Geometry  **Circles**

Very Important Stuff

Alright here's a list of things you should always try when looking at circles.

1. If there are tangents, please draw in the lines connecting them. Makes your life like 2,000 times easier. The reason they include tangents is so you can draw in the tangent lines.

2. Know your Power of a Point stuff. This helps A LOT. Know both the ones that relate to sides and angles because those are flippin op.

3. As a result of having a lot of circles, stuff like right angles tends to naturally show up (ex. from tangents, angles subtending the diameter). As a result, keep in mind the Pythagorean Theorem and Similar Triangles, because Right Angles=This stuff.

4. **Ptolemy's Theorem**, if you have a cyclic quad and need side lengths.

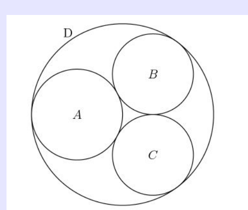
5. Drawing lengths from the center of a circle to intersection points can help a ton as well.

6. **Kissing Circles**
   The curvature k of a circle is the reciprocal of its radius. Give 4 circles, if they are all tangent to each other (so we have exactly 6 tangent points), then the following equation holds:

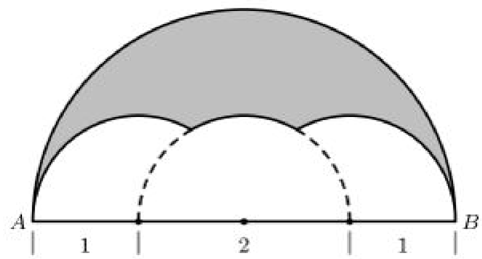
$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_1k_3 + k_2k_3}$$

   Where each k represents the curvature of a different circle. When solving this equation, you will get two solutions because of the plus minus- take the one that makes the most sense.
   Note that the curvature will always be positive unless the other three circles are all internally tangent to it. For example, in the following diagram, the curvatures of the circles would be

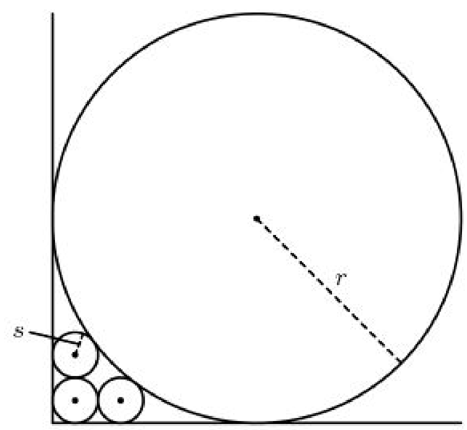
$$\frac{1}{r_A}, \frac{1}{r_B}, \frac{1}{r_C}, -\frac{1}{r_D}$$



Problem 1. (3$\star$)
Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. Let $\frac{a\pi - b\sqrt{c}}{d}$ be the area of the shaded region that lies within the large semicircle but outside the smaller semicircles, where $\gcd(a, b, d) = 1$ and c is square-free. What is the value of $a + b + c + d$?



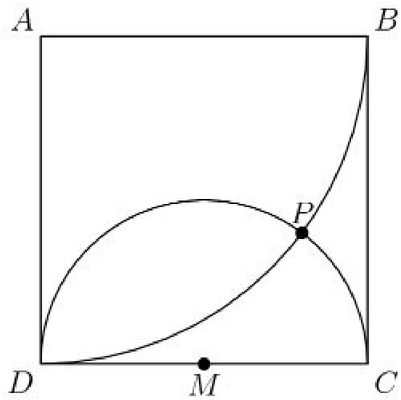
Problem 2. (3$\star$)
Three circles of radius 1 are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > 1$ is tangent to both axes and to the second and third circles. What is r?



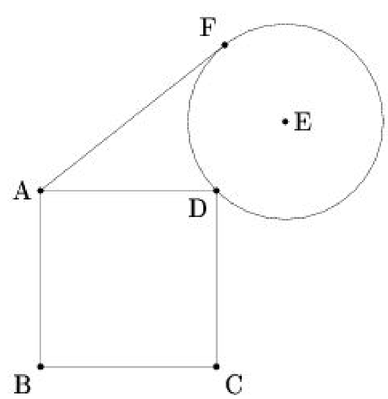
Problem 3. (2$\star$)
Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent line intersects the circles at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD?

Problem 4. (5$\star$)
Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

Problem 5. (5$\star$)
Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?

