqslant n),$$

$\Delta_{n-1}^{(0,1,1)}$ and $\Delta_{1:n}^{(0,0,1)}$ are defined in (7).

Eq. (21) shows the relationship between $\Delta_{n-1}^{(0,1,1)}$ and $\Delta_{1:n}^{(0,0,1)}$ but we shall find a formula that express the relationship between $\Delta_n^{(0,1,1)}$ and $\Delta_n^{(0,0,1)}$. For this purpose nothing the definition of $\Delta_{n-1}^{(0,1,1)}$ and $\Delta_{1:n}^{(0,0,1)}$, we introduce matrix D as follows (E is an upper triangular matrix and its diagonal elements is not zero):

$$D = \begin{bmatrix} 0 & E^{-1} \\ 0 & 0 \end{bmatrix}.$$

Now from (21) and definition of $\Delta_n^{(i,j,k)}$ it is easy to show that

$$\Delta_n^{(0,1,1)} = D\Delta_n^{(0,0,1)}.$$

Repeating this procedure for $y^{(j)}(x)$, $j = 2, 3, \ldots$ concludes:

$$\begin{aligned} &y^{(j)}(x) \simeq \sum_{i=0}^{n} \delta_i^{(0,j,1)} P_i^{(\alpha,\beta)}(x) = \left(\Delta_n^{(0,j,1)}\right)^T P^{(\alpha,\beta)}(x), \quad j = 2, 3, \ldots, \\ &\Delta_n^{(0,j,1)} = D^j \Delta_n^{(0,0,1)}. \qquad \square \end{aligned} \tag{22}$$

Now for solving the problem of Step 2 (The production of moments and derivatives) we consider the following theorem.

Theorem 2. *If we consider the Jacobi approximation of $x^i y^{(j)}(x)$ from (6) in the following form*

$$x^i y^{(j)}(x) \simeq \sum_{k=0}^{n} \delta_k^{(i,j,1)} P_k^{(\alpha,\beta)}(x) = \left(\Delta_n^{(i,j,1)}\right)^T P^{(\alpha,\beta)}(x), \tag{23}$$

then

$$\Delta_n^{(i,j,1)} = G^i D^j \Delta_n^{(0,0,1)}, \tag{24}$$

where

$$G_{pq} = \begin{cases} \frac{2(p+\alpha)(p+\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+1)}, & p = q-1, \\ \frac{\beta^2-\alpha^2}{(\alpha+\beta+2p+1)(\alpha+\beta+2p+2)}, & p = q, \\ \frac{2(p)(\alpha+\beta+p)}{(\alpha+\beta+2p)(\alpha+\beta+2p-1)}, & p = q+1, \\ 0, & o.w., \end{cases} \quad (1 \leqslant p, q \leqslant n), \tag{25}$$

and D is defined in (17).

Proof. If $i = 0$ and j be a free parameter then from Theorem 1, the proof of Theorem 2 is obvious. Now if we suppose $i = 1$ and j be a free parameter, then by j times derivative of both sides of (5) multiply by x and noting (6), there exist $\delta_k^{(1,0,1)}$ and $\delta_k^{(1,j,1)}, k = 0, 1, \ldots, n$ such that:

$$xy^{(j)}(x) \simeq x \sum_{k=0}^{n} \delta_k^{(0,j,1)} P_k^{(\alpha,\beta)}(x) \simeq \sum_{k=0}^{n} \delta_k^{(1,j,1)} P_k^{(\alpha,\beta)}(x), \tag{26}$$

then from (26) we obtain:

$$xy^{(j)}(x) \simeq \sum_{k=0}^{n} \delta_k^{(0,j,1)} x P_k^{(\alpha,\beta)}(x) \simeq \sum_{k=0}^{n} \delta_k^{(1,j,1)} P_k^{(\alpha,\beta)}(x). \tag{27}$$

Now combining (13) and (27), yields:

$$\sum_{k=0}^{n} \delta_k^{(0,j,1)} x P_k^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \delta_k^{(0,j,1)} \left(D_k^{(\alpha,\beta)} P_{k+1}^{(\alpha,\beta)} + E_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) + F_k^{(\alpha,\beta)} P_{k-1}^{(\alpha,\beta)}(x)\right) \simeq \sum_{k=0}^{n} \delta_k^{(1,j,1)} P_k^{(\alpha,\beta)}(x). \tag{28}$$

