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A new general eighth-order family of iterative methods for solving nonlinear equations

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ABSTRACT

In this work, we present a family of iterative methods for solving nonlinear equations. It is proved that these methods have convergence order 8. These methods require three evaluations of the function, and only use one evaluation of the first derivative per iteration. The efficiency of the method is tested on a number of numerical examples. On comparison with the eighth-order methods, the iterative methods in the new family behave either similarly or better for the test examples.

1. Introduction

The nonlinear equations are regarded as among the most important problems in physical and engineering. Therefore, this problem has been studied by many authors, such as: Parhi et al. in [1], who suggested a sixth-order method with efficiency index 1.565; Kou et al., who derived in [2] a seventh-order method with efficiency index 1.6266; Liu et al., who in [3] developed an eighth-order method with efficiency index 1.682; and Cordero et al., who in [4] proposed families of iterative methods with sixth and seventh orders of convergence, the efficiency indices of these methods being 1.565 and 1.6266, respectively. This work is concerned with the iterative methods for finding a zero x^* of a nonlinear equation $f(x) = 0$, where $f : D \subseteq \Re \to \Re$ is a smooth function, and D is an open interval.

2. Preliminaries and notation

2.1. The efficiency index (EI)

Definition 2.1.1. The efficiency index (EI) is defined as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method [5].

2.2. The computational order of convergence (COC)

Definition 2.2.1. The computational order of convergence (COC) ρ is computed by using [5]

$$\rho \approx \frac{\ln(\| X_{n+1} - X_n \|_\infty / \| X_n - X_{n-1} \|_\infty)}{\ln(\| X_n - X_{n-1} \|_\infty / \| X_{n-1} - X_{n-2} \|_\infty)}, \tag{1}$$

where X_{n+2}, X_{n+1}, X_n and X_{n-1} are iterations close to a zero of the nonlinear system.

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2.3. The order of convergence

Definition 2.3.1. Suppose $\{X_n\}_{n=0}^{\infty}$ is a sequence that converges to α, with $X_n \neq \alpha$ for all n. If positive constants c and r exist with

$$\lim_{n\to\infty} \frac{\| X_{n+1} - \alpha \|}{\| X_n - \alpha \|^r} = c \neq 0, \tag{2}$$

then $\{X_n\}_{n=0}^{\infty}$ converges to α of order r, with asymptotic error constant c.

3. The main results

Let $f(x)$ have a simple root x^* and be analytic in a small neighborhood of x^*. We consider the following three-step iteration scheme:

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - G\left(\dfrac{f(y_m)}{f(x_m)}\right) \dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{\mu}{\lambda + \nu q_m^2} \dfrac{f(z_m)}{f'(z_m)}, \end{cases} \tag{3}$$

where $\lambda, \mu, \nu \in \Re$, $G(t)$ represents a real-valued function and $q_m = \frac{f(z_m)}{f(x_m)}$.

In order to improve the efficiency index, we modify the family (1) approximating $f'(z_m)$. In [6], the authors constructed the following interpolatory polynomial of degree 3 to approximate $f(x)$:

$$P(x) = a + b(x - z_m) + c(x - z_m)^2 + d(x - z_m)^3, \tag{4}$$

satisfying $P(x_m) = f(x_m)$, $P'(x_m) = f'(x_m)$, $P(y_m) = f(y_m)$ and $P(z_m) = f(z_m)$. They obtained that

$$f'(z_m) \approx K - C(y_m - z_m) - D(y_m - z_m)^2, \tag{5}$$

where

$$\begin{cases} H = \dfrac{f(x_m) - f(y_m)}{x_m - y_m}, \\ K = \dfrac{f(y_m) - f(z_m)}{y_m - z_m}, \\ D = \dfrac{f'(x_m) - H}{(x_m - y_m)(x_m - z_m)} - \dfrac{H - K}{(x_m - z_m)^2}, \\ C = \dfrac{H - K}{(x_m - y_m)(x_m - z_m)} - D(x_m + y_m - 2z_m). \end{cases} \tag{6}$$

Substituting (5) into (1), we can construct a new family of methods as follows:

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - G\left(\dfrac{f(y_m)}{f(x_m)}\right) \dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{\mu}{\lambda + \nu q_m^2} \dfrac{f(z_m)}{K - C(y_m - z_m) - D(y_m - z_m)^2}. \end{cases} \tag{7}$$

4. Convergence analysis

Hereunder, the convergence analysis of the family defined by (7) is studied by using the following theorem.

