Chapter 2
Simple Comparative Experiments
Solutions

2-1 The breaking strength of a fiber is required to be at least 150 psi. Past experience has indicated that the standard deviation of breaking strength is $\sigma = 3$ psi. A random sample of four specimens is tested. The results are y_1=145, y_2=153, y_3=150 and y_4=147.

(a) State the hypotheses that you think should be tested in this experiment.

$$H_0\text{: } \mu = 150 \qquad H_1\text{: } \mu > 150$$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$n = 4$, $\sigma = 3$, $\bar{y} = 1/4\ (145 + 153 + 150 + 147) = 148.75$

$$z_o = \frac{\bar{y} - \mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{148.75 - 150}{\frac{3}{\sqrt{4}}} = \frac{-1.25}{\frac{3}{2}} = -0.8333$$

Since $z_{0.05} = 1.645$, do not reject.

(c) Find the *P*-value for the test in part (b).

From the *z*-table: $P \cong 1 - \left[0.7967 + \left(2/3\right)\left(0.7995 - 0.7967\right)\right] = 0.2014$

(d) Construct a 95 percent confidence interval on the mean breaking strength.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
$$148.75 - (1.96)(3/2) \le \mu \le 148.75 + (1.96)(3/2)$$

$$145.81 \le \mu \le 151.69$$

2-2 The viscosity of a liquid detergent is supposed to average 800 centistokes at 25°C. A random sample of 16 batches of detergent is collected, and the average viscosity is 812. Suppose we know that the standard deviation of viscosity is $\sigma = 25$ centistokes.

(a) State the hypotheses that should be tested.

$$H_0\text{: } \mu = 800 \qquad H_1\text{: } \mu \ne 800$$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$$z_o = \frac{\bar{y}-\mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{812-800}{\frac{25}{\sqrt{16}}} = \frac{12}{\frac{25}{4}} = 1.92$$

Since $z_{\alpha/2} = z_{0.025} = 1.96$, do not reject.

(c) What is the *P*-value for the test? $P = 2(0.0274) = 0.0549$

(d) Find a 95 percent confidence interval on the mean.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
$$812-(1.96)(25/4) \le \mu \le 812+(1.96)(25/4)$$
$$812-12.25 \le \mu \le 812+12.25$$
$$799.75 \le \mu \le 824.25$$

2-3 The diameters of steel shafts produced by a certain manufacturing process should have a mean diameter of 0.255 inches. The diameter is known to have a standard deviation of $\sigma = 0.0001$ inch. A random sample of 10 shafts has an average diameter of 0.2545 inches.

(a) Set up the appropriate hypotheses on the mean μ.

H_0: $\mu = 0.255$ $\quad H_1$: $\mu \neq 0.255$

(b) Test these hypotheses using $\alpha = 0.05$. What are your conclusions?

$$n = 10,\quad \sigma = 0.0001,\quad \bar{y} = 0.2545$$

$$z_o = \frac{\bar{y}-\mu_o}{\frac{\sigma}{\sqrt{n}}} = \frac{0.2545-0.255}{\frac{0.0001}{\sqrt{10}}} = -15.81$$

Since $z_{0.025} = 1.96$, reject H_0.

(c) Find the *P*-value for this test. $P = 2.6547\text{x}10^{-56}$

(d) Construct a 95 percent confidence interval on the mean shaft diameter.

The 95% confidence interval is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
$$0.2545-(1.96)\left(\frac{0.0001}{\sqrt{10}}\right) \le \mu \le 0.2545+(1.96)\left(\frac{0.0001}{\sqrt{10}}\right)$$

$$0.254438 \le \mu \le 0.254562$$

2-4 A normally distributed random variable has an unknown mean μ and a known variance $\sigma^2 = 9$. Find the sample size required to construct a 95 percent confidence interval on the mean, that has total length of 1.0.

Since $y \sim N(\mu,9)$, a 95% two-sided confidence interval on μ is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
$$\bar{y} - (1.96)\frac{3}{\sqrt{n}} \le \mu \le \bar{y} + (1.96)\frac{3}{\sqrt{n}}$$

If the total interval is to have width 1.0, then the half-interval is 0.5. Since $z_{\alpha/2} = z_{0.025} = 1.96$,

$$(1.96)\left(3/\sqrt{n}\right) = 0.5$$
$$\sqrt{n} = (1.96)(3/0.5) = 11.76$$
$$n = (11.76)^2 = 138.30 \cong 139$$

2-5 The shelf life of a carbonated beverage is of interest. Ten bottles are randomly selected and tested, and the following results are obtained:

Days	
108	138
124	163
124	159
106	134
115	139

(a) We would like to demonstrate that the mean shelf life exceeds 120 days. Set up appropriate hypotheses for investigating this claim.

H_0: $\mu = 120$ H_1: $\mu > 120$

(b) Test these hypotheses using $\alpha = 0.01$. What are your conclusions?

$$\bar{y} = 131$$
$$S^2 = 3438 / 9 = 382$$
$$S = \sqrt{382} = 19.54$$

$$t_o = \frac{\bar{y} - \mu_o}{S/\sqrt{n}} = \frac{131-120}{19.54/\sqrt{10}} = 1.78$$

since $t_{0.01,9} = 2.821$; do not reject H_0

Minitab Output

```
T-Test of the Mean

Test of mu = 120.00 vs mu > 120.00

Variable     N      Mean    StDev   SE Mean        T          P
Shelf Life  10    131.00    19.54      6.18     1.78      0.054

T Confidence Intervals

Variable     N      Mean    StDev  SE Mean       99.0 % CI
Shelf Life  10    131.00    19.54     6.18  (  110.91,  151.09)
```

(c) Find the *P*-value for the test in part (b). P=0.054

(d) Construct a 99 percent confidence interval on the mean shelf life.

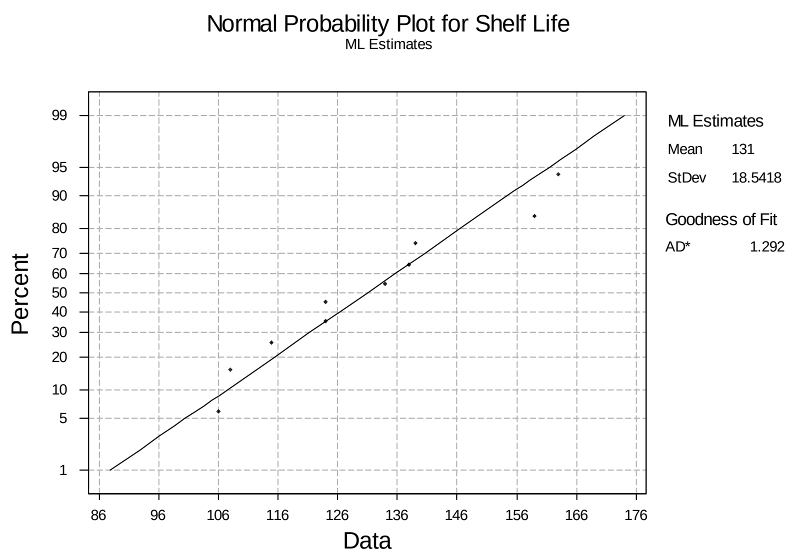
The 99% confidence interval is $\bar{y} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \le \mu \le \bar{y} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}$ with $\alpha = 0.01$.

$$131 - (3.250)\left(\frac{1954}{\sqrt{10}}\right) \le \mu \le 131 + (3.250)\left(\frac{1954}{\sqrt{10}}\right)$$

$$110.91 \le \mu \le 151.09$$

2-6 Consider the shelf life data in Problem 2-5. Can shelf life be described or modeled adequately by a normal distribution? What effect would violation of this assumption have on the test procedure you used in solving Problem 2-5?

A normal probability plot, obtained from Minitab, is shown. There is no reason to doubt the adequacy of the normality assumption. If shelf life is not normally distributed, then the impact of this on the *t*-test in problem 2-5 is not too serious unless the departure from normality is severe.



2-7 The time to repair an electronic instrument is a normally distributed random variable measured in hours. The repair time for 16 such instruments chosen at random are as follows:

Hours

159	280	101	212
224	379	179	264
222	362	168	250
149	260	485	170

(a) You wish to know if the mean repair time exceeds 225 hours. Set up appropriate hypotheses for investigating this issue.

H_0: $\mu = 225$ H_1: $\mu > 225$

(b) Test the hypotheses you formulated in part (a). What are your conclusions? Use $\alpha = 0.05$.

$$\bar{y} = 247.50$$
$$S^2 = 146202 / (16 - 1) = 9746.80$$

$$S = \sqrt{9746.8} = 98.73$$

$$t_o = \frac{\bar{y} - \mu_o}{\frac{S}{\sqrt{n}}} = \frac{241.50 - 225}{\frac{98.73}{\sqrt{16}}} = 0.67$$

since $t_{0.05,15} = 1.753$; do not reject H_0

Minitab Output

```
T-Test of the Mean

Test of mu = 225.0 vs mu > 225.0

Variable     N      Mean    StDev   SE Mean        T          P
Hours       16     241.5     98.7      24.7     0.67       0.26

T Confidence Intervals

Variable     N      Mean    StDev  SE Mean       95.0 % CI
Hours       16     241.5     98.7     24.7  (   188.9,   294.1)
```

(c) Find the *P*-value for this test. P=0.26

(d) Construct a 95 percent confidence interval on mean repair time.

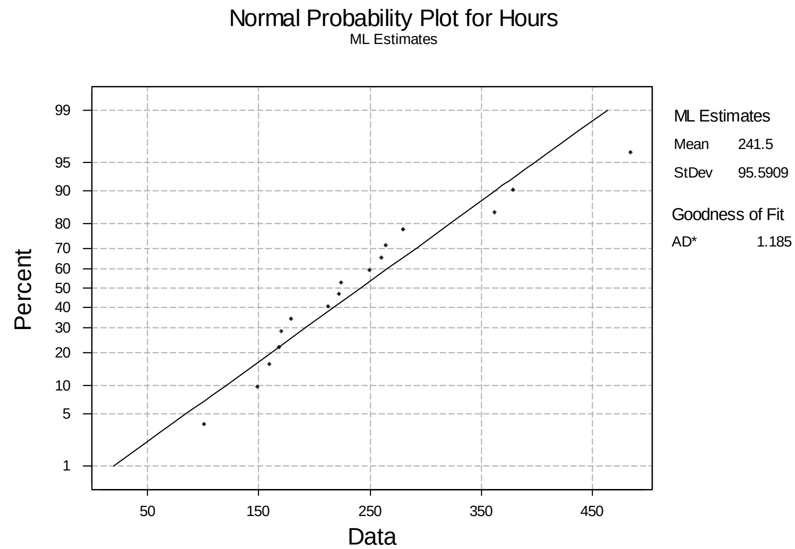
The 95% confidence interval is $\bar{y} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \le \mu \le \bar{y} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}$

$$241.50 - (2.131)\left(\frac{98.73}{\sqrt{16}}\right) \le \mu \le 241.50 + (2.131)\left(\frac{98.73}{\sqrt{16}}\right)$$

$$188.9 \le \mu \le 294.1$$

2-8 Reconsider the repair time data in Problem 2-7. Can repair time, in your opinion, be adequately modeled by a normal distribution?

The normal probability plot below does not reveal any serious problem with the normality assumption.



2-9 Two machines are used for filling plastic bottles with a net volume of 16.0 ounces. The filling processes can be assumed to be normal, with standard deviation of σ_1 = 0.015 and σ_2 = 0.018. The quality engineering department suspects that both machines fill to the same net volume, whether or not this volume is 16.0 ounces. An experiment is performed by taking a random sample from the output of each machine.

Machine 1		Machine 2	
16.03	16.01	16.02	16.03
16.04	15.96	15.97	16.04
16.05	15.98	15.96	16.02
16.05	16.02	16.01	16.01
16.02	15.99	15.99	16.00

(a) State the hypotheses that should be tested in this experiment.

$$H_0\text{: } \mu_1 = \mu_2 \qquad H_1\text{: } \mu_1 \neq \mu_2$$

(b) Test these hypotheses using α=0.05. What are your conclusions?

$$\begin{array}{ll} \bar{y}_1 = 16.015 & \bar{y}_2 = 16.005 \\ \sigma_1 = 0.015 & \sigma_2 = 0.018 \\ n_1 = 10 & n_2 = 10 \end{array}$$

$$z_o = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{16.015 - 16.018}{\sqrt{\frac{0.015^2}{10} + \frac{0.018^2}{10}}} = 1.35$$

$z_{0.025}$ = 1.96; do not reject

(c) What is the *P*-value for the test? P = 0.1770

(d) Find a 95 percent confidence interval on the difference in the mean fill volume for the two machines.

The 95% confidence interval is

$$\bar{y}_1 - \bar{y}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{y}_1 - \bar{y}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$(16.015-16.005)-(19.6)\sqrt{\frac{0.015^2}{10}+\frac{0.018^2}{10}} \le \mu_1 - \mu_2 \le (16.015-16.005)+(19.6)\sqrt{\frac{0.015^2}{10}+\frac{0.018^2}{10}}$$

$$-0.0045 \le \mu_1 - \mu_2 \le 0.0245$$

2-10 Two types of plastic are suitable for use by an electronic calculator manufacturer. The breaking strength of this plastic is important. It is known that $\sigma_1 = \sigma_2 = 1.0$ psi. From random samples of $n_1 = 10$ and $n_2 = 12$ we obtain $\bar{y}_1 = 162.5$ and $\bar{y}_2 = 155.0$. The company will not adopt plastic 1 unless its breaking strength exceeds that of plastic 2 by at least 10 psi. Based on the sample information, should they use plastic 1? In answering this questions, set up and test appropriate hypotheses using $\alpha = 0.01$. Construct a 99 percent confidence interval on the true mean difference in breaking strength.

H_0: $\mu_1 - \mu_2 = 10$ $\quad$ H_1: $\mu_1 - \mu_2 > 10$

$\bar{y}_1 = 162.5$ $\quad$ $\bar{y}_2 = 155.0$

$\sigma_1 = 1$ $\quad$ $\sigma_2 = 1$

$n_1 = 10$ $\quad$ $n_2 = 10$

$$z_o = \frac{\bar{y}_1 - \bar{y}_2 - 10}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = \frac{162.5-155.0-10}{\sqrt{\frac{1^2}{10}+\frac{1^2}{12}}} = -5.85$$

$z_{0.01} = 2.225$; do not reject

The 99 percent confidence interval is

$$\bar{y}_1 - \bar{y}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{y}_1 - \bar{y}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$(162.5-155.0)-(2.575)\sqrt{\frac{1^2}{10}+\frac{1^2}{12}} \le \mu_1 - \mu_2 \le (162.5-155.0)+(2.575)\sqrt{\frac{1^2}{10}+\frac{1^2}{12}}$$

$$6.40 \le \mu_1 - \mu_2 \le 8.60$$

2-11 The following are the burning times (in minutes) of chemical flares of two different formulations. The design engineers are interested in both the means and variance of the burning times.