Equating coefficients of the same terms in both sides of (28), we have:

$$\underbrace{[\delta_0^{(0,j,1)}, \ldots, \delta_n^{(0,j,1)}]}_{[\Delta_n^{(0,j,1)}]^T} G^T \begin{bmatrix} P_0^{(\alpha,\beta)}(x) \\ P_1^{(\alpha,\beta)}(x) \\ \vdots \\ P_n^{(\alpha,\beta)}(x) \end{bmatrix} \simeq \underbrace{[\delta_0^{(1,j,1)}, \ldots, \delta_n^{(1,j,1)}]}_{[\Delta_n^{(1,j,1)}]^T} \begin{bmatrix} P_0^{(\alpha,\beta)}(x) \\ P_1^{(\alpha,\beta)}(x) \\ \vdots \\ P_n^{(\alpha,\beta)}(x) \end{bmatrix},$$

or noting the linear independence of Jacobi polynomials we get

$$\Delta_n^{(1,j,1)} = G\Delta_n^{(0,j,1)},$$

where

$$G_{pq} = \begin{cases} \frac{2(p+\alpha)(p+\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+1)}, & p = q-1, \\ \frac{\beta^2-\alpha^2}{(\alpha+\beta+2p+1)(\alpha+\beta+2p+2)}, & p = q, \\ \frac{2(p)(\alpha+\beta+p)}{(\alpha+\beta+2p)(\alpha+\beta+2p-1)}, & p = q+1, \\ 0, & o.w. \end{cases} \quad (1 \leqslant p, q \leqslant n), \tag{29}$$

Now if we repeat this procedure for obtaining the matrix relation of $x^i y^{(j)}(x)$; $i = 2, 3, \ldots$ then we get:

$$\Delta_n^{(i,j,1)} = G^i \Delta_n^{(0,j,1)}. \tag{30}$$

where $\Delta_n^{(i,j,1)}$ given in (6) satisfies in

$$x^i y^{(j)}(x) \simeq \sum_{k=0}^{n} \delta_k^{(i,j,1)} P_k^{(\alpha,\beta)}(x) = \left(\Delta_n^{(i,j,1)}\right)^T P^{(\alpha,\beta)}(x), \quad i = 2, 3, \ldots,$$

and finally combining (30) and (22) we obtain

$$\Delta_n^{(i,j,1)} = G^i D^j \Delta_n^{(0,0,1)}. \qquad \square$$

Finally we want to study the last Step (Step 3). To do this, we present Theorem 3 that computes $x^i[y^{(j)}(x)]^k$ as a linear combination of Jacobi polynomials.

Theorem 3. *If we consider the Jacobi approximation of $x^i[y^{(j)}(x)]^k$ in (6) in the following form*

$$x^i (y^{(j)}(x))^k \simeq \sum_{t=0}^{n} \delta_t^{(i,j,k)} P_t^{(\alpha,\beta)}(x) = \left(\Delta_n^{(i,j,k)}\right)^T P^{(\alpha,\beta)}(x),$$

then we have:

$$\Delta_n^{(i,j,k)} = \left(eG^i D^j \Delta_n^{(0,0,1)}\right)(P)^{k-1} L \left(\sum_{r=0}^{n} eG^r D^j \Delta_n^{(0,0,1)}\right)^{k-2}, \quad (k \geqslant 2), \tag{31}$$

where

$$e = [1, 1, \ldots, 1], \quad P = \sum_{r=0}^{n} \gamma_r, \quad \gamma_r = \sum_{s=r}^{n} E_s^{(\alpha,\beta,r)}, \quad L = \sum_{s=0}^{n} G^s D^j \Delta_n^{(0,0,1)},$$

$$E_s^{(\alpha,\beta,r)} = \sum_{k=0}^{s}(-1)^{s-k}B_k^{(\alpha,\beta,r)}\binom{s}{k}; \quad \alpha,\beta > -1, \quad 0 \leqslant s \leqslant r,$$

$$B_k^{(\alpha,\beta,s)} = 2^{-k}\binom{s+\alpha+\beta+k}{k}\binom{s+\alpha}{s-k}; \quad k = 0,1,2,\ldots,n,$$

and D, G are defined in (17) and (25) respectively.

