Algebra  **Polynomials**

Very Important Stuff

1. A number r is a zero of the polynomial $P(x)$ if and only if $P(x)$ has a factor of the form $(x - r)$.

2. The graph of a quadratic equation is a parabola who's vertex is at the value of x that makes the squared term 0.

$$x = -\frac{b}{2a}$$

   Plugging it back in gives its y-coordinate (please double check if you need the x or y coordinate in problems).

   - If $a > 0$ then the parabola opens upwards and the vertex gives the MINIMAL value.
   - If $a < 0$ then the parabola opens downwards and the vertex gives the MAXIMAL value.
   - Very good for finding maximum or minimum of a function!

3. And yes, discriminant is op. It is $b^2 - 4ac$ if given a quadratic $P(x) = ax^2 + bx + c$.

   - If $b^2 - 4ac > 0$ there are two DISTINCT, REAL roots.
   - If $b^2 - 4ac > 0$ there are two IDENTICAL, REAL roots.
   - If $b^2 - 4ac < 0$ there are two DISTINCT, COMPLEX roots.

Problem 1. $(2\star)$
Suppose that f is a polynomial such that

$$(x - 1) \cdot f(x) = 3x^4 + x^3 - 25x^2 + 38x - 17$$

(a) What is the degree of f?
(b) What is the leading term of $f(x)$?
(c) What is the constant term of $f(x)$?
(d) Find $f(x)$.

Problem 2. $(2\star)$
A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$ and $(2, 0)$ is on the graph of the parabola. What is abc?

Problem 3. $(3\star)$
The polynomial functions F, G satisfy $F(y) = 3y^2 - y + 1$ and $F(G(y)) = 12y^4 - 62y^2 + 81$.

What are all possible values for the sum of the coefficients of $G(y)$?

Problem 4. ($5\star$)
The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

Problem 5. ($5\star$)
For how many integers x is the number $x^4 - 51x^2 + 50$ negative?

More Very Important Stuff

1. If you have a polynomial with degree above 2 and you need to get the roots, don't give up and immediately assume you did it wrong. Try plugging in some random possible integer roots into the polynomial, and if $P(r) = 0$, you can divide $P(x)$ by $(x - r)$ to lower the degree of your polynomial (hopefully to a quadratic!)

   - Roots to Try $-2, -1, 0, 1, 2$. And whatever else might work.
   - Rational Root Theorem To make sure you don't guess randomly, use the Rational Root Theorem! Any rational root that divides the polynomial must be of the form
   $$\frac{\text{factor of } a_0}{\text{factor of } a_n}$$
   - Where a_0 is coefficient of constant term and a_n is coefficient of leading term. Note that the factors don't have to be positive.

2. Vieta's Formulas (but not just the 2-degree version bruh)

   - If we have $y = ax^2 + bx + c$, sum of roots is $-\frac{b}{a}$, product is $\frac{c}{a}$.
   - If we have $y = ax^3 + bx^2 + cx + d$, sum of roots is $-\frac{b}{a}$, $r_1r_2 + r_2r_3 + r_1r_3$ is $\frac{c}{a}$, product is $-\frac{d}{a}$.
   - For $y = ax^4 + bx^3 + cx^2 + dx + e$, I'm far too lazy to type it out, but sum of roots will always be $-\frac{b}{a}$ and the continue by switching signs. Since b neg, c pos, d neg, e pos, product of roots is $\frac{e}{a}$.
   - And so on for polynomials of higher degrees.

Problem 6. ($3\star$)
Find a constant c such that there is no remainder when $x^3 + cx^2 + 4x - 21$ is divided by $x - 3$. (Hint: Plug in 3).

Problem 7. (5⋆)
Find a if the remainder is constant (aka an integer) when $x^3 + 3x^2 + ax + 13$ is divided by $x^2 + 3x - 2$.

Problem 8. (7⋆)
YOU MUST DO THIS PROBLEM FIRST BEFORE AND PROBLEMS OF HIGHER PROBLEM NUMBERS. Find the remainder when x^{100} is divided by $(x-1)(x-2)$.

Problem 9. (5⋆)
Find the remainder when $x^{100} - 4x^{98} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$.

Problem 10. (5⋆)
$f(x)$ is a polynomial of degree greater than 3. If $f(1) = 2, f(2) = 3, f(3) = 5$, find the remainder when $f(x)$ is divided by $(x-1)(x-2)(x-3)$.

Problem 11. (3⋆)
Find the remainder when $x^{13} + 1$ is divided by $x - 1$.

Problem 12. (5⋆)
Find the remainder when $13x^6 + 3x^4 + 9x^3 + 2x^2 + 17$ is divided by $x^2 - 1$.

Problem 13. Required. (7⋆)
When

$$P(x) = x^{81} + Lx^{57} + Gx^{41} + Hx^{19} + 2x + 1$$

Is divided by $x - 1$, the remainder is 5, and when $P(x)$ is divided by $x - 2$, the remainder is -4. However,

$$x^{81} + Lx^{57} + Gx^{41} + Hx^{19} + Kx + 4$$

is exactly divisible by $(x-1)(x-2)$. If L, G, H, K, R are all real, compute the ordered pair (K, R).

Problem 14. (5⋆)
Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 + x$.

Problem 15. (7⋆)
Find a polynomial $f(x)$ of degree 5 such that $f(x) - 1$ is divisible by $(x-1)^3$ and $f(x)$ is divisible by x^3.

Problem 16. Required. (5⋆)
Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

Problem 17. (7⋆)
Find the sum of all roots, real and nonreal, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001}$, given that there are no multiple roots.

Problem 18. (11⋆)
Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c, c + a$. Find t.

Problem 19. (11⋆)
Find A^2, where A is the sum of the ABSOLUTE VALUES of all roots of the following equation

$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{x}}}}}$$

Problem 20. (11⋆)
Determine $w^2 + x^2 + y^2 + z^2$ given

$$\frac{x^2}{2^2 - 1} + \frac{y^2}{2^2 - 3^2} + \frac{z^2}{2^2 - 5^2} + \frac{w^2}{2^2 - 7^2} = 1$$

$$\frac{x^2}{4^2 - 1} + \frac{y^2}{4^2 - 3^2} + \frac{z^2}{4^2 - 5^2} + \frac{w^2}{4^2 - 7^2} = 1$$

$$\frac{x^2}{6^2 - 1} + \frac{y^2}{6^2 - 3^2} + \frac{z^2}{6^2 - 5^2} + \frac{w^2}{6^2 - 7^2} = 1$$

$$\frac{x^2}{8^2 - 1} + \frac{y^2}{8^2 - 3^2} + \frac{z^2}{8^2 - 5^2} + \frac{w^2}{8^2 - 7^2} = 1$$