Type 1		Type 2	
65	82	64	56
81	67	71	69
57	59	83	74
66	75	59	82
82	70	65	79

(a) Test the hypotheses that the two variances are equal. Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2 \qquad S_1 = 9.264$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2 \qquad S_2 = 9.367$$

$$F_0 = \frac{S_1^2}{S_2^2} = \frac{85.82}{87.73} = 0.98$$

$$F_{0.025,9,9} = 4.03 \qquad F_{0.975,9,9} = \frac{1}{F_{0.025,9,9}} = \frac{1}{4.03} = 0.248 \quad \text{Do not reject.}$$

(b) Using the results of (a), test the hypotheses that the mean burning times are equal. Use α = 0.05. What is the *P*-value for this test?

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{1561.95}{18} = 86.775$$
$$S_p = 9.32$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{70.4-70.2}{9.32\sqrt{\frac{1}{10}+\frac{1}{10}}} = 0.048$$
$$t_{0.025,18} = 2.101 \quad \text{Do not reject.}$$

From the computer output, *t*=0.05; do not reject. Also from the computer output *P*=0.96

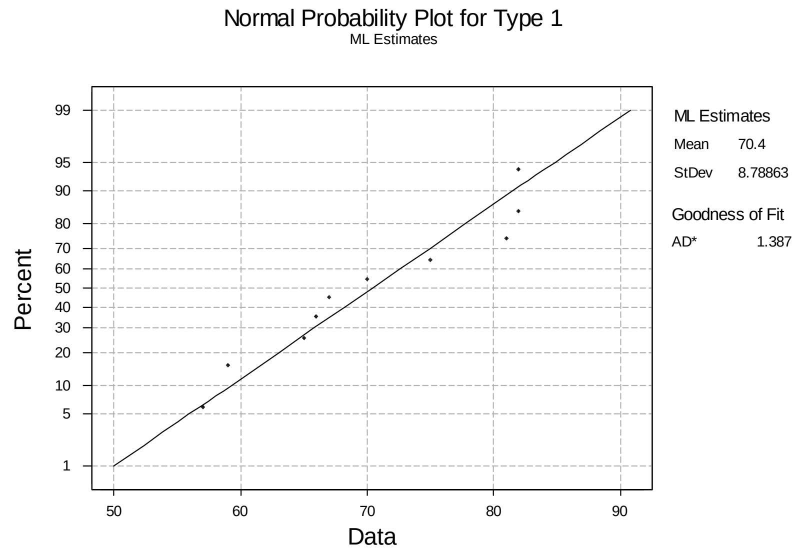
Minitab Output

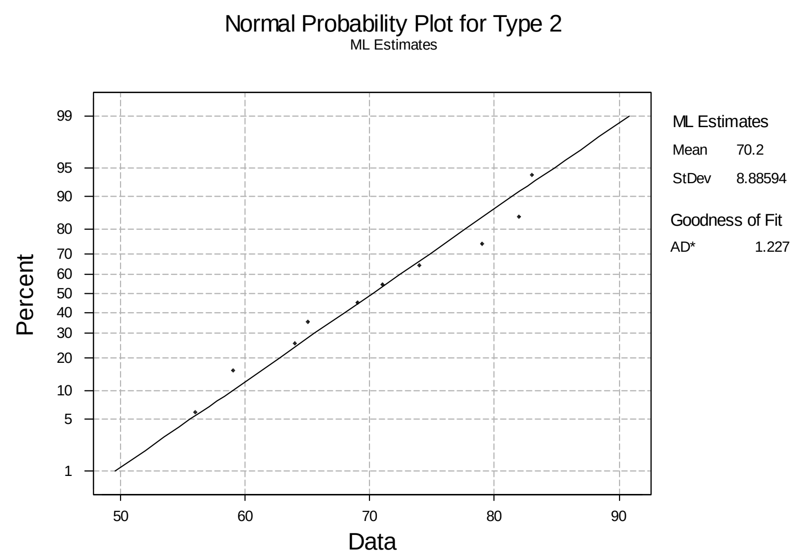
Two Sample T-Test and Confidence Interval

```
Two sample T for Type 1 vs Type 2

          N      Mean     StDev   SE Mean
Type 1  10     70.40      9.26       2.9
Type 2  10     70.20      9.37       3.0

95% CI for mu Type 1 - mu Type 2: ( -8.6,  9.0)
T-Test mu Type 1 = mu Type 2 (vs not =): T = 0.05  P = 0.96  DF = 18
Both use Pooled StDev = 9.32
```

(c) Discuss the role of the normality assumption in this problem. Check the assumption of normality for both types of flares.

The assumption of normality is required in the theoretical development of the *t*-test. However, moderate departure from normality has little impact on the performance of the *t*-test. The normality assumption is more important for the test on the equality of the two variances. An indication of nonnormality would be of concern here. The normal probability plots shown below indicate that burning time for both formulations follow the normal distribution.





2-12 An article in *Solid State Technology*, "Orthogonal Design of Process Optimization and Its Application to Plasma Etching" by G.Z. Yin and D.W. Jillie (May, 1987) describes an experiment to determine the effect of C_2F_6 flow rate on the uniformity of the etch on a silicon wafer used in integrated circuit manufacturing. Data for two flow rates are as follows:

C_2F_6 (SCCM)	Uniformity Observation 1	2	3	4	5	6
125	2.7	4.6	2.6	3.0	3.2	3.8
200	4.6	3.4	2.9	3.5	4.1	5.1

(a) Does the C_2F_6 flow rate affect average etch uniformity? Use $\alpha = 0.05$.

No, C_2F_6 flow rate does not affect average etch uniformity.

Minitab Output

```
Two Sample T-Test and Confidence Interval

Two sample T for Uniformity

Flow Rat    N      Mean     StDev   SE Mean
125         6     3.317     0.760      0.31
200         6     3.933     0.821      0.34

95% CI for mu (125) - mu (200): ( -1.63,  0.40)
T-Test mu (125) = mu (200) (vs not =): T = -1.35  P = 0.21  DF = 10
Both use Pooled StDev = 0.791
```

(b) What is the *P*-value for the test in part (a)? From the computer printout, *P*=0.21

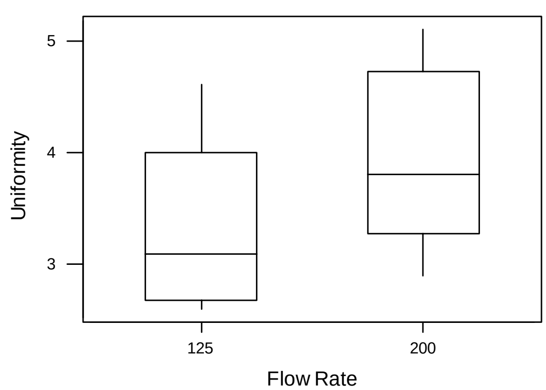
(c) Does the C_2F_6 flow rate affect the wafer-to-wafer variability in etch uniformity? Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2$$
$$F_{0.05,5,5} = 5.05$$
$$F_0 = \frac{0.5776}{0.6724} = 0.86$$

Do not reject; C_2F_6 flow rate does not affect wafer-to-wafer variability.

(d) Draw box plots to assist in the interpretation of the data from this experiment.

The box plots shown below indicate that there is little difference in uniformity at the two gas flow rates. Any observed difference is not statistically significant. See the *t*-test in part (a).



2-13 A new filtering device is installed in a chemical unit. Before its installation, a random sample yielded the following information about the percentage of impurity: $\bar{y}_1$ = 12.5, S_1^2 =101.17, and n_1 = 8. After installation, a random sample yielded $\bar{y}_2$ = 10.2, S_2^2 = 94.73, n_2 = 9.

(a) Can you concluded that the two variances are equal? Use $\alpha = 0.05$.

$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2$$
$$F_{0.025,7,8} = 4.53$$
$$F_0 = \frac{S_1^2}{S_2^2} = \frac{101.17}{94.73} = 1.07$$

Do Not Reject. Assume that the variances are equal.

(b) Has the filtering device reduced the percentage of impurity significantly? Use α = 0.05.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$
$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{(8-1)(101.17)+(9-1)(94.73)}{8+9-2} = 97.74$$
$$S_p = 9.89$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{12.5-10.2}{9.89\sqrt{\frac{1}{8}+\frac{1}{9}}} = 0.479$$
$$t_{0.05,15} = 1.753$$

Do not reject. There is no evidence to indicate that the new filtering device has affected the mean

2-14 Photoresist is a light-sensitive material applied to semiconductor wafers so that the circuit pattern can be imaged on to the wafer. After application, the coated wafers are baked to remove the solvent in the photoresist mixture and to harden the resist. Here are measurements of photoresist thickness (in kÅ) for eight wafers baked at two different temperatures. Assume that all of the runs were made in random order.

95 ºC	100 ºC
11.176	5.263
7.089	6.748
8.097	7.461
11.739	7.015
11.291	8.133
10.759	7.418
6.467	3.772
8.315	8.963

(a) Is there evidence to support the claim that the higher baking temperature results in wafers with a lower mean photoresist thickness? Use α = 0.05.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$
$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{(8-1)(4.41)+(8-1)(2.54)}{8+8-2} = 3.48$$
$$S_p = 1.86$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{9.37-6.89}{1.86\sqrt{\frac{1}{8}+\frac{1}{8}}} = 2.65$$
$$t_{0.05,14} = 1.761$$

Since $t_{0.05,14}$ = 1.761, reject H_0. There appears to be a lower mean thickness at the higher temperature. This is also seen in the computer output.

Minitab Output

Two-Sample T-Test and CI: Thickness, Temp

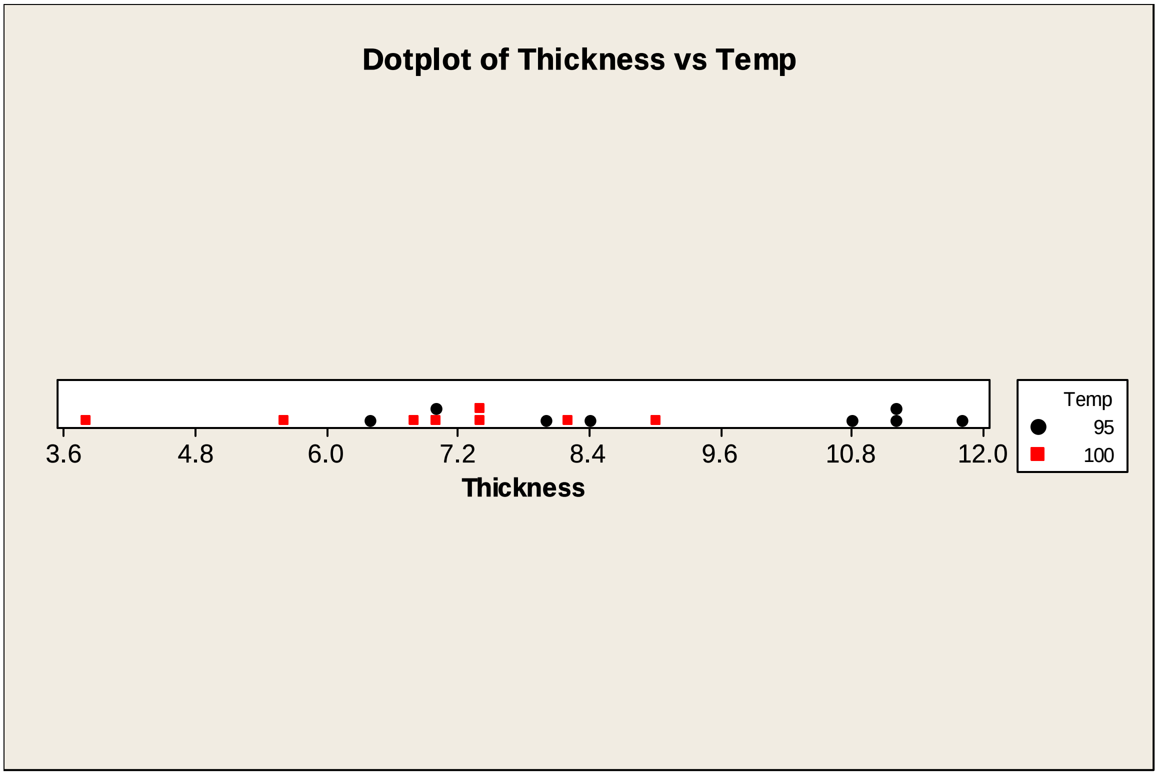
```
Two-sample T for Thickness

Temp        N      Mean     Stdev   SE Mean
 95         8      9.37      2.10      0.74
100         8      6.89      1.60      0.56

Difference = mu ( 95) - mu (100)
Estimate for difference:  2.475
95% CI for difference: (0.476, 4.474)
T-Test of difference = 0 (vs not =): T-Value = 2.65  P-Value = 0.019  DF = 14
Both use Pooled StDev = 1.86
```

(b) What is the *P*-value for the test conducted in part (a)? *P* = 0.019

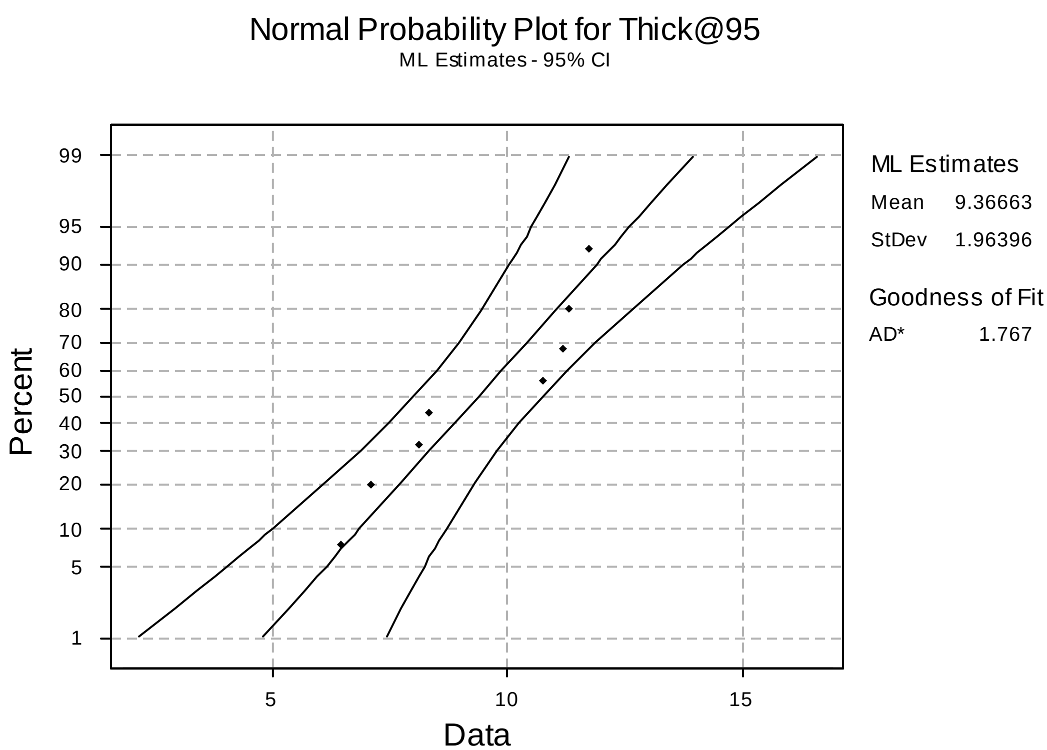
(c) Find a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

