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Steffensen type methods for solving nonlinear equations☆

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ABSTRACT

In the present paper, by approximating the derivatives in the well known fourth-order Ostrowski's method and in a sixth-order improved Ostrowski's method by central-difference quotients, we obtain new modifications of these methods free from derivatives. We prove the important fact that the methods obtained preserve their convergence orders 4 and 6, respectively, without calculating any derivatives. Finally, numerical tests confirm the theoretical results and allow us to compare these variants with the corresponding methods that make use of derivatives and with the classical Newton's method.

1. Introduction

In the last few years, a lot of papers have developed the idea of removing derivatives from the iteration function in order to avoid defining new functions such as the first or second derivative, and calculate iterates only by using the function that describes the problem, obviously trying to preserve the convergence order. In this sense, in the literature of nonlinear equations there can frequently be found the expression "derivative free", referring in most cases to the second derivative (see [1–3]). The interest of these methods is for being applied with nonlinear equations $f(x) = 0$, when there are many problems, in order to obtain and evaluate the derivatives involved.

The procedure of removing the derivatives usually increases the number of functional evaluations per iteration. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $p^{1/d}$, where p is the order of convergence and d is the total number of functional evaluations per step.

There are different methods for computing a zero α of a nonlinear equation $f(x) = 0$; the most well known of these methods is the classical Newton's method (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{1}$$

that, under certain conditions, has quadratic convergence.

Newton's method has been modified in a number of ways to avoid the use of derivatives without affecting the order of convergence. For example, on replacing in (1) the derivative by the forward approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}.$$

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Newton's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

which is called Steffensen's method (*SM*). This method still has quadratic convergence, in spite of being derivative free and using only two functional evaluations per step.

When an iterative method is free from first derivatives, authors refer to it as a "Steffensen like method". Some of these methods use forward differences for approximating the derivatives. For example, in [4] Jain proposed a Steffensen-secant method (*JM*) deformed from the Newton-secant one. This method only uses three functional evaluations per step and gets third-order convergence. Another Steffensen like method of third order, based on the homotopy perturbation theory, is presented in [5] (*FM*). It uses three functional evaluations per step.

By applying forward-difference approximation to the Weerakoon–Fernando formula [6], Zheng et al. derived in [7] a family of Steffensen like methods (*ZM*) which have order of convergence 3 and use four functional evaluations per iteration.

In order to control the approximation of the derivative and the stability of the iteration, a Steffensen type method, with quadratic convergence and two functional evaluations per step, has been proposed in [8], (*AM*). The recent paper [9] has extended it to Banach spaces, obtaining its semilocal and local convergence theorems.

If we try to use forward-difference approximation, with the fourth-order Ostrowski's method [10]:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \end{aligned} \tag{2}$$

the order of convergence of the new method goes down to 3. For this reason, we have used the central-difference form in (2), obtaining a variant of Ostrowski's method that preserves the convergence order 4 and is derivative free. Recently, Dehghan and Hajarian [11] proposed a derivative free iterative method (*DM*) by replacing the forward-difference approximation in Steffensen's method by the central-difference approximation. However, it is still a method of third order and requires four functional evaluations per iteration.

In the same way, we consider the sixth-order method proposed in [12] as an improvement on the Ostrowski root-finding method, whose iteration scheme is

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n), \\ x_{n+1} &= z_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(z_n). \end{aligned} \tag{3}$$

We are going to replace in (3) the first derivative by a symmetric difference in order to obtain a new method that preserves the sixth convergence order and is derivative free.

The rest of this paper is organized as follows. In Section 2, we describe our methods that are free from derivatives as variants of Ostrowski's method and the improved Ostrowski's method, respectively. In Section 3, we establish the convergence order of these methods. Finally, in Section 4 different numerical tests confirm the theoretical results and allow us to compare these variants with the original methods (which make use of derivatives) and also with Newton's method.

