Geometry

Triangles

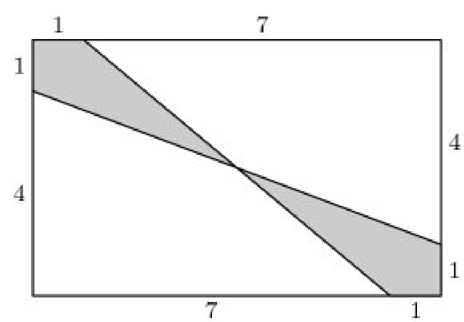
Very Important Stuff

Alright here's a list of things you should always try when looking at triangles.

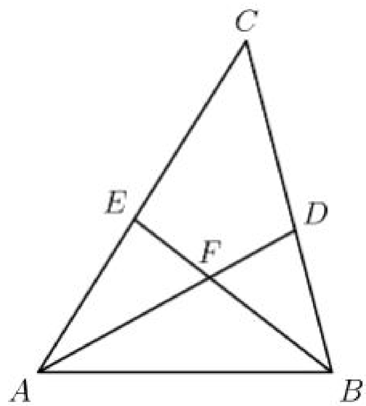
1. Rule Number 1: Similarity works 100% of the time, so there's really no point in trying anything else. As long as you use Rule 1 correctly, you will always get it right, so just try similarity! In the case that you somehow forget Rule 1 (works 100% of the time if you remember it!) like an idiot, here's a couple more rules that might be helpful.
2. They will (almost) never give you extra information in a problem. Make sure you utilize everything they give you.
3. Look out for right-triangles and 30-60-90 shenanigans.
4. If angles aren't working out, try using lengths, and vice versa.
5. Triangle Inequality, sometimes.
6. Mass Points/Angle Bisector Theorem are nice for ratios.
7. Law of Cosines and Law of Sines if you have nice angles.
8. If you are given a point in a triangle and distances to the vertices, rotation + Law of Cosines works every time.
9. And of course, Pythagorean Theorem.

Problem 1. ($2\star$)
All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

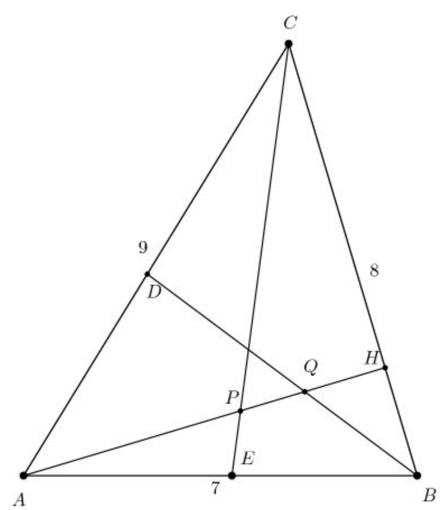
Problem 2. ($3\star$)
Find the area of the shaded region.



Problem 3. (5⋆)
In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $\frac{AF}{FD}$?

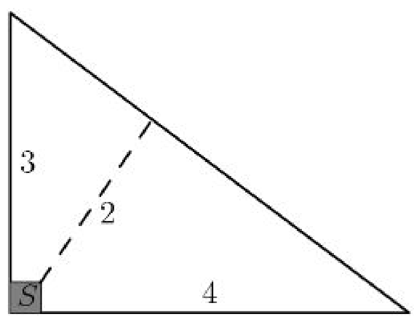


Problem 4. Required. (5⋆)
In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?