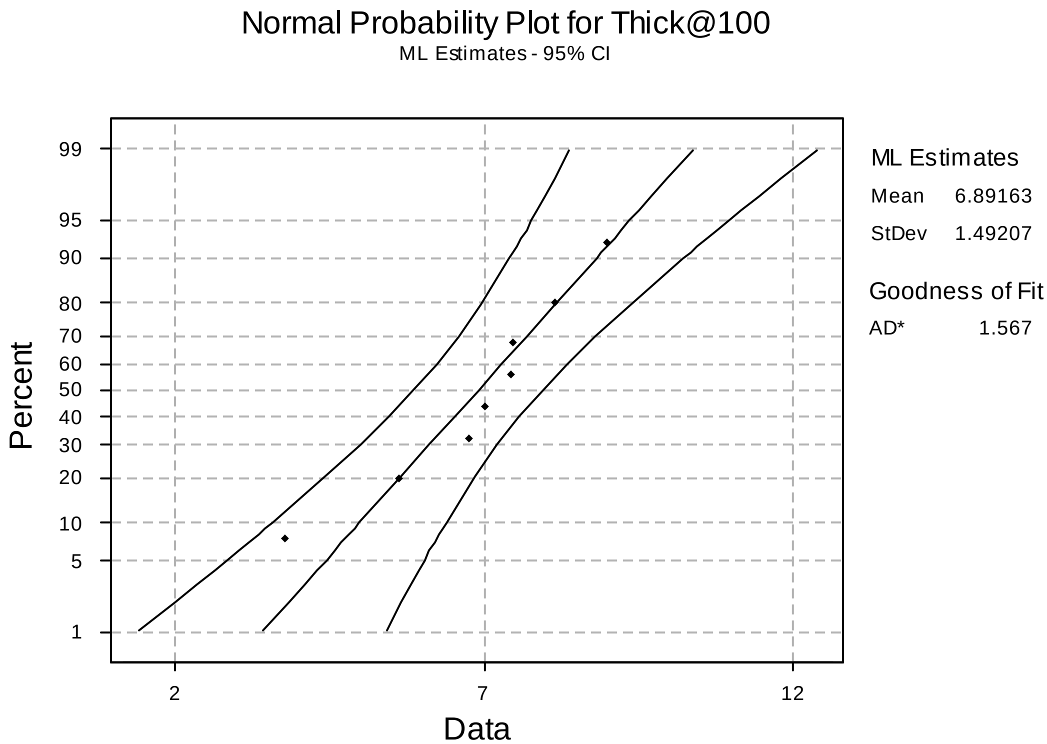
From the computer output the 95% confidence interval is $0.476 \le \mu_1 - \mu_2 \le 4.474$. This confidence interval doesnot include 0 in it, there for there is a difference in the two temperatures on the thickness of the photo resist.

(d) Draw dot diagrams to assist in interpreting the results from this experiment.



(e) Check the assumption of normality of the photoresist thickness.





There are no significant deviations from the normality assumptions.

(f) Find the power of this test for detecting an actual difference in means of 2.5 kÅ.

Minitab Output

Power and Sample Size

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Sigma = 1.86

            Sample
Difference    Size   Power
       2.5       8  0.7056
```

(g) What sample size would be necessary to detect an actual difference in means of 1.5 kÅ with a power of at least 0.9?.

Minitab Output

Power and Sample Size

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Sigma = 1.86

            Sample  Target  Actual
Difference    Size   Power   Power
       1.5      34  0.9000  0.9060
```

This result makes intuitive sense. More samples are needed to detect a smaller difference.

2-15 Front housings for cell phones are manufactured in an injection molding process. The time the part is allowed to cool in the mold before removal is thought to influence the occurrence of a particularly troublesome cosmetic defect, flow lines, in the finished housing. After manufacturing, the housings are inspected visually and assigned a score between 1 and 10 based on their appearance, with 10 corresponding to a perfect part and 1 corresponding to a completely defective part. An experiment was conducted using two cool-down times, 10 seconds and 20 seconds, and 20 housings were evaluated at each level of cool-down time. The data are shown below.

10 Seconds		20 Seconds	
1	3	7	6
2	6	8	9
1	5	5	5
3	3	9	7
5	2	5	4
1	1	8	6
5	6	6	8
2	8	4	5
3	2	6	8
5	3	7	7

(a) Is there evidence to support the claim that the longer cool-down time results in fewer appearance defects? Use $\alpha = 0.05$.

Minitab Output

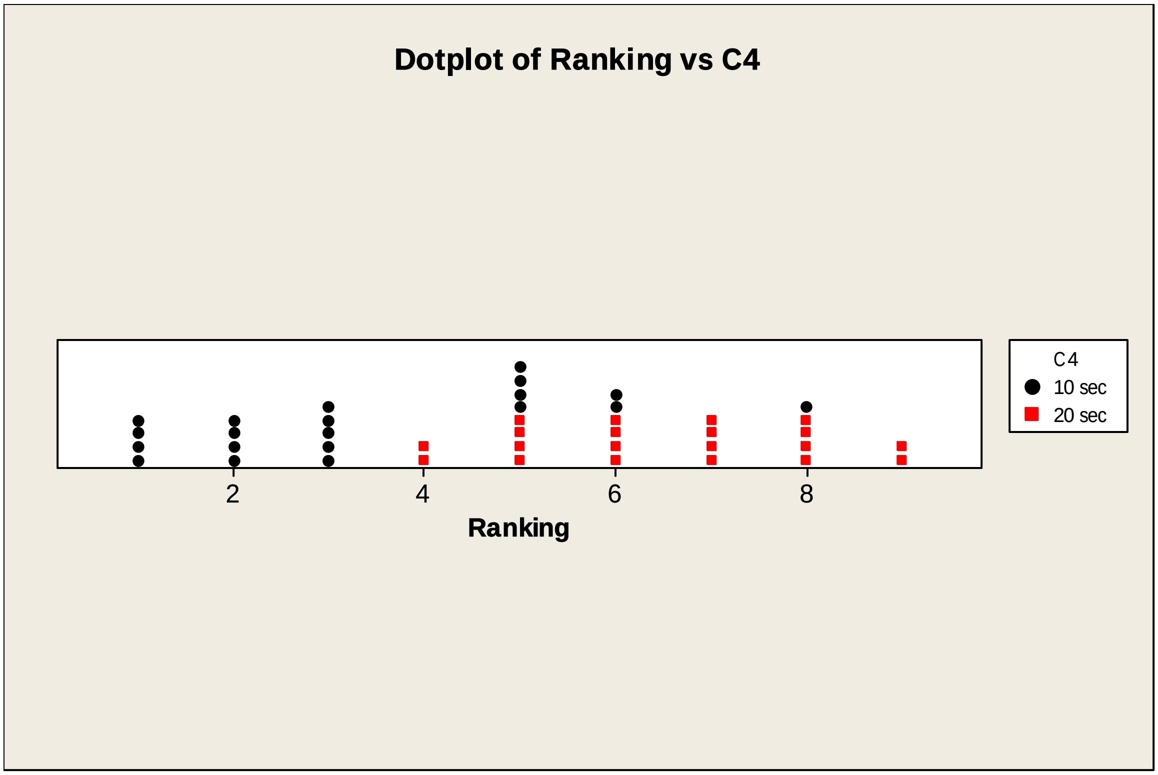
```
Two-Sample T-Test and CI: 10 seconds, 20 seconds


Two-sample T for 10 seconds vs 20 seconds

           N      Mean     StDev   SE Mean
10 secon  20      3.35      2.01      0.45
20 secon  20      6.50      1.54      0.34

Difference = mu 10 seconds - mu 20 seconds
Estimate for difference:  -3.150
95% CI for difference: (-4.295, -2.005)
T-Test of difference = 0 (vs not =): T-Value = -5.57  P-Value = 0.000  DF = 38
Both use Pooled StDev = 1.79
```

(b) What is the *P*-value for the test conducted in part (a)? From the Minitab output, $P = 0.000$

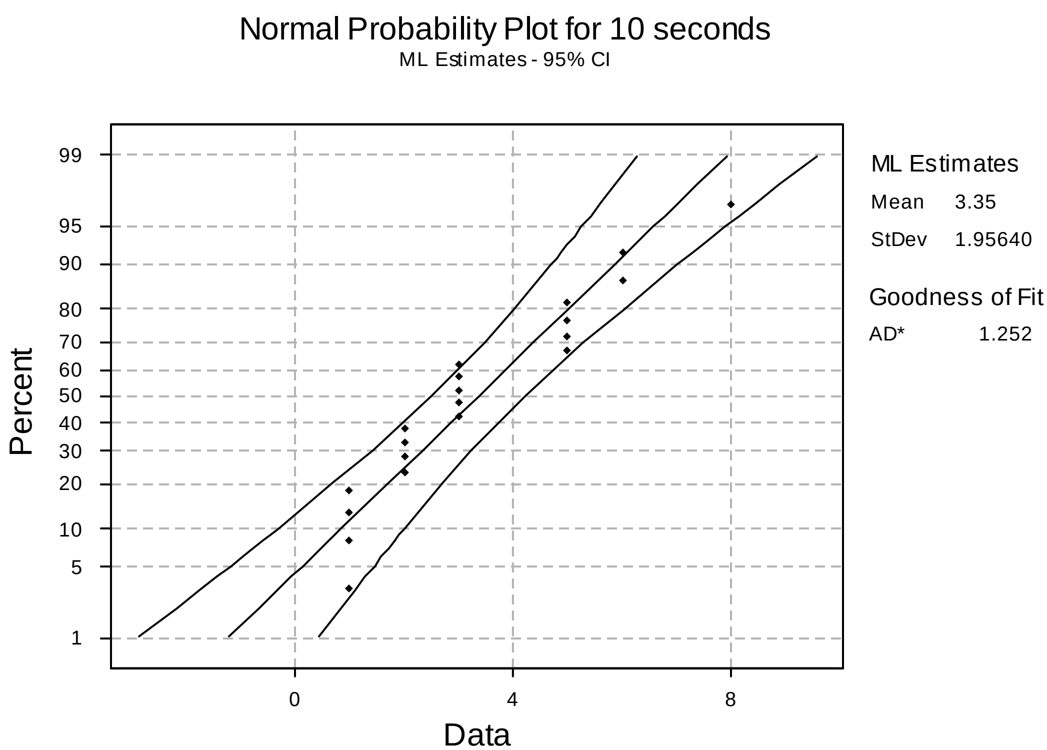
(c) Find a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

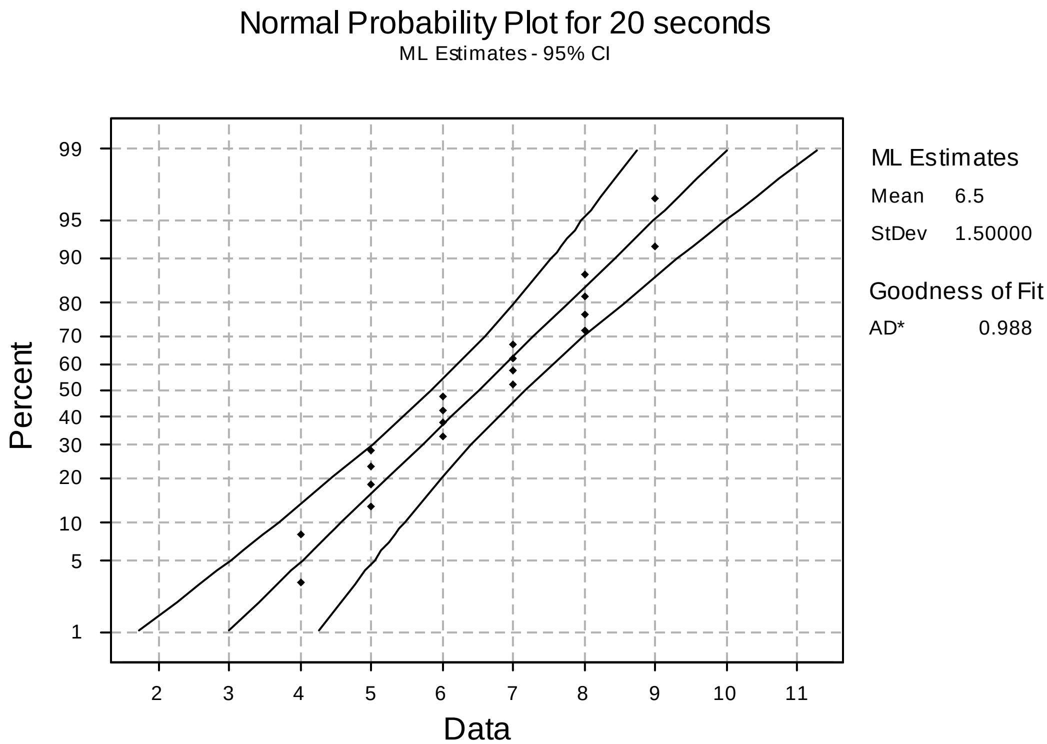
From the computer output, $-4.295 \le \mu_1 - \mu_2 \le -2.005$. This interval does not contain 0. The two samples are different. The 20 second cooling time gives a cosmetically better housing.

(d) Draw dot diagrams to assist in interpreting the results from this experiment.



(e) Check the assumption of normality for the data from this experiment.





There are no significant departures from normality.

2-16 Twenty observations on etch uniformity on silicon wafers are taken during a qualification experiment for a plasma etcher. The data are as follows:

5.34	6.65	4.76	5.98	7.25
6.00	7.55	5.54	5.62	6.21
5.97	7.35	5.44	4.39	4.98
5.25	6.35	4.61	6.00	5.32

(a) Construct a 95 percent confidence interval estimate of σ^2.

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{(1-\alpha/2),n-1}}$$

$$\frac{(20-1)(0.88907)^2}{32.852} \le \sigma^2 \le \frac{(20-1)(0.88907)^2}{8.907}$$

$$0.457 \le \sigma^2 \le 1.686$$

(b) Test the hypothesis that $\sigma^2 = 1.0$. Use $\alpha = 0.05$. What are your conclusions?

$$H_0: \sigma^2 = 1$$

$$H_1: \sigma^2 \ne 1$$

$$\chi^2_0 = \frac{SS}{\sigma^2_0} = 15.019$$

$$\chi^2_{0.025,19} = 32.852 \qquad \chi^2_{0.975,19} = 8.907$$

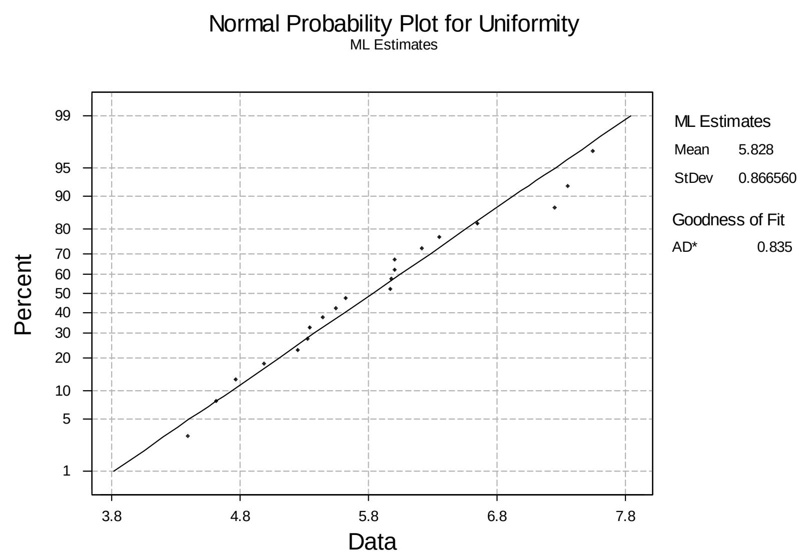
Do not reject. There is no evidence to indicate that $\sigma^2 \neq 1$

(c) Discuss the normality assumption and its role in this problem.