2. Description of the methods

By using a symmetric difference quotient

$$f'(x_n) \simeq \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)},$$

to approximate the derivative in the fourth-order Ostrowski's method (2), we obtain a new method free from derivatives that we call the *modified Ostrowski's method free from derivatives* (*ODF*):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}. \end{aligned} \tag{4}$$

As we have said, in [12], Grau et al. proposed an improvement of Ostrowski's method (3) and proved that it has sixth order of convergence. By approximating the derivative by the central-difference method we obtain a new method free from derivatives, that we call the *improved Ostrowski's method free from derivatives* (*IODF*):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n+f(x_n)) - f(x_n - f(x_n))}, \\ z_n &= y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n), \\ x_{n+1} &= z_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(z_n). \end{aligned} \tag{5}$$

In the next section, we are going to prove that the methods *ODF* and *IODF* have orders of convergence 4 and 6, respectively.

3. Analysis of convergence

In this section we analyze the orders of convergence of the methods described previously.

Theorem 1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the modified Ostrowski's method free from derivatives defined by (4) has order of convergence 4 and satisfies the error equation*

$$e_{n+1} = -c_2 \left(-\frac{c_2^2}{c_1^3} + c_3 + \frac{c_3}{c_1^2} \right) e_n^4 + O(e_n^5).$$

Proof. Let $e_n = x_n - \alpha$. The Taylor series of $f(x_n)$ about α is

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5), \tag{6}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!},\ k = 1, 2, \ldots$.

Computing the Taylor series of $f(x_n + f(x_n))$ and substituting $f(x_n)$ by (6) we have

$$\begin{aligned} f(x_n + f(x_n)) = {} & c_1(1+c_1)e_n + (c_1c_2 + (1+c_1)^2c_2)e_n^2 + (2(1+c_1)c_2^2 + c_1c_3 + (1+c_1)^3c_3)e_n^3 \\ & + (3(1+c_1)^2c_2c_3 + c_2(c_2^2 + 2(1+c_1)c_3) + c_1c_4 + (1+c_1)^4c_4)e_n^4 + O(e_n^5). \end{aligned} \tag{7}$$

Analogously, the Taylor series of $f(x_n - f(x_n))$ is

$$\begin{aligned} f(x_n - f(x_n)) = {} & (1-c_1)c_1e_n + ((1-c_1)^2c_2 - c_1c_2)e_n^2 + (-2(1-c_1)c_2^2 + (1-c_1)^3c_3 - c_1c_3)e_n^3 \\ & + (-3(1-c_1)^2c_2c_3 + c_2(c_2^2 - 2(1-c_1)c_3) + (1-c_1)^4c_4 - c_1c_4)e_n^4 + O(e_n^5). \end{aligned} \tag{8}$$

Then, the quotient that appears in the expression of y_n in (4) is

$$\begin{aligned} \frac{2f(x_n)^2}{f(x_n+f(x_n)) - f(x_n-f(x_n))} = {} & e_n - \frac{c_2e_n^2}{c_1} + \frac{(2c_2^2 - c_1(2+c_1^2)c_3)e_n^3}{c_1^2} \\ & + \left(-\frac{4c_2^3}{c_1^3 + c_2c_3} + \frac{7c_2}{c_3}c_1^2 - \frac{3c_4}{c_1 - 4c_1c_4} \right) e_n^4 + O(e_n^5). \end{aligned} \tag{9}$$

We obtain $y_n - \alpha$ taking into account (9):

$$\begin{aligned} y_n - \alpha &= e_n - \frac{2f(x_n)^2}{f(x_n+f(x_n)) - f(x_n-f(x_n))} \\ &= \frac{c_2e_n^2}{c_1} - \frac{(2c_2^2 - c_1(2+c_1^2)c_3)e_n^3}{c_1^2} + \left(\frac{4c_2^3}{c_1^3 - c_2c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4} \right) e_n^4 + O(e_n^5). \end{aligned} \tag{10}$$

