Sequences

Very Important Stuff

1. Writing out the first few cases/terms of a sequence and finding a pattern sometimes works (but please do not do that on this handout :D)
2. Please organize your work, it is very important. Don't fricking write all over the paper like I see you doing.
3. New rule: You cannot use a sheet of paper for scratch work if you used it for a different problem.

Problem 1. $(2\star)$
An increasing series $2, 3, 5, 6, 7, 10, 11$ is formed by removing all perfect squares and perfect cubes. What is the 1991st term?

Problem 2. $(2\star)$
Given that $a_2 = 2$ and for $n \geq 1$, $a_{n+1} = \frac{1+a_n}{1-a_n}$, find a_{2017}.

Problem 3. $(3\star)$
Given that $a_1 = 7$ and $a_{n+1} = \sqrt{|a_n^2 - 16|}$, find the value of a_{2018}.

Problem 4. $(5\star)$
Given that $a_1 = 1$ and for a series of numbers $a_1, a_2, \cdots a_n$, we have that

$$a_{n+1} = \left(1 + \frac{1}{n}\right) a_n + \frac{1}{n}$$

Find a_{1990}.

Problem 5. $(5\star)$
The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

Problem 6. $(7\star)$
A sequence of integers $a_1, a_2, a_3, \cdots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

More Very Important Stuff

1. The formula for an arithmetic series is

$$\frac{(x+y)\cdot n}{2}$$

Where x is the first term, y is the last term, and n is the number of terms.

2. If you're given three consecutive terms x, y, z of an arithmetic sequence, please please please please please please don't look at the common ratio or some garbage (unless you really have to, because common ratio does something important). Do $2y = x + z$. Same goes for if we have like w, x, y, z. Please do $2x = w + y, 2y = x + z$.

Problem 7. ($2\star$)
Determine the sum of the first 50 terms of an arithmetic sequence (a_n) with $a_2 = -2$ and $a_7 = 28$.

Problem 8. ($3\star$)
The fourth term of a particular infinite arithmetic sequence is 203, and the thirteenth term is 167. What is the smallest value of n such that the n^{th} term of the sequence is negative?

Problem 9. Required. ($7\star$)
The first four terms of an arithmetic sequence, in order, are $x + y, x - y, xy, \frac{x}{y}$, in that order. What is the fifth term?

Problem 10. ($5\star$)
The roots of the polynomial $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. Find them.

Problem 11. ($5\star$)
Find all value of k such that $x^4 - (3k+4)x^2 + k^2 = 0$ has 4 real roots in arithmetic progression.

Problem 12. ($3\star$)
Find the value of $a_2 + a_4 + \cdots + a_{98}$ if $a_1, a_2, \cdots, a_{98}$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + \cdots + a_{98} = 137$.

Problem 13. (3⋆)
Let $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be arithmetic progressions such that $a_1 = 25, b_1 = 75$, and $a_{100} + b_{100} = 100$. Find the sum of the first 100 terms of the sequence $a_1 + b_1, a_2 + b_2, \cdots$.

Problem 14. (3⋆)
In a certain arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is $\frac{r^2}{s^2}$ for any r, s. Find the ratio of the 8^{th} term to the 23^{rd} term.

Problem 15. (11⋆)
Find the 8^{th} term of the sequence $1440, 1716, 1848, \cdots$ whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.

Problem 16. (3⋆)
If the integer k is added to each of the numbers $36, 300, 596$, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k.

More More Very Important Stuff

1. The formula for a geometric series is

$$\frac{a \cdot (r^n - 1)}{r - 1}$$

   Where a is the first term, r is the common difference. You can derive the formula by letting $n = \infty$, and since in infinite geometric series we have $|r| < 1$, $r^\infty = 0$. I assume you can do the rest from here :)

2. If you're given three consecutive terms x, y, z of a geometric sequence, please please please please please please don't look at the common ratio or some garbage (unless you really have to, because common ratio does something important). Do $y^2 = xz$. Same goes for if we have like w, x, y, z. Please do $x^2 = wy, y^2 = xz$.

Problem 17. (3⋆)
Call a 3-digit number geometric if it has 3 distinct digits, which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

Problem 18. Required. (5$\star$)
Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If $x, 2y, 3z$ is an arithmetic sequence, find the value of r.

Problem 19. (5$\star$)
The roots $2x^3 - 19x^2 + kx - 54$ are in geometric progression for some constant k. Find k.

Problem 20. (5$\star$)
An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is m where m and n are relatively prime integers. Find $m + n$.

Problem 21. Required. (5$\star$)
The sum of the first 2011 terms of a geometric series is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

Problem 22. (3$\star$)
The sum of three consecutive terms in a geometric series is 39, and the sum of their squares is 741. Find the three terms.

Problem 23. (2$\star$)
Find the sum

$$1 + 2 + 3 + 6 + 9 + 18 + \cdots + 729 + 1458$$

Problem 24. (2$\star$)
Find the value of x satisfying $1 + x + x^2 + \cdots = 4$

Problem 25. (2$\star$)
Find all real values of a such that

$$\frac{(a^8 + a^4 + 1)(a^3 + a^2 + a + 1)}{a^9 + a^6 + a^3 + 1} = 21$$

Problem 26. (3$\star$)
If a and b are the roots of $11x^2 - 4x - 2 = 0$, then compute the product

$$(1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots)$$

Problem 27. (5$\star$)
The roots of the equation

$$x^5 - 90x^4 + Px^3 + Qx^2 + Rx + S = 0$$

Are in a geometric progression. The sum of their reciprocals is 5. Find $|S|$.

Problem 28. (3$\star$)
Let S_n be the sum of an infinite geometric series with first term a^n and common ratio r^n for $n \geq 1$. If $|r| < 1, |a| > 1$, determine the sum

$$\frac{1}{S_1} + \frac{1}{S_2} + \cdots$$

In terms of a, r.

Problem 29. (5$\star$)
In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.