The normality assumption is much more important when analyzing variances then when analyzing means. A moderate departure from normality could cause problems with both statistical tests and confidence intervals. Specifically, it will cause the reported significance levels to be incorrect.

(d) Check normality by constructing a normal probability plot. What are your conclusions?

The normal probability plot indicates that there is not any serious problem with the normality assumption.



2-17 The diameter of a ball bearing was measured by 12 inspectors, each using two different kinds of calipers. The results were:

Inspector	Caliper 1	Caliper 2	Difference	Difference^2
1	0.265	0.264	.001	.000001
2	0.265	0.265	.000	0
3	0.266	0.264	.002	.000004
4	0.267	0.266	.001	.000001
5	0.267	0.267	.000	0
6	0.265	0.268	-.003	.000009
7	0.267	0.264	.003	.000009
8	0.267	0.265	.002	.000004
9	0.265	0.265	.000	0
10	0.268	0.267	.001	.000001
11	0.268	0.268	.000	0
12	0.265	0.269	-.004	.000016
			$\sum = 0.003$	$\sum = 0.000045$

(a) Is there a significant difference between the means of the population of measurements represented by the two samples? Use $\alpha = 0.05$.

$$\begin{array}{l} H_0 : \mu_1 = \mu_2 \\ H_1 : \mu_1 \neq \mu_2 \end{array} \quad \text{or equivalently} \quad \begin{array}{l} H_0 : \mu_d = 0 \\ H_1 : \mu_d \neq 0 \end{array}$$

Minitab Output

Paired T-Test and Confidence Interval

```
Paired T for Caliper 1 - Caliper 2

                  N      Mean     StDev   SE Mean
Caliper          12  0.266250  0.001215  0.000351
Caliper          12  0.266000  0.001758  0.000508
Difference       12  0.000250  0.002006  0.000579

95% CI for mean difference: (-0.001024, 0.001524)
T-Test of mean difference = 0 (vs not = 0): T-Value = 0.43  P-Value = 0.674
```

(b) Find the *P*-value for the test in part (a). *P*=0.674

(c) Construct a 95 percent confidence interval on the difference in the mean diameter measurements for the two types of calipers.

$$\bar{d} - t_{\alpha/2, n-1}\frac{S_d}{\sqrt{n}} \leq \mu_D \left(= \mu_1 - \mu_2\right) \leq \bar{d} + t_{\alpha/2, n-1}\frac{S_d}{\sqrt{n}}$$

$$0.00025 - 2.201\frac{0.002}{\sqrt{12}} \leq \mu_d \leq 0.00025 + 2.201\frac{0.002}{\sqrt{12}}$$

$$-0.00102 \leq \mu_d \leq 0.00152$$

2-18 An article in the *Journal of Strain Analysis* (vol.18, no. 2, 1983) compares several procedures for predicting the shear strength for steel plate girders. Data for nine girders in the form of the ratio of predicted to observed load for two of these procedures, the Karlsruhe and Lehigh methods, are as follows:

Girder	Karlsruhe Method	Lehigh Method	Difference	Difference^2
S1/1	1.186	1.061	0.125	0.015625
S2/1	1.151	0.992	0.159	0.025281
S3/1	1.322	1.063	0.259	0.067081
S4/1	1.339	1.062	0.277	0.076729
S5/1	1.200	1.065	0.135	0.018225
S2/1	1.402	1.178	0.224	0.050176
S2/2	1.365	1.037	0.328	0.107584
S2/3	1.537	1.086	0.451	0.203401
S2/4	1.559	1.052	0.507	0.257049
		Sum =	2.465	0.821151
		Average =	0.274	

(a) Is there any evidence to support a claim that there is a difference in mean performance between the two methods? Use $\alpha = 0.05$.

$$\begin{array}{l} H_0 : \mu_1 = \mu_2 \\ H_1 : \mu_1 \neq \mu_2 \end{array} \quad \text{or equivalently} \quad \begin{array}{l} H_0 : \mu_d = 0 \\ H_1 : \mu_d \neq 0 \end{array}$$

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{9}(2.465) = 0.274$$

$$s_d = \left[\frac{\sum_{i=1}^{n} d_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2}{n-1}\right]^{1/2} = \left[\frac{0.821151 - \frac{1}{9}(2.465)^2}{9-1}\right]^{1/2} = 0.135$$

$$t_0 = \frac{\bar{d}}{\frac{S_d}{\sqrt{n}}} = \frac{0.274}{\frac{0.135}{\sqrt{9}}} = 6.08$$

$t_{\alpha/2,n-1} = t_{0.025,9} = 2.306$, reject the null hypothesis.

Minitab Output

Paired T-Test and Confidence Interval

Paired T for Karlsruhe - Lehigh

	N	Mean	StDev	SE Mean
Karlsruh	9	1.3401	0.1460	0.0487
Lehigh	9	1.0662	0.0494	0.0165
Difference	9	0.2739	0.1351	0.0450

95% CI for mean difference: (0.1700, 0.3777)
T-Test of mean difference = 0 (vs not = 0): T-Value = 6.08 P-Value = 0.000

(b) What is the *P*-value for the test in part (a)? *P*=0.0002

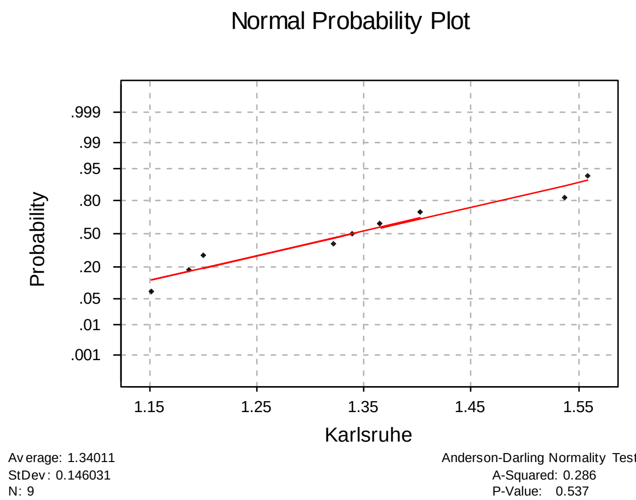
(c) Construct a 95 percent confidence interval for the difference in mean predicted to observed load.

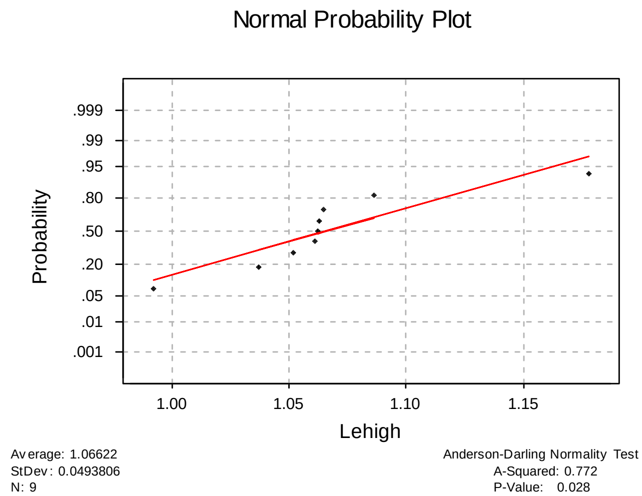
$$\bar{d} - t_{\alpha/2,n-1}\frac{S_d}{\sqrt{n}} \le \mu_d \le \bar{d} + t_{\alpha/2,n-1}\frac{S_d}{\sqrt{n}}$$

$$0.274 - 2.306\frac{0.135}{\sqrt{9}} \le \mu_d \le 0.274 + 2.306\frac{0.135}{\sqrt{9}}$$

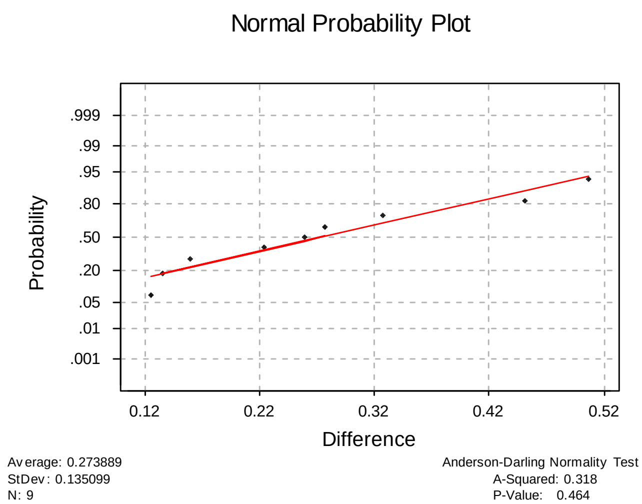
$$0.17023 \le \mu_d \le 0.37777$$

(d) Investigate the normality assumption for both samples.





(e) Investigate the normality assumption for the difference in ratios for the two methods.



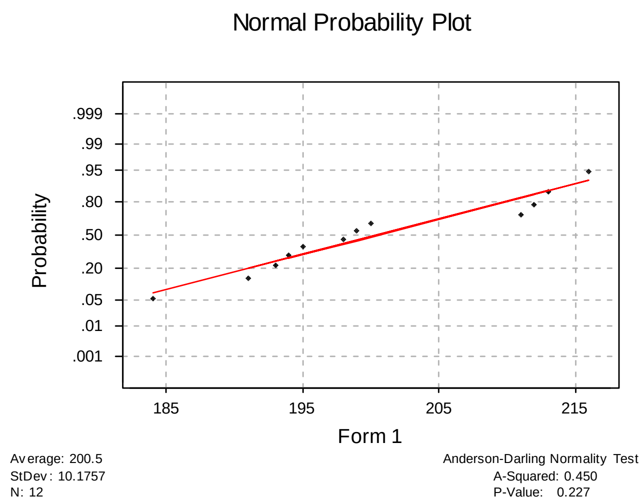
(f) Discuss the role of the normality assumption in the paired *t*-test.

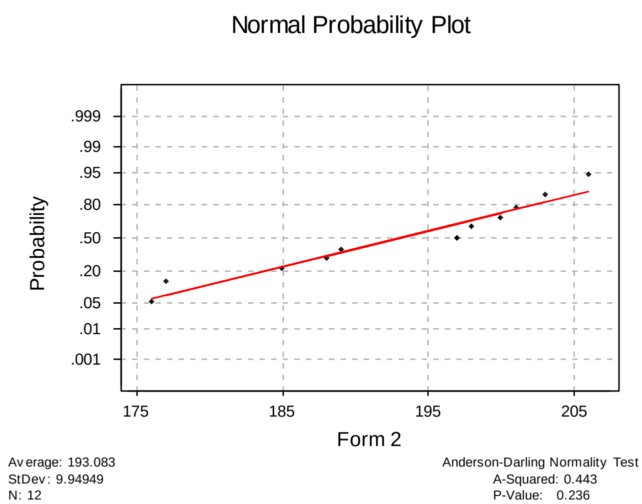
As in any *t*-test, the assumption of normality is of only moderate importance. In the paired *t*-test, the assumption of normality applies to the distribution of the differences. That is, the individual sample measurements do not have to be normally distributed, only their difference.

2-19 The deflection temperature under load for two different formulations of ABS plastic pipe is being studied. Two samples of 12 observations each are prepared using each formulation, and the deflection temperatures (in °F) are reported below:

Formulation 1			Formulation 2		
212	199	198	177	176	198
194	213	216	197	185	188
211	191	200	206	200	189
193	195	184	201	197	203

(a) Construct normal probability plots for both samples. Do these plots support assumptions of normality and equal variance for both samples?





(b) Do the data support the claim that the mean deflection temperature under load for formulation 1 exceeds that of formulation 2? Use $\alpha = 0.05$.

Minitab Output

Two Sample T-Test and Confidence Interval

```
Two sample T for Form 1 vs Form 2

         N      Mean     StDev   SE Mean
Form 1  12     200.5      10.2       2.9
Form 2  12    193.08      9.95       2.9

95% CI for mu Form 1 - mu Form 2: ( -1.1,  15.9)
T-Test mu Form 1 = mu Form 2 (vs >): T = 1.81  P = 0.042  DF = 22
Both use Pooled StDev = 10.1
```

(c) What is the *P*-value for the test in part (a)? $P = 0.042$

2-20 Refer to the data in problem 2-19. Do the data support a claim that the mean deflection temperature under load for formulation 1 exceeds that of formulation 2 by at least 3 °F?
Yes, formulation 1 exceeds formulation 2 by at least 3 °F.

Minitab Output

Two-Sample T-Test and CI: Form1, Form2

```
Two-sample T for Form1 vs Form2

        N      Mean     StDev   SE Mean
Form1  12     200.5      10.2       2.9
Form2  12    193.08      9.95       2.9r
Difference = mu Form1 - mu Form2
Estimate for difference:  7.42
95% lower bound for difference: 0.36
T-Test of difference = 3 (vs >): T-Value = 1.08  P-Value = 0.147  DF = 22
Both use Pooled StDev = 10.1
```

2-21 In semiconductor manufacturing, wet chemical etching is often used to remove silicon from the backs of wafers prior to metalization. The etch rate is an important characteristic of this process. Two different etching solutionsare being evaluated. Eight randomly selected wafers have been etched in each solution and the observed etch rates (in mils/min) are shown below:

Solution 1		Solution 2	
9.9	10.6	10.2	10.6
9.4	10.3	10.0	10.2
10.0	9.3	10.7	10.4
10.3	9.8	10.5	10.3

(a) Do the data indicate that the claim that both solutions have the same mean etch rate is valid? Use α = 0.05 and assume equal variances.

See the Minitab output below.