Now, substituting (10) in the Taylor series of $f(y_n)$, we have

$$f(y_n) = c_2e_n^2 - \frac{(2c_2^2 - c_1(2+c_1^2)c_3)e_n^3}{c_1} + \left(\frac{c_2^3}{c_1^2} + c_1 \left(\frac{4c_2^3}{c_1^3 - c_2c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4} \right) \right) e_n^4 + O(e_n^5). \tag{11}$$

From (6) and (11) we obtain

$$f(y_n) - f(x_n) = -c_1 e_n + \left(-c_3 - \frac{2c_2^2 - c_1(2+c_1^2)c_3}{c_1}\right) e_n^3 + \left(\frac{c_2^3}{c_1^2 - c_4 + c_1}\left(\frac{4c_2^3}{c_1^3 - c_2c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4}\right)\right) e_n^4 + O(e_n^5) \quad (12)$$

and

$$2f(y_n) - f(x_n) = -c_1 e_n + c_2 e_n^2 + \left(-c_3 - \frac{2(2c_2^2 - c_1(2+c_1^2)c_3)}{c_1}\right) e_n^3 + \left(-c_4 + 2\left(\frac{c_2^3}{c_1^2} + c_1\left(\frac{4c_2^3}{c_1^3} - c_2c_3 - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1} + 4c_1c_4\right)\right)\right) e_n^4 + O(e_n^5).$$

Taking into account (9), (12) and the last expression, we finally obtain

$$e_{n+1} = -c_2\left(-\frac{c_2^2}{c_1^3} + c_3 + \frac{c_3}{c_1^2}\right) e_n^4 + O(e_n^5).$$

This proves that the method is of fourth order. □

Theorem 2. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the improved Ostrowski's method free from derivatives defined by (5) has order of convergence 6 and satisfies the error equation*

$$e_{n+1} = \frac{(-2c_2^2 + c_1(1+c_1^2)c_3)(-c_2^3 + c_1(1+c_1^2)c_2c_3)}{c_1^5} e_n^6 + O(e_n^7).$$

Proof. Let $e_n = x_n - \alpha$. The Taylor series of $f(x_n)$ about α is

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7), \quad (13)$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!}$, $k = 1, 2, \ldots$.

Computing the Taylor series of $f(x_n + f(x_n))$ and substituting $f(x_n)$ by (13) we have

$$\begin{aligned} f(x_n + f(x_n)) = {} & c_1(1+c_1)e_n + (c_1 + (1+c_1)^2)c_2 e_n^2 + (2(1+c_1)c_2^2 + c_1c_3 + (1+c_1)^3 c_3)e_n^3 \\ & + (3(1+c_1)^2 c_2c_3 + c_2(c_2^2 + 2(1+c_1)c_3) + c_1c_4 + (1+c_1)^4 c_4)e_n^4 \\ & + (3(1+c_1)c_3(c_2^2 + c_3 + c_1c_3) + 4(1+c_1)^3 c_2c_4 + 2c_2(c_2c_3 + c_4 + c_1c_4) + c_1c_5 \\ & + (1+c_1)^5 c_5)e_n^5 + (2(1+c_1)^2(3c_2^2 + 2(1+c_1)c_3)c_4 + c_3(c_2^3 + 6(1+c_1)c_2c_3 + 3(1+c_1)^2 c_4) \\ & + 5(1+c_1)^4 c_2c_5 + c_2(c_3^2 + 2(c_2c_4 + c_5 + c_1c_5)) + c_1c_6 + (1+c_1)^6 c_6)e_n^6 + O(e_n^7). \end{aligned} \quad (14)$$

