Perfect competition

large number of buyers and sellers → nobody has a market power (cannot influence prices)

Method: minimize the ~~#~~ # exog. var.

compute a price system that clear all the markets simultaneously

determine assoc. production/consumption decisions of all agents.

$x = (x_1, \dots, x_L) \in \mathbb{R}^L$

$y = (y_1, \dots, y_L)$

$$\begin{cases} x \geq y \leftrightarrow x_i \geq y_i \quad \forall i = 1, \dots, L \\ x \gg y \leftrightarrow x_i > y_i \quad \forall i = 1, \dots, L \\ x > y \leftrightarrow x \geq y, x \neq y \end{cases}$$

- I consumers $i = 1, \dots, I$
- L goods $l = 1, \dots, L$
- X_i - possible consumption bundles for the consumer i; typically: vector $X_i = \mathbb{R}^L_+$
- $x_i = (x_{1i}, \dots, x_{Li}) \in X_i$ is a consumption bundle

x_{li} - amount of commodity l in the bundle x_i

- $\succsim_i$ the preferences: complete, transitive and continuous $\rightarrow \exists$ a utility func.

$u: X_i \rightarrow \mathbb{R}$: $x_i \succsim_i y_i \leftrightarrow u(x_i) \geq u(y_i)$

- $x_i \sim_i y_i \leftrightarrow x_i \succsim_i y_i$ and $y_i \succsim_i x_i$

$x_i \succ_i y_i \leftrightarrow x_i \succsim_i y_i$ and $\neg\, y_i \succsim_i x_i$

- Strictly monotonic if

$x_i \geq y_i \rightarrow x_i \succsim_i y_i$

- weakly mon.

$x_i \gg y_i \rightarrow x_i \succsim_i y_i$

- local non-satiation: $\forall\, x_i \in X_i$ and $\forall\, \varepsilon > 0\ \exists\, x_i' \in X_i$: $\|x_i' - x_i\| < \varepsilon$ and $x_i' \succsim_i x_i$

- Assume X_i is convex

$\succsim_i$ is said to be convex if

$x_i \succsim_i y_i$, $\alpha \in (0,1) \rightarrow \alpha x_i + (1-\alpha) y_i \succsim_i y_i$

$\leftrightarrow$ upper ~~contour~~ contourset is convex (if $\succsim$ is transitive)

- Strictly convex if $x_i \succsim_i y_i$ and $\boxed{x_i \neq y_i} \rightarrow \alpha x_i + (1-\alpha) y_i \succ y_i$ for $\alpha \in (0,1)$

Exchange economy

- no production (only exchange)
- $w_i = (w_{1i}, \dots, w_{Li}) \in X_i$ initial endowment of the consumer i
- an allocation $x = (x_1, \dots, x_I) \in \mathbb{R}^{LI}$ assigns the cons. bundle $x_i \in X_i$ to individ. i
- $\bar{w} = \sum_{i=1}^{I} w_i \in \mathbb{R}^L$ is the total endowment vector
- An allocation x is feasible if $\sum_{i=1}^{I} x_i \leq \bar{w}$ (assuming free disposal)
- Since we have goods we can also say that an allocation x is nonwasteful if $\sum_{i=1}^{I} x_i = \bar{w}$
- wealth of individuals i is endogeneous

given a price-vector $p = (p_1, \dots, p_L)$ the wealth of i is $p w_i = \sum_{l=1}^{L} p_l w_{li}$

the expenditure to buy x is $p x_i = \sum_{l=1}^{L} p_l x_{li}$

the budget set of i: $\{x_i \in X_i : p x_i \leq p w_i\} = B_i(p)$

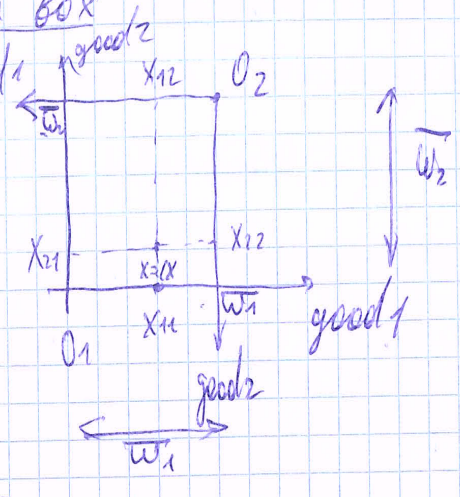
Chapter 15.B

Edgeworth box

$L = 2$

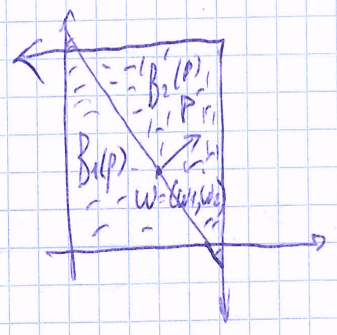
$I = 2$

$X_i = \mathbb{R}^2_+$



$X_{11} + X_{12} = \overline{\omega}_1$

$X_{21} + X_{22} = \overline{\omega}_2$ → non wasteful allocation



Monotonicity Pref: • cons. 1 becomes happier as we go ↗

• -11- cons 2 -11-

↙

Consumer's prb: Max $\succsim_i$ or (u_i) over the budget set $B_i(p)$ given p

Claim: If $\succsim_i$ is LNS (loc. non-sati.) then $\forall x_i^*$ that solves cons i's prb $px_i^* = p\omega_i$

Proof if $px_i^* < p\omega_i$, then $\exists \varepsilon > 0$: $px_i < p\omega_i$ $\forall x_i \in X_i$ with $\|x_i - x_i^*\| < \varepsilon$

moreover by LNS there $\exists$ $x_i \in X_i$ with $x_i \succ_i x_i^*$ and $\|x_i - x_i^*\| < \varepsilon$. So, such x_i^* cannot solve the cons i's prb. □

→ Walras law: $p \sum_i x_i^* = p\overline{\omega} = p \sum_i \omega_i$

$= \sum_i (px_i^*)$

Total expenditure = total income

Defn: A Walrasian (or a competitive) equilibr. in this model consists of all $x^* = (x_1^*, x_2^*)$ and a p-vector p s.t.

1) x_i^* solves the cons i's prb at a given p $\forall i = 1, 2$

2) $x_1^* + x_2^* \leq \overline{\omega}$ feasibility cond.

Claim: if $\succsim_i$ of at least one indiv is strongly monotonic and every other have LNS pref. for some ℓ then $\forall$ equil. allocation $x^* = (x_1^*, x_2^*)$ we have $x_1^* + x_2^* = \bar{\omega}$ (non-wasteful allocation)

Proof: if $\succsim_i$ is strictly mon. then $p \gg 0$ $\forall$ equil. price vector

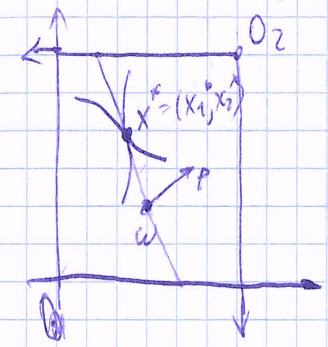
indeed suppose $p_\ell \leq 0$ for some ℓ then select any $x_i \in B_i(p)$ then $p(x_i + \varepsilon \mathbb{1}_\ell) - p x_i = p \varepsilon \mathbb{1}_\ell = \varepsilon p_\ell \leq 0$ for $\varepsilon > 0$ $\rightarrow$ no x_i can optimize utility

moreover $x_i + \varepsilon \mathbb{1}_\ell \succ_i x_i$ by strong mon. $\rightarrow$ no point in $B_i(p)$ solves cons i's prob $\rightarrow p \gg 0$ should be in equilibr.

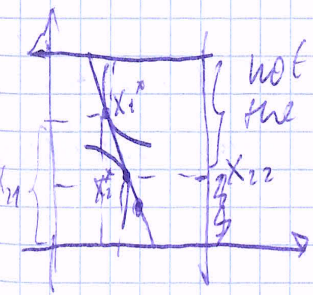
Now, suppose by contrad. that equilibr alloc. x^* ~~is distinct~~ satisfies $\sum_j x_j^* \neq \bar{\omega}$. Since x^* is feasible $\rightarrow \sum_j x_j^* < \bar{\omega}$. Since $p \gg 0 \rightarrow p \sum_j x_j^* < p\bar{\omega} \rightarrow$ violates the Walras's law.

$p(\bar{\omega} - \sum_j x_j) > 0$

So, we must have $\sum_j x_j^* = \bar{\omega}$ $\square$



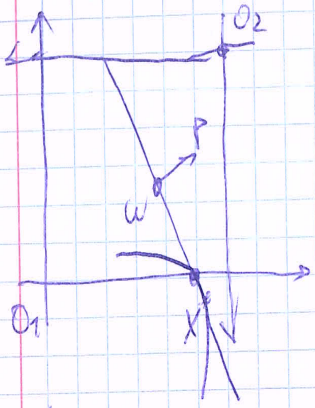
$\leftarrow$ interior equilibr.



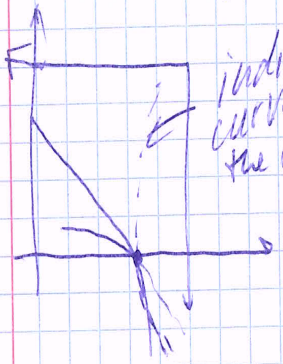
not clearing the market $\rightarrow$ not an equilibrium

$x_{22} + x_{21}$ exceeds the total endowment $\omega_2 \rightarrow$ excess demand for good 2

$x_{11}^* + x_{12}^* < \bar{\omega}_1 \rightarrow$ excess supply of good 1



for cons 2 full tangency



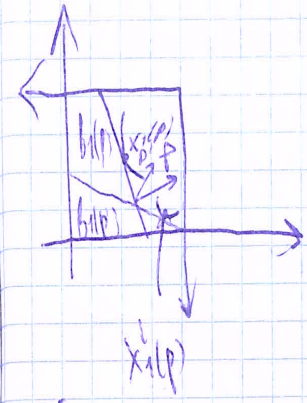
indif. curve of the cons 1

$\boxed{\text{boundary equilibrium}}$

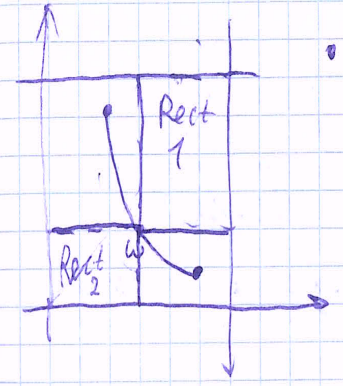
Offer curve

- A graphical illustr. of consumers' demand functions (as functions of $p \gg 0$)
- Suppose $\succsim_i$ is strongly monotonic, continuous + strictly convex

$\exists$ a unique $x_i^*(p)$ that solves i's prob at $p \gg 0$

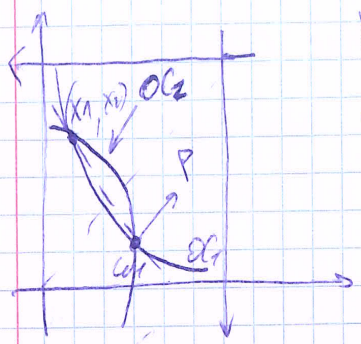


Graph ~~for~~ of $x_1(p)$ for $p \gg 0$ → offer curve for the cons 1



- Offer curve 1 cannot $x_1(\cdot)$ touch a rect 1 because any point x_1 in this rect. costs more than $p\omega_1$ whatever p
- ~~Offer~~ ω_1 is always in $B_1(p)$ → → $x_1(p) \succsim_1 \omega_1$ → since pref are strongly monotone → any point in rect 2 cannot touch the offer curve 1.
- When p_1/p_2 is very small $x_1(p)$ will ~~be~~ lie on the right on ω_1 similarly when p_2/p_1 is very small $x_1(p)$ -||- on the left of ω_1

→ OC should go through the ω,



$x_i \in OC_1 \to x_1 = x_1(p)$ is the optimal cons. bundle for some $p \gg 0$

$x_1 = x_1(p)$ and ω_1 determine a unique budget line that is orthogonal to p

- Suppose OC_1 and OC_2 intersect at $(x_1, x_2) \neq \omega$ then it is an equilibr. allocation

indeed the unique p that det. by $x_1 = x_1(p)$ and $x_2 = x_2(p)$ clears the market.

