Very Important Stuff

Alright here's a list of things you should always try when doing NT.

1. Divisibility Rules Help!
2. Factoring is VERY VERY OP.
3. Budget Euclidean Algorithm- Like if we have that 5 divides $N(N-2)$, then since the numbers differ by 2 they can't both be divisible by 5, so either $5 \mid N$ or $5 \mid N-2$. Similarly, if we have 5 divides $N(N^2+1)$, since N^2+1 is 1 more than N times N, if N is divisible by 5 then obviously N^2+1 cannot be, and if N^2+1 is divisible by 5 then N cannot be.
4. Taking mods, I guess, although this has a 50% chance of doing nothing.
5. Honestly idk man, just do these problems. Like literally just try whatever comes to mind.

Problem 1. (2$\star$)
Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{\text{321 digits}}.$$

Find the sum of the digits of N.

Problem 2. (3$\star$)
A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

Problem 3. (5$\star$)
Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, $42,020$ is in S because 4 is a divisor of $42,020$. Find the sum of all the digits of all the numbers in S. For example, the number $42,020$ contributes $4+2+0+2+0=8$ to this total.

Problem 4. (3$\star$)
Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$\begin{aligned} abc &= 70, \\ cde &= 71, \\ efg &= 72. \end{aligned}$$

Problem 5. (7$\star$)
Call a positive integer n k-pretty if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

Problem 6. (5$\star$)
There are positive integers x and y that satisfy the system of equations

$$\begin{aligned} \log_{10} x + 2\log_{10}(\gcd(x, y)) &= 60 \\ \log_{10} y + 2\log_{10}(\operatorname{lcm}(x, y)) &= 570. \end{aligned}$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

Problem 7. (5$\star$)
Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

Problem 8. (3$\star$)
The number n can be written in base 14 as $\underline{a}\ \underline{b}\ \underline{c}$, can be written in base 15 as $\underline{a}\ \underline{c}\ \underline{b}$, and can be written in base 6 as $\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}$, where $a > 0$. Find the base-10 representation of n.

Problem 9. (5$\star$)
A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are 189, 320, 287, 234, x, and y. Find the greatest possible value of $x + y$.

Problem 10. (5$\star$)
Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer.

Problem 11. (3⋆)
When each of 702, 787, and 855 is divided by the positive integer m, the remainder is always the positive integer r. When each of 412, 722, and 815 is divided by the positive integer n, the remainder is always the positive integer $s \neq r$. Find $m + n + r + s$.

Problem 12. (2⋆)
For a positive integer n, let d_n be the units digit of $1 + 2 + \cdots + n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

Problem 13. Required. (3⋆)
A rational number written in base eight is $\underline{ab}.\underline{cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb}.\underline{ba}$. Find the base-ten number $\underline{abc}$.

Problem 14. Required. (7⋆)
Let $a_{10} = 10$, and for each positive integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least positive $n > 10$ such that a_n is a multiple of 99. (Hint: take the recursion sequence mod 99)

Problem 15. (3⋆)
An $a \times b \times c$ rectangular box is built from $a \cdot b \cdot c$ unit cubes. Each unit cube is colored red, green, or yellow. Each of the a layers of size $1 \times b \times c$ parallel to the $(b \times c)$ faces of the box contains exactly 9 red cubes, exactly 12 green cubes, and some yellow cubes. Each of the b layers of size $a \times 1 \times c$ parallel to the $(a \times c)$ faces of the box contains exactly 20 green cubes, exactly 25 yellow cubes, and some red cubes. Find the smallest possible volume of the box.

Problem 16. (2⋆)
Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by 1000.

Problem 17. (2⋆)
Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Problem 18. (5⋆)
The expressions $A = 1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 37 \times 38 + 39$ and $B = 1 + 2 \times 3 + 4 \times 5 + \cdots + 36 \times 37 + 38 \times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B.

Problem 19. (11⋆)
There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p. (Hint: factoring and parity)

Problem 20. (5⋆)
Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and the denominator have a sum of 1000.

Problem 21. Required. (11⋆)
The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

Problem 22. (3⋆)
A positive integer divisor of 12! is chosen at random. What is the probability that the divisor chosen is a perfect square?

Problem 23. Required. (7⋆)
The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!...a_m!}{b_1!b_2!...b_n!}$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

Problem 24. (2⋆)
What is the hundreds digit of $(20! - 15!)$?