

Very Important Stuff

Alright here's a list of things you should always try when doing NT.

1. Divisibility Rules Help!
2. Factoring is VERY VERY OP.
3. Budget Euclidean Algorithm- Like if we have that 5 divides $N(N - 2)$, then since the numbers differ by 2 they can't both be divisible by 5, so either $5 \mid N$ or $5 \mid N - 2$. Similarly, if we have 5 divides $N(N^2 + 1)$, since $N^2 + 1$ is 1 more than N times N , if N is divisible by 5 then obviously $N^2 + 1$ cannot be, and if $N^2 + 1$ is divisible by 5 then N cannot be.
4. Taking mods, I guess, although this has a 50% chance of doing nothing.
5. Honestly idk man, just do these problems. Like literally just try whatever comes to mind.

Problem 1. (2★)

Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99 \dots 99}_{321 \text{ digits}}.$$

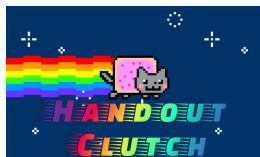
Find the sum of the digits of N .

Problem 2. (3★)

A positive integer N has base-eleven representation $\underline{a}\underline{b}\underline{c}$ and base-eight representation $\underline{1}\underline{b}\underline{c}\underline{a}$, where a, b , and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

Problem 3. (5★)

Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N . For example, 42,020 is in S because 4 is a divisor of 42,020. Find the sum of all the digits of all the numbers in S . For example, the number 42,020 contributes $4 + 2 + 0 + 2 + 0 = 8$ to this total.



Problem 4. (3★)

Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$abc = 70,$$

$$cde = 71,$$

$$efg = 72.$$

Problem 5. (7★)

Call a positive integer n k -pretty if n has exactly k positive divisors and n is divisible by k . For example, 18 is 6-pretty. Let S be the sum of positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

Problem 6. (5★)

There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2 \log_{10}(\gcd(x, y)) = 60$$

$$\log_{10} y + 2 \log_{10}(\text{lcm}(x, y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x , and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y . Find $3m + 2n$.

Problem 7. (5★)

Find the sum of all positive integers $b < 1000$ such that the base- b integer 36_b is a perfect square and the base- b integer 27_b is a perfect cube.

Problem 8. (3★)

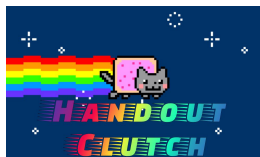
The number n can be written in base 14 as $\underline{a} \underline{b} \underline{c}$, can be written in base 15 as $\underline{a} \underline{c} \underline{b}$, and can be written in base 6 as $\underline{a} \underline{c} \underline{a} \underline{c}$, where $a > 0$. Find the base-10 representation of n .

Problem 9. (5★)

A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are 189, 320, 287, 234, x , and y . Find the greatest possible value of $x + y$.

Problem 10. (5★)

Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer.



Problem 11. (3★)

When each of 702, 787, and 855 is divided by the positive integer m , the remainder is always the positive integer r . When each of 412, 722, and 815 is divided by the positive integer n , the remainder is always the positive integer $s \neq r$. Find $m + n + r + s$.

Problem 12. (2★)

For a positive integer n , let d_n be the units digit of $1 + 2 + \cdots + n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

Problem 13. Required. (3★)

A rational number written in base eight is $\underline{ab.cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb.ba}$. Find the base-ten number \underline{abc} .

Problem 14. Required. (7★)

Let $a_{10} = 10$, and for each positive integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least positive $n > 10$ such that a_n is a multiple of 99. (Hint: take the recursion sequence mod 99)

Problem 15. (3★)

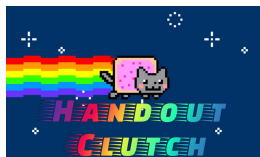
An $a \times b \times c$ rectangular box is built from $a \cdot b \cdot c$ unit cubes. Each unit cube is colored red, green, or yellow. Each of the a layers of size $1 \times b \times c$ parallel to the $(b \times c)$ faces of the box contains exactly 9 red cubes, exactly 12 green cubes, and some yellow cubes. Each of the b layers of size $a \times 1 \times c$ parallel to the $(a \times c)$ faces of the box contains exactly 20 green cubes, exactly 25 yellow cubes, and some red cubes. Find the smallest possible volume of the box.

Problem 16. (2★)

Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by 1000.

Problem 17. (2★)

Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m .


Problem 18. (5★)

The expressions $A = 1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 37 \times 38 + 39$ and $B = 1 + 2 \times 3 + 4 \times 5 + \cdots + 36 \times 37 + 38 \times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B .

Problem 19. (11★)

There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p . (Hint: factoring and parity)

Problem 20. (5★)

Find the number of rational numbers r , $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and the denominator have a sum of 1000.

Problem 21. Required. (11★)

The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc .

Problem 22. (3★)

A positive integer divisor of $12!$ is chosen at random. What is the probability that the divisor chosen is a perfect square?

Problem 23. Required. (7★)

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!\cdots a_m!}{b_1!b_2!\cdots b_n!}$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

Problem 24. (2★)

What is the hundreds digit of $(20! - 15!)$?