<u>Conclusion</u>: any allocation in the intersection of 2 OC is an equilibr. allocation. $(x \neq \omega)$

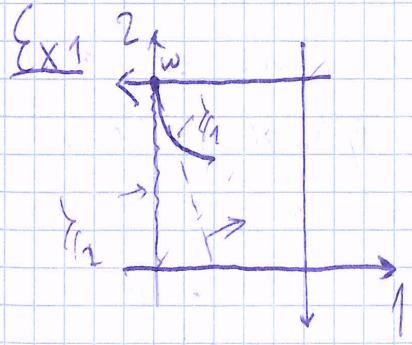
<u>Note</u> that ω does not need to be an equilibrium despite the fact that it lies on the intersection of 2 offer curves

We know that $\exists\, p$: ω_1 is optimal at p and $\exists\, p'$ s.t ω_2 can be equal to $x_2(p')$ but p can be dist. from p'

<u>Existence of equilibrium</u>:

requires <u>strong monoton.</u> and <u>convexity</u>

<u>Ex 1</u>



$u_2(x_2) = x_{12}$ not strongly monotonic

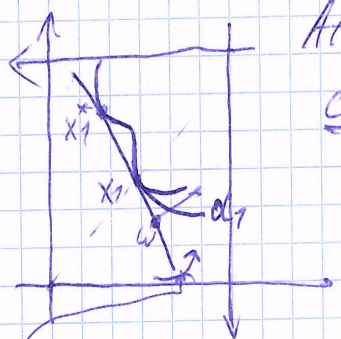
$\succ_1$ strongly mon + tangency to vertical axes

<u>Claim</u> ↑ there are no equilibria

- $\succ_1$ is strongly mon → any equil p should be $\gg 0$
- Since 2 cares only about good 1 and has this good → ω is an

optimal point for 2 i.e. $W_2 - X_2(p)$
$\rightarrow X_1(p) = w_1$ but the tangency cond. implies that w_1 is a ↑ optimal for cons 1. at any $p >> 0$. □

<u>Ex 2</u> Non-convex indiff. curves

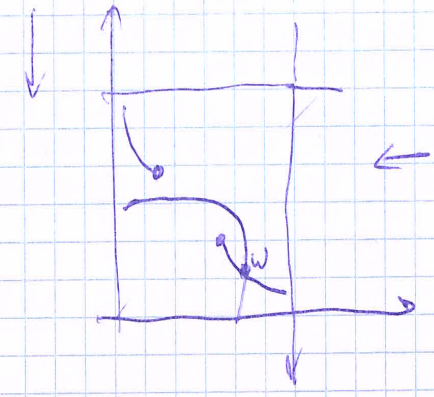


At p the demand <u>correspondence</u> is given by 2 points x_1, x_1^* $X_1(p) = \{x_1, x_1^*\}$

rotate budget line ~~to~~ in this direction optimal point will be on the right of x_1
rotate —||— in ↺ → —||— on the left of x_1^*

But huge hole in between x_1^* x_1 appears →

argmax corresp. is not convex valued → OC is a ↑ connected curve

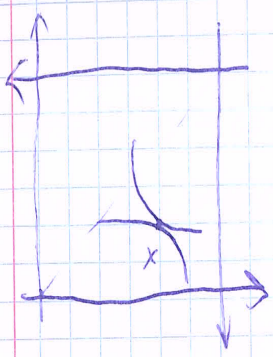


← no equilibrium

Depending on slope of OC_2 we may have no equilibrium if the 2 offer curves do not intersect in a point that is distinct from w

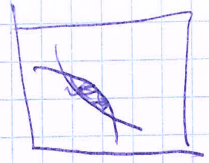
<u>Welfare Properties of Equilibria</u>

A feasible allocation $x = (x_1, \dots, x_I)$ is said to be Pareto efficient (optimal) if there is not ∃ a feasible allocation y such that all individ. $\forall i$ → $y_i \succsim x_i$ and at least one indiv. i $y_i \succ x_i$

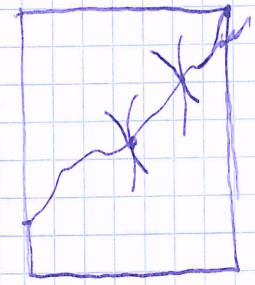


x- efficient allocation

→ indiff. curves should be tangent to each other

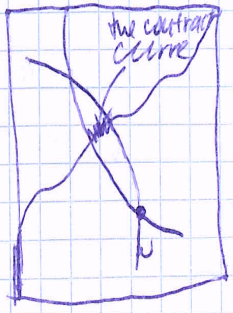


- not efficient can go to the shaded area and make both indiv. better-off.



Pareto set of all pareto efficient allocations

$\boxed{\text{Contract curve}}$. The part of pareto set that both consumers prefer to their initial endowment.

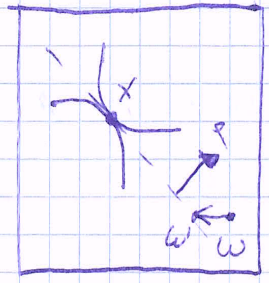


$\boxed{\text{1st fundam. theorem of welfare economics}}$

Assuming LNS pref. a Walrasian equilibr. is Pareto efficient

$\boxed{\text{2nd -II- theorem -II-}}$

Assuming convex preferences upon a suitable transfer of resources among the agents any pareto efficient allocations can be supported as a Walrasian equilibrium.



x- efficient alloc.

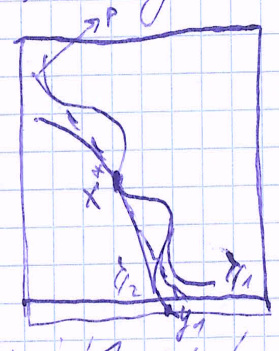
p - any price vector that could generate x in a Walrasian equilibr.

moving w to w'. Take some of goods from indiv 1 give it to indiv 2 so that the new endowment after the transfer is given by w'. Then let the market work via

will end up with equilibr. alloc. we wanted

Pref. not LNS → 1st Fund. theorem may fail

2nd Fund theorem requires convexity of pref.



$\succ_1$ is not convex

x^* is efficient we cannot make 1 better off without hurting 2

x_2^* can be optimal for $\succ_2$ only at the budget line so the only possible eq. price vector is p, but x^* isn't optimal for $\succ_1$

~~for example $y_1 \succ_1 x_1^*$~~

Note $y_1 \sim_1 x_1^*$ and y_1 is clearly not optimal in the $B_1(p)$ so we cannot support x^* as an equil. allocation.

→ 2nd Fund theorem fails if $\succ$ is not convex.

(strictly convexity or not?)

<u>Ex</u> (equilibr. comp. with Cobb-Douglas pref.)

$$U_i(X_i) = X_{1i}^{\alpha} X_{2i}^{1-\alpha} \qquad i = 1, 2$$

Let $\omega_1 = (1, 2)$ $\omega_2 = (2, 1)$

$$X_1(p) = \begin{pmatrix} \alpha \dfrac{p_1 + 2p_2}{p_1} \\ (1-\alpha)\dfrac{p_1+2p_2}{p_2} \end{pmatrix}$$

$$X_2(p) = \begin{pmatrix} \alpha \dfrac{2p_1+p_2}{p_1} \\ 1-\alpha \dfrac{2p_1+p_2}{p_2} \end{pmatrix}$$

Market clear. cond

<u>Market 1</u>: $X_{11}(p) + X_{12}(p) = 3$

$\to \alpha\left(1 + \frac{p_2}{p_1}\right) = 1 \to$

$\to \boxed{\frac{p_2}{p_1} = \frac{1-\alpha}{\alpha}}$

Plug into demand func → equil. allocation

$X_1(p) = (1+\alpha, 2-\alpha)$

$X_2(p) = (2-\alpha, 1+\alpha)$

No need to clear the 2nd market separately

Let $\bar{X}(p) = X_1(p) + X_2(p)$

By derivation $p X_1(p) + p X_2(p) =$

$= p\omega_1 + p\omega_2 = p\bar{\omega} =$

$= p_1 \bar{X}_1(p) + p_2 \bar{X}_2(p) = p_1\bar{\omega}_1 + p_2\bar{\omega}_2$

→ if we clear 1st market

i.e. $p_1 \bar{X}_1(p) = \bar{\omega}_1$ then automatically

<u>we clear the 2nd</u>

Ch15C $\boxed{\text{One consumer - one producer}}$

$\boxed{\text{Robinson Crusoe model}}$

Consumer and producer = the same person

- z - labour input, $q = f(z)$ is the firm's output

$f: \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave and strictly increas fnc.

prod fnc.

- w - the wage rate (cost of labour)
- p - price of output

Firm's prb is to max $\boxed{p f(z) - wz}$

$\boxed{\text{s.t. } z \geq 0}$

↳ $z(w,p)$ optimal labour demand

$q(w,p) = f(z(w,p))$ - output

$\pi(w,p) = p f(w,p) - w z(w,p)$

- The consumer owns all the labour input $\bar{L}$ units of time to be allocated between leisure x_1 and labour $\bar{L} - x_1$

The consumer also likes to consume the consumption good produced by the firm. Denote this as x_2. The consumer also owns the firm; collects the profit $\pi(w,p)$

Cons's prb is max $u(x_1, x_2)$ s.t.

$x_1, x_2 \geq 0$

$0 \leq x_1 \leq \bar{L}$

$$p x_2 \leq \underbrace{w(\bar{L} - x_1)}_{\text{sells labour}} + \pi(w, p)$$

Denote by $x_1(w,p)$, $x_2(w,p)$ the s/n to cons prb

<u>Defn</u>: An equil is a price vector (w,p) such that $z(w,p) = \bar{L} - x_1(w,p)$ and $q(w,p) = x_2(w,p)$

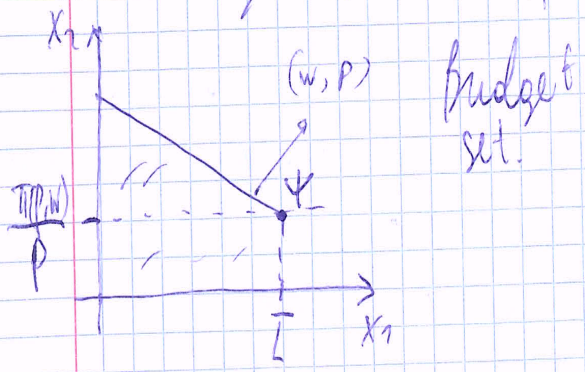


Fig. 1

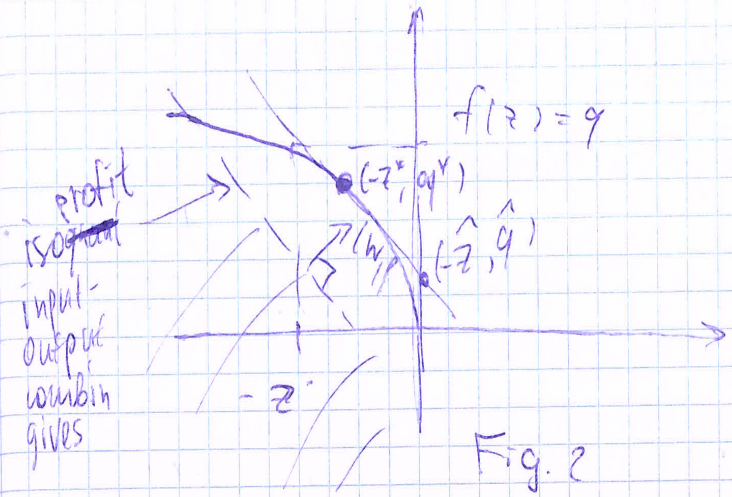


Fig. 2

The shaded area - production set gives the feasible inputs/output combinations.

Each point on the dashed line leads to the certain level of profit (isoprofit line)

Firm pushes isoprofit line towards tangency to the prod. funct. curve $f(z)$ to max. profit

$\rightarrow (-z^*, q^*)$ maximizes firm's profit

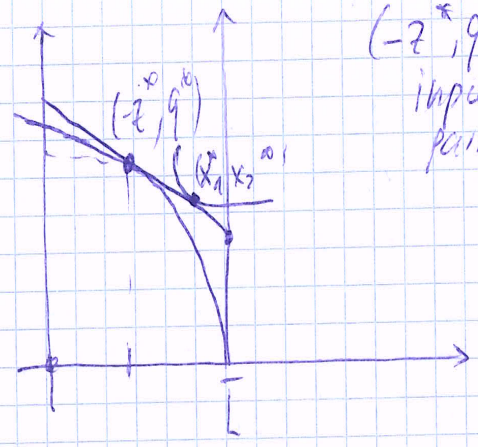
The vertical intercept $(-\hat{z}, \hat{q}) = (0, \hat{q})$ at the same isoprofit line

$\rightarrow (0, \hat{q}) \cdot (w, p) = \pi(w, p)$

$\rightarrow \hat{q}\, p = \pi(w, p) \rightarrow \hat{q} = \frac{\pi(w, p)}{p}$

the same as y

→ put two figures Fig 1 and 2 on one



$(-z^*, q^*)$ - optimal input-output pair for the firm

↳ consumer's demand $x^* = (x_1^*, x_2^*)$ is as in the picture

excess labour demand + excess supply ~~the~~ of good

↳ not the equilibrium

In equil x^* must coincide with $-z^*$

~~Well~~ ~~2nd~~ 1st Welfare theorem says that equil. should be efficient. But there is only 1 guy → we should max utility s.t. ~~to~~ resource constraint + technol. constraint.