Proof. The proof of this theorem is based on induction. Firstly we prove that (31) is true for $k = 2$ (base of induction). For this aim we use the following formula [21]:

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} E_j^{(\alpha,\beta,n)}x^j; \quad \alpha,\beta > -1,$$

and we have

$$\begin{aligned} x^i(y^{(j)}(x))^2 &\simeq x^i\left(\sum_{r=0}^{n}\delta_r^{(0,j,1)}P_r^{(\alpha,\beta)}(x)\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,1)}P_s^{(\alpha,\beta)}(x)\right) = \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=0}^{r}E_l^{(\alpha,\beta,r)}x^l\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,1)}P_s^{(\alpha,\beta)}(x)\right)\right) \\ &= \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=r}^{n}E_l^{(\alpha,\beta,r)}\right)x^r\left(\sum_{s=0}^{n}\delta_s^{(0,j,1)}P_s^{(\alpha,\beta)}(x)\right)\right) = \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=r}^{n}E_r^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(r,j,1)}P_s^{(\alpha,\beta)}(x)\right)\right). \end{aligned} \tag{32}$$

Simplifying (32) we get:

$$\begin{aligned} x^i(y^{(j)}(x))^2 &= \delta_0^{(i,j,1)}\left(\sum_{l=0}^{n}E_0^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,1)}P_s^{(\alpha,\beta)}(x)\right) + \delta_1^{(i,j,1)}\left(\sum_{l=1}^{n}E_1^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(1,j,1)}P_s^{(\alpha,\beta)}(x)\right) \\ &\quad + \cdots + \delta_n^{(i,j,1)}\left(\sum_{l=n}^{n}E_n^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(n,j,1)}P_s^{(\alpha,\beta)}(x)\right). \end{aligned}$$

Therefore we obtain

$$x^i(y^{(j)}(x))^2 \simeq \sum_{t=0}^{n}\delta_t^{(i,j,2)}P_t^{(\alpha,\beta)}(x),$$

where

$$\begin{aligned} \delta_t^{(i,j,2)} &= \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\right)\left(\sum_{l=0}^{n}\gamma_l\right)\left(\sum_{s=0}^{n}\delta_t^{(s,j,1)}\right), \\ \gamma_l &= \sum_{s=l}^{n}E_l^{(\alpha,\beta,s)}. \end{aligned} \tag{33}$$

The equivalent matrix form of (33) is

$$\Delta_n^{(i,j,2)} = \left([1,\ldots,1]G^iD^j\Delta_n^{(0,0,1)}\right)\left(\sum_{l=0}^{n}\gamma_l\right)\sum_{s=0}^{n}G^sD^j\Delta_n^{(0,0,1)},$$

or

$$\Delta_n^{(i,j,2)} = \left(eG^iD^j\Delta_n^{(0,0,1)}\right)(P)^1L,$$

the proof of this part is completed for $k = 2$.

Now suppose that (31) be true for $k-1$ and we prove it for k. For this purpose, we have

$$\begin{aligned} x^i(y^{(j)}(x))^k &= x^i(y^{(j)}(x))(y^{(j)}(x))^{k-1} \simeq x^i\left(\sum_{r=0}^{n}\delta_r^{(0,j,1)}P_r^{(\alpha,\beta)}(x)\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,k-1)}P_s^{(\alpha,\beta)}(x)\right) \\ &= \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=0}^{r}E_l^{(\alpha,\beta,r)}x^l\right)\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,k-1)}P_s^{(\alpha,\beta)}(x)\right) = \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=r}^{n}E_l^{(\alpha,\beta,r)}\right)x^r\left(\sum_{s=0}^{n}\delta_s^{(0,j,k-1)}P_s^{(\alpha,\beta)}(x)\right)\right) \\ &= \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\left(\sum_{l=r}^{n}E_r^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(r,j,k-1)}P_s^{(\alpha,\beta)}(x)\right)\right). \end{aligned} \tag{34}$$

Expanding (34) we have:

$$x^i\left(y^{(j)}(x)\right)^k = \delta_0^{(i,j,1)}\left(\sum_{l=0}^{n}E_0^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(0,j,k-1)}P_s^{(\alpha,\beta)}(x)\right) + \delta_1^{(i,j,1)}\left(\sum_{l=1}^{n}E_1^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(1,j,k-1)}P_s^{(\alpha,\beta)}(x)\right) + \cdots + \delta_n^{(i,j,1)}\left(\sum_{l=n}^{n}E_n^{(\alpha,\beta,l)}\right)\left(\sum_{s=0}^{n}\delta_s^{(n,j,k-1)}P_s^{(\alpha,\beta)}(x)\right).$$