Minitab Output

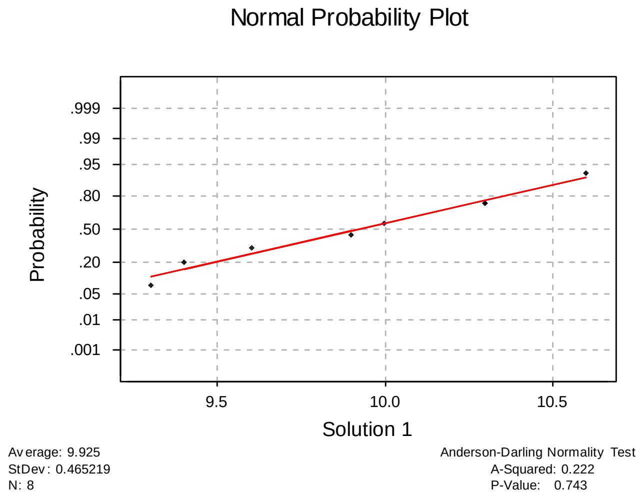
Two Sample T-Test and Confidence Interval

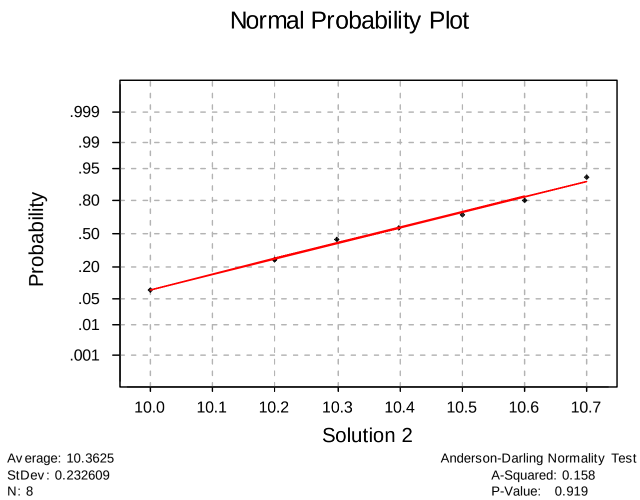
```
Two sample T for Solution 1 vs Solution 2

          N      Mean     StDev   SE Mean
Solution  8     9.925     0.465      0.16
Solution  8    10.362     0.233     0.082

95% CI for mu Solution - mu Solution: ( -0.83,  -0.043)
T-Test mu Solution = mu Solution (vs not =):T = -2.38 P = 0.032 DF = 14
Both use Pooled StDev = 0.368
```

(b) Find a 95% confidence interval on the difference in mean etch rate.

From the Minitab output, -0.83 to –0.043.

(c) Use normal probability plots to investigate the adequacy of the assumptions of normality and equal variances.





Both the normality and equality of variance assumptions are valid.

2-22 Two popular pain medications are being compared on the basis of the speed of absorption by the body. Specifically, tablet 1 is claimed to be absorbed twice as fast as tablet 2. Assume that σ_1^2 and σ_2^2 are known. Develop a test statistic for

$$H_0\text{: } 2\mu_1 = \mu_2$$
$$H_1\text{: } 2\mu_1 \neq \mu_2$$

$2\bar{y}_1 - \bar{y}_2 \sim N\left(2\mu_1 - \mu_2, \dfrac{4\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)$, assuming that the data is normally distributed.

The test statistic is: $z_o = \dfrac{2\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{4\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$, reject if $|z_o| > z_{\alpha/2}$

2-23 Suppose we are testing

$$H_0\text{: } \mu_1 = \mu_2$$
$$H_1\text{: } \mu_1 \neq \mu_2$$

where σ_1^2 and σ_2^2 are known. Our sampling resources are constrained such that $n_1 + n_2 = N$. How should we allocate the N observations between the two populations to obtain the most powerful test?

The most powerful test is attained by the n_1 and n_2 that maximize z_o for given $\bar{y}_1 - \bar{y}_2$.
Thus, we chose n_1 and n_2 to $\max z_o = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$, subject to $n_1 + n_2 = N$.

This is equivalent to min $L = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2} = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{N - n_1}$, subject to $n_1 + n_2 = N$.

Now $\dfrac{dL}{dn_1} = \dfrac{-\sigma_1^2}{n_1^2} + \dfrac{\sigma_2^2}{\left(N - n_1\right)^2} = 0$, implies that $n_1 / n_2 = \sigma_1 / \sigma_2$.

Thus n_1 and n_2 are assigned proportionally to the ratio of the standard deviations. This has intuitive appeal, as it allocates more observations to the population with the greatest variability.

2-24 Develop Equation 2-46 for a 100(1 - α) percent confidence interval for the variance of a normal distribution.

$\dfrac{SS}{\sigma^2} \sim \chi^2_{n-1}$. Thus, $P\left\{\chi^2_{1-\frac{\alpha}{2},n-1} \le \dfrac{SS}{\sigma^2} \le \chi^2_{\frac{\alpha}{2},n-1}\right\} = 1-\alpha$. Therefore,

$$P\left\{\frac{SS}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma^2 \le \frac{SS}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right\} = 1-\alpha,$$

so $\left[\dfrac{SS}{\chi^2_{\frac{\alpha}{2},n-1}}, \dfrac{SS}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$ is the 100(1 - α)% confidence interval on σ^2.

2-25 Develop Equation 2-50 for a 100(1 - α) percent confidence interval for the ratio σ_1^2 / σ_2^2, where σ_1^2 and σ_2^2 are the variances of two normal distributions.

$$\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F_{n_2-1,n_1-1}$$

$$P\left\{F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \le F_{\alpha/2,n_2-1,n_1-1}\right\} = 1-\alpha \quad \text{or}$$

$$P\left\{\frac{S_1^2}{S_2^2} F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha/2,n_2-1,n_1-1}\right\} = 1-\alpha$$

2-26 Develop an equation for finding a 100(1 - α) percent confidence interval on the difference in the means of two normal distributions where $\sigma_1^2 \neq \sigma_2^2$. Apply your equation to the portland cement experiment data, and find a 95% confidence interval.

$$\frac{(\bar{y}_1-\bar{y}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} \sim t_{\alpha/2,\upsilon}$$

$$t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le (\bar{y}_1-\bar{y}_2)-(\mu_1-\mu_2) \le t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$(\bar{y}_1-\bar{y}_2)-t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le (\mu_1-\mu_2) \le (\bar{y}_1-\bar{y}_2)+t_{\alpha/2,\upsilon}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$\text{where} \quad \upsilon = \frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1-1}+\frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2-1}}$$

Using the data from Table 2-1

$$n_1 = 10 \qquad n_2 = 10$$
$$\bar{y}_1 = 16.764 \qquad \bar{y}_2 = 17.343$$
$$S_1^2 = 0.100138 \qquad S_2^2 = 0.0614622$$

$$(16.764-17.343)-2.110\sqrt{\frac{0.100138}{10}+\frac{0.0614622}{10}} \le (\mu_1-\mu_2) \le$$
$$(16.764-17.343)+2.110\sqrt{\frac{0.100138}{10}+\frac{0.0614622}{10}}$$

$$\text{where} \quad \upsilon = \frac{\left(\frac{0.100138}{10}+\frac{0.0614622}{10}\right)^2}{\frac{\left(\frac{0.100138}{10}\right)^2}{10-1}+\frac{\left(\frac{0.0614622}{10}\right)^2}{10-1}} = 17.024 \cong 17$$

$$-1.426 \le (\mu_1-\mu_2) \le -0.889$$

This agrees with the result in Table 2-2.

2-27 Construct a data set for which the paired *t*-test statistic is very large, but for which the usual two-sample or pooled *t*-test statistic is small. In general, describe how you created the data. Does this give you any insight regarding how the paired *t*-test works?

A	B	delta
7.1662	8.2416	1.07541
2.3590	2.4555	0.09650
19.9977	21.1018	1.10412
0.9077	2.3401	1.43239
-15.9034	-15.0013	0.90204
-6.0722	-5.5941	0.47808

9.9501	10.6910	0.74085
-1.0944	-0.1358	0.95854
-4.6907	-3.3446	1.34615
-6.6929	-5.9303	0.76256

Minitab Output

Paired T-Test and Confidence Interval

```
Paired T for A - B
                  N      Mean     StDev   SE Mean
A                10      0.59     10.06      3.18
B                10      1.48     10.11      3.20
Difference       10    -0.890     0.398     0.126

95% CI for mean difference: (-1.174, -0.605)
T-Test of mean difference = 0 (vs not = 0): T-Value = -7.07  P-Value = 0.000
```

Two Sample T-Test and Confidence Interval

```
Two sample T for A vs B

    N      Mean     StDev   SE Mean
A  10       0.6      10.1       3.2
B  10       1.5      10.1       3.2

95% CI for mu A - mu B: ( -10.4,  8.6)
T-Test mu A = mu B (vs not =): T = -0.20  P = 0.85  DF = 18
Both use Pooled StDev = 10.1
```

These two sets of data were created by making the observation for *A* and *B* moderately different within each pair (or block), but making the observations between pairs very different. The fact that the difference between pairs is large makes the pooled estimate of the standard deviation large and the two-sample *t*-test statistic small. Therefore the fairly small difference between the means of the two treatments that is present when they are applied to the same experimental unit cannot be detected. Generally, if the blocks are very different, then this will occur. Blocking eliminates the variabiliy associated with the nuisance variable that they represent.

2-28 Consider the experiment described in problem 2-11. If the mean burning times of the two flames differ by as much as 2 minutes, find the power of the test. What sample size would be required to detect an actual difference in mean burning time of 1 minute with a power of at least 0.90?

Minitab Output

Power and Sample Size

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Sigma = 9.32

            Sample  Target  Actual
Difference    Size   Power   Power
         2     458  0.9000  0.9004
```

2-29 Reconsider the bottle filling experiment described in Problem 2-9. Rework this problem assuming that the two population variances are unknown but equal.

Minitab Output

```
Two-Sample T-Test and CI: Machine 1, Machine 2


Two-sample T for Machine 1 vs Machine 2

          N      Mean     StDev   SE Mean
Machine  10   16.0150    0.0303    0.0096
Machine  10   16.0050    0.0255    0.0081

Difference = mu Machine 1 - mu Machine 2
Estimate for difference:  0.0100
95% CI for difference: (-0.0163, 0.0363)
T-Test of difference = 0 (vs not =): T-Value = 0.80  P-Value = 0.435  DF = 18
Both use Pooled StDev = 0.0280
```

The hypothesis test is the same: H_0: $\mu_1 = \mu_2$ H_1: $\mu_1 \neq \mu_2$
The conclusions are the same as Problem 2-9, do not reject H_0. There is no difference in the machines. The *P*-value for this anlysis is 0.435.
The confidence interval is (-0.0163, 0.0363). This interval contains 0. There is no difference in machines.

2-29 Consider the data from problem 2-9. If the mean fill volume of the two machines differ by as much as 0.25 ounces, what is the power of the test used in problem 2-9? What sample size could result in a power of at least 0.9 if the actual difference in mean fill volume is 0.25 ounces?

Minitab Output

```
Power and Sample Size

2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Sigma = 0.028

            Sample
Difference    Size   Power
      0.25      10  1.0000
```

Minitab Output

```
Power and Sample Size

2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Sigma = 0.028

            Sample  Target  Actual
Difference    Size   Power   Power
      0.25       2  0.9000  0.9805
```

Chapter 3
Experiments with a Single Factor: The Analysis of Variance
Solutions

3-1 The tensile strength of portland cement is being studied. Four different mixing techniques can be used economically. The following data have been collected:

Mixing Technique	Tensile Strength (lb/in^2)			
1	3129	3000	2865	2890
2	3200	3300	2975	3150
3	2800	2900	2985	3050
4	2600	2700	2600	2765

(a) Test the hypothesis that mixing techniques affect the strength of the cement. Use $\alpha = 0.05$.

Design Expert Output

Response: Tensile Strengthin lb/in^2
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	4.897E+005	3	1.632E+005	12.73	0.0005	significant
A	*4.897E+005*	*3*	*1.632E+005*	*12.73*	*0.0005*	
Residual	1.539E+005	12	12825.69			
Lack of Fit	*0.000*	*0*				
Pure Error	*1.539E+005*	*12*	*12825.69*			
Cor Total	6.436E+005	15				

The Model F-value of 12.73 implies the model is significant. There is only a 0.05% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

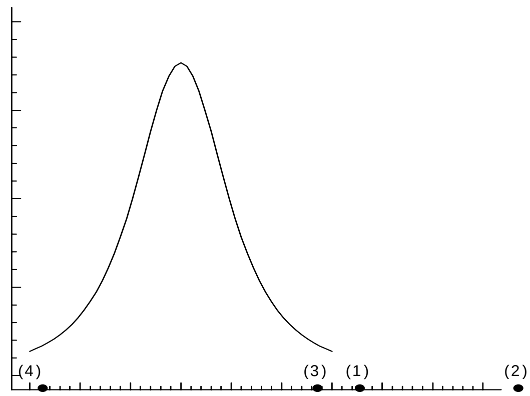
	Estimated Mean	Standard Error
1-1	2971.00	56.63
2-2	3156.25	56.63
3-3	2933.75	56.63
4-4	2666.25	56.63

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-185.25	1	80.08	-2.31	0.0392
1 vs 3	37.25	1	80.08	0.47	0.6501
1 vs 4	304.75	1	80.08	3.81	0.0025
2 vs 3	222.50	1	80.08	2.78	0.0167
2 vs 4	490.00	1	80.08	6.12	< 0.0001
3 vs 4	267.50	1	80.08	3.34	0.0059

The *F*-value is 12.73 with a corresponding *P*-value of .0005. Mixing technique has an effect.

(b) Construct a graphical display as described in Section 3-5.3 to compare the mean tensile strengths for the four mixing techniques. What are your conclusions?

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{12825.7}{4}} = 56.625$$



Based on examination of the plot, we would conclude that μ_1 and μ_3 are the same; that μ_4 differs from μ_1 and μ_3, that μ_2 differs from μ_1 and μ_3, and that μ_2 and μ_4 are different.

(c) Use the Fisher LSD method with α=0.05 to make comparisons between pairs of means.

$$LSD = t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}$$

$$LSD = t_{0.025,16-4}\sqrt{\frac{2(12825.7)}{4}}$$

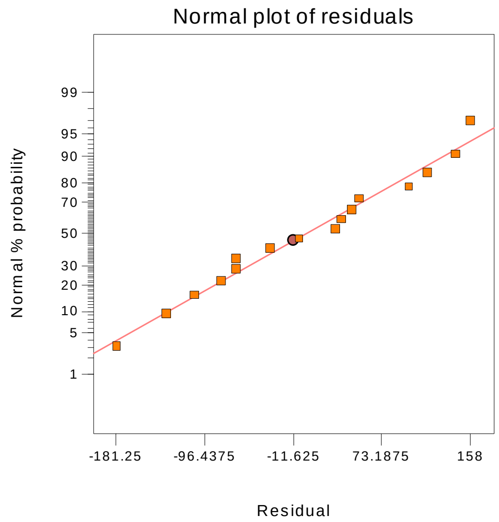
$$LSD = 2.179\sqrt{6412.85} = 174.495$$

Treatment 2 vs. Treatment 4 = 3156.250 - 2666.250 = 490.000 > 174.495
Treatment 2 vs. Treatment 3 = 3156.250 - 2933.750 = 222.500 > 174.495
Treatment 2 vs. Treatment 1 = 3156.250 - 2971.000 = 185.250 > 174.495
Treatment 1 vs. Treatment 4 = 2971.000 - 2666.250 = 304.750 > 174.495
Treatment 1 vs. Treatment 3 = 2971.000 - 2933.750 = 37.250 < 174.495
Treatment 3 vs. Treatment 4 = 2933.750 - 2666.250 = 267.500 > 174.495

The Fisher LSD method is also presented in the Design-Expert computer output above. The results agree with the graphical method for this experiment.

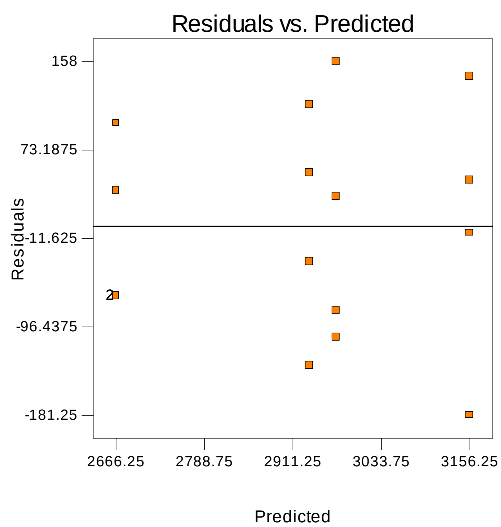
(d) Construct a normal probability plot of the residuals. What conclusion would you draw about the validity of the normality assumption?

There is nothing unusual about the normal probability plot of residuals.



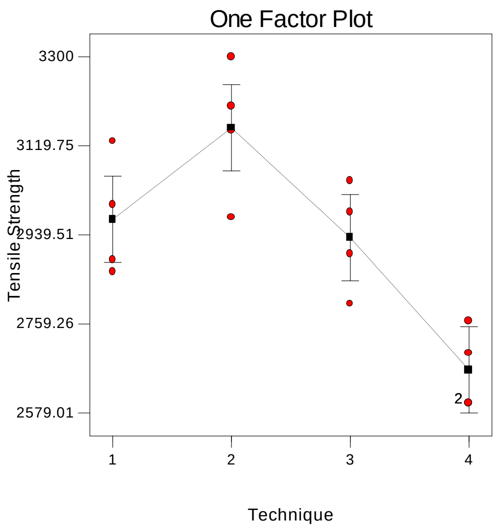
(e) Plot the residuals versus the predicted tensile strength. Comment on the plot.

There is nothing unusual about this plot.



(f) Prepare a scatter plot of the results to aid the interpretation of the results of this experiment.

Design-Expert automatically generates the scatter plot. The plot below also shows the sample average for each treatment and the 95 percent confidence interval on the treatment mean.



3-2 (a) Rework part (b) of Problem 3-1 using Tukey's test with α = 0.05. Do you get the same conclusions from Tukey's test that you did from the graphical procedure and/or the Fisher LSD method?

Minitab Output