$0 \le x_1 \le \bar{L}$ $\qquad$ $0 \le x_2 \le f(\bar{L} - x_1)$

if u is strictly monotonic we can let $x_2 = f(\bar{L} - x_1)$, then solve $\max u(x_1, f(\bar{L} - x_1))$

s.t. $0 \le x_1 \le \bar{L}$

↳ x_1^*, x_2^*

This points should ~~be~~ ~~opti~~ also max u over $B(w, p)$

$MRS = MRT \rightarrow$ ~~$u(x_1^*, x_2^*)$~~

$$\frac{\frac{\partial u}{\partial x_1}(x_1^*, x_2^*)}{\frac{\partial u}{\partial x_2}(x_1^*, x_2^*)} = \frac{w}{p} \rightarrow \text{equil. price ratio}$$

Ch15D A more generall model of production

- J firms with prod. fnc: $f_j : \mathbb{R}^L_+ \to \mathbb{R}$
  f_j is concave (not necessary str. concave) strictly q-concave, str. diff. str. increasing
- L inputs/factors $\bar{z}_\ell$ - total endowment of factor ℓ, fully owned by the households; (inelastically) supplied to the firms
- Each firm is producing a cons. goods traded freely in the intern. market. The price p_j is taken as given
- the economy is small compare to the rest of the world. Open in cons. goods We thus take $(p_1, \dots, p_J)$ as given. Consequently we have no market clearing cond
- By contrast factor markets are closed. Thus, we need to determine domestic factor ~~market~~ prices $w = (w_1, \dots, w_L)$ and factors' allocation to clear the ~~mar~~ domestic factor market
- Firm j's prb: $\max_{z_j \in \mathbb{R}^L_+} p_j f_j(z_j) - w z_j$

$\to z_j^*, q_j^* = f(z_j^*)$

~~Cost function~~

Alternatively we may be given a cost fnc for the firm j

$C_j(w, q_j)$

by def: $C_j(w, q_j) \equiv \min w z_j$ s.t. $f(z_j) \geq q_j$

$z_j^*(w)$ - unconditional factor demand
uncondition to q

$z_j^*(w, q_j)$ - conditional factor dem.
condition to q

Thus $C_j(w, q_j) = w\, z_j(w, q_j)$

by Shepard's lemma: $D_w C_j(w, q_j) = z_j(w, q_j)$

$$\max_{q_j \geq 0} \; p_j q_j - C_j(w, q_j) \to q_j^*$$

$$D_w C_j(w, q_j^*) = z_j(w, q_j^*) = z_j^*(w)$$

<u>Defn</u>: An equilibrium ~~of fac~~ consists of factor prices $w \in \mathbb{R}^L_+$, factor allocation $(z_1^*, \dots, z_J^*) \in \mathbb{R}^{LJ}$ and output levels $(q_1^*, \dots, q_J^*) \in \mathbb{R}^J$ such that

(i) $(z_j^* q_j^*)$ solves the firm j's prb $\forall j$

(ii) $\sum_{j=1}^{J} z_j^* = (\bar{z}_1, \dots, \bar{z}_J)$ so, factor market should be cleared

2x2 production model CRS

2 firms, 2 factors

$J = 2$, $L = 2$

f_j - CRS function $\to C_j(w, q_j) = q_j \underbrace{C_j(w, 1)}_{C_j(w)}$

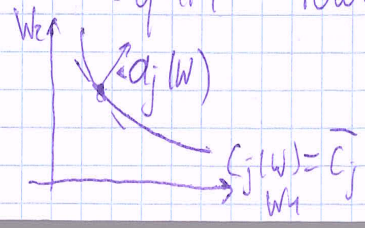
Firm's prb. $\max_{q_j} (p_j - C_j(w)) q_j$ s.t.

$\to p_j \leq C_j(w)$ in equil.

moreover if $p_j = C_j(w)$ firm is indifferent. It can optimally produce any amount

if $p_j > C_j(w)$ firm shuts down

It follows that if $q_j > 0$ in equil then $\boxed{C_j(w) = p_j}$



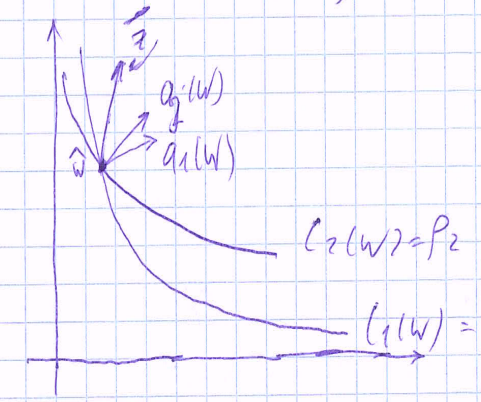
$C_j(w)$ is concave in w

Recall $D_w C_j(w) = \underbrace{z_j(w,1)}_{q_j(w)}$

<u>Interrior equil</u>: both firms produce positive levels of output

$\hookrightarrow C_1(w) = p_1 \qquad C_2(w) = p_2$

$\hookrightarrow \hat{w} = (\hat{w}_1, \hat{w}_2)$

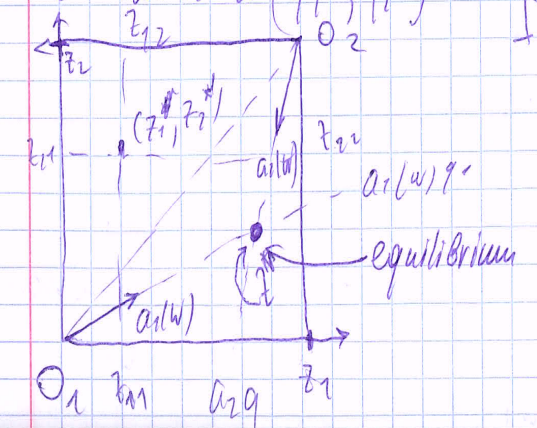


For market clearing:

$$q_1^* a_1(\hat{w}) + q_2^* a_2(\hat{w}) = (\bar{z}_1, \bar{z}_2)$$

2 equations,

2 unknowns (q_1^*, q_2^*)



In this fig.

$z^* = (z_1^*, z_2^*)$

is an equil.

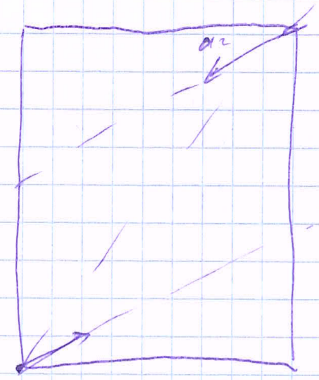
factor allocation

$z_j^* = q_i^* \cdot a_i(\hat{w})$

In an interrior equil we must have

$$\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})}$$

<u>note</u>: we will assume $\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})}$



<u>boundary equilibria</u>:

Occur if $\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{a_{12}(\hat{w})}{a_{12}(\hat{w})} \geq \frac{\bar{z}_1}{\bar{z}_2}$

2nd firm produces

or $\frac{\bar{z}_1}{\bar{z}_2} > \dots >$

$\hookrightarrow$ 1st firm produces

If 1st firm shuts down

$C_2(W^*) = p_2$ and it uses all resources i.e. $\frac{a_{12}(W^*)}{a_{22}(W^*)} = \frac{\bar{z}_1}{\bar{z}_2}$ $\longrightarrow$

$\rightarrow$ solve for $W^* = (W_1^*, W_2^*)$

Check that ~~it~~ really $C_1(W^*) > p_1$ so that it shuts down

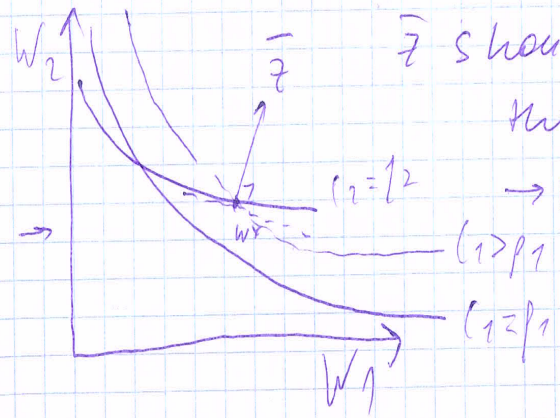
If yes $\rightarrow$ equilibrium W^*

$z^* = (\bar{z}_1, z_2^*)$ where $z_1^* = (0,0)$

$z_2^* = (\bar{z}_1, \bar{z}_2)$

$q^* = (q_1^*, q_2^*)$; $q_1^* = 0$

$q_2^* = f(\bar{z}_1, \bar{z}_2)$



$\bar{z}$ should be $\perp$ to the deriv.

Ex

$f_j(z_j) = z_{1j}^{\alpha_j} z_{2j}^{1-\alpha_j}$ $\quad \alpha_j \in (0,1)$

$C_j(W) = \left(\frac{W_1}{\alpha_j}\right)^{\alpha_j} \left(\frac{W_2}{1-\alpha_j}\right)^{1-\alpha_j}$

$\lrcorner$ $\alpha_1 > \alpha_2$

In practice we will assume either $\alpha_j = \frac{1}{2}$ for some j or else $\alpha_1 = 1 - \alpha_2$

Recall that $C_1(\hat{W}) = p_1$ $\quad C_2(\hat{W}) = p_2$

if $\alpha_j = \frac{1}{2} \rightarrow C_j(W) = 2\sqrt{W_1 W_2} \rightarrow C_j(\hat{W}) = 2\sqrt{\hat{W}_1 \hat{W}_2} = p_j$

(1)

$\alpha_1 = 1-\alpha_2 \rightarrow \dfrac{C_1(W)}{C_2(W)} = \left(\dfrac{W_1}{W_2}\right)^{\alpha_1}\left(\dfrac{W_2}{W_1}\right)^{1-\alpha} =$

$= \left(\dfrac{W_1}{W_2}\right)^{2\alpha_1-1}$

$\alpha_1 > \alpha_2$

$\alpha_1 > \alpha_2 \Rightarrow \alpha_1$

$$\frac{C_1(\hat{W})}{C_2(\hat{W})} = \left(\frac{\hat{W}_1}{\hat{W}_2}\right)^{2\alpha_1-1} = \frac{p_1}{p_2} \quad (2)$$

Ex $\alpha_1 = \frac{2}{3}$ $\alpha_2 = \frac{1}{2}$ $\bar{z}_1 = 2$ $\bar{z}_2 = 24$

$p_1 = 3$ $p_2 = 2$

Since $\alpha_1 = \frac{1}{2}$ By (1) $\rightarrow C_2(\hat{W}) = 2\sqrt{\hat{W}_1\hat{W}_2} = \frac{p_2}{p_1} = 2$

$\rightarrow \hat{W}_1\hat{W}_2 = 1 \rightarrow \hat{W}_1 = \dfrac{1}{\hat{W}_2}$

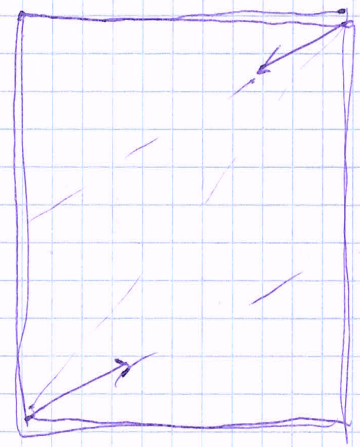
Plug this into $C_1(\hat{W}) = p_1$

$C_1(\hat{W}) = \left(\dfrac{W_1}{2/3}\right)^{2/3}\left(\dfrac{W_2}{1/3}\right)^{1/3} = 3\left(\dfrac{W_1}{2}\right)^{2/3} W_2^{1/3} \rightarrow$

$\rightarrow C_1(\hat{W}) = 3\left(\dfrac{\hat{W}_1}{2}\right)^{\frac{2}{3}}\left(\dfrac{1}{\hat{W}_1}\right)^{1/3} = \underset{p_1}{3} \rightarrow$

$\rightarrow \underbrace{\hat{W}_1^{\frac{1}{3}}}_{4^{1/3}} = 1 \rightarrow \boxed{\hat{W}_1 = 4}$

$\boxed{\hat{W}_2 = \frac{1}{4}}$



$q_1 a_1(\hat{W}) + q_2 a_2(\hat{W}) = \bar{z}$

if q_1 or q_2 are negative $\rightarrow$ no interior equil.

General remarks

$$a_j(W) = \begin{pmatrix} \dfrac{\partial C_j(W)}{\partial W_1} \\ \dfrac{\partial C_j(W)}{\partial W_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha_j}{\alpha_j}\left(\dfrac{W_1}{\alpha_j}\right)^{\alpha_j-1}\left(\dfrac{W_2}{1-\alpha_j}\right)^{1-\alpha_j} \\ \dfrac{1-\alpha_j}{1-\alpha_j}\left(\dfrac{W_2}{1-\alpha_j}\right)^{-\alpha_j}\left(\dfrac{W_1}{\alpha_j}\right)^{\alpha_j} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha_j}{w_1} c_j(w) \\ \frac{1-\alpha_j}{w_2} c_j(w) \end{pmatrix} \rightarrow \frac{a_{1j}(w)}{a_{2j}(w)} = \frac{\alpha_j}{1-\alpha_j} \frac{w_2}{w_1} \quad (3)$$

$$\rightarrow a_j(\hat{w}) = \left(\frac{\alpha_j}{\hat{w}_1} p_j , \frac{1-\alpha_j}{\hat{w}_2} p_j \right) \quad (4)$$

Ex 1 continued

by eq. 3 $\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} = \frac{\alpha_1}{1-\alpha_1} \frac{\hat{w}_2}{\hat{w}_1} = \frac{2/3}{1/3} \cdot \frac{1}{16} =$

$$= \frac{1}{8}$$

$$\frac{a_{12}(\hat{w})}{a_{22}(\hat{w})} = \frac{\alpha_2}{1-\alpha_2} \frac{\hat{w}_2}{\hat{w}_1} = \frac{1}{16}$$

need to compare these ratios with $\frac{\bar{z}_1}{\bar{z}_2} = \frac{1}{12} > \frac{1}{16}$

$$\rightarrow \frac{1}{8} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{1}{16} \rightarrow \text{interior equilib.}$$

$$q_1 a_1(\hat{w}) + q_2 a_2(\hat{w}) = \bar{z} = (2, 24)$$

by eq. 4 $a_j(\hat{w}) = \left(\frac{\alpha_j}{4} p_j , \frac{1-\alpha_j}{1/4} p_j \right)$

$$\rightarrow a_1(\hat{w}) = \left(\frac{2/3}{4} \cdot 3 , \frac{1/3}{1/4} \cdot 3 \right) = \left(\frac{1}{2}, 4 \right)$$

$$a_2(\hat{w}) = \left(\frac{1/2}{4} \cdot 2 \quad \frac{1/2}{1/4} \cdot 2 \right) \left(\frac{1}{4}, 4 \right)$$

Market clearing

$$\begin{cases} q_1 \frac{1}{2} + q_2 \frac{1}{4} = 2 \\ 4 \quad\quad 4 \quad\quad 24 \end{cases} \rightarrow \quad 10 - 8$$

$$\rightarrow \quad 2q_2 = 8 \rightarrow \boxed{\begin{matrix} q_2^* = 4 \\ q_1^* = 2 \end{matrix}}$$