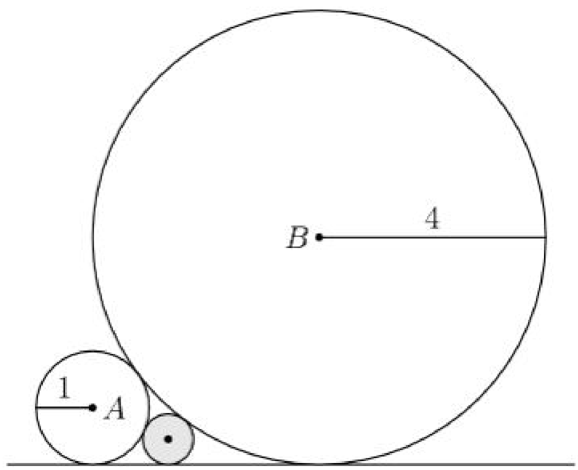


Problem 6. (11$\star$)
Square $ABCD$ has side length s, a circle centered at E has radius r, and r and s are both rational. The circle passes through D, and D lies on $\overline{BE}$. Point F lies on the circle, on the same side of $\overline{BE}$ as A. Segment AF is tangent to the circle, and $AF = \sqrt{9 + 5\sqrt{2}}$. What is $\frac{r}{s}$?



Problem 7. (7$\star$)
Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency? Hint: Use sine area formula to find unwanted chunks.

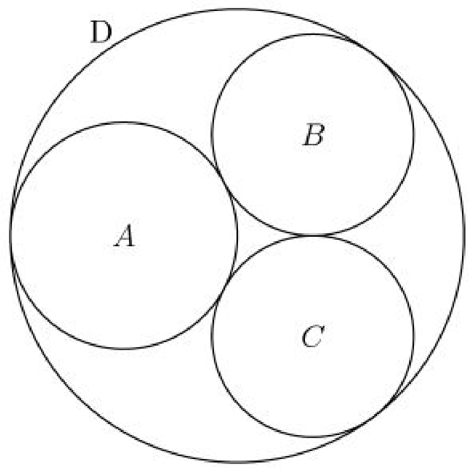
Problem 8. (5$\star$)
A circle centered at A with a radius of 1 and a circle centered at B with a radius of 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as shown. What is the radius of the third circle? Hint: Lines have curvature 0 and can technically be thought of as circles with infinite radius.



Problem 9. (7$\star$)
Let C_1 and C_2 be circles defined by $(x-10)^2+y^2=36$ and $(x+15)^2+y^2=81$ respectively. What is the length of the shortest line segment PQ that is tangent to C_1 at P and to C_2 at Q?

Problem 10. (5$\star$)
Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?

Problem 11. (5⋆)
Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?



Problem 12. (11⋆)
Andrea inscribed a circle inside a regular pentagon, circumscribed a circle around the pentagon, and calculated the area of the region between the two circles. Bethany did the same with a regular heptagon. The areas of the two regions were A and B, respectively. Each polygon had a side length of 2. Let k be a real number such that $A = k \cdot B$. What is the value of k?

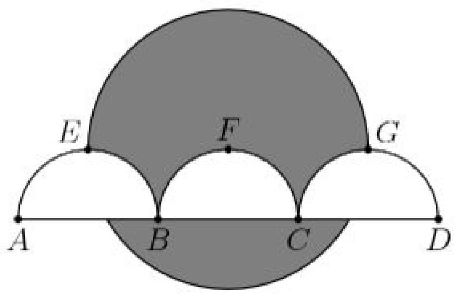
Problem 13. (3⋆)
In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have **integer lengths**. What is BC? Hint: Triangle Inequality, Power of a Point, Number Theory tricks.

Problem 14. Required. (7⋆)
In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. What is the length of segment $\overline{DF}$? Hint: If opposite angles add up to 180 degrees in a quadrilateral, it spreads diseases through its tongue (sick lick).

Problem 15. Required. (11⋆)
As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, defined by the points $A\overset{\frown}{E}B, B\overset{\frown}{F}C$, and $C\overset{\frown}{G}D$, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

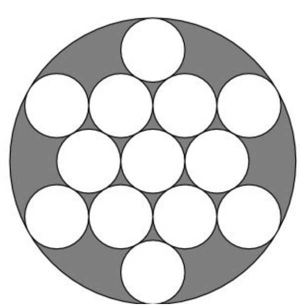
$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where a, b, c, and d are positive integers and a and b are relatively prime. What is $a+b+c+d$?

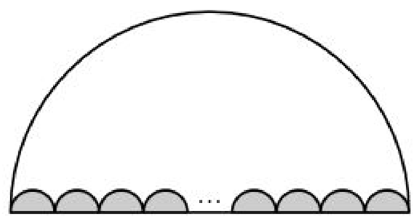


Problem 16. (3⋆)
Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

Problem 17. (3⋆)
The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. The area of the shaded region in the figure, inside the larger circle but outside all the circles of radius 1, can be expressed as $a\pi\sqrt{b}$, where a and b are integers and b is square-free. Find the value of $a + b$.



Problem 18. Required. (7⋆)
In the figure below, N congruent semicircles lie on the diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the semicircles. The ratio $A : B$ is $1 : 18$. What is N? (Don't you dare guess random values or I WILL find you).



Problem 19. (5⋆)
In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

Problem 20. (3⋆)
The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

Problem 21. (3⋆)
Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. If the fraction of the area of $\triangle ABC$ that lies outside the circle can be expressed as $\frac{a-b\pi\sqrt{c}}{d}$, where $\gcd(a, b, d) = 1$ and c is square-free, find the value of $a + b + c + d$.