```
Tukey's pairwise comparisons

   Family error rate = 0.0500
Individual error rate = 0.0117
Critical value = 4.20

Intervals for (column level mean) - (row level mean)

            1          2          3

   2      -423
           53

   3      -201       -15
          275        460

   4        67       252        30
          543        728       505
```

No, the conclusions are not the same. The mean of Treatment 4 is different than the means of Treatments 1, 2, and 3. However, the mean of Treatment 2 is not different from the means of Treatments 1 and 3 according to the Tukey method, they were found to be different using the graphical method and the Fisher LSD method.

(b) Explain the difference between the Tukey and Fisher procedures.

Both Tukey and Fisher utilize a single critical value; however, Tukey's is based on the studentized range statistic while Fisher's is based on *t* distribution.

3-3 Reconsider the experiment in Problem 3-1. Find a 95 percent confidence interval on the mean tensile strength of the portland cement produced by each of the four mixing techniques. Also find a 95 percent confidence interval on the difference in means for techniques 1 and 3. Does this aid in interpreting the results of the experiment?

$$\bar{y}_{i.} - t_{\alpha/2,N-a}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i.} + t_{\alpha/2,N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Treatment 1: } 2971 \pm 2.179\sqrt{\frac{1282837}{4}}$$

$$2971 \pm 123.387$$

$$2847.613 \le \mu_1 \le 3094.387$$

$$\text{Treatment 2: } 3156.25 \pm 123.387$$

$$3032.863 \le \mu_2 \le 3279.637$$

$$\text{Treatment 3: } 2933.75 \pm 123.387$$

$$2810.363 \le \mu_3 \le 3057.137$$

$$\text{Treatment 4: } 2666.25 \pm 123.387$$

$$2542.863 \le \mu_4 \le 2789.637$$

$$\text{Treatment 1 - Treatment 3: } \bar{y}_{i.} - \bar{y}_{j.} - t_{\alpha/2,N-a}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i.} - \bar{y}_{j.} + t_{\alpha/2,N-a}\sqrt{\frac{2MS_E}{n}}$$

$$2971.00 - 2933.75 \pm 2.179\sqrt{\frac{2(12825.7)}{4}}$$

$$-137.245 \le \mu_1 - \mu_3 \le 211.745$$

3-4 A product developer is investigating the tensile strength of a new synthetic fiber that will be used to make cloth for men’s shirts. Strength is usually affected by the percentage of cotton used in the blend of materials for the fiber. The engineer conducts an experiment with five levels of cotton content and replicated the experiment five times. The data are shown in the following table.

Cotton Weight Percentage	Observations				
15	7	7	15	11	9
20	12	17	12	18	18
25	14	19	19	18	18
30	19	25	22	19	23
35	7	10	11	15	11

(a) Is there evidence to support the claim that cotton content affects the mean tensile strength? Use α = 0.05.

Minitab Output

One-way ANOVA: Tensile Strength versus Cotton Percentage

```
Analysis of Variance for Tensile
Source     DF        SS        MS        F        P
Cotton P    4    475.76    118.94    14.76    0.000
Error      20    161.20      8.06
Total      24    636.96
```

Yes, the *F*-value is 14.76 with a corresponding *P*-value of 0.000. The percentage of cotton in the fiber appears to have an affect on the tensile strength.

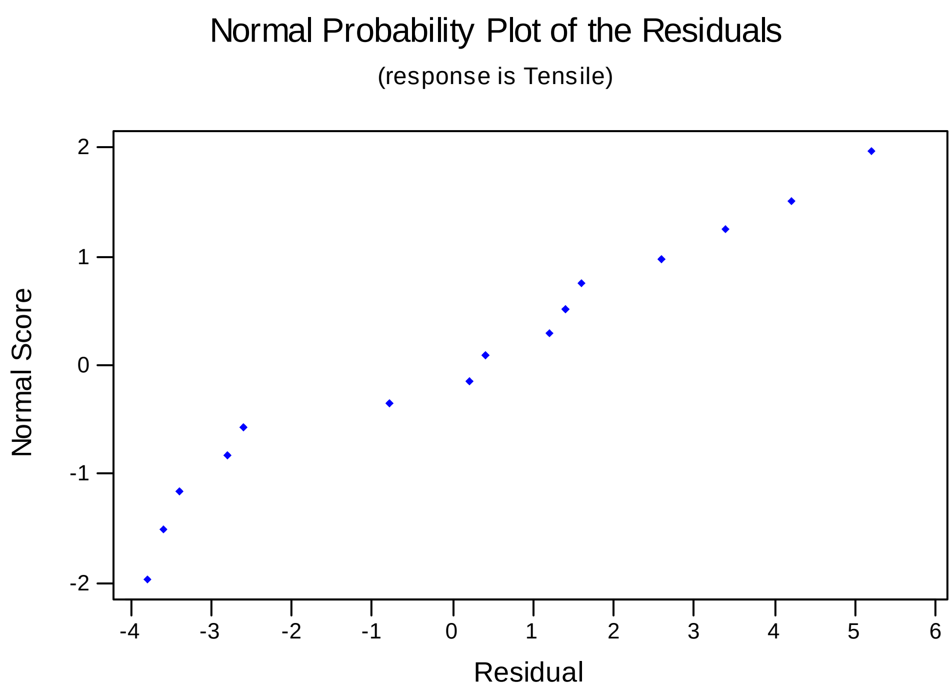
(b) Use the Fisher LSD method to make comparisons between the pairs of means. What conclusions can you draw?

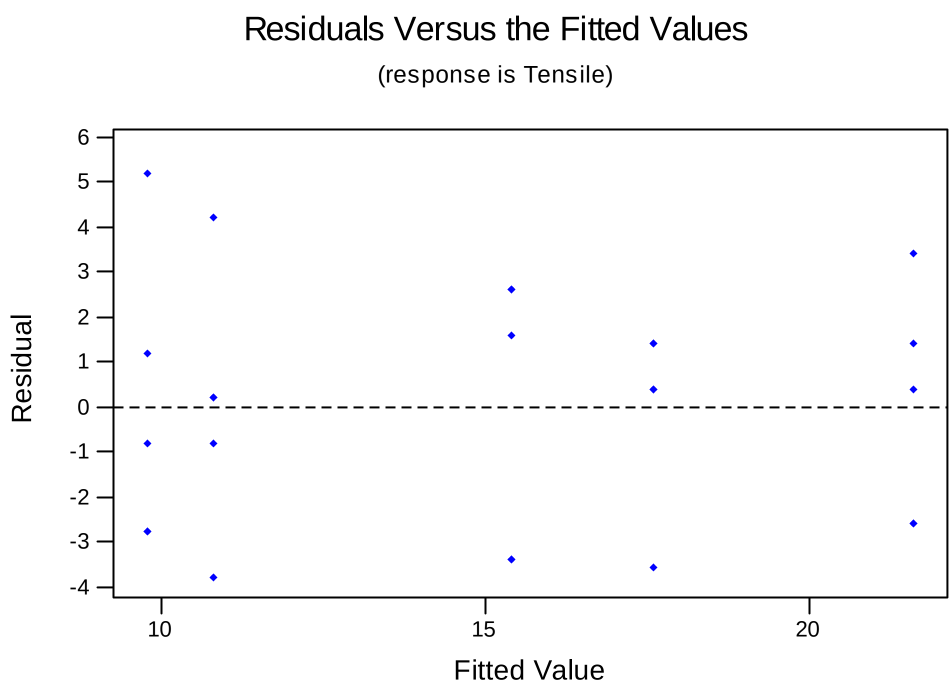
Minitab Output

```
Fisher's pairwise comparisons

    Family error rate = 0.264
Individual error rate = 0.0500

Critical value = 2.086

Intervals for (column level mean) - (row level mean)

                  15          20          25          30

      20      -9.346
              -1.854

      25     -11.546      -5.946
              -4.054       1.546

      30     -15.546      -9.946      -7.746
              -8.054      -2.454      -0.254

      35      -4.746       0.854       3.054       7.054
               2.746       8.346      10.546      14.546
```

In the Minitab output the pairs of treatments that do not contain zero in the pair of numbers indicates that there is a difference in the pairs of the treatments. 15% cotton is different than 20%, 25% and 30%. 20% cotton is different than 30% and 35% cotton. 25% cotton is different than 30% and 35% cotton. 30% cotton is different than 35%.

(c) Analyze the residuals from this experiment and comment on model adequacy.





The residuals show nothing unusual.

3-5 Reconsider the experiment described in Problem 3-4. Suppose that 30 percent cotton content is a control. Use Dunnett's test with $\alpha = 0.05$ to compare all of the other means with the control.

For this problem: $a = 5$, a-1 = 4, f=20, n=5 and $\alpha = 0.05$

$$d_{0.05}(4,20)\sqrt{\frac{2MS_E}{n}} = 2.65\sqrt{\frac{2(8.06)}{n}} = 4.76$$

$$\begin{aligned}
1 \text{ vs. } 4: \bar{y}_{1.} - \bar{y}_{4.} &= 9.8 - 21.6 = -11.8^* \\
2 \text{ vs. } 4: \bar{y}_{2.} - \bar{y}_{4.} &= 15.4 - 21.6 = -6.2^* \\
3 \text{ vs. } 4: \bar{y}_{3.} - \bar{y}_{4.} &= 17.6 - 21.6 = -4.0 \\
5 \text{ vs. } 4: \bar{y}_{5.} - \bar{y}_{4.} &= 10.8 - 21.6 = -10.6^*
\end{aligned}$$

The control treatment, treatment 4, differs from treatments 1,2 and 5.

3-6 A pharmaceutical manufacturer wants to investigate the bioactivity of a new drug. A completely randomized single-factor experiment was conducted with three dosage levels, and the following results were obtained.

Dosage	Observations			
20g	24	28	37	30
30g	37	44	31	35

40g	42	47	52	38

(a) Is there evidence to indicate that dosage level affects bioactivity? Use $\alpha = 0.05$.

Minitab Output

```
One-way ANOVA: Activity versus Dosage

Analysis of Variance for Activity
Source     DF        SS        MS        F        P
Dosage      2     450.7     225.3     7.04    0.014
Error       9     288.3      32.0
Total      11     738.9
```

There appears to be a different in the dosages.

(b) If it is appropriate to do so, make comparisons between the pairs of means. What conclusions can you draw?

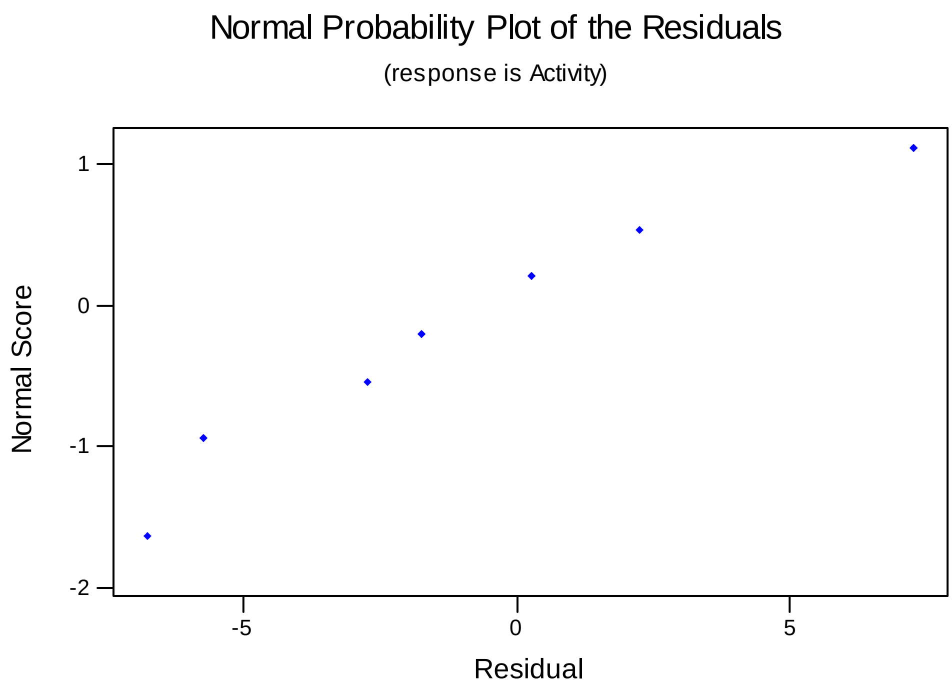
Because there appears to be a difference in the dosages, the comparison of means is appropriate.

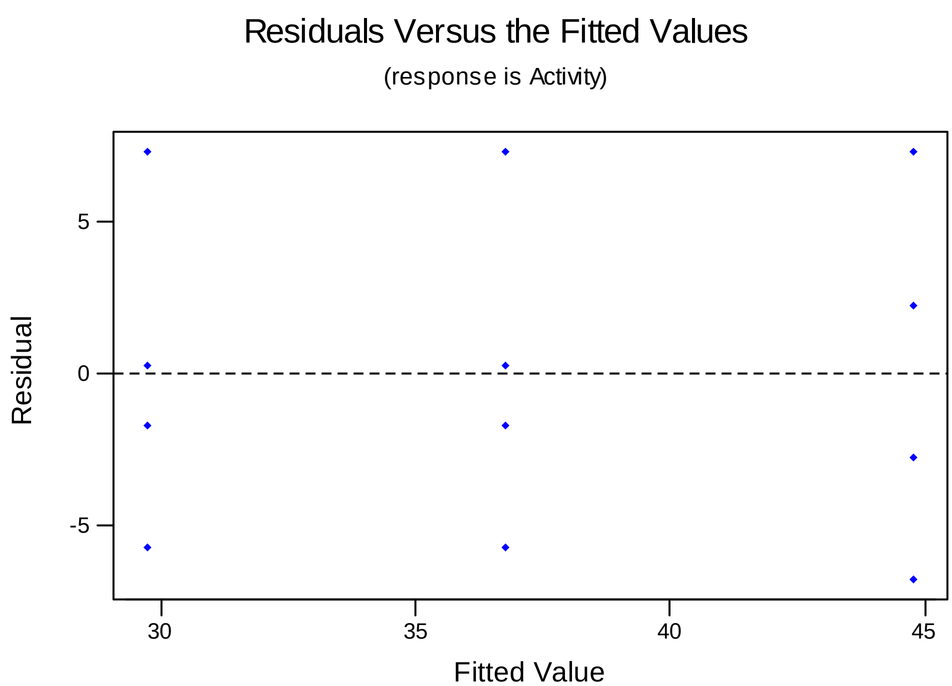
Minitab Output