So, interior equil $\hat{w} = (4, 1/4)$

output $q_1^* = (2, 4)$ $\quad z_1^* = q_1 \hat{a}_1 = 2 \cdot \begin{pmatrix} \frac{1}{2} \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

$$z_2^* = q_2 a_2 = 4 \cdot \begin{pmatrix} \frac{1}{4} \\ 4 \end{pmatrix} = \begin{matrix} 1 \\ 16 \end{matrix}$$

<u>Ex 2</u> $\alpha_1 = \frac{2}{3}$ $\alpha_2 = \frac{1}{3}$ $z_1 = 1$ $z_2 = 2$

$$\alpha_1 = 1 - \alpha_2$$

$$p_2 = \frac{9}{2^{2/3}} \quad p_1 \geq 0$$

For which values of p_1 would the firm 1 shut down in equil

$$\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})} > \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})} > \frac{\bar{z}_1}{\bar{z}_2} = \frac{1}{2} \quad (*)$$

By eq(3) we know

$$\frac{a_{12}(\hat{w})}{a_{22}(\hat{w})} = \frac{\alpha_2}{1-\alpha_2} \frac{\hat{w}_2}{\hat{w}_1}$$

Moreover $\alpha_1 = 1 - \alpha_2 \longrightarrow (2)$

$$\left(\frac{\hat{w}_1}{\hat{w}_2}\right)^{2\alpha_1 - 1} = \frac{p_1}{p_2}$$

Subst this into prev. equality

$$\frac{\alpha_2}{1-\alpha_2} \left(\frac{p_2}{p_1}\right)^{\frac{1}{2\alpha_1 - 1}} = \frac{a_{12}(\hat{w})}{a_{22}(\hat{w})}$$

So, $(*)$ iff $\frac{\alpha_2}{1-\alpha_2} \left(\frac{p_2}{p_1}\right)^{\frac{1}{2\alpha_1 - 1}} > \frac{\bar{z}_1}{\bar{z}_2} = \frac{1}{2}$

$$\rightarrow \frac{1/3}{2/3} \left(\frac{9}{2^{2/3} p_1}\right)^3 > \frac{1}{2}$$

$$\frac{\cancel{2}}{2 \cancel{2}^2} \frac{9^3}{p_1^3} > \frac{1}{\cancel{2}} \rightarrow \boxed{}$$

$$\boxed{p_2 \geq p_1}$$

So 1 shuts down $\Leftrightarrow p_1 \leq p_2$

Let's compute z^* and w^*

We need $\begin{cases} c_2(w^*) = p_2 \\ \dfrac{a_{11}(w^*)}{a_{22}(w^*)} = \dfrac{\bar{z}_1}{\bar{z}_2} \end{cases} \rightarrow$

$\rightarrow$

By (3): $\dfrac{a_{12}(w^*)}{a_{22}(w^*)} = \dfrac{\alpha_2}{1-\alpha_2} \dfrac{w_2}{w_1} \Rightarrow$

$\Rightarrow \dfrac{a_{12}(w^*)}{a_{22}(w^*)} = \dfrac{1/3}{2/3} \cdot \dfrac{w_2^*}{w_1^*} = \dfrac{1}{2} \rightarrow$ $\dfrac{\bar{z}_1}{\bar{z}_2}$

$\rightarrow \dfrac{w_2^*}{w_1^*} = 1 \quad w_2^* = w_1^*$

$c = p_2 = \dfrac{9}{2^{2/3}}$

$c_2(w^*) = \dfrac{w_1^*}{\alpha_1} \left(\dfrac{w_1^*}{1/3}\right)^{1/3} \left(\dfrac{w_2^*}{2/3}\right)^{2/3} =$

$= \dfrac{9}{2^{2/3}} \rightarrow w_1^* = w_2^* = 3$

$z_2^* = (1,2) \qquad z_1^* = (0,0)$

<u>Some theoretical observations on interior equilibria of the 2x2 model</u>

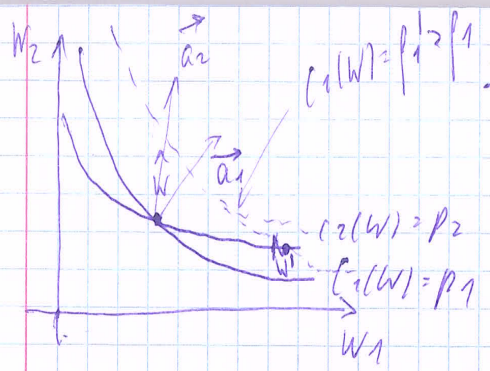
[comparative statics]

$c_1(w) = p_1, \; c_2(w) = p_2 \rightarrow \hat{w}$ equil prices

- i.e. in the interior equilibria initial endowments do not ~~depen~~ influence on factors' prices
- $\hat{w}$ $\boxed{\text{only}}$ depends on production technologies c_1, c_2 and p_1, p_2

<u>FACTOR PRICE equalization theorem</u>

2 countries which have access to the same technologies must have <u>the same factors' prices</u> irrespective of factors' endowments



$c_1(w) = p_1' > p_1 \quad \frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)} \quad \forall w \rightarrow \underbrace{\frac{\partial c_2}{\partial w_1}}_{a_{12}} \frac{\partial w_1}{\partial p_1} + \underbrace{\frac{\partial c_2}{\partial w_2}}_{a_{22}} \frac{\partial w_2}{\partial p_1} = 0$

It can be shown (since $\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$)

$\frac{\partial w_1}{\partial p_1} > 0 \qquad \frac{\partial w_2}{\partial p_1} < 0$

$p_1' > p_1 \rightarrow \hat{w} \rightarrow w' \leftarrow$ new equilibrium

$w_1' > \hat{w}_1, \; w_2' < \hat{w}_2$

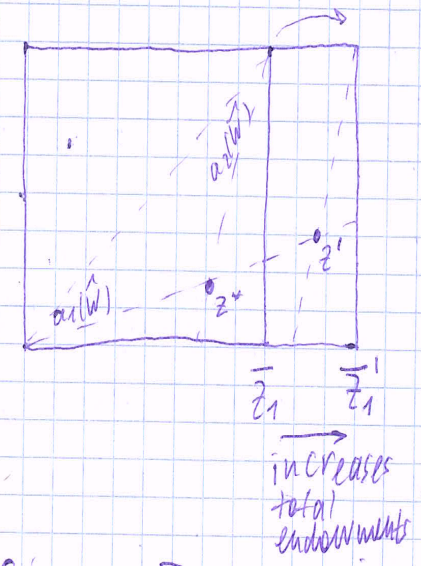
Stolper-Samuelson theorem

if the price of a consumption good j increases then the price of the factor that is more intensively used in production of j also increases, the price of other factor declines

Pf idea: $c_1(w(p_1)) \equiv p_1 \rightarrow$

$c_2(w(p_1)) = p_2$

$\rightarrow \underbrace{\frac{\partial c_1}{\partial w_1}}_{a_{11}} \frac{\partial w_1}{\partial p_1} + \underbrace{\frac{\partial c_1}{\partial w_2}}_{a_{21}} \frac{\partial w_2(p_1)}{\partial p_1} = 1$



$\bar{z}_1 \uparrow \bar{z}_1'$

$z_{11}^* > z_{11}^*$

$z_{21}^* > z_{21}^*$

So firm 1 uses more of both inputs → firm 2

Since $\bar{z}$ remains the same ~~the increase~~ Firm 2 should be using less of factor 2 since $a_2(\hat{w})$ remains the same it also follows that firm 2 should be using less of factor 1 as well

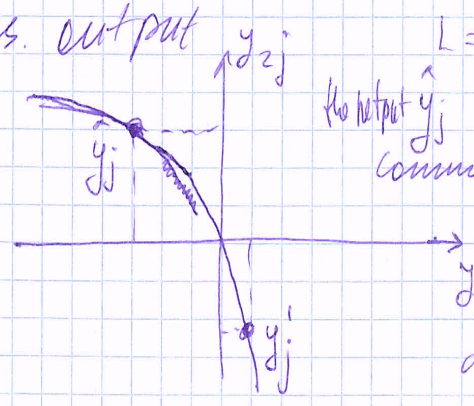
$\rightarrow q_1 \uparrow, q_2 \downarrow$

Chapter 16

Welfare properties of equilibrium

- J firms I consumers L goods
- $X_i \subseteq \mathbb{R}^L$ is the consumption set of consumer i
- $Y_j \subseteq \mathbb{R}^L$ is a production set of firm j (technologically feasible outputs)

An element $y_j \in Y_j$ is reffered to as netput or input-output vector. The negative coordinates of y_j represent inputs, positive coordin. repres. output



the netput $\hat{y}_j$ uses the 1st commodity as an input to produce the second as an output

the same firm could also select y_j' to use change role of inputs and outputs

- Let $p = (p_1, \dots, p_L)$ be the price vector then $p y_j = \sum_{l=1}^{L} p_l y_{lj}$ is the profit by y_j
- The firm's j prob can be written as $p y_j \rightarrow \max$ s.t. $y_j \in Y_j$
- An allocation is a list (x, y) s.t. $x = (x_1, \dots, x_I) \in X_1 \times \dots \times X_I$ and $y = (y_1, \dots, y_J) \in Y_1 \times \dots \times Y_J$

Allocation is feasible if the total consumption equals total endowments plus total production

$$\sum_{i=1}^{I} x_i = \bar{w} + \sum_{j=1}^{J} y_j$$

$\bar{\omega} = \sum_{i=1}^{I} \omega_i$ where $\omega_i \in \mathbb{R}^L$ is the endowment of consumers

- Let θ_{ij} denote the share of cons i's share in the firm j. Naturally we assume $\forall j \ \sum_{i=1}^{I} \theta_{ij} = 1$

- If the firm j ~~produces~~ selects the output level y_j^* for $j = 1 \dots J$ the total income of the cons i is $\leq \{ p\omega_i + \sum_{j=1}^{J} \theta_{ij} p y_j \}$ - budget set

$\{x_i \in X_i : px_i$

The budget set of the cons. i is

The consumers' problem is to max. the pref. relations over the they budget set. $B_i(p)$

<u>Definition 1</u>: An equilibrium consists of an allocation (x^*, y^*) and a price vector p:

(1) for each i, x_i^* max $\succsim_i$ over $B_i(p)$, given $y_1^*, \dots, y_J^*$ and the price vector p

(2) ~~Each firm max~~ $p y_j^* \geq p y_j \ \forall y_j \in Y_j$

(3) Market clearing: $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

<u>Def. 2</u> An equilibrium with transfers consists of an allocation x^*, y^*, prices $p = (p_1, \dots, p_L)$ and wealth levels $(w_1, \dots, w_I)$ s.t. (2) and (3) holds in addition to

(1') x_i^* maximises $\succsim_i$ over $\{x_i \in X_i : px_i \leq w_i\}$

Moreover we must have balanced

budget i.e. $\sum_{i=1}^{I} W_i = p\bar{w} + \sum_{j=1}^{J} p y_j^*$

<u>LNS</u>: For any $x_i \in X_i$ and any $\varepsilon > 0$ $\exists$ $x_i' \in X_i$ s.t. $x_i' \succ_i x_i$ and $\| x_i - x_i' \| < \varepsilon$

<u>1st fundam. theorem of welfare economics</u>

If each pref. rel. are LNS $\forall i \in I$ then $\forall$ equilibrium with transfers is pareto efficient

---

feasible allocation (x, y) is pareto efficient if ~~the~~ $\nexists$ other feasible allocation $(\hat{x}, \hat{y})$: $\hat{x}_i \succsim_i x_i$ $\forall i$ and $\hat{x}_i \succ_i x_i$ for some i.

<u>Proof</u>: Let (x^*, y^*) and $(w_1, \dots, w_I)$ is some equilibr. with transfers.

] by contradiction that $\exists$ a feasible allocation $(\hat{x}, \hat{y})$ s.t. $\hat{x}_i \succsim_i x_i^*$ $\forall i$ and for a person k $\hat{x}_k \succ_k x_k^*$

$\rightarrow p\hat{x}_k > w_k$ (1) because x_k^* max the pref rel. over $x_k : \{p x_k \leq w_k\}$

$\rightarrow p\hat{x}_i \geq \hat{w}_i$ $\forall i$ (2) (LNS, otherwise by LNS we can find $x_i' \succ_i x_i^*$, $x_i' \in B_i(p)$

(1) and (2) $\longrightarrow \sum_{i=1}^{I} p\hat{x}_i > \sum_{i=1}^{I} w_i$

$\rightarrow$ balanced budget requir $\sum_{i=1}^{I} w_i = p\bar{w} + \sum_{j=1}^{J} p y_j^*$. It follows that $p \sum_i \hat{x}_i > p\bar{w} + \sum_{j=1}^{J} p y_j^*$

Since $p y_j^* \geq p \hat{y}_j$ $\forall j \rightarrow p \sum_i \hat{x}_i > p\bar{w} + \sum_{j=1}^{J} p \hat{y}_j$

$= p(\bar{w} + \sum_{j=1}^{J} \hat{y}_j) \rightarrow$ total consump exceed total endowment $\Rightarrow (\hat{x}, \hat{y})$ ~~b~~ allocation isn't feasible

$\rightarrow$ contradiction.