Therefore we obtain

$$x^i(y^{(j)}(x))^k \simeq \sum_{t=0}^{n}\delta_t^{(i,j,k)}P_t^{(\alpha,\beta)}(x),$$

where

$$\delta_t^{(i,j,k)} = \left(\sum_{r=0}^{n}\delta_r^{(i,j,1)}\right)\left(\sum_{l=0}^{n}\gamma_l\right)\left(\sum_{s=0}^{n}\delta_t^{(s,j,k-1)}\right), \quad \gamma_l = \sum_{s=l}^{n}E_l^{(\alpha,\beta,s)}. \tag{35}$$

We can present (35) in the equivalent matrix form as follows:

$$\Delta_n^{(i,j,k)} = \left([1,\ldots,1]G^iD^j\Delta_n^{(0,0,1)}\right)\left(\sum_{l=0}^{n}\gamma_l\right)\sum_{s=0}^{n}\Delta_n^{(s,j,k-1)}, \tag{36}$$

by consecutive substitution in (36) for $k \geqslant 1$ we obtain:

$$\Delta_n^{(i,j,k)} = \left(eG^iD^j\Delta_n^{(0,0,1)}\right)(P)^{k-1}L\left(\sum_{r=0}^{n}eG^rD^j\Delta_n^{(0,0,1)}\right)^{k-2}, \quad (k \geqslant 2). \qquad \square$$

Now we assume that the function $f(x)$ in (1) can be expanded as

$$f(x) \simeq \sum_{i=0}^{n}f_iP_i^{(\alpha,\beta)}(x),$$

or in the matrix form we can write:

$$f(x) \simeq (F)^TP^{(\alpha,\beta)}(x), \tag{37}$$

where

$$F = [f_0, f_1, \ldots, f_n]^T.$$

If we use the matrix representation of the Jacobi polynomial expansions as

$$x^i[y^{(j)}(x)]^{s_j} = \left(\Delta_n^{(i,j,s_j)}\right)^TP^{(\alpha,\beta)}(x), \ldots, x^i[y(x)]^{s_0} = \left(\Delta_n^{(i,0,s_0)}\right)^TP^{(\alpha,\beta)}(x), \quad i = 0,1,2,\ldots, \quad s_j = 1,2,\ldots, \quad j = 0,1,2,\ldots, \tag{38}$$

then by substituting the expressions (37) and (38) in (4), we have the matrix equation:

$$(P^{(\alpha,\beta)}(x))^T\left(\sum_{k=1}^{s_j}\sum_{i=0}^{m_j}r_{i,k}^{(j)}\Delta_n^{(i,j,k)} + \sum_{k=1}^{s_{j-1}}\sum_{i=0}^{m_{j-1}}r_{i,k}^{(j-1)}\Delta_n^{(i,j-1,k)} + \cdots + \sum_{k=1}^{s_0}\sum_{i=0}^{m_0}r_{i,k}^{(0)}\Delta_n^{(i,0,k)}\right) = (P^{(\alpha,\beta)}(x))^TW \simeq (P^{(\alpha,\beta)}(x))^TF, \tag{39}$$

and noting the linear independence of $P_i^{(\alpha,\beta)}(x)$ we have:

$$W = F, \tag{40}$$

where

$$W = [w_0, w_1, \ldots, w_n]^T,$$

and w_i are dependent to $\delta_k^{(0,0,1)}, k = 0,1,\ldots n$. Therefore from (40) we have a system of $(n+1)$ algebraic equations for the $n+1$ unknown coefficients $\delta_i^{(0,0,1)}, i = 0,1,2,\ldots,n$. Finally, we must obtain the corresponding matrix form for the boundary conditions. For this purpose from (2) the values $y^{(l)}(a_i)$ $i = 1,2,\ldots,j$ can be written as:

$$y^{(l)}(a_i) = (P^{(\alpha,\beta)}(a_i))^TD^l\Delta_n^{(0,0,1)}, \quad a_i \in [-1,1], \quad l = 0,\ldots,j, \quad i = 0,\ldots,j. \tag{41}$$