Theorem. *Let $f : D \subseteq \Re \to \Re$ have a single root $x^* \in D$, for an open interval D. If the initial point x_0 is sufficiently close to x^*, then the sequence $\{x_m\}$ generated by any method of the family (7) converges to x^*. If G is any function with $M_0 = G(0) = 1$, $M_1 = G'(0) = 2$, $M_2 = G''(0) < \infty$ and $\lambda = \mu \neq 0$ then the methods defined by (7) have convergence order at least 8.*

Proof. We denote the error in each iteration by $e_m = x_m - x^*$. We define $c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$, $k = 2, 3, \ldots$. Using Taylor's expansion, we have

$$f(x_m) = f'(x^*)[e_m + c_2e_m^2 + c_3e_m^3 + c_4e_m^4 + c_5e_m^5 + c_6e_m^6 + c_7e_m^7 + c_8e_m^8 + O(e_m^9)], \tag{8}$$

$$f'(x_m) = f'(x^*)[1 + 2c_2e_m + 3c_3e_m^2 + 4c_4e_m^3 + 5c_5e_m^4 + 6c_6e_m^5 + 7c_7e_m^6 + 8c_8e_m^7 + 9c_9e_m^8 + O(e_m^9)], \tag{9}$$

$$\begin{aligned} y_m - x^* = {} & c_2e_m^2 + (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 3c_2c_3 - 2(2c_3 - 2c_2^2)c_2)e_m^4 + (4c_5 - 10c_2c_4 - 6c_3^2 \\ & + 20c_3c_2^2 - 8c_2^4)e_m^5 + (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 33c_2c_3^2 + 5c_6 - 52c_3c_2^3 + 16c_2^5)e_m^6 \\ & + (-22c_5c_3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 + 92c_4c_2c_3 - 12c_4^2 - 72c_4c_2^3 + 18c_3^3 - 126c_3^2c_2^2 \\ & + 128c_3c_2^4 - 32c_2^6)e_m^7 + (7c_8 - 348c_4c_3c_2^2 + 118c_5c_2c_3 + 64c_2^7 - 19c_2c_7 - 31c_4c_5 \\ & + 64c_2c_4^2 + 75c_4c_3^2 + 176c_4c_2^4 - 92c_5c_2^3 - 27c_6c_3 + 44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 \\ & - 304c_3c_2^5e_m^8 + O(e_m^9)]. \end{aligned} \tag{10}$$

From (10), we get

$$\begin{aligned} f(y_m) = {} & f'(x^*)[c_2e^2 + (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_m^4 + (-6c_3^2 + 24c_3c_2^2 - 10c_2c_4 \\ & + 4c_5 - 12c_2^4)e^5 + (-17c_4c_3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_m^6 \\ & + (-22c_5c_3 + 44c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 104c_4c_2c_3 - 104c_4c_2^3 + 18c_3^3 - 160c_3^2c_2^2 \\ & + 206c_3c_2^4 - 64c_2^6)e_m^7 + (7c_8 - 455c_4c_3c_2^2 + 134c_5c_2c_3 + 144c_2^7 - 19c_2c_7 - 31c_4c_5 \\ & + 73c_2c_4^2 + 75c_4c_3^2 + 297c_4c_2^4 - 134c_5c_2^3 - 27c_6c_3 + 54c_6c_2^2 - 147c_2c_3^3 + 582c_3^2c_2^3 \\ & - 552c_3c_2^5)e_m^8 + O(e_m^9)]. \end{aligned} \tag{11}$$

From (8), (9) and (11), it follows that

$$\begin{aligned} \frac{f(y_m)}{f(x_m)} = {} & c_2e_m + (2c_3 - 3c_2^2)e_m^2 \\ & + (3c_4 - 10c_2c_3 + 8c_2^3)e_m^3 + (4c_5 - 14c_2c_4 - 8c_3^2 + 37c_3c_2^2 - 20c_2^4)e_m^4 + O(e_m^5), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \frac{f(y_m)}{f'(x_m)} = {} & c_2e_m^2 + (2c_3 - 4c_2^2)e_m^3 \\ & + (3c_4 - 14c_2c_3 + 13c_2^3)e_m^4 + (4c_5 - 20c_2c_4 - 12c_3^2 + 64c_3c_2^2 - 38c_2^4)e_m^5 + O(e_m^6). \end{aligned} \tag{13}$$

Using the Taylor expansion $G\left(\frac{f(y_m)}{f(x_m)}\right) = M_0 + M_1\frac{f(y_m)}{f(x_m)} + \frac{1}{2}M_2\frac{f^2(y_m)}{f^2(x_m)} + O\left(\left(\frac{f(y_m)}{f(x_m)}\right)^3\right)$, and with (8)–(10), (12) and (13), we have

$$\begin{aligned} z_m - x^* = {} & (c_2 - M_0c_2)e_m^2 + (-2M_0c_3 + 4M_0c_2^2 - M_1c_2^2 + 2c_3 - 2c_2^2)e_m^3 \\ & + (-3M_0c_4 + 14M_0c_2c_3 - 13M_0c_2^3 - 4M_1c_2c_3 + 7M_1c_2^3 - M_2c_2^3 \\ & + 3c_4 - 7c_2c_3 + 4c_2^3)e_m^4 + (-33M_0c_2^4 + 38M_1c_3c_2^2 - 6M_1c_2c_4 \\ & - 8c_2^4 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 4M_0c_5 + 12M_0c_3^2 + 38M_0c_2^4 + 10M_2c_2^4 \\ & - 64M_0c_3c_2^2 - 6M_2c_2^2c_3 + 4c_5 - 4M_1c_3^2 + 20M_0c_2c_4)e_m^5 + O(e_m^6). \end{aligned} \tag{14}$$