The Taylor series of $f(x_n - f(x_n))$ is

$$\begin{aligned} f(x_n - f(x_n)) = {} & -(-1+c_1)c_1 e_n + (1 - 3c_1 + c_1^2)c_2 e_n^2 + (2(-1+c_1)c_2^2 - (-1+c_1)^3 c_3 - c_1c_3)e_n^3 \\ & + (-3(-1+c_1)^2 c_2c_3 + c_2(c_2^2 + 2(-1+c_1)c_3) + (-1+c_1)^4 c_4 - c_1c_4)e_n^4 \\ & + (-3(-1+c_1)c_3(c_2^2 + (-1+c_1)c_3) + 4(-1+c_1)^3 c_2c_4 \\ & + 2c_2(c_2c_3 + (-1+c_1)c_4) - (-1+c_1)^5 c_5 - c_1c_5)e_n^5 + (-c_2^3 c_3 + 2(4 - 6c_1 + 3c_1^2)c_2^2 c_4 \\ & + (-1+c_1)^2(-7+4c_1)c_3c_4 + c_2((7-6c_1)c_3^2 + (-7 + 22c_1 - 30c_1^2 + 20c_1^3 - 5c_1^4)c_5) \\ & + (1 - 7c_1 + 15c_1^2 - 20c_1^3 + 15c_1^4 - 6c_1^5 + c_1^6)c_6)e_n^6 + O(e_n^7). \end{aligned} \quad (15)$$

Substituting (14) and (15) in the expression for y_n in (5) gives us

$$\begin{aligned} y_n - \alpha &= x_n - \alpha - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \\ &= e_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \\ &= \frac{c_2 e_n^2}{c_1} - \frac{(2c_2^2 - c_1(2+c_1^2)c_3)e_n^3}{c_1^2} + \left(\frac{4c_2^3}{c_1^3} - c_2c_3 - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1} + 4c_1c_4\right)e_n^4 \\ &\quad - \frac{1}{c_1^4}(8c_2^4 - c_1(20 + 3c_1^2)c_2^2c_3 + 2c_1^2(5 + 2c_1^2)c_2c_4 + c_1^2((6 + 3c_1^2 + c_1^4)c_3^2 - c_1(4 + 10c_1^2 + c_1^4)c_5))e_n^5 \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{c_1^5}(-16c_2^5+c_1(52+7c_1^2)c_2^3c_3-4c_1^2(7+3c_1^2)c_2^2c_4-c_1^2c_2((33+12c_1^2+c_1^4)c_3^2\\
&+c_1(-13-10c_1^2+c_1^4)c_5)+c_1^3((17+17c_1^2+8c_1^4)c_3c_4-c_1(5+20c_1^2+6c_1^4)c_6))e_n^6+O(e_n^7).
\end{aligned}\tag{16}
$$

Now, substituting (16) in the Taylor series of $f(y_n)$ we have

$$
\begin{aligned}
f(y_n) = {}& c_2e_n^2+\left(-\frac{2c_2^2}{c_1}+2c_3+c_1^2c_3\right)e_n^3+\left(\frac{5c_2^3}{c_1^2}-\frac{7c_2c_3}{c_1}-c_1c_2c_3+3c_4+4c_1^2c_4\right)e_n^4\\
&+\frac{1}{c_1^3}(-12c_2^4+c_1(24+5c_1^2)c_2^2c_3-2c_1^2(5+2c_1^2)c_2c_4+c_1^2(-(6+3c_1^2+c_1^4)c_3^2\\
&+c_1(4+10c_1^2+c_1^4)c_5))e_n^5+\frac{1}{c_1^4}(28c_2^5-c_1(73+13c_1^2)c_2^3c_3+2c_1^2(17+10c_1^2)c_2^2c_4\\
&+c_1^2c_2((37+16c_1^2+2c_1^4)c_3^2+c_1(-13-10c_1^2+c_1^4)c_5)+c_1^3(-(17+17c_1^2+8c_1^4)c_3c_4\\
&+c_1(5+20c_1^2+6c_1^4)c_6))e_n^6+O(e_n^7).
\end{aligned}\tag{17}
$$