<u>Remark</u>: Let $J=1$ $Y_1 = -\mathbb{R}^L_+ = \mathbb{R}^L_-$

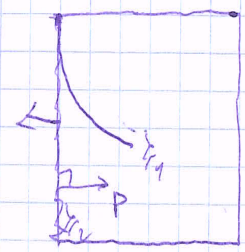
So $y_1 \in Y_1 \Leftrightarrow y_1 \leq 0$ then allocation (x, y) is feasible

$\Leftrightarrow \sum_i x_i \leq \bar{w}$. $\leftarrow$ earlier feasibility condition in the exchange model)

<u>Defn</u>: A quasi-equil. with transfers consists of an allocation (x^*, y^*), price-vector $p \in \mathbb{R}^L_{\neq 0}$ and wealth levels $(w_1, \dots, w_I)$ s.t. with $\sum_{i=1}^{I} w_i = p\bar{w} + \sum_{j=1}^{J} p y_j^*$ such that the properties (2) and (3) holds (profit max + market clear.) in addition to

(1'') $\forall x_i \in X_i : x_i \succsim_i x_i^* \rightarrow p x_i \geq w_i \ \forall i$ (and $p x_i^* \leq w_i$)

<u>Remark</u>: if each pref. relat. are strongly monotonic, continuous and $X_i = \mathbb{R}^L_+$ then under some additional regularity requirements $\forall$ q-equil. with transf is a full equil with transf



$\leftarrow p$, q-equil but not equil since no strictly monotonicity

<u>2nd fundam. theorem of welfare economics</u>

1 • X_i and Y_j are convex sets $\forall i, j$

• $\succsim_i$ is convex ($x_i \succsim_i \hat{x}_i \rightarrow \alpha x_i + (1-\alpha)\hat{x}_i \succsim_i \hat{x}_i$ $\forall \alpha \in (0,1)$) $\forall i$

• $\succsim_i$ LNS $\forall i$

Then $\forall$ Pareto efficient allocation (x^*, y^*) we can make this allocation a q-equil. with transfers for some p & wealth levels $(w_1, \dots, w_I)$

$A, B \subseteq \mathbb{R}^L$; $A+B = \{a+b : a \in A, b \in B\}$

$A_1, .., A_n \subseteq \mathbb{R}^L$: $\sum_i A_i = \{\sum a_i : a_i \in A_i \text{ for } i=1..n\}$

<u>Pf. sketch</u>

Let (x^*, y^*) be an efficient allocation

Define $V_i = \{x_i \in X_i : x_i \succsim_i x_i^*\}$. This is a convex set.

Then set $V = V_1 + .. + V_I$ is also convex

$Y_0 = \{\bar{\omega}\} + Y_1 + .. + Y_J$ is also convex

convex as a point (vector)

$\rfloor$ $z \in V$, then $z = x_1 + .. + x_I$ for some $x_1, .., x_I$, $x_i \succsim_i x_i^* \ \forall i$

If $z \in Y_0$ then $\exists\, y_1, .., y_J \in Y_1 \times .. \times Y_J$:

$z = \bar{\omega} + y_1 + .. + y_J$

$\rightarrow$ the obtained feasible allocation (x, y) : $x_i \succsim x_i^* \ \forall i \rightarrow (x^*, y^*)$ is not efficient. $\rightarrow$ V and Y_0 ~~and~~ do have no intersection

$V \cap Y_0 = \emptyset \rightarrow$ by the separating hyperplane theorem $\exists$ a non-zero $p \in \mathbb{R}^L_+$ and $r \in \mathbb{R}$ s.t. $pz \geq r \geq p z_0 \ \forall$ $z \in V$, $z_0 \in Y_0$

That is $\boxed{p \sum_{i=1}^{I} x_i \geq r \text{ whenever } x_i \succsim_i x_i^* \ \forall i}$ A

and $\boxed{p(\bar{\omega} + \sum_j y_j) \leq r}$ B $\forall\, y_1, .., y_J \in Y_1 \times .. \times Y_J$

Note that by LNS $\forall\, x_i \succsim_i x_i^* \ \exists\, x_i'$ which is arbitrary close to x_i : $x_i' \succsim x_i^*$

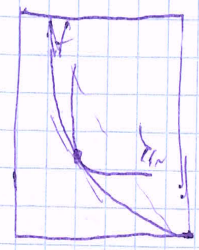
$\rightarrow$ A can be strengthened to read as

$\boxed{p \sum_i x_i \geq r \text{ whenever } x_i \succsim_i x_i^* \ \forall i}$ A'

Some algebra $\rightarrow$ define $w_i = p x_i^* \ \forall i$ $\square$

$(x^*, y^*, p, w_{i_{1..I}})$ - q-equil.

Counter ex.



We cannot support the eff. allocation since prod. set is not convex

16.E.F Characterization of efficiency

(A1) • Suppose $X_i = \mathbb{R}^L_+$, $\succsim_i$ are strongly monotone and continuous

<u>Prop</u> feasible alloc (x^*, y^*) is eff.

$\iff$ it is $\boxed{\text{weakly eff}}$, in the sense that there is no feasible alloc (x, y) : $x_i \succsim_i x_i^*$ $\forall i$

<u>Pf</u> Eff. $\to$ weakly eff. obviously

For the converse, show that $\neg$efficient $\to$ $\neg$ weakly eff. $\neg(x^*, y^*)$ is not eff. $\to \exists$ feasible alloc. (x, y) : $x_i \succsim_i x_i^*$ $\forall i$ and for some $x_k \succ x_k^*$. ~~This implies~~

$\hat{x}_k = (1-\varepsilon)x_k$, redistribute equally εx_k among all other individuals $\to$ increase $x_i + \frac{\varepsilon \cdot x_k}{I-1} = \hat{x}_i$

$\to$ new consump. $(\hat{x}, y)$ is feasible, $\hat{x}_i \succ x_i^*$ $\forall i$, ~~[illegible]~~ $\forall$ suff. small ε

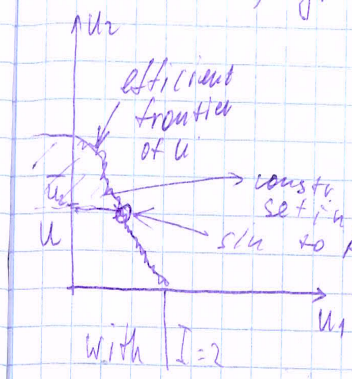
u_i - utility fnc for person i

<u>Prop 2</u> An feasible alloc. (x^*, y^*) is eff. $\iff \exists$ param. $\bar{u}_2, \dots, \bar{u}_I$ s.t. (x^*, y^*) solves

prb 1 $\max_{(x,y)} u_1(x_1)$ s.t. $u_i(x_i) \geq \bar{u}_i$ $\forall i = 2, \dots, I$

& $\sum_i x_i = \bar{\omega} + \sum_{i=1}^{I} y_i$

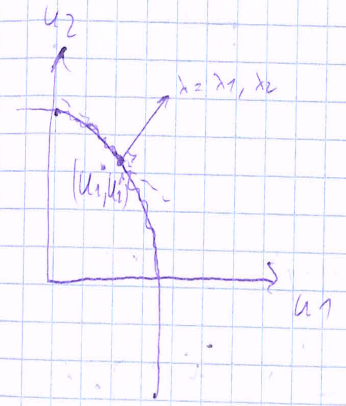
& $x_i \geq 0$ $\forall i$, $y_i \in Y_i$ $\forall i$



utility possibility set

Let $\boxed{u}$ be set of all vectors $(\hat{u}_1, \dots, \hat{u}_I)$ s.t. for some feasible alloc (x, y) $\hat{u}_i \leq u_i(x, y)$ $\forall i = 1, \dots, I$

$\to$ we can recover this eff. frontier by changing $\bar{u}_2$



if $u_1, \dots, u_I$ are concave fncs and $X_1, \dots, X_I$, $Y_1, \dots, Y_I$ are convex $\to$ UPS is convex

$\lambda \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}$ will maximize $\lambda(\hat{u}_1, \hat{u}_2)$ over the UPS

<u>Prop 3</u>

Assume in addition that u_i is concave $\forall i$

Then an alloc (x^*, y^*) is eff $\iff \exists (\lambda_1, \dots, \lambda_I) \geq 0$, $\neq 0$

s.t. $\lambda_i > 0$ for at least one i

s.t. (x^*, y^*) solves:

prb 2: $\max_{(x,y)} \lambda_1 u_1(x_1) + \dots + \lambda_n u_n(x_I)$ s.t.

$\sum_i x_i = \bar{\omega} + \sum_i y_i$

$x_i \geq 0$, $y_i \in Y_i$ $\forall i, j$

if $Y_j = \{y_j \in \mathbb{R}^L : \underbrace{F_j(y_j)}_{\text{production transf function}} \leq 0\}$

then the constr. $y_j \in Y_j$ takes a form $\boxed{F_j(y_j) \leq 0}$

We can then use FOC in Prb 1 and prb 2

Reminder **Prop 2** (x^*, y^*) is eff. iff $\exists \bar{u}_2, .., \bar{u}_I$ s.t. (x^*, y^*) solves

$$(P1) \quad \text{Max } u_1(x_1) \text{ s.t. } u_i(x_i) \geq \bar{u}_i \quad i=2,..,I$$
$$\sum_i x_i = \bar{\omega} + \sum_j y_j$$
$$x_i \geq 0 \quad \forall i=1,..,I$$
$$y_j \in Y_j \quad \forall j=1,..,J$$

Proof] (x^*, y^*) is eff. Let $\bar{u}_i = u_i(x_i^*)$ for $i=2,..,I$ → (x^*, y^*) satisfies the constr. in (P1)

] by contrad. that (x^*, y^*) does not solve the prb → $\exists (\hat{x}, \hat{y})$: satisfies all the constraints in P1 and $u_1(\hat{x}_1) > u_1(x_1^*)$. Constr hold imply that $(\hat{x}, \hat{y})$ is feasible & $u_i(\hat{x}_i) \geq \bar{u}_i = u_i(x_i^*)$ $\forall i=2,..,I$ →

→ $(\hat{x}, \hat{y})$ Pareto dominates (x^*, y^*)

→ (x^*, y^*) was not efficient.

Converse:] (x^*, y^*) solves the prb (P1) for some $\bar{u}_2, .., \bar{u}_I$

By the constraints (x^*, y^*) is feasible

] (x^*, y^*) is not efficient → $\exists$ another ~~feasible alloc~~ $(\hat{x}, \hat{y})$ → it is not even weakly eff. i.e. $\exists$ a feasible alloc $(\hat{x}, \hat{y})$: $u_i(\hat{x}) > u_i(x^*)$ $\forall i=1,..,I$

Since & $u_i(x_i^*) \geq \bar{u}_i$ → $u_i(\hat{x}_i) > \bar{u}_i$ $\forall i=2,..,I$. All constraints hold for $(\hat{x}, \hat{y})$ and moreover $u_1(\hat{x}_1) > u_1(x_1^*)$ which contradicts that x^*, y^* actually solves the prb. □

Prop] $u_1, .., u_I$ are concave so that the UPS is convex. Then an allocation x^*, y^* is efficient ⟺ $\exists \lambda = (\lambda_1 ... \lambda_I) > 0$ s.t. (x^*, y^*) solves

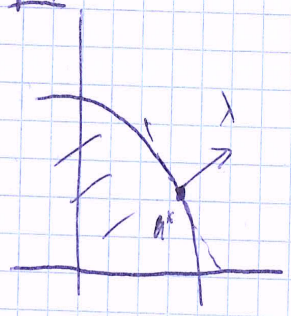
the prb

(P2) $\max \lambda_1 u_1(x_1) + \dots + \lambda_I u_I(x_I)$

$$\text{s.t. } \sum_i x_i = \bar{\omega} + \sum_j y_j$$

$$x_i \geq 0 \quad y_j \in Y_j \ \forall i, j$$

Pf



The "only if" part comes from the supporting hyperplane theorem + convexity of ups

For the converse implication $\exists\, x^*, y^*$ solves the prb P2 for some $\lambda > 0$. if x^*, y^* were not eff. → it is not even weakly eff. → $\exists$ a feasible alloc $(\hat{x}, \hat{y})$: $u_i(\hat{x}_i) > u_i(x_i^*)$ feasibility means that it satisfies constraints in P2. Since $\lambda_i \geq 0 \ \forall i$ with strict > for some

$$\rightarrow \sum_{i=1}^{I} \lambda_i u_i(\hat{x}_i) > \sum_i \lambda_i u_i(x_i^*)$$

which contradicts that (x^*, y^*) solves (P2)

Efficiency is equiv to weak eff only under str. monot + continuity of preferences.

$$Y_j = \{y_j \in \mathbb{R}^L : f_j(y_j) \leq 0\}$$

Check that FOCs in both prb (P1) and (P2) can be written as (assuming interior slns)

1) $$MRS_i(\ell, \hat{\ell}) \equiv \frac{\partial u_i(x_i)/\partial x_{i\ell}}{\partial u_i(x_i)/\partial x_{i\hat{\ell}}} = \frac{\partial u_{\hat{i}}(x_{\hat{i}})/\partial x_{\hat{i}\ell}}{\partial u_{\hat{i}}(x_{\hat{i}})/\partial x_{\hat{i}\hat{\ell}}} \equiv MRS_\ell$$

2) $$MRTS_j(\varphi, \hat{\varphi}) \equiv \frac{\partial F_j(y_j)/\partial y_{j\varphi}}{\partial F_j(y_j)/\partial y_{j\hat{\varphi}}} = \frac{\partial F_{\hat{j}}(y_{\hat{j}})/\partial y_{\hat{j}\varphi}}{\partial F_{\hat{j}}(y_{\hat{j}})/\partial y_{\hat{j}\hat{\varphi}}} \equiv MRT_{\hat{j}}(\varphi, \hat{\varphi})$$

3) $MRTS_j(\ell,\hat{\ell}) = MRS_i(\ell,\hat{\ell})$ $\forall j, i, \ell, \hat{\ell}$

Chapter 17

Existence of equilibrium

Consider exchange economy with I consumers, L goods, strongly monotone, continuous + strictly convex pref.