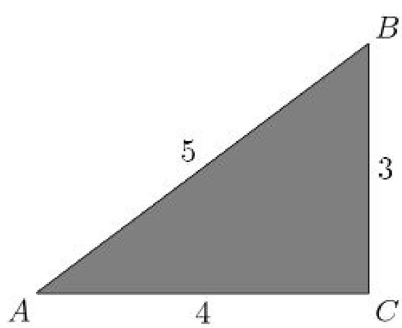


Problem 5. (11⋆)
A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

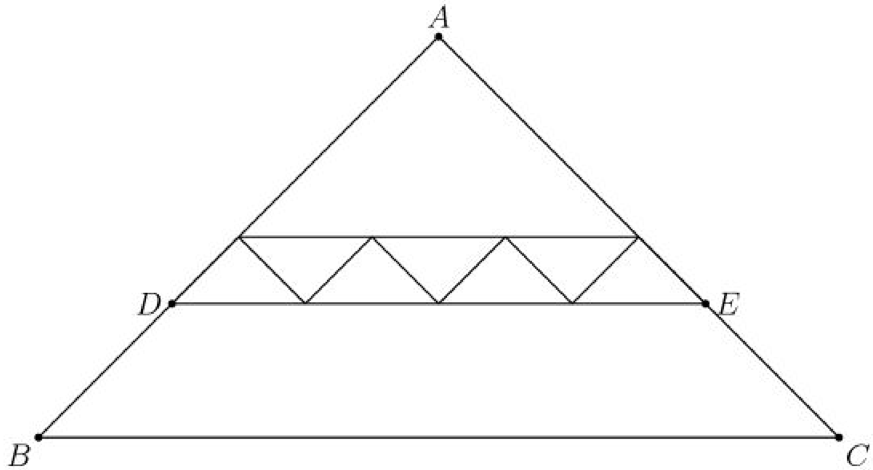
Problem 6. (3⋆)
Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



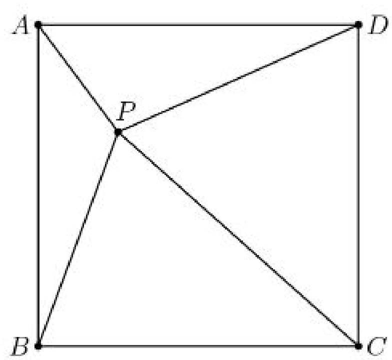
Problem 7. (5⋆)
A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



Problem 8. ($2\star$)
All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?

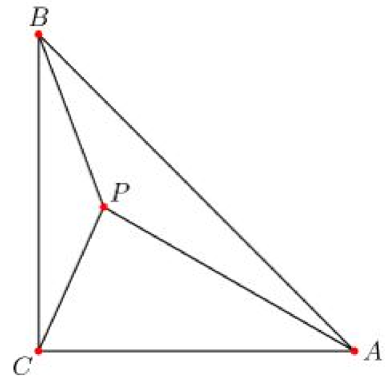


Problem 9. ($7\star$)
Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral? (Hint: Scale the quadrilateral and use the only property of centroids that you know.)



Problem 10. Required. ($7\star$)
Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

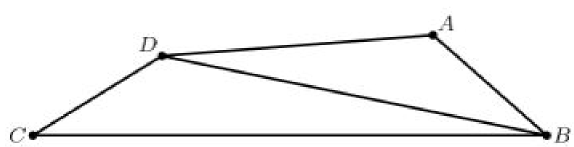
Problem 11. ($7\star$)
Isosceles $\triangle ABC$ has a right angle at C. Point P is inside $\triangle ABC$, such that $PA = 11$, $PB = 7$, and $PC = 6$. Legs $\overline{AC}$ and $\overline{BC}$ have length $s = \sqrt{a + b\sqrt{2}}$, where a and b are positive integers. What is $a + b$?



Problem 12. Required. ($11\star$)
Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB? (Hint: Use a variation of the proof of Area$=s \cdot r$).

Problem 13. ($7\star$)
On each side of a unit square, an equilateral triangle of side length 1 is constructed. On each new side of each equilateral triangle, another equilateral triangle of side length 1 is constructed. The interiors of the square and the 12 triangles have no points in common. Let R be the region formed by the union of the square and all the triangles, and S be the smallest convex polygon that contains R. What is the area of the region that is inside S but outside R?

Problem 14. ($2\star$)
In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



Problem 15. (3⋆)
Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

Problem 16. Required. (3⋆)
Equiangular (not necessarily equilateral) hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

Problem 17. (3⋆)
Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$.

Problem 18. Required. (5⋆)
In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.

Problem 19. (5⋆)
Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

Problem 20. (11⋆)
In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$,$DF = 8$,$\overline{BE}||\overline{DF}$,$\overline{EF}||\overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m,n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

Problem 21. (5⋆)
In triangle ABC, $AB = 125$, $AC = 117$ and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

Problem 22. (3⋆)
On square $ABCD$, point E lies on side AD and point F lies on side BC, so that $BE = EF = FD = 30$. Find the area of the square $ABCD$.

Problem 23. (7⋆)
In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. What is the ratio of CP to PA?