Substituting (41) in boundary conditions (2) and then simplifying it, we obtain the following matrix form:

$$\sum_{i=0}^{j} b_{i,l} y^{(l)}(a_i) = (P_x^{(\alpha,\beta)}(a_i))^T \left\{ \sum_{i=0}^{j} b_{i,l} D^i \Delta_n^{(0,0,1)} \right\} = \theta_l, \quad a_i \in [-1,1], \quad l = 0,\ldots,j-1. \tag{42}$$

Now from (40) and (42) we have $n+j+1$ algebraic equations and $n+1$ unknown coefficients. Thus for obtaining the unknown coefficients, we must eliminate j arbitrary equations from these $n+j+1$ equations. But because of the necessarily of holding the boundary conditions, we eliminate the last j equations from (40). Finally, replacing the last j equations of (40) by the j equations of (42), we obtain a system of $n+1$ equations and $n+1$ unknowns ($\delta_i^{(0,0,1)}, i = 0,1,\ldots,n$). This nonlinear system which is of the size $(n+1)\times(n+1)$ can be solved using a suitable iterative scheme such as Newton-type methods [22,23]. However, designing an effective iterative scheme capable of solving a large nonlinear system is a very challenging task in applied mathematics. In this paper we solve this nonlinear system via the well-known Newton method using the well-known software MATLAB. In the next section we employ three test problems to illustrate the efficiency of the new method.

4. Test problems

Example 1. Consider the following nonlinear differential equation

$$\begin{aligned} &4(y'') - 2(y')^2 + y = 0, \\ &y(0) = -1, \quad y'(0) = -1, \end{aligned} \tag{43}$$

with the exact solution $y = \frac{x^2}{8} - 1$. We want to solve this problem via the new method. For this purpose if we suppose

$$y(x) \simeq (P^{(\alpha,\beta)}(x))^T \Delta_3^{(0,0,1)},$$

then the matrix form of (43) is:

$$\left(4D^2\Delta_3^{(0,0,1)} - 2\Delta_3^{(0,1,2)} + \Delta_3^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{44}$$

where from (17) we have

$$D = \begin{bmatrix} 0 & \frac{(\alpha+\beta)}{2}+1 & \frac{(\alpha-\beta)(\alpha+\beta+3)(\alpha+\beta+4)}{8} & \frac{(\alpha-\beta)(\alpha^2-\beta^2)(\alpha+\beta+6)(\beta+5)}{48(\alpha+\beta+2)^2} \\ 0 & 0 & \frac{(\alpha+\beta+3)(\alpha+\beta+4)}{(\alpha+\beta+2)} & \frac{(\alpha+\beta+5)(\alpha+\beta+6)(\alpha^2-\beta^2)}{(\alpha+\beta+2)^2} \\ 0 & 0 & 0 & \frac{(\alpha+\beta+5)(\alpha+\beta+6)}{(\alpha+\beta+3)} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

Now by applying the new method for arbitrary (α,β), we obtain a corresponding approximate solution of (43). For example for Legendre polynomials ($\alpha = 0, \beta = 0$) we obtain

$$\begin{aligned} &\delta_0^{(0,0,1)} = -.9583, \quad \delta_1^{(0,0,1)} = 0, \quad \delta_2^{(0,0,1)} = .1237, \quad \delta_3^{(0,0,1)} = 0, \\ &y = .1237x^2 - .9989, \end{aligned}$$

and for the first kind of Chebyshev polynomials ($\alpha = -0.5, \beta = -0.5$), we get

$$\begin{aligned} &\delta_0^{(0,0,1)} = -1, \quad \delta_1^{(0,0,1)} = 0, \quad \delta_2^{(0,0,1)} = \frac{1}{8}, \quad \delta_3^{(0,0,1)} = 0, \\ &y = \frac{1}{8}x^2 - 1. \end{aligned}$$

Figs. 1 and 2 present the exact and approximate solutions for ($\alpha = 0, \beta = 0$) and ($\alpha = -0.5, \beta = -0.5$) respectively.