$B_i(p_i) = \{X_i \in \mathbb{R}^L_+ : pX_i \le p\,w_i\}$

$X_i(p)$ the optimal bundle $p \gg 0$

Excess demand func. $\sum_{i=1}^{I} (X_i(p) - w_i) = \left(\sum_{i=1}^{I} X_i(p)\right) - \bar{w} \quad z(p)$

A price vector p is an equilibrium iff $p \gg 0$ and excess demand

$z(p) = 0$

Proposition 17.B.2

$\exists\, \bar{w} \gg 0$. Then

(i) $z(p)$ is continuous in p. by the theorem of the maximum

(ii) $z(\gamma p) = z(p)$ $\forall \gamma > 0$ homogenity of deg 0

(iii) $p\,z(p) = 0$ Walras law $\forall p \gg 0$

(iv) $X_i = \mathbb{R}^L_+$ $\forall i$

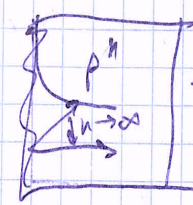
$z(p)$ is bounded from below in this particular setting

$z(p) \ge -\bar{w}$ $\forall p \gg 0$

(v) if $p^n \to p$, $p \neq 0$, $p_\ell = 0$ for some ℓ whereas $p^n \gg 0$ then

$\max\{z_1(p^n), \dots, z_\ell(p^n)\} \xrightarrow[n\to\infty]{} \infty$

i.e. the demand for some good is going to ∞



this part fails here because there is no strict monotonicity of pref.

Prof. 17.C.1

$\exists\, p^* \gg 0$ that clears the market s.t. $z(p^*)=0$

Pf We will use Kakutani's fixed point theorem after defining correspondence that maps prices into prices.

Since $z(\cdot)$ is homogen of deg. 0 we can focus on the unit simplex $\Delta =$
$= \{p \in \mathbb{R} : \sum_{\ell=1}^{L} p_\ell = 1\}$

Let $\mathring{\Delta} = \{p \in \Delta : p_\ell > 0 \ \forall \ell\}$

Define a correspondence $\Psi : \Delta \to \Delta$ as follows.

for $p \in \mathring{\Delta}$ $\quad \Psi(p) = \{q \in \Delta : q \cdot z(p) \geq \hat{q}\, z(p) \ \forall \hat{q} \in \Delta\}$

For $p \in \Delta \setminus \mathring{\Delta}$ $\quad \Psi(p) = \{q \in \Delta : q \cdot p = 0\}$

Note that there could not be a fixed point at a boundary since p cannot be $\perp$ to itself.

i.e. $\nexists\, p \in \Delta \setminus \mathring{\Delta}$ s.t. $p \in \Psi(p)$

$\lrcorner$ $p \in \mathring{\Delta}$. Check that $\Psi(p) = \{q \in \Delta :$
$: q_\ell > 0 \to z_\ell(p) \geq z_{\hat{\ell}}(p) \ \forall \hat{\ell}\}$

~~Here~~ claim

We know that $\forall$ fixed point of Ψ clears the market.

i.e. $p^* \in \Psi(p^*) \to z(p^*) = 0$

$\lrcorner$ p^* is a fixed point $p^* \in \Psi(p^*)$

$\to p^* \gg 0$ as we noted earlier

$\to \Psi(p^*) = \{q \in \Delta : q_\ell > 0 \to z_\ell(p^*) \geq z_{\hat{\ell}}(p^*)$
$\forall \hat{\ell}\}$

Since $p^* \gg 0$ and $\in \Psi(p^*) \to$

$\to \forall \ell \ z_\ell(p^*) \geq z_{\hat{\ell}}(p^*) \ \forall \hat{\ell}$. Thus,

$\exists\, \varepsilon \in \mathbb{R} : z(p^*) = (\varepsilon, \dots, \varepsilon)$. Since

$p^* z(p^*) = 0$ by Walras law

$\to \varepsilon = 0 \to z(p) = 0$. ~~same~~

It can be shown that ψ is closed, convex-valued and non-empty valued and usc $\rightarrow \exists$ a fixed point $p^* \in \psi(p^*)$ and this completes the proof.

usc follows from (V) of Prop 17.B.2.

if there is prod. sector

$$z(p) = \sum_i' x_i(p) - \bar{\omega} - \sum_i' y_j(p)$$

<u>Uncertainty. Equilibrium under uncert.</u>

Ch.19

- Two periods $t = 0, 1$
- L goods (physical goods)
- S states
- I indiv. (consumers)
- J firms
- At $t=0$ people are traiding forward (future) contracts on state contingent commodities
- At $t=1$ uncertainty resolves. Depending on the realization of the state the contracts are executed
- X_i is the consumption plan (set of all possible cons plans) for cons. i Subset of $\mathbb{R}^{LS}$. $X_i = \mathbb{R}^{LS}_+$ typically
- $x_i \in X_i = (x_{esi})_{es}$. With $x_{si} = (x_{1si}, \dots, x_{Lsi})$, $x_i = (x_{1i}, \dots, x_{Si}) = (x_{si})_s$
- In state s, the person i is entitled to ~~receive~~ consume x_{esi} units of the good (this the content of a partic. plan x_i)
- $\omega_i \in (\omega_{esi}) \in \mathbb{R}^{LS}$ consumer's i initial endowment. In state s cons i will receive ω_{esi} units of good ℓ
- if $x_{esi} < \omega_{esi}$, consumer i will give up some of his endowment of good ℓ Otherwise he recoives good

- $Y_j \subseteq \mathbb{R}^{LS}$ is the set of all possible prod. plans for firm j $\quad y_j = (y_{esj})$

if $y_{esj} > 0$ the firm j is promising to produce y_{esj} units of good e in state s

Otherwise it is entitled to receive $-y_{esj}$ units of goods as an input

- $p = (p_{es}) \in \mathbb{R}^{LS}$ a price vector

p_{es} - time 0 price of receiving a good e at state s at time 1

or price at $t=0$ of right to receiving one unit of good e in state

- py_j induced by the prod. plan y_j

Firms' prb.

$$\max_{y_j \in Y_j} py_j \rightarrow py_j^*$$

- Firms are owned by individuals

$$B_i(p) = \{x_i \in X_i \mid px_i \leq pw_i + \sum_{j=1}^{J} \theta_{ij} p y_j^*\}$$

- All activities (production, delivery, consumption) takes place at $t=1$ after the resolution of uncertainty.

- Cons i evaluates a cons plan X_i with expected utility funct.

$$U_i(X_i) = \sum_{s=1}^{S} \pi_{si} u_{si}(X_{si})$$

π_{si} - probability of states, according to person's i beliefs; $\pi_i = (\pi_{1i}, \dots, \pi_{Si})$ - prior belief of person i

- u_{si} is the Bernoulli utility index in state s

if $u_{si} = u_i \ \forall s \rightarrow$ utility function is state independent

If u_{si} is concave over physical cons. bundles on $\mathbb{R}^L_+ \ \forall s$ then U_i is concave

fnc over cons. plans

- if u_{si} is concave $\Rightarrow$ individ is risk-averse $\mapsto$ try to smoothen consumption over diff. states s (does not like volatility)

The trade at $t=0$ allows people to transfer their wealth across diff. states for consumpt. smoothing. In particular, one can sell $w_{\hat{s}i}$ to increase consumpt. of good $\hat{\ell}$ in state $\hat{s}$

<u>Defn</u> An Arrow-Debreu equilibrium is a Walrasian equil. of this model with state-contingence commodity. It consists of price vector $p \in \mathbb{R}^{LS}$, cons. plans $(x_1^*, \dots, x_I^*)$ and prod. plan $(y_1^*, \dots, y_J^*)$

s.t.

(1) $\xi \; p y_j^* \geq p y_j \quad \forall y_j \in Y_j$
$(y_j^* \in Y_j)$

(2) x_i^* maximizes u_i over $B_i(p)$

(3) $\sum_{i=1}^{I} x_i = \bar{\omega} + \sum_{j=1}^{J} \xi y_j$

$\boxed{\text{Example 19.C.1}}$ $L=1, \; I=2, \; S=2$

$x_i = (x_{1i}, x_{2i})$

$\omega_i = (\omega_{1i}, \omega_{2i})$

$\omega_1 = (1, 0)$
$\omega_2 = (0, 1)$ $\Rightarrow \bar{\omega} = (1, 1)$

No aggregate uncertainty

Also assume strictly concave ~~indep.~~ utilities

$u_i(x_i) = \pi_{1i} u_i(x_{1i}) + \pi_{2i} u_i(x_{2i})$

$MRS = \dfrac{\pi_{1i} u_i'(x_{1i})}{\pi_{2i} u_i'(x_{2i})}$; Efficiency: $MRS_1 = MRS_2 \rightarrow \bar{\omega}_1 = x_{11}$

$\rightarrow \dfrac{\pi_{11} u_1'(x_{11})}{\pi_{21} u_1'(x_{21})} = \dfrac{\pi_{12} u_2'(x_{12})}{\pi_{22} u_2'(x_{22})}$

$$\frac{\pi_{11} u_1'(x_{11})}{\pi_{21} u_2'(x_{21})} = \frac{\pi_{12} u_2'(1-x_{11})}{\pi_{22} u_2'(1-x_{22})}$$

if $\pi_{11} = \pi_{12}$ common beliefs then

$x_{11} = x_{21}$
$x_{22} = x_{12}$

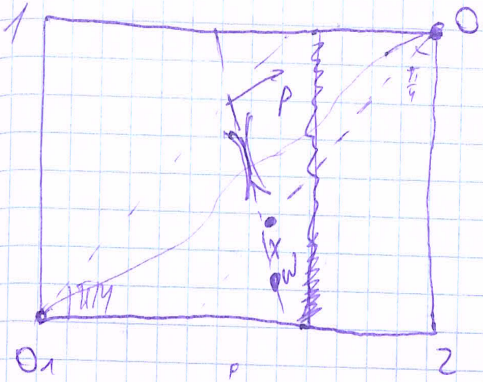
Efficiency requires perfect consumpt. smoothing

$\rfloor$ $\pi_{11} < \pi_{12}$, $\rightarrow \frac{x_{11}}{x_{21}} < 1 < \frac{x_{12}}{x_{22}}$ in any eff. alloc.

$\boxed{\text{Ex. 19.C2}}$ $\rfloor$ nonuncertain everything the same, but

$w_1 + w_2 = \bar{w} = (2, 1)$ (there is agg. uncert.)

$\rfloor$ $\pi_{s1} = \pi_{s2} = \pi_s$



if x lies on the 2nd 45° line

$MRS_2 = \frac{\pi_{12}}{\pi_{22}} = \frac{\pi_1}{\pi_2}$

$MRS_1 = \frac{\pi_1}{\pi_2} \frac{u_1'(x_{11})}{u_1'(x_{21})} =$

since $x_{11} > x_{21} \rightarrow MRS_1 < \frac{\pi_1}{\pi_2}$

$\rightarrow$ x is not eff.

Similarly $\nexists$ eff. alloc on the 1st 45° line

• All eff. alloc. should lie in between these two lines

$\rightarrow x_{1i} > x_{2i}$ for $i = 1, 2$ in an eff. alloc. $\rightarrow \frac{u_1'(x_{11})}{u_1'(x_{21})} < 1$

$MRS_1 < \frac{\pi_1}{\pi_2}$, as well as MRS_2

A typical equil. as in graph

$MRS_i(x^*) = \frac{p_1}{p_2} < \frac{\pi_1}{\pi_2} \rightarrow$

$\rightarrow$ if $\pi_1 = \pi_2 \rightarrow p_1 < p_2$ i.e. cons.

in state 2 (poor state) is more valuable

- More generally $\frac{p_1}{\pi_1} < \frac{p_2}{\pi_2}$,

After correcting for probabilities the prices are negatively corr with the total wealth $\bar{\omega}_s$

AD model

$t=0$

trade state contingent commodity to max exp. utility of future cons

$t=1$

- state realises
- cons/prod takes place

(1) $\max \; u(x_i) = \sum_s \pi_{si} u(x_{si}) \quad s.t \; x_i \geq 0$

$$p x_i = \sum_e \sum_s p_{es} x_{es} \leq p \omega_i$$

$$\sum_e \sum_s p_{es} x_{es} = \sum_s p_s x_{si}$$

If x_i^* solves (1) then $\forall s : \pi_{si} > 0$ x_{si}^* solves the prob

$$\max \; u_{si}(x_{si}) \quad s.t \; p_s x_{si} \leq p_s x_{si}^*$$

i.e. once state is resolved consumers have no need to change their consumption plans (because of separability of the UF)

Cons: Let $p = (p_s)$, $x^* = (x_1^*, \dots, x_I^*)$ be an equilibr. in the AD exchange model + $\pi_{si} > 0 \; \forall s \; \forall i$. Then $\forall s$ the price vector p_s & alloc. induced by consumpt. plan to that part. state constitute a Walrasian equil. in a spot market after reopening trading sessions where indiv i has x_{si}^* as her endowment.

Sequential trading (Ch. 19D)

$t=0$

- trade assets to max exp utility of cons

- fewer assets than before (in AD model)

~~at~~ $t=1$
- state realises
- a spot market opens for physical com
- consumpt. takes place

1st physical good ($\ell \neq 1$) acts as a numeraire (money). All assets are denominated in good 1

- Basic setup with <u>Arrow security</u>: there are S assets per state. A given one indexed by s pays 1 unit of good 1 in state s and 0 in other states

$z_i = (z_{1i}, \dots, z_{Si}) \in \mathbb{R}^S$ denotes the asset portfolio of person i

if $z_{si} > 0 \rightarrow$ person i is entitled to receive z_{si} units of physical good 1 in state s

if $z_{si} < 0$ then person i is obliged to deliver z_{si} units of good 1 output at state s.