Using (13), (16) and (17) in (5) gives

$$
\begin{aligned}
z_n-\alpha = {}& y_n-\mu_nf(y_n)=\frac{c_2(c_2^2-c_1(1+c_1^2)c_3)e_n^4}{c_1^3}\\
&-\frac{(4c_2^4-2c_1(4+c_1^2)c_2^2c_3+c_1^2(2+3c_1^2+c_1^4)c_3^2+2c_1^2(1+2c_1^2)c_2c_4)e_n^5}{c_1^4}\\
&-\frac{1}{c_1^5}(-10c_2^5+2c_1(15+2c_1^2)c_2^3c_3-4c_1^2(3+2c_1^2)c_2^2c_4+c_1^3(7+17c_1^2+8c_1^4)c_3c_4\\
&+c_1^2c_2((-18-8c_1^2+c_1^4)c_3^2+c_1(3+10c_1^2+c_1^4)c_5))e_n^6+O(e_n^7)
\end{aligned}\tag{18}
$$

and substituting (18) in the Taylor series of $f(z_n)$ we have

$$
\begin{aligned}
f(z_n) = {}& \frac{c_2(c_2^2-c_1(1+c_1^2)c_3)e_n^4}{c_1^2}-\frac{(4c_2^4-2c_1(4+c_1^2)c_2^2c_3+c_1^2(2+3c_1^2+c_1^4)c_3^2+2c_1^2(1+2c_1^2)c_2c_4)e_n^5}{c_1^3}\\
&-\frac{1}{c_1^4}(-10c_2^5+2c_1(15+2c_1^2)c_2^3c_3-4c_1^2(3+2c_1^2)c_2^2c_4+c_1^3(7+17c_1^2+8c_1^4)c_3c_4\\
&+c_1^2c_2((-18-8c_1^2+c_1^4)c_3^2+c_1(3+10c_1^2+c_1^4)c_5))e_n^6+O(e_n^7).
\end{aligned}\tag{19}
$$

Taking into account (18) and (19), we finally obtain

$$
\begin{aligned}
e_{n+1} &= z_n-\alpha-\mu_nf(z_n)\\
&=\frac{(-2c_2^2+c_1(1+c_1^2)c_3)(-c_2^3+c_1(1+c_1^2)c_2c_3)}{c_1^5}e_n^6+O(e_n^7).
\end{aligned}\tag{20}
$$

This proves that the method is of sixth order. □

It is easy to observe that the method *ODF* uses four functional evaluations per step, whereas the *IODF* needs five. There are many techniques for obtaining high order iterative methods, but the complexity of the iterative expressions increase considerably. So we have introduced in [13], in the context of nonlinear systems, a new index in order to compare the different methods, taking into account not only the number of functional evaluations, but also the number of products and quotients involved in each step of the iterative process. The *computational efficiency index* is defined as $CI = p^{1/(d+op)}$, where p is the order of convergence, d is the number of functional evaluations per step and op is the number of products and quotients per iteration.

In the next table we present the order of convergence, the efficiency index and the computational efficiency index of the Steffensen's like methods mentioned in Section 1 and our new methods (see Table 1).

We can observe the position of our methods in relation to the other ones, taking into account the efficiency index:

$$I_{FM} = I_{JM} > I_{IODF} > I_{ODF} = I_{SM} = I_{AM} > I_{ZM} = I_{DM}$$

and the computational efficiency index:

$$CI_{IODF} > CI_{ODF} = CI_{SM} = CI_{AM} > CI_{DM} > CI_{JM} > CI_{ZM} > CI_{FM}.$$

Table 1
Order and efficiency indices of some derivative free methods.

Method	Order	Efficiency index	Comp. efficiency index
Steffensen (*SM*)	2	$2^{1/2}$	$2^{1/(2+2)}$
Jain (*JM*)	3	$3^{1/3}$	$3^{1/(3+6)}$
Feng–He (*FM*)	3	$3^{1/3}$	$3^{1/(3+8)}$
Zheng et al. (*ZM*)	3	$3^{1/4}$	$3^{1/(4+6)}$
Amat–Busquier (*AM*)	2	$2^{1/2}$	$2^{1/(2+2)}$
Dehghan–Hajarian (*DM*)	3	$3^{1/4}$	$3^{1/(4+4)}$
ODF	4	$4^{1/4}$	$4^{1/(4+4)}$
IODF	6	$6^{1/5}$	$6^{1/(5+5)}$

Table 2
Numerical results for nonlinear equations from (a)–(j).