```
Tukey's pairwise comparisons

    Family error rate = 0.0500
Individual error rate = 0.0209

Critical value = 3.95

Intervals for (column level mean) - (row level mean)

               20g         30g

     30g     -18.177
               4.177

     40g     -26.177     -19.177
              -3.823       3.177
```

The Tukey comparison shows a difference in the means between the 20g and the 40g dosages.

(c) Analyze the residuals from this experiment and comment on the model adequacy.





There is nothing too unusual about the residuals.

3-7 A rental car company wants to investigate whether the type of car rented affects the length of the rental period. An experiment is run for one week at a particular location, and 10 rental contracts are selected at random for each car type. The results are shown in the following table.

Type of Car	Observations									
Sub-compact	3	5	3	7	6	5	3	2	1	6
Compact	1	3	4	7	5	6	3	2	1	7
Midsize	4	1	3	5	7	1	2	4	2	7
Full Size	3	5	7	5	10	3	4	7	2	7

(a) Is there evidence to support a claim that the type of car rented affects the length of the rental contract? Use $\alpha = 0.05$. If so, which types of cars are responsible for the difference?

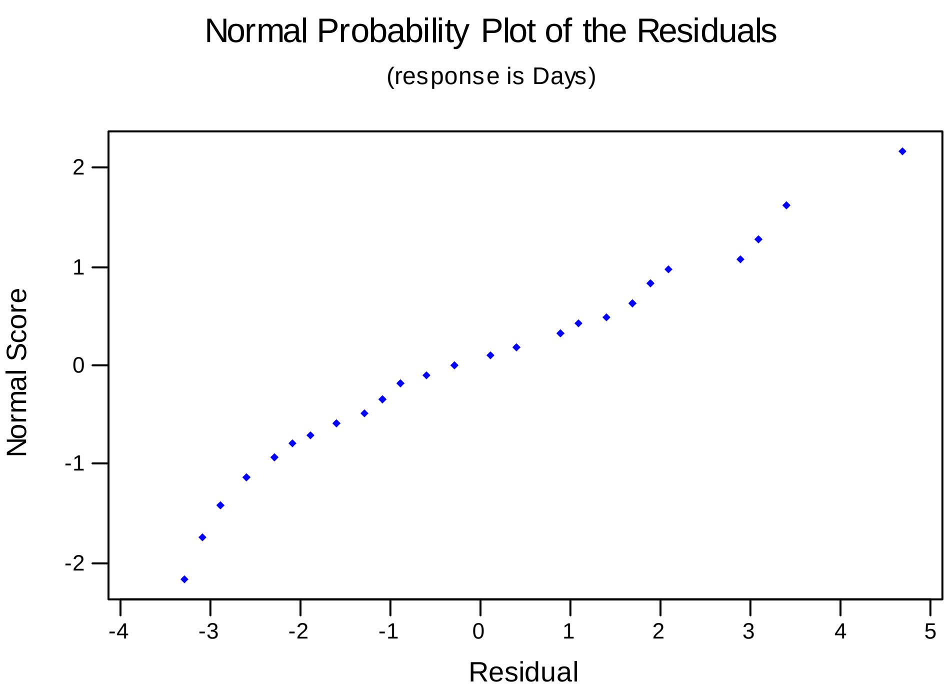
Minitab Output

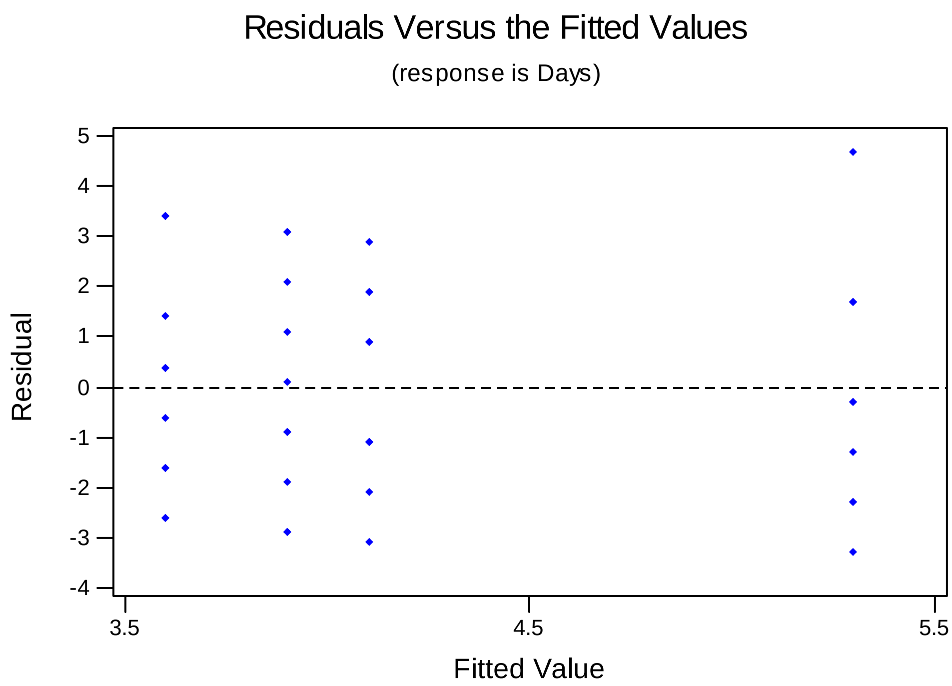
One-way ANOVA: Days versus Car Type

```
Analysis of Variance for Days
Source     DF        SS        MS        F        P
Car Type    3     16.68      5.56     1.11    0.358
Error      36    180.30      5.01
Total      39    196.98
```

There is no difference.

(b) Analyze the residuals from this experiment and comment on the model adequacy.





There is nothing unusual about the residuals.

(c) Notice that the response variable in this experiment is a count. Should the cause any potential concerns about the validity of the analysis of variance?

Because the data is count data, a square root transformation could be applied. The analysis is shown below. It does not change the interpretation of the data.

Minitab Output

One-way ANOVA: Sqrt Days versus Car Type

```
Analysis of Variance for Sqrt Day
Source      DF        SS        MS        F        P
Car Type     3     1.087     0.362     1.10    0.360
Error       36    11.807     0.328
Total       39    12.893
```

3-8 I belong to a golf club in my neighborhood. I divide the year into three golf seasons: summer (June-September), winter (November-March) and shoulder (October, April and May). I believe that I play my best golf during the summer (because I have more time and the course isn't crowded) and shoulder (because the course isn't crowded) seasons, and my worst golf during the winter (because all of the part-year residents show up, and the course is crowded, play is slow, and I get frustrated). Data from the last year are shown in the following table.

Season	Observations									
Summer	83	85	85	87	90	88	88	84	91	90
Shoulder	91	87	84	87	85	86	83			
Winter	94	91	87	85	87	91	92	86		

(a) Do the data indicate that my opinion is correct? Use $\alpha = 0.05$.

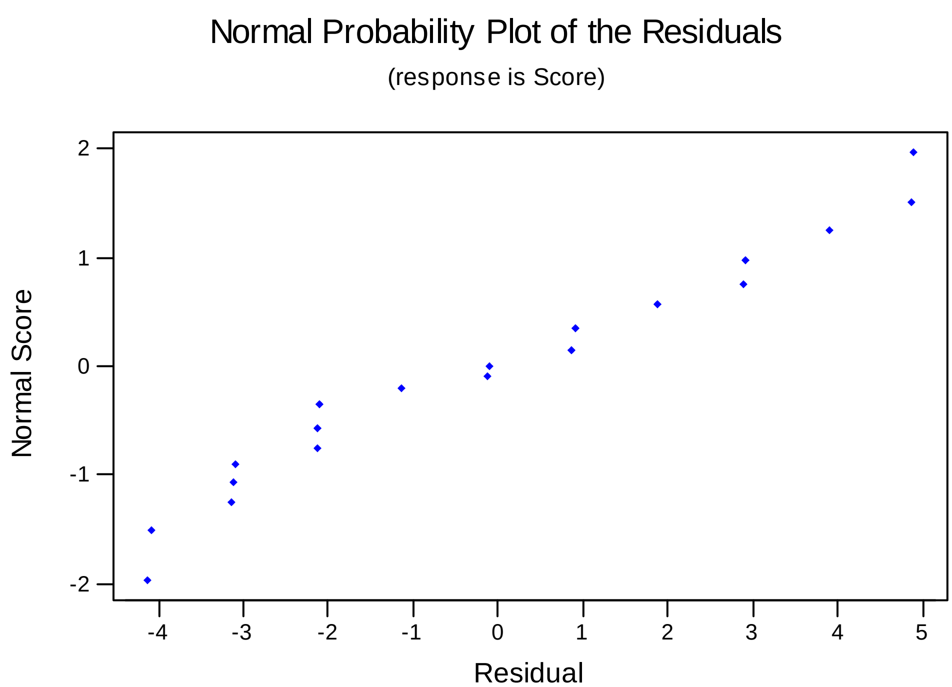
Minitab Output

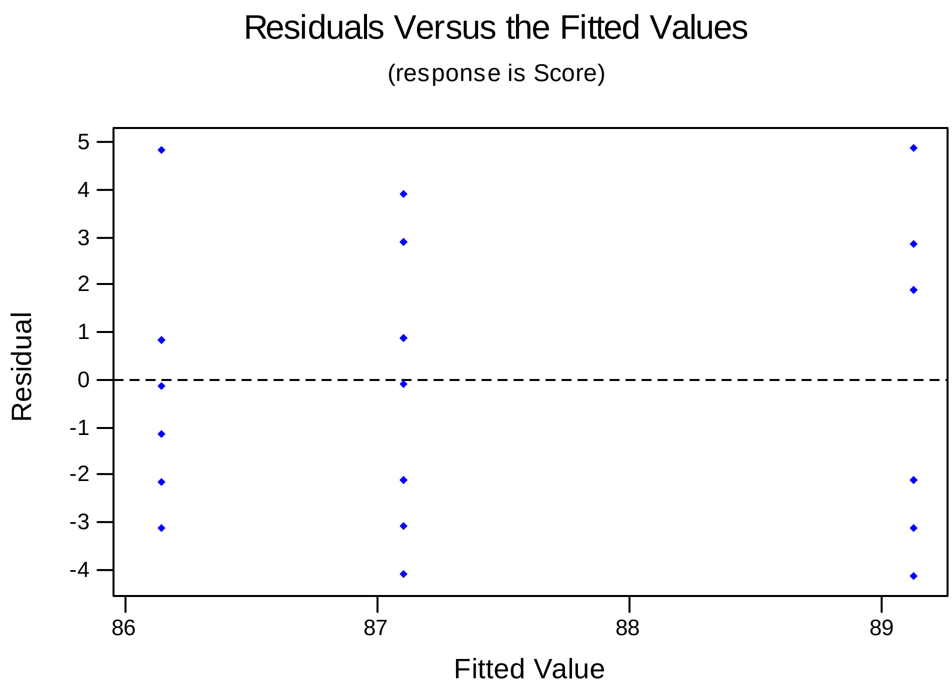
One-way ANOVA: Score versus Season

```
Analysis of Variance for Score
Source     DF        SS        MS        F        P
Season      2     35.61     17.80     2.12    0.144
Error      22    184.63      8.39
Total      24    220.24
```

The data do not support the author's opinion.

(b) Analyze the residuals from this experiment and comment on model adequacy.





There is nothing unusual about the residuals.

3-9 A regional opera company has tried three approaches to solicit donations from 24 potential sponsors. The 24 potential sponsors were randomly divided into three groups of eight, and one approach was used for each group. The dollar amounts of the resulting contributions are shown in the following table.

Approach	Contributions (in $)							
1	1000	1500	1200	1800	1600	1100	1000	1250
2	1500	1800	2000	1200	2000	1700	1800	1900
3	900	1000	1200	1500	1200	1550	1000	1100

(a) Do the data indicate that there is a difference in results obtained from the three different approaches? Use $\alpha = 0.05$.

Minitab Output

One-way ANOVA: Contribution versus Approach