- $w_i = (w_{si}) \in \mathbb{R}^{LS}$ is the endowment of i
- if $z_{si} + w_{1si} < 0 \rightarrow$ the person i is short selling good 1 at state s, if that state realizes she will need to buy it in the spot market at $t=1$
- $q = (q_s) \in \mathbb{R}^S$ is the price vector for the assets at $t=0$.
- q_s is the time 0 price of the sth Arrow asset
- $p = (p_s) \in \mathbb{R}^{LS}$ is the price vector for the spot markets at $t=1$
- $x_i = (x_{si}) \in \mathbb{R}^{LS}_+$ is a cons. plan

Cons i's prb at $t=0$: (P) $\max U_i(x_i) = \sum_s \pi_{si} u_{si}(x_{si})$

s.t. $x_i \in \mathbb{R}^{LS}_+$, $z_i \in \mathbb{R}^S$; s.t. $q_i z_i \leq 0$

$p_s x_s \leq p_s w_{s1} + z_{si} p_{s1}$ (1st coord of p_s)

Remark: consumer can predict future prices at $t=0$ (Rationale expect. assump.)

<u>Defn</u> A Radner equil. (basic sequential exchange model) consists of asset prices $q \in \mathbb{R}^S$, spot prices $p \in \mathbb{R}^{LS}$, consumpt. plans $X_i \in \mathbb{R}^{LS}_+$ and asset portfolios $z_i^* \in \mathbb{R}^S$ for each consumer

(1) X_i^* solves the (P) $\forall i$

(2) Market clearing $\sum_i X_i^* \leq \overline{\omega}$

(3) $\sum_i z_i^* \leq 0$

<u>Define</u> $B_i^R(p,q) = \{X \in \mathbb{R}^{LS}_+ : \exists z_i \in \mathbb{R}^S : qz_i \leq 0$ and $p_s X_{si} \leq p_s \omega_s + p_{1s} z_{si} \ \forall s\}$

- Check that $X_i^*, z_i^* \in \mathbb{R}^{LS}_+$ jointly solves (P) $\Leftrightarrow$ X_i^* max $U_i(X_i)$ over the $B_i^R(p,q)$ (Indeed z_i does not influence utility it's just the tool to support consump. plans. This role is already <u>included in the defn of $B_i^R(p,q)$.</u>)

i.e. z_i^* should satisfy

$qz_i^* \leq 0$ & $p_s X_{si}^* \leq p_s \omega_{si} + p_{1s} z_{si}^* \ \forall s$

- $B_i^{AD}(p) = \{X_i \in \mathbb{R}^{LS}_+ : pX_i^* \leq p\omega_i\}$

<u>Prop: 19.D.1</u> (1) If $X^*(p)$ is an AD equil. with $p >> 0$. Then $\exists$ a Radner equil. (X^*, z^*, p, q) where $q_s = p_{1s} \ \forall s$

(2) If (X^*, z^*, p, q) is a Radner equil. with $p >> 0$, $q >> 0$. Then Define γ_s as $q_s = \gamma_s p_{1s} \ \forall s$ & let $\tilde{p} = (\tilde{p}_s)$ with $\tilde{p}_s = \gamma_s p_s \ \forall s$. Then $(X^*, \tilde{p})$ is an AD equil.

<u>Proof</u> (1): If (X^*, p^*) is an AD equil $p >> 0$. Let $q_s = p_{1s} \ \forall s$. 1st step: to show that Budget set remains the same

$B_i^R(p,q) = B_i^{AD}(p)$. Let X_i - cons. plan

which $\in B_i^{AD}(p)$ (3): $px_i = \sum_s' p_s x_{si} \leq$

$\leq \sum_s' p_s w_{si}$. Define $z_i \in \mathbb{R}^S$

as $z_{si} = \frac{1}{p_{1s}}(p_s(x_{si} - w_{si}))$ $\forall s$. Note

that $qz_i = \sum_s' q_s z_{si} = \sum_s' p_{1s} z_{si} =$

$= \sum_s' p_s(x_{si} - w_{si}) \leq 0$

by (3)

By construction, $p_{1s} z_{si} + p_s w_{si} \geq p_s x_{si}$ $\forall$

Thus, $x_i \in B_i^R(p,q)$, as we sought.

Converse: $\exists$ $x_i \in B_i^R(p,q)$ then

$\exists$ $z_i \in \mathbb{R}^S$ s.t. $qz_i \leq 0$ & $p_s x_{si} \leq p w_{si}$

$+ p_{1s} z_{si}$ $\forall s$ (4)

Sum in (4) $\sum_s' p_s x_{si} \leq p w_i + \sum_s' p_{1s} z_{si}$

$\sum_s' p_{1s} z_{si} = \sum_s' q_s z_{si} = q z_i \leq 0$

left

$\rightarrow$ $\sum_s' p x_i \leq p w_i$ i.e. $x_i \in B_i^{AD}(p)$

Since $B_i^{AD}(p) = B_i^R(p,q)$ it follows

that x_i^* max $U_i(x_i)$ over $B_i^R(p,q)$.

Select z_i^* as $z_{si}^* = \frac{1}{p_{1s}} p_s(x_{si} - w_{si})$, so

that z_i^* supports x_i^*. Thereby $\forall i$

we found a solution (x_i^*, z_i^*) to

prob (P). The allocation $(x_1^*, .., x_I^*)$

is feasible by defn of AD equil.

Market clearing need to be

checked $\sum_i' z_{si}^* = \sum_i' \frac{1}{p_{1s}} p_s(x_{si}^* - w_{si})$

$= \frac{1}{p_{1s}} \sum_i' p_s(x_{si}^* - w_{si}) = \frac{1}{p_{1s}} \underbrace{p_s}_{\text{assump}} \underbrace{\sum_i' (x_{si}^* - w_{si})}_{\leq 0 \text{ feasibility in AD}}$

$\rightarrow$ $\sum_i' z_{si}^* \leq 0$

Pf of the ($\supseteq$): Check that

$B^R(p,q) = B_i^R(\hat{p},q)$ because we com

scale in state s prices by a number >0, γ_s in particular

And $B_i^R(\bar{p}, q) = B_i^{AD}(\bar{p})$ by the 1st part. And that's enough

A more general assets structure

- K assets, $k = 1, \dots, K$. The asset k is characterized by return vector $r_k = (r_{si}) \in \mathbb{R}^S$, where r_{si} is the amount of physical good 1 that assets k pays in state s.
- $z_i = (z_{1i}, \dots, z_{ki})$ - asset portfolio of consumer i. And z_{ki} amount of asset k in the portfolio z_i

In state s, $p_s \sum_{k=1}^{K} z_{ki} r_{sk}$ - return of the portfolio z_i

- Spot market Budget inequality:

$$p_s x_{si} \le p_s w_{si} + p_{1s} \sum_{k=1}^{K} z_{ki} r_{sk}$$

- $q = (q_1, \dots, q_K) \in \mathbb{R}^K$ is the asset price vector. Given q and spot prices $p \in \mathbb{R}^S$ the cons. prb is

$$\max u(x_i)$$

$$\text{s.t. } x_i \ge 0,\ z_i \in \mathbb{R}^K \text{ s.t. } q z_i \le 0$$

$$+ \ p_s(x_{si} - w_{si}) \le p_{1s} \sum_{i=1}^{K} z_{ki} r_{sk} \ \forall s$$

<u>Defn</u> In a Radner equil $p, q, x_1^*, \dots, x_I^*, z_1^*, \dots, z_I^*$ satisfying:

(1) $(x_1^*, \dots, x_I^*)$, $(z_1^*, \dots, z_I^*)$ solve consumers' problems (x_i^*, z_i^*) solves

(2) $\sum_i x_i^* \le \bar{\omega}$

(3) $\sum_i z_i^* \le 0$

Normalize spot price ~~at~~ as $p_{1s}=1 \ \forall s$

Let $R=[r_1 \dots r_k]$ be the returns

column vector $S \times K$ matrix

$$Rz_i = \begin{bmatrix} \sum_{k=1}^{K} r_{1k} z_{ki} \\ \dots \\ \sum_{k=1}^{K} r_{Sk} z_{ki} \end{bmatrix}$$ is the wealth distr across the states induced by portfolio z_i

The budget inequality in the spot market: $[p_s(x_{si}-w_{si})]_s \leq Rz_i$

Accordingly, the budget of cons plans is

$$B_i^R(p,q) = \{x_i \in \mathbb{R}_+^{LS} : \exists z_i \in \mathbb{R}^K : qz_i \leq 0 \text{ and } [p_s(x_{si}-w_{si})]_s \leq Rz_i\}$$

Assume: A1 $\bar{w}_{si} > 0$ and u_{si} is LNS $\forall s \in S$

Claim: In any Radner-equil the asset prices satisfy the following No-arbitrage condition:

$$\nexists z_i \in \mathbb{R}^K \text{ s.t. } qz_i \leq 0 \text{ and } Rz_i > 0$$

Proof: Let (x_i^*, z_i^*) be the solution to the cons i prb and by contradict. that $\exists z_i \in \mathbb{R}^K : qz_i \leq 0$ & $Rz_i > 0$. Let $\hat{z}_i = z_i + z_i^*$. Then

$q\hat{z}_i = qz_i^* + qz_i \leq qz_i^* \leq 0$ feasible (affordable)

$R\hat{z}_i = Rz_i^* + Rz_i > Rz_i^*$. Thus $\hat{z}_i$ allows the consumer to increase her spot market expenditure in some states without reducing exp. in other states → by A1 expected utility is increased ~~with~~ → x_i^* cannot be the maximizer.

~~If all returns~~

Corollary: if $r_k > 0$ then $q_k > 0$

Proof Let $z_i = \mathbb{1}_k$. Then $qz_i = q_k$. Moreover $Rz_i = r_k > 0$ → by No-arbitr. cond

$q_k > 0$ $\square$

<u>Prop 19.E.1</u>: $\exists\, r_k > 0\ \forall k$, and $\forall s\ \exists k$ s.t. $r_{sk} > 0$. Then $\exists\, \mu = (\mu_1 .. \mu_S)$ s.t. $\mu^T R = q^T$ i.e. $\mu^T r_k = \sum_{s=1}^{S} \mu_s r_{sk} = q_k$ $\forall k = 1, .., K$.

In many cases

μ_s - price time 0 price of physical good 1 at state s. (s Arrow security)

<u>Proof skipped</u> but follows from the no-arbitr. cond. and sep. hyp. theorem

~~In many states you can think of~~ s

Let Range of $R = \{Rz_i : z_i \in \mathbb{R}^K\}$

<u>Prop 19.E.3</u>: $\exists\, R$: satisfies 19.E.1 and let $\mu > 0$ the vector from 19.E.1 i.e. $q = \mu^T R$. In a Radner equil. (x^*, z^*, p, q). Consider any other return matrix $\tilde{R}$ s.t. Range $\tilde{R}$ = ~~R and~~ Range R

Then the alternative model with the return matrix $\hat{R}$ has a Radner equil $(x^*, \tilde{z}^*, p, \tilde{q})$ s.t. $\tilde{q} = \mu^T \tilde{R}$ i.e. prices of Arrow sec do not change

<u>Defn</u> The asset market is said to be complete if Range $R = \mathbb{R}^S$

Note: completeness requires $K \geq S$ assets the return matrix R should have an $S \times S$ invertible submatrix or equiv. there should be at least S assets with linearly indep return vectors.

Note: In a model with S Arrow securities, The return matrix is $R^A = I_{S \times S}$. This asset market is complete. Moreover, $q^A = \mu^A R^A = \mu^A I_{S \times S} = \mu^A$ i.e. the prices of Arrow securities q^A equal

the μ-vector from the Prop 19.E.1. Thus we can invoke Prop 19.E.3 to compute the asset price ~~of the~~ in any other complete markets $q = q^A R$
(consumpt. plans and etc do not change)

Recall that by Prop 19.D.1 we also know ~~to~~ how to convert an AD equil to a Radner equil with Arrow securities and vice versa

<u>Note</u>: If the asset market is complete ~~and~~ adding the new assets will not change the range. The price of new asset q_{k+1} can be computed in 2 ways:
1st: Solve μ in $q = \mu R$ and let $q_{k+1} = \mu^T r_{k+1}$

(ii) Solve the equation r_{k+1} as a linear comb. of the other return vect.

Say $r_{k+1} = \alpha_1 r_1 + \dots + \alpha_k r_k$

Then $q_{k+1} = \alpha_1 q_1 + \dots + \alpha_k q_k$

Indeed $q_{k+1} = \mu^T(\alpha_1 r_1 + \dots + \alpha_k r_k) =$
$= \alpha_1 q_1 + \dots + \alpha_k q_k$

<u>Proof of Prop. 19.E.3</u>

Step 1: $\sum_{i=1}^{I} z_i^+ = 0$ (omitted)

Step 2: $B_i^R(p, q) = B_i^{\hat{R}}(p, \hat{q})$, $\hat{q} = \mu^T \hat{R}$

Let $x_i \in B_i^R(p, q) \to \exists z_i \in \mathbb{R}^k$ s.t. $q z_i \leq 0$ and $[p_s(x_{si} - w_{si})] \leq R z_i$. Since Range of R = range of $\hat{R}$ $\to \exists$ equiv. portfolio $\tilde{z}_i \in \mathbb{R}^{\tilde{k}}$: $\tilde{R}\tilde{z}_i = R z_i$. Thus $[p_s(x_{si} - w_{si})] \leq$
$\leq \hat{R}\tilde{z}_i$. Moreover $\tilde{q}\tilde{z}_i = (\mu^T \tilde{R})\tilde{z}_i =$

$= \mu^T(\hat{R}\hat{z}_i) = \mu^T R z_i = (\mu^T R) z_i = q z_i \leq 0$

Since $q z_i \leq 0 \rightarrow \hat{q}\hat{z}_i \leq 0 \rightarrow X_i \in B^h_i(p, \hat{z}_i)$

as we sought

Conclusion: the original equil. cons. plan maximises the utility fnc. over the new budget

It remains to define the asset portfolios $\hat{z}_1^*, \dots, \hat{z}_I^*$ in the new market for $i < I$ pick any $\hat{z}_i^*$ s.t. $\hat{R}\hat{z}_i^* = R z_i^*$ (We can select such $\hat{z}_i^*$ because Range $\hat{R}$ = Range R)

For I define z_I^* to clear portfolio market i.e.

$$\hat{z}_I^* = -\sum_{i<I} \hat{z}_i^*$$

By defn $\sum_i \hat{z}_i^* = 0$.