Example 2. Consider the initial value problem [24,25]:

$$\begin{cases} \frac{dx}{dt} + \frac{dy}{dt} + y = t + e^t, \quad x(0) = 1, \\ \frac{dx}{dt} + 4\frac{dy}{dt} + x = 1 + 2e^{-2t}, \quad y(0) = 0. \end{cases} \tag{46}$$

We compare our results for $n = 4$ and different values of (α, β) with the obtained results in [24], and Stehfest's method that is given by Davies and Crann in [25]. First we assume

$$\begin{aligned} &x(t) \simeq (P^{(\alpha,\beta)}(t))^T \Delta_3^{(0,0,1)}, \\ &y(t) \simeq (P^{(\alpha,\beta)}(t))^T \Omega_3^{(0,0,1)}, \end{aligned} \tag{47}$$

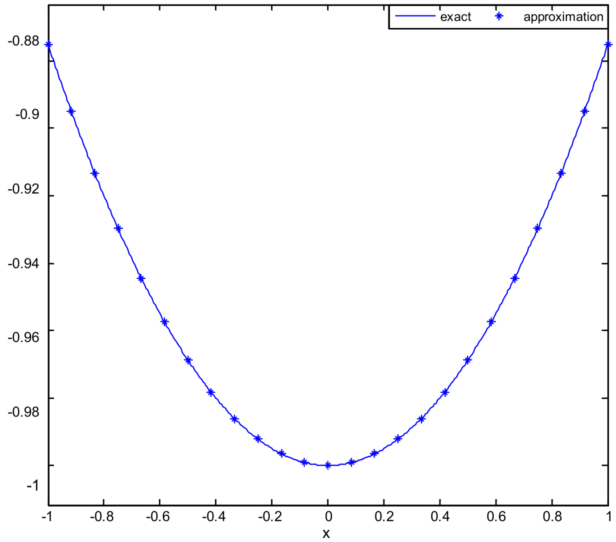

Fig. 1. The comparison of exact and approximate solutions for ($\alpha = 0, \beta = 0$).

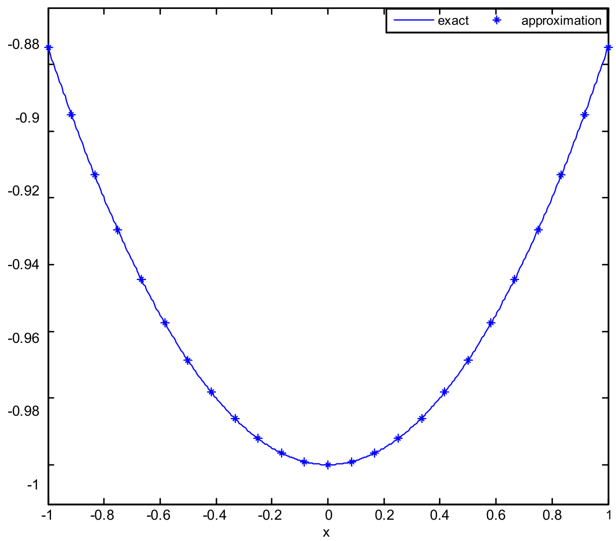

Fig. 2. The comparison of exact and approximate solutions for ($\alpha = -.5, \beta = -.5$).

where

$$\Delta_3^{(0,0,1)} = \left[\delta_0^{(0,0,1)}, \ldots, \delta_3^{(0,0,1)}\right]^T, \quad \Omega_3^{(0,0,1)} = \left[\varepsilon_0^{(0,0,1)}, \ldots, \varepsilon_3^{(0,0,1)}\right]^T.$$

Also we expand $t + e^t$ and $1 + 2e^{-2t}$ in Jacobi basis:

$$\begin{aligned} t + e^t &\simeq \sum_{i=0}^{3} \rho_i P_i^{(\alpha,\beta)}(t), \\ 1 + 2e^{-2t} &\simeq \sum_{i=0}^{3} \mu_i P_i^{(\alpha,\beta)}(t), \end{aligned} \tag{48}$$

where

$$\rho_i = \frac{\int_{-1}^{-1}(1-t)^\alpha(1+t)^\beta(t+e^t)P_i^{(\alpha,\beta)}(t)dt}{\int_{-1}^{-1}(1-t)^\alpha(1+t)^\beta(P_i^{(\alpha,\beta)}(t))^2dt}, \quad i=0,\ldots,3.$$

$$\mu_i = \frac{\int_{-1}^{-1}(1-t)^\alpha(1+t)^\beta(1+2e^{-2t})P_i^{(\alpha,\beta)}(t)dt}{\int_{-1}^{-1}(1-t)^\alpha(1+t)^\beta(P_i^{(\alpha,\beta)}(t))^2dt}, \quad i=0,\ldots,3.$$