If $M_0 = 1$, $M_1 = 2$ and $M_2 < \infty$, it is obvious that

$$\begin{aligned} z_m - x^* = {} & (-c_2c_3 + 5c_2^3 - M_2c_2^3)e_m^4 \\ & + (-2c_2c_4 + 32c_3c_2^2 - 36c_2^4 + 10M_2c_2^4 - 6M_2c_2^2c_3 - 2c_3^2)e_m^5 + O(e_m^6). \end{aligned} \tag{15}$$

Now, expanding $f(z_m)$ about x^* and applying (14),

$$\begin{aligned} f(z_m) = {} & f'(x^*)[(c_2 - M_0c_2)e_m^2 + (-2M_0c_3 + 4M_0c_2^2 - M_1c_2^2 + 2c_3 - 2c_2^2)e_m^3 \\ & + (-3M_0c_4 + 14M_0c_2c_3 - 15M_0c_2^3 - 4M_1c_2c_3 + 7M_1c_2^3 - M_2c_2^3 + 3c_4 - 7c_2c_3 \\ & + 5c_2^3 + G^2(0)c_2^3)e_m^4 + (4c_5 - 6M_1c_2c_4 + 38M_1c_3c_2^2 - 10c_2c_4 - 12c_2^4 - 6c_3^2 \\ & + 24c_3c_2^2 - 4M_0c_5 + 12M_0c_3^2 + 50M_0c_2^4 + 10M_2c_2^4 + 20M_0c_2c_4 - 72M_0c_3c_2^2 \\ & - 6M_2c_2^2c_3 - 4M_1c_3^2 - 35M_1c_2^4 + 4G^2(0)c_2^2c_3 - 8G^2(0)c_2^4 + 2M_0c_2^4M_1)e_m^5 + O(e_m^6)]. \end{aligned} \tag{16}$$

From (8)–(16), we obtain

$$f'(z_m) = f'(x^*)[1 - 2c_2^2(-1 + M_0)e_m^2 + 2(-2M_0c_3 + 4M_0c_2^2 - M_1c_2^2 + 2c_3 - 2c_2^2)c_2e_m^3 + O(e_m^4)]. \quad (17)$$

Thus, substituting (8)–(17) in (6), we get

$$e_{m+1} = K_1e_m^4 + K_2e_m^5 + K_3e_m^6 + K_4e_m^7 + K_5e_m^8 + O(e_m^9), \quad (18)$$

where $K_i, i = 1, 2, \ldots, 5$, are multivariate polynomials in λ, μ, M_2; for instance

$$K_1 = \frac{c_2((-\lambda + \mu)c_3 + (-5\mu + \mu M_2 + 5\lambda - M_2\lambda)c_2^2)}{\lambda}. \quad (19)$$

It can be easily seen that K_1 can vanish whenever $\lambda = \mu \neq 0$. Substituting $\lambda = \mu$ into $K_2, i = 2, \ldots, 4$, we have $K_i = 0, i = 2, \ldots, 4$. This means that the convergence order of any method of the family (6) is 8, and the error equation is

$$\begin{aligned} e_{m+1} &= c_2^2(-c_4c_3 + c_2c_3^2 - 10c_3c_2^3 \\ &\quad + 2M_2c_2^3c_3 + 5c_4c_2^2 + 25c_2^5 - 10M_2c_2^5 - M_2c_2^2c_4 + M_2^2c_2^5)e_m^8 + O(e_m^9). \end{aligned} \quad (20)$$

It is obvious that each iteration of any method of the family (6) requires three evaluations of the function and one evaluation of its first derivative. We consider the definition of the efficiency index [7] as $P^{\frac{1}{d}}$, where P is the order of the method and d is the number of function evaluations per iteration required by the method. Then, we have that the family of methods (6) has the efficiency index $8^{\frac{1}{4}} = 1.682$ which is better than the 1.414 for Newton's method, and better than the 1.565 in [1,4] and the 1.6266 in [2,4], and also equal to efficiency index of methods described in [3,4]. □

5. Numerical implementations

Now, we consider two special cases of the family defined by (6) for numerical experiments.