$f(x)$	x_0	Iterations					ρ				
		NM	*OM*	*IOM*	*ODF*	*IODF*	*NM*	*OM*	*IOM*	*ODF*	*IODF*
(a)	1	9	5	5	5	5	2.00	4.00	6.00	4.00	6.00
(b)	0.7	7	5	4	5	6	2.00	4.00	6.00	4.00	5.99
(c)	1	8	5	4	5	5	2.00	4.00	6.00	3.80	6.00
(d)	1.5	11	6	5	6	6	2.00	4.00	6.00	4.00	6.00
(e)	2	8	5	4	5	6	2.00	4.00	6.00	4.00	5.99
(f)	1	9	5	4	6	NC	2.00	4.00	6.00	4.00	–
(g)	1	9	5	4	5	5	2.00	4.00	6.00	4.00	6.00
(h)	1.5	8	5	4	6	6	2.00	4.00	6.00	4.00	6.01
(i)	1	9	5	4	5	6	2.00	4.00	6.00	4.00	5.99
(j)	1	8	5	5	5	5	3.00	5.00	7.00	5.00	7.00
(j)	2.5	NC	NC	5	8	6	–	–	7.00	5.00	7.00

4. Numerical results

In this section we check the effectiveness of the new methods *ODF* and *IODF* applied to obtain the solutions of several nonlinear equations. We use equations (a)–(j) to compare the methods described with their counterparts that make use of derivatives, that is, Ostrowski's method (*OM*) and the improved Ostrowski's method (*IOM*) and the classical Newton's method (*NM*).

(a) $f(x) = \sin^2 x - x^2 + 1, \alpha \approx 1.404492$,
(b) $f(x) = x^2 - e^x - 3x + 2, \alpha \approx 0.257530$,
(c) $f(x) = \cos x - x, \alpha \approx 0.739085$,
(d) $f(x) = (x-1)^3 - 1, \alpha = 2$,
(e) $f(x) = x^3 - 10, \alpha \approx 2.154435$,
(f) $f(x) = \cos(x) - xe^x + x^2, \alpha \approx 0.639154$,
(g) $f(x) = e^x - 1.5 - \arctan(x), \alpha \approx 0.767653$,
(h) $f(x) = x^3 + 4x^2 - 10, \alpha \approx 1.365230$,
(i) $f(x) = 8x - \cos(x) - 2x^2, \alpha \approx 0.128077$,
(j) $f(x) = \arctan(x), \alpha = 0$.

Numerical computations have been carried out using variable precision arithmetic with 256 digits in MATLAB 7.1. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_k)| < 10^{-100}$; therefore, we check that the iterates in succession converge to an approximation to the solution of the nonlinear equation. For every method, we count the number of iterations needed to reach the wished for tolerance and estimate the computational order of convergence (ACOC), according to (see [14])

$$\rho = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}. \tag{21}$$

The value of ρ that appears in Table 2 is the last coordinate of vector ρ when the variation between its values is small. A comparison between methods using derivatives and derivative free methods can be established. The behavior of the new methods is similar to that of the classical ones of the same order of convergence, as theoretical results show. It can be observed that the new methods need more iterations than their counterparts, in some cases, but when the initial estimation is not good and methods using derivatives diverge, derivative free methods *ODF* and *IODF* converge quickly.

5. Conclusions

We have used central-difference approximations for the first derivative in Ostrowski's method, that has order of convergence 4, and in an improved version of Ostrowski's method with sixth order of convergence, obtaining two new iterative methods for nonlinear equations free from derivatives, and we have proven that they preserve their convergence order. The theoretical results have been checked with some numerical examples, comparing our algorithms with Newton's method and with the corresponding methods that make use of derivatives. We have compared some Steffensen like methods with our methods from the point of view of the efficiency index and computational efficiency index.

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