The exact values of dominators of ρ_i and μ_i are obtained from (9) for $m = n = i$ and the approximate values of numerators of them are calculated with n-point Gauss–Jacobi quadrature formula in the form

$$\int_{-1}^{-1}(1-t)^\alpha(1+t)^\beta f(t)dt \simeq \sum_{i=0}^{n} w_i f(t_i),$$

where the explicit forms of w_i and t_i for $i = 1,2,\ldots,n$ are given in [26].

Now we convert (46) into this matrix form:

$$\begin{aligned} &\left(D\left(\Omega_3^{(0,0,1)}+\Delta_3^{(0,0,1)}\right)+\Omega_3^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(x) = \begin{bmatrix}\rho_0\\ \rho_1\\ \rho_2\\ \rho_3\end{bmatrix},\\ &\left(D\left(\Omega_3^{(0,0,1)}+4\Delta_3^{(0,0,1)}\right)+\Delta_3^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(x) = \begin{bmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\end{bmatrix}, \end{aligned} \tag{49}$$

where D is defined in (45).

Now using the new method, we obtain the corresponding approximate solutions of (46). Table 2 shows the obtained results for Legendre polynomials ($\alpha = 0, \beta = 0$). Also Table 3 shows the obtained results for ($\alpha = -0.4, \beta = 0.3$).

Example 3. Consider the Riccati equation [27–29]

$$y'(x) = 2y(x) - y^2(x) + 1, \quad 0 \leqslant x \leqslant 1, \tag{50}$$

with boundary condition $y(0) = 0$.

The exact solution of (50) is:

$$y_e(x) = 1 + \sqrt{2}\tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right].$$

This equation was solved via different methods such as the homotopy perturbation method [27], the standard Adomian's decomposition method [28] and Chebyshev matrix method [29]. The approximate solutions of the mentioned methods respectively are:

$$\begin{aligned} y_h(x) &\cong x + x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{7}{15}x^5 - \frac{7}{45}x^6 - \frac{53}{315}x^7 + \frac{221}{1260}x^8,\\ y_A(x) &\cong x + x^2 + \frac{1}{3}x^3 - \frac{2}{3}x^4 + \frac{2}{15}x^5,\\ y_C(x) &\cong \sum_{i=0}^{5} a_i T_i^*(x), \end{aligned}$$

where $T_i^*(x)$ are shifted Chebyshev polynomials [30] and:

Table 2
Error analysis of Example 2 for the x values ($\alpha = 0, \beta = 0$).

t_i	Presented method ($N = 4$)		Method of [24] ($N = 5$)		Stehfest's method ($M = 8$)	
	Absolute error	Relative error	Absolute error	Relative error	Absolute error	Relative error
.1	9.3106×10^{-5}	28.301×10^{-5}	4.5105×10^{-5}	5.5928×10^{-5}	6.7614×10^{-5}	25.318×10^{-5}
.2	7.7020×10^{-5}	0.5734×10^{-5}	7.9850×10^{-5}	1.2770×10^{-5}	8.4949×10^{-5}	31.809×10^{-5}
.5	119.8×10^{-5}	912.30×10^{-5}	9.7190×10^{-5}	6.9580×10^{-5}	318.97×10^{-5}	1194.4×10^{-5}

Table 3
Error analysis of Example 2 for the x values ($\alpha = -.4, \beta = .3$).

t_i	Presented method ($N = 4$)		Method of [24] ($N = 5$)		Stehfest's method ($M = 8$)	
	Absolute error	Relative error	Absolute error	Relative error	Absolute error	Relative error
.1	2.0214×10^{-5}	0.10215×10^{-5}	4.5105×10^{-5}	5.5928×10^{-5}	6.7614×10^{-5}	25.318×10^{-5}
.2	0.32090×10^{-5}	0.51324×10^{-5}	7.9850×10^{-5}	1.2770×10^{-5}	8.4949×10^{-5}	31.809×10^{-5}
.5	135.70×10^{-5}	929.50×10^{-5}	9.7190×10^{-5}	6.9580×10^{-5}	318.97×10^{-5}	1194.4×10^{-5}