Moreover, $\hat{R}\hat{z}_I^* = \hat{R}\left(-\sum_{i<I} \hat{z}_i^*\right) =$

$= R\left(-\sum_i z_i^*\right) = R z_I^*$ (by step 1)

defn of $\hat{z}_i^*$ for $i<I$

Thus, $\forall i = 1, \dots, I$ we did not change distr. → original consumpt. plan remains affordable in spot market $\forall i$

$R\hat{z}_i^* = R z_i^*$

Also check that $\hat{q}\hat{z}_i^* \leq 0 \ \forall i$, as in step 2 □

Incomplete markets (19 F)

$K < S$; Range $R \subsetneq \mathbb{R}^S$

Incomplete markets → inefficiency
maybe

Ex 19.C.4

$L = 1$; $S = 2$; $I = 2$; consumer 1 is risk-neutral
cons. 2 is risk-averse

$\lrcorner$ $\omega_1 = \omega_2 = \frac{\overline{\omega}}{2} = \frac{1}{2}(\overline{\omega_1}, \overline{\omega_2})$

$\overline{\omega_1} > \overline{\omega_2}$

Efficiency: $\rightarrow$ $X_{12} = X_{22}$

$K = 0$

Since $L = 1$ in spot market no trade. Thus, people simply consum endowments at equil. which violates efficiency condition

<u>Constraint efficiency</u>: can we improve upon the market outcome without introducing new assets?

Yes, if only 1 phys. good

No, if > 1 goods

$\lrcorner$ $L = 1$ Then the portfolio z_i uniquely determines cons. in state $(t=1)$ in the spot market. In state s the cons is $\omega_{si} + \sum_{k=1}^{K} z_{ki} r_{sk}$

Corresponding <u>indirect</u> utility $u_i^*(z_i) =$

$= u_i(\omega_{1i} + \sum_{k=1}^{K} z_{ki} r_{sk}, \dots, \omega_{Si} + \sum_{k=1}^{K} z_{ki} r_{sk})$

Cons 1 Prb: $\max u_i^*(z_i)$

s.t. $q z_i \leq 0$

$z_i \in \mathbb{R}^K$

$\lrcorner$ $r_k > 0 \ \forall k$, $\pi_{si} > 0 \ \forall i$, $u_{si}(\,)$ is $\uparrow$

Then $q \gg 0$

if $\hat{z}_i \succsim_i z_i^*$ then $q\hat{z}_i \geq 0$ if $\hat{z}_i \succ_i z_i^*$ then $q\hat{z}_i > 0$

So, $\hat{z}_i \succsim_i z_i^* \ \forall i$ and $\exists i: \hat{z}_i \succ_i z_i^*$

then $\cancel{q\hat{z}_i \geq 0}$ $q \sum_{i=1}^{I} \hat{z}_i > 0 \rightarrow \sum_i \hat{z}_i > 0$

$\rightarrow$ it is not feasible $(q \gg 0)$

Thus, z^* is Pareto efficient (constrained efficient)

• If $L > 1$?

<u>Example</u>: No assets. Spots markets are identical, but admit multiple Walrasian equilibria, say p' and p''

$\lceil$ Some equilibria can Pareto dominate

Other equilibria.

Let $S=2$

Indirect utility: $v_1(p') > v_1(p'')$, hence $v_2(p') < v_2(p'')$

Radner equilibria (p',p'), (p',p'') (p'',p') (p'',p'')

if π_{11} is large i.e. agent 1 prefers (p',p'') to (p'',p')

Similarly π_{22} is large -//- 2 prefers (p'',p') to (p',p'')

(p',p'') Pareto dominates (p'',p')

Imperfect information (19-H)

- What if there is further uncertainty ~~before~~ at the time of spot trading
- Suppose individuals get private signals as a function of state $\sigma_i(s)$ signal of indiv. i in state s
- if $\sigma_i(s) \neq \sigma_i(s')$ then indiv i can privately distinguish between s and s' $\sigma_i(s) = \sigma_i(s') \to \omega_i(s) = \omega_i(s)$ [endowments are part of signals]
- if the true state is $\bar{s}$ indiv i knows that the state belongs to $S_i(\bar{s}) = \{s' \in S : \sigma_i(s') = \sigma_i(\bar{s})\}$
- If Bernouilli utility indices are state dependent. Then a cons. bundle $x_{\bar{s}i}$ should be evaluated with condition of expected utility (1) $u_i(x_{\bar{s}i} \mid Inform_i(\bar{s}))$
- In a rational expectation equilibrium $Inf_i(\bar{s})$ should take into account the information carried by prices as a public signal

Defn a Rational expect. equil. is a price function $p(\cdot): S \to \mathbb{R}^L : \forall \bar{s} \in S$ $p(\bar{s})$ clears the spot market in state $\bar{s}$, assuming that each indiv. i evaluates

a consumption bundle x_{si} according to (1), where $In_i(\bar{s}) = \{s' \in S : \sigma_i(s') \in \sigma_i(\bar{s}), p(s') = p(\bar{s})\}$

Cyclic situation: p influence utility, but in utility determine prices

- Check if there is an equil in which prices make public all info available to diff. indiv.

] everybody can observe ~~the other~~ ~~others~~ each others signals. That is, $\forall i$ can observe

$\hat{\sigma}_i \equiv (\sigma_1(s), \dots, \sigma_I(s))$

↳ Symmetric information $\hat{\sigma}_1(s) = \hat{\sigma}_2(s) = \dots = \hat{\sigma}_I(s)$

$\hat{S}_i(\bar{s}) = \{s' \in S : \hat{\sigma}_i(s') = \hat{\sigma}_i(\bar{s})\}$

Let $s' \in \hat{S}_i(\bar{s}) \Rightarrow$ ~~s~~ $\hat{S}_i(s') = \hat{S}_i(\bar{s})$

$\rightarrow s' \in \hat{S}_j(\bar{s}) \forall j \rightarrow w_j(s') = w_j(\bar{s})$

→ Thus, ∃ a rationale expect. equil $\hat{p}(\cdot)$: $\hat{p}(\bar{s}) = \hat{p}(s')$ $\forall s' \in S_i(\bar{s})$ $\forall \bar{s} \in S$ and i

Need to check that those prices indeed reveal all the information

i.e. need to check if $\hat{p}(s) \neq \hat{p}(s')$ whenever $\hat{\sigma}_i(s) \neq \hat{\sigma}_i(s')$

if yes, we have an equil. (true rational expect. equil.)

No → troubles

Exp 19.H.3

2 states, 2 consumers

	cons 1	cons 2
s=1	u	v
s=2	v	u

L=2

- $\pi_1 = \pi_2 = 1/2$
- $\omega_1 = \omega_2$, $\omega_{11} = \omega_{21}$ Endowments carry no info
- $\sigma_1(1) \neq \sigma_1(2)$, $\sigma_2(1) = \sigma_2(2)$

Case 1: $P(1) \neq P(2)$ (Symmetric info)

Both indiv. can distinguish between states

$S=1 \rightarrow \bar{P}$ (unique equil of the corresp. spot market)

$S=2 \rightarrow \bar{P}$ (just change indices $1 \rightarrow 2$)

Contradict to

→ no equil. with sym. inf.

Case 2 $p(1) = p(2)$

$S=1 \longrightarrow$ cons 1 uses u

cons. 2 cannot tell the diff. between the states; $\sigma_2(1) = \sigma_2(2)$ and $p(1) = p(2)$. Thus, he uses the expected utility $\frac{1}{2}u + \frac{1}{2}v$

$S=2 \rightarrow$ cons 1 uses v

cons 2 $\frac{1}{2}u + \frac{1}{2}v$

→ in $S=1$ and $S=2$ situations are different → typically we will end up with diff. price

→ contradiction.

↳ NO RATIONAL EXPECT. EQUIL. ☒

Public goods and externalities (11.B-C)

Defn Externalities are present when actions of economic agents directly influence the well-being of others.

For public goods consumption by one indiv. does not the amount available ~~to~~ to others.

Externalities → Public

→ Private (bilateral & in extreme case)

Bilateral example:

- 2 indiv.
- indiv 1 is taking an action that creates externalities
- h - amount of the action
- utility of person i is q-linear i.e. $\varphi_i(h) + m_i$
- Total amount of money is fixed $\bar{\omega} = \omega_1 + \omega_2$

φ_1 is strictly concave

φ_2 is declining with h (negative externality)

<u>Efficiency</u>: An allocation is of the form (h, m_1, m_2) is feasible if $m_1 + m_2 = \bar{\omega}$. Check that (h^*, m_1^*, m_2^*) is efficient iff h^* maximizes $\varphi_1(h) + \varphi_2(h)$ (s.t $m_1^* + m_2^* = \bar{\omega}$)

$$\max_{h \geq 0} \varphi_1(h) + \varphi_2(h) \rightarrow h^0 : \boxed{\varphi_1'(h^0) = -\varphi_2'(h^0)}$$

<u>Private solution</u>: (Indiv 1 only cares about himself.

$$FOC \rightarrow \varphi_1'(h^*) = 0$$

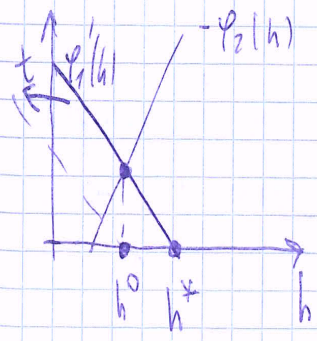
$$\varphi_2'(h^0) < 0 \rightarrow -\varphi_2'(h^0) > 0 \rightarrow$$

$$\rightarrow \varphi_1(h^0) > 0 \rightarrow \boxed{h^0 < h^*}$$

$\rightarrow$ no efficient alloc.

<u>How to fix it</u>: • A quota h^0

- A Pigouvian tax; tax indiv 1 per unit of cons. by amount t.



The objective func of indiv 1 becomes $\varphi_1(h) - th$ (shift φ_1' by t)

$$\varphi_1'(h) - t = 0$$

$$\rightarrow \boxed{t = \varphi_1'(h^0)}$$

A bargaining based solution

- Give indiv 2 a right for externalities free environment and let him sell it to ind 1 if he wishes to do so.
- Suppose 1 & 2 bargain for the amount of h as follows
  - ind 2 makes take it or leave offer to 1.
  - if he does not accept he produces $h=0$
- An offer is a pair (h,T) where T is a transfer from 1 to 2 that allows 1 to produce h
- Indiv. 1 would accept the offer $(h,T) \Leftrightarrow \varphi_1(h) - T \geq \varphi_1(0)$ (1)

Problem of ind 2 is

$$\max_{(h,T)} \varphi_2(h) + T \quad \text{s.t.} \quad \varphi_1(h) - T \geq \varphi_1(0)$$

binds $\rightarrow T = \varphi_1(h) - \varphi_1(0)$

$$\rightarrow \max \varphi_2(h) + \varphi_1(h) - \varphi_1(0)$$

$$\text{s.t.} \quad h \geq 0$$

$\rightarrow$ solution will be h^0

<u>Coase theorem</u> outcome will be eff. if property rights are properly defined

<u>Example</u>: Private provision of public goods

- n consumers, 1 firm, 2 goods
- Good 1 is a private good, good 2 is a public good
- The firm is producing q units of public goods at a cost $c(q)$ units of private good

A Firm sells public good at price p per unit (denominated in private good)

↳ Firm's pro is

$$\max_{q \geq 0} \; pq - c(q) \rightarrow \text{FOC: } p = c'(q^*)$$

- Consumers select their contrib to the total amount of private good. Let x_i be the contrib. of indiv i. Given others contrib x_j^* $j \neq i$ the con's pro is

$$\max_{x_i \geq 0} \; \varphi_i\left(x_i + \sum_{j \neq i} x_j^*\right) + W_i - x_i p$$

FOC: $\varphi_i'\left(x_i^* + \sum_{j \neq i} x_j^*\right) = p$ if $x_i^* > 0$

$\leq p$, $x_i^* = 0$

at the equil $x_i^* + \sum_{j \neq i} x_j^* = q^* \rightarrow$

$\rightarrow \varphi_i'(q^*) = p$, $x_i^* > 0$

Conclusion: if indiv i contributes

$\rightarrow \varphi_i'(q^*) \geq \varphi_j'(q^*)$ $\forall j$

otherwise he free rides

<u>In equil.</u> $p = c'(q^*) = \varphi_i'(q^*)$, $\forall$ contributor i i.e. $x_i > 0$

Is it efficient?

Efficiency requires to

$$\max_{q \geq 0} \sum_{i=1}^{I} \varphi_i(q) - c(q)$$

FOC: $\sum_{i=1}^{I} \varphi_i'(q^0) = c'(q^0)$

$$\sum_{i=1}^{I} \varphi_i'(q^0) - c'(q^0) = 0 < \sum \varphi_i'(q^*) - c'(q^*)$$

$\rightarrow q^0 > q^*$ underproduction