Method 1 ($M - 1$). Let $\lambda = \mu = 1$. For the function G defined by

$$G(t) = \frac{1}{1 - 2t + \omega t^2} \quad (21)$$

where $\omega \in \Re$, it can easily be seen that $G(0) = 1$, $G'(0) = 2$ and $G''(0) < \infty$. Hence we get a new eighth-order method:

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \dfrac{f^2(x_m)}{f^2(x_m) - 2f(x_m)f(y_m) + \omega f^2(y_m)} \dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{1}{1 + \nu q_m^2} \dfrac{f(z_m)}{K - C(y_m - z_m) - D(y_m - z_m)^2}. \end{cases} \quad (22)$$

Method 2 ($M - 2$). Let $\lambda = \mu = 1$. For the function G defined by

$$G(t) = \frac{1 + \alpha t}{1 + \beta t}, \quad (23)$$

where $\beta, \alpha \in \Re$ and $\alpha - \beta = 2$, it can easily be seen that $G(0) = 1$, $G'(0) = 2$ and $G''(0) < \infty$. Hence we get a new eighth-order method:

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \dfrac{f(x_m) + \alpha f(y_m)}{f(x_m) + \beta f(y_m)} \dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{1}{1 + \nu q_m^2} \dfrac{f(z_m)}{K - C(y_m - z_m) - D(y_m - z_m)^2}. \end{cases} \quad (24)$$

Now, Method 1, $(M - 1)(\omega = 1, \nu = 1)$, and Method 2, $(M - 2)(\alpha = 3, \beta = 1, \nu = 1)$, are employed to solve some nonlinear equations and compared with the seventh-order method G_7 defined in [2], the eighth-order method M_8 defined in Example 2.3 of [3] and the methods MK_6 and MK_7 of [4].

Table 1 shows, the number of iterations required to obtain $|x_{n+1} - x_n| \leq 10^{-200}$ and the computational order of convergence (ρ) that ρ is defined with [5]:

$$\rho \approx \frac{\ln(|x_{m+1} - x_m|/|x_m - x_{m-1}|)}{\ln(|x_m - x_{m-1}|/|x_{m-1} - x_{m-2}|)}, \quad (25)$$

Table 1
(Iterations, ρ) for the various iterative methods.

f_i	x_0	$M-1$	$M-2$	MK_6	MK_7	G_7	M_8
f_1	$x_0=1$	(5, 8)	(5, 8)	(5, 6)	(5, 9)	(5, 7)	(5, 8)
f_2	$x_0=2$	(4, 8)	(4, 8)	(4, 6)	(4, 7)	(4, 7)	(4, 8)
f_3	$x_0=1$	(4, 8)	(4, 8)	(4, 6)	(4, 7)	(4, 7)	(4, 8)
f_4	$x_0=1$	(5, 8)	(5, 8)	(5, 5)	(5, 7)	(5, 7)	(5, 8)
f_5	$x_0=2$	(4, 8)	(4, 8)	(4, 7)	(5, 7)	(5, 7)	(5, 8)

Table 2
The value of the function f in the last iteration.

f_i, x_0	$M-1$	$M-2$	MK_6	MK_7	G_7	M_8
$f_1, x_0=1$	9.43e−191	1.43e−185	4.47e−106	1.61e−078	2.74e−109	5.78e−180
$f_2, x_0=2$	0	0	7.37e−264	0	0	0
$f_3, x_0=1$	0	0	1.86e−254	0	0	0
$f_4, x_0=1$	8.51e−178	6.44e−171	1.58e−112	3.17e−153	8.71e−144	1.91e−160
$f_5, x_0=2$	1.23e−140	7.34e−136	5.03e−085	1.12e−141	9.02e−081	6.23e−145

where x_{m+1}, x_m, x_{m-1} and x_{m-2} are iterations close to a zero of the nonlinear equation; also, Table 2 shows the value of the function f in the last iteration, computed with 350 significant digits.

We use the following functions, some of them taken from Refs. [5,8]:

Example 1. $f_1(x) = \sin^2(x) - x^2 + 1$; $x^* \approx 1.4044916482153$;

Example 2. $f_2(x) = \sin(x) - \frac{x}{2}$; $x^* \approx 1.8954942670339$;

Example 3. $f_3(x) = \cos(x) - x$; $x^* \approx 0.73908513321516$;

Example 4. $f_4(x) = 10xe^{-x^2} - 1$; $x^* \approx 1.6796306104285$;

Example 5. $f_5(x) = x^2 - e^x - 3x + 2$; $x^* \approx 0.2575302854398608$.

6. Conclusions

Our study presents a family of iterative methods with some parameters for solving nonlinear equations. On choosing the proper parameters, we get a new family of iterative methods with eighth-order convergence. These iterative methods were compared in their efficiency and performance to various other iteration methods, and it was observed that the new iterative methods behaved either similarly or better for the test examples.

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