Table 4
The results obtained using the new method and the method of [27].

x	Exact solution	Present method ($\alpha = -.4, \beta = .3$)	Error of present method	Method of [27]	Error of Method of [27]
0	0	0	0	0	0
0.1	0.1103	0.1105	2.00×10^{-4}	0.1078	2.50×10^{-3}
0.2	0.2420	0.2410	1.00×10^{-3}	0.2438	1.80×10^{-3}
0.3	0.3951	0.3967	1.60×10^{-3}	0.3967	1.60×10^{-3}
0.4	0.5678	0.5677	1.00×10^{-3}	0.5696	1.80×10^{-3}
0.5	0.7560	0.7540	2.00×10^{-3}	0.7581	2.10×10^{-3}
0.6	0.9536	0.9533	3.00×10^{-4}	0.9558	2.20×10^{-3}
0.7	1.1529	1.1520	9.00×10^{-4}	1.1552	2.30×10^{-3}
0.8	1.3464	1.3412	52.0×10^{-4}	1.3485	2.10×10^{-3}
0.9	1.5269	1.5256	13.0×10^{-4}	1.5289	2.00×10^{-3}
1	1.6895	1.6890	5.00×10^{-4}	1.6912	1.70×10^{-3}

Table 5
The results obtained using the methods of [28,29].

x	Exact solution	Method of [28]	Error of Method of [28]	Method of [29]	Error of Method of [29]
0	0	0	0	0	0
0.1	0.1103	0.1053	5.00×10^{-3}	0.1053	5.00×10^{-3}
0.2	0.2420	0.2216	20.4×10^{-3}	0.2220	2.00×10^{-2}
0.3	0.3951	0.3489	90.4×10^{-3}	0.3500	4.51×10^{-2}
0.4	0.5678	0.4856	82.2×10^{-3}	0.4872	8.06×10^{-2}
0.5	0.7560	0.6292	17.7×10^{-2}	0.6282	12.7×10^{-2}
0.6	0.9536	0.7760	3.00×10^{-4}	0.7635	19.0×10^{-2}
0.7	1.1529	0.9217	23.1×10^{-2}	0.8788	27.4×10^{-3}
0.8	1.3464	1.0613	33.7×10^{-2}	0.9546	38.1×10^{-2}
0.9	1.5269	1.1893	58.7×10^{-3}	0.9661	56.0×10^{-2}
1	1.6895	1.3000	0.38×10^{-4}	0.8849	80.4×10^{-2}

$$\begin{aligned} a_0 &= 0.80541332764471685, \quad a_1 = 0.87401175937508757, \\ a_2 &= 0.04376673754405507, \quad a_3 = -0.0299944533379256, \\ a_4 &= -0.0035681483673057, \quad a_5 = 0.00154469082563499. \end{aligned}$$

For solving this example by the new method, we transform the interval $[-1,1]$ to $[0,1]$ by $z = \frac{x+1}{2}$. Substituting this transform in (50) we obtain:

$$2y'(2z-1) = 2y(2z-1) - y^2(2z-1) + 1, \quad -1 \leqslant z \leqslant 1. \tag{51}$$

Now we use the new method for $n = 5$ and different values of (α, β) and compare our results with the obtained numerical results in [27–29]. The matrix form of (51) and its boundary condition respectively are:

$$\left(4\Delta_5^{(0,0,1)} - 2D\Delta_5^{(0,1,2)} - \Delta_5^{(0,0,2)}\right)^T P^{(\alpha,\beta)}(2z-1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{52}$$

$$y(0) = 0 \Rightarrow \left(\Delta_5^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(-1) = 0, \tag{53}$$

where D can be obtained from (17) for $n = 4$. Now using the new method we obtain the approximate solutions of (48) for different values of (α, β).

Tables 4 and 5 show the obtained results from using the different numerical methods and the exact solution. Exact and approximate solutions are presented in both Tables 4 and 5 to show the efficiency of the new method.

5. Conclusion

In this study, we obtained the approximate solutions for some nonlinear ordinary differential equations by using the general Jacobi matrix method. Explicit formulae which express the Jacobi expansion coefficients for the powers of derivatives and moments of any differentiable function in terms of the original expansion coefficients of the function itself are presented in the matrix form. Finally three test problems are given to illustrate the efficiency of the proposed method.

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