```
Analysis of Variance for Contribution
Source     DF        SS        MS        F        P
Approach    2   1362708    681354     9.41    0.001
Error      21   1520625     72411
Total      23   2883333
```

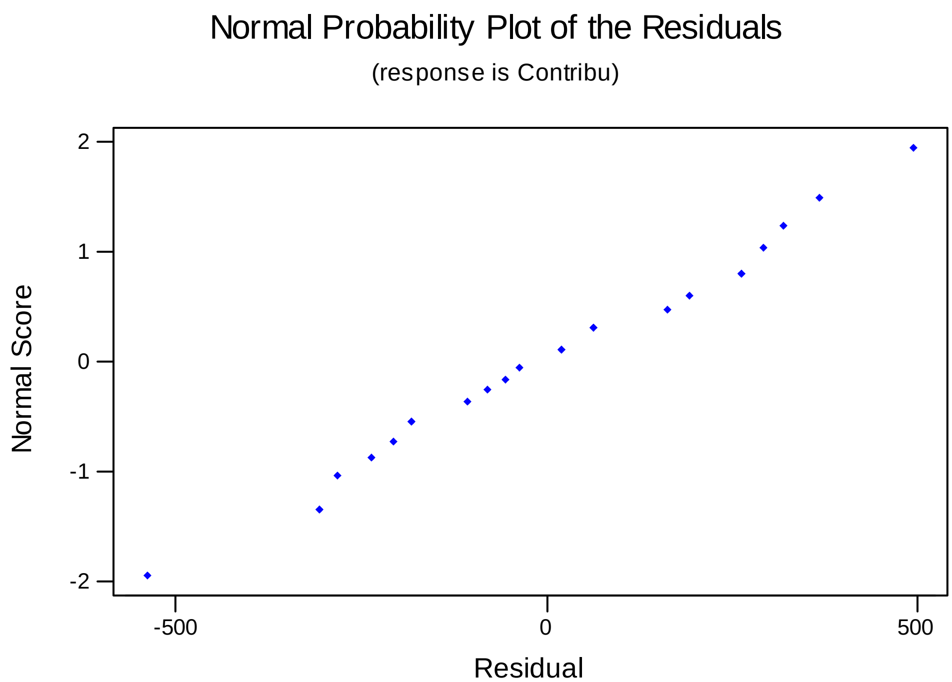
There is a difference between the approaches. The Tukey test will indicate which are different. Approach 2 is different than approach 3.

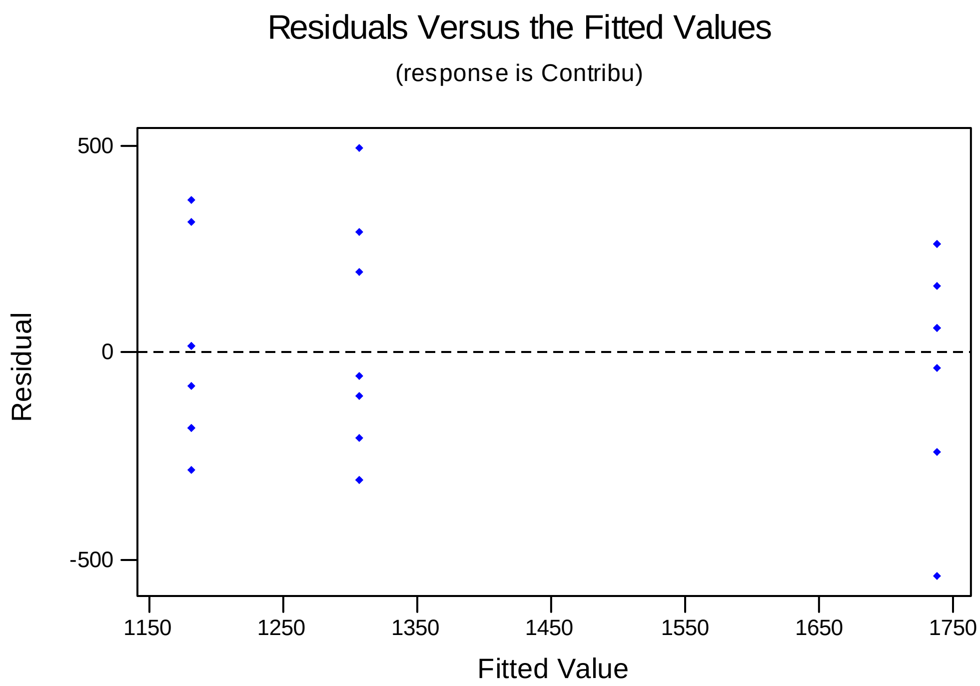
Minitab Output

```
Tukey's pairwise comparisons

    Family error rate = 0.0500
Individual error rate = 0.0200

Critical value = 3.56
```

```
Intervals for (column level mean) - (row level mean)

                 1           2

       2        -770
                 -93

       3        -214         218
                 464         895
```

(b) Analyze the residuals from this experiment and comment on the model adequacy.





There is nothing unusual about the residuals.

3-10 An experiment was run to determine whether four specific firing temperatures affect the density of a certain type of brick. The experiment led to the following data:

Temperature	Density				
100	21.8	21.9	21.7	21.6	21.7
125	21.7	21.4	21.5	21.4	
150	21.9	21.8	21.8	21.6	21.5
175	21.9	21.7	21.8	21.4	

(a) Does the firing temperature affect the density of the bricks? Use $\alpha = 0.05$.

No, firing temperature does not affect the density of the bricks. Refer to the Design-Expert output below.

Design Expert Output

Response: Density

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.16	3	0.052	2.02	0.1569	not significant
A	*0.16*	*3*	*0.052*	*2.02*	*0.1569*	
Residual	0.36	14	0.026			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.36*	*14*	*0.026*			
Cor Total	0.52	17				

The "Model F-value" of 2.02 implies the model is not significant relative to the noise. There is a 15.69 % chance that a "Model F-value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

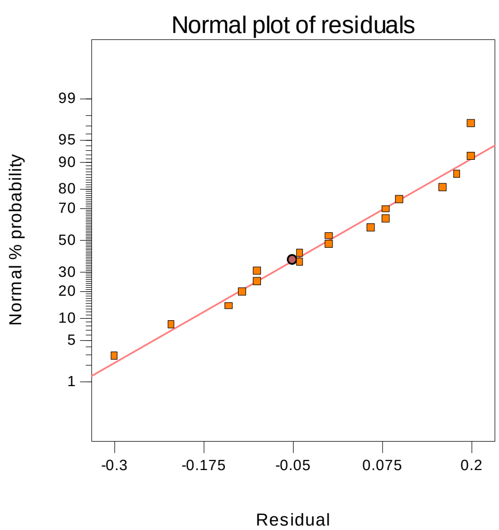
	Estimated Mean	Standard Error
1-100	21.74	0.072
2-125	21.50	0.080
3-150	21.72	0.072
4-175	21.70	0.080

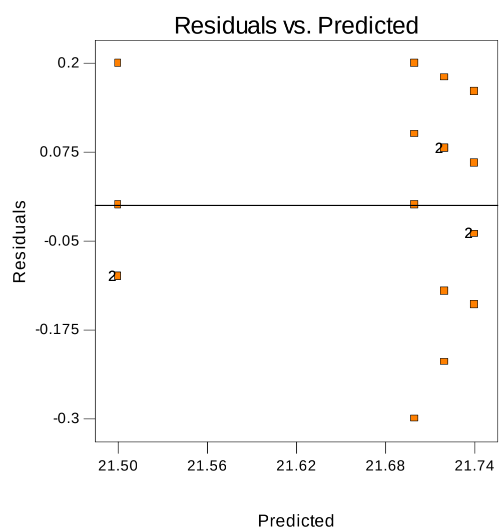
Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	0.24	1	0.11	2.23	0.0425
1 vs 3	0.020	1	0.10	0.20	0.8465
1 vs 4	0.040	1	0.11	0.37	0.7156
2 vs 3	-0.22	1	0.11	-2.05	0.0601
2 vs 4	-0.20	1	0.11	-1.76	0.0996
3 vs 4	0.020	1	0.11	0.19	0.8552

(b) Is it appropriate to compare the means using the Fisher LSD method in this experiment?

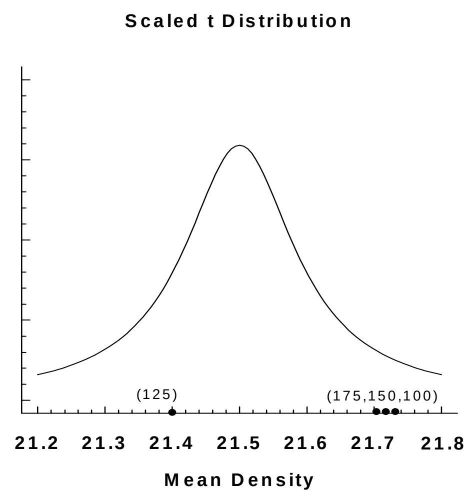
The analysis of variance tells us that there is no difference in the treatments. There is no need to proceed with Fisher's LSD method to decide which mean is difference.

(c) Analyze the residuals from this experiment. Are the analysis of variance assumptions satisfied? There is nothing unusual about the residual plots.





(d) Construct a graphical display of the treatments as described in Section 3-5.3. Does this graph adequately summarize the results of the analysis of variance in part (b). Yes.



3-11 Rework Part (d) of Problem 3-10 using the Tukey method. What conclusions can you draw? Explain carefully how you modified the procedure to account for unequal sample sizes.

When sample sizes are unequal, the appropriate formula for the Tukey method is

$$T_\alpha = \frac{q_\alpha(a, f)}{\sqrt{2}}\sqrt{MS_E\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

Treatment 1 vs. Treatment 2 = 21.74 – 21.50 = 0.24 < 0.994
Treatment 1 vs. Treatment 3 = 21.74 – 21.72 = 0.02 < 0.937
Treatment 1 vs. Treatment 4 = 21.74 – 21.70 = 0.04 < 0.994
Treatment 3 vs. Treatment 2 = 21.72 – 21.50 = 0.22 < 1.048
Treatment 4 vs. Treatment 2 = 21.70 – 21.50 = 0.20 < 1.048
Treatment 3 vs. Treatment 4 = 21.72 – 21.70 = 0.02 < 0.994

All pairwise comparisons do not identify differences. Notice that there are different critical values for the comparisons depending on the sample sizes of the two groups being compared.

Because we could not reject the hypothesis of equal means using the analysis of variance, we should **never** have performed the Tukey test (or any other multiple comparison procedure, for that matter). If you ignore the analysis of variance results and run multiple comparisons, you will likely make type I errors.

3-12 A manufacturer of television sets is interested in the effect of tube conductivity of four different types of coating for color picture tubes. The following conductivity data are obtained:

Coating Type	Conductivity			
1	143	141	150	146
2	152	149	137	143
3	134	136	132	127
4	129	127	132	129

(a) Is there a difference in conductivity due to coating type? Use $\alpha = 0.05$.

Yes, there is a difference in means. Refer to the Design-Expert output below..

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	844.69	3	281.56	14.30	0.0003	significant
A	*844.69*	*3*	*281.56*	*14.30*	*0.0003*	
Residual	236.25	12	19.69			
Lack of Fit	*0.000*	*0*				
Pure Error	*236.25*	*12*	*19.69*			
Cor Total	1080.94	15				

The Model F-value of 14.30 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	145.00	2.22
2-2	145.25	2.22
3-3	132.25	2.22
4-4	129.25	2.22

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.25	1	3.14	-0.080	0.9378
1 vs 3	12.75	1	3.14	4.06	0.0016
1 vs 4	15.75	1	3.14	5.02	0.0003
2 vs 3	13.00	1	3.14	4.14	0.0014
2 vs 4	16.00	1	3.14	5.10	0.0003
3 vs 4	3.00	1	3.14	0.96	0.3578

(b) Estimate the overall mean and the treatment effects.

$$\hat{\mu} = 2207/16 = 137.9375$$
$$\hat{\tau}_1 = \bar{y}_{1.} - \bar{y}_{..} = 145.00 - 137.9375 = 7.0625$$
$$\hat{\tau}_2 = \bar{y}_{2.} - \bar{y}_{..} = 145.25 - 137.9375 = 7.3125$$
$$\hat{\tau}_3 = \bar{y}_{3.} - \bar{y}_{..} = 132.25 - 137.9375 = -5.6875$$
$$\hat{\tau}_4 = \bar{y}_{4.} - \bar{y}_{..} = 129.25 - 137.9375 = -8.6875$$

(c) Compute a 95 percent interval estimate of the mean of coating type 4. Compute a 99 percent interval estimate of the mean difference between coating types 1 and 4.

$$\text{Treatment 4: } 129.25 \pm 2.179\sqrt{\frac{19.69}{4}}$$
$$124.4155 \le \mu_4 \le 134.0845$$
$$\text{Treatment 1 - Treatment 4: } (145 - 129.25) \pm 3.055\sqrt{\frac{(2)19.69}{4}}$$
$$6.164 \le \mu_1 - \mu_4 \le 25.336$$

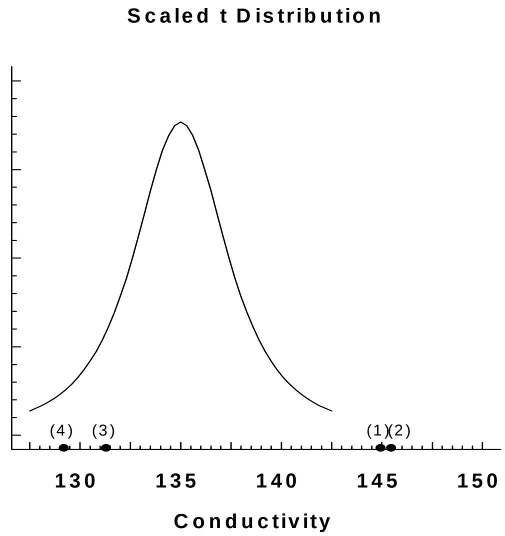
(d) Test all pairs of means using the Fisher LSD method with α=0.05.

Refer to the Design-Expert output above. The Fisher LSD procedure is automatically included in the output.

The means of Coating Type 2 and Coating Type 1 are not different. The means of Coating Type 3 and Coating Type 4 are not different. However, Coating Types 1 and 2 produce higher mean conductivity that does Coating Types 3 and 4.

(e) Use the graphical method discussed in Section 3-5.3 to compare the means. Which coating produces the highest conductivity?

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{16.96}{4}} = 2.219$$ Coating types 1 and 2 produce the highest conductivity.

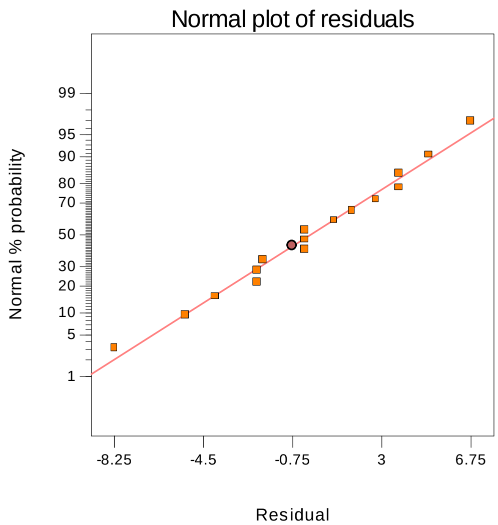


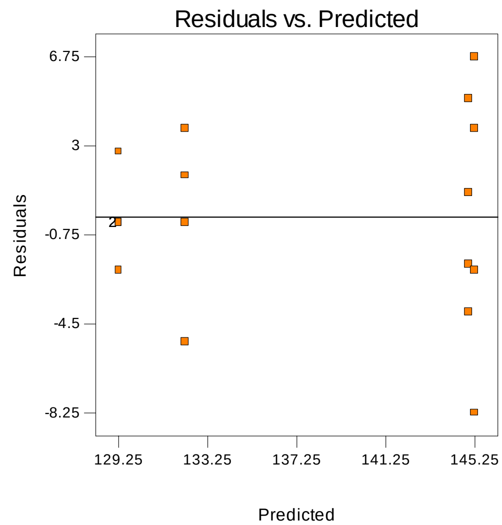
(f) Assuming that coating type 4 is currently in use, what are your recommendations to the manufacturer? We wish to minimize conductivity.

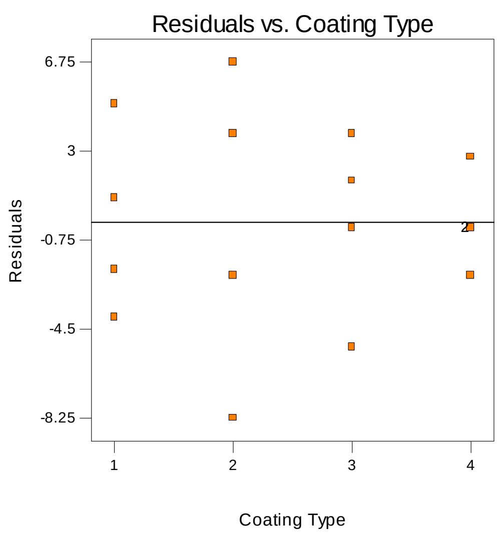
Since coatings 3 and 4 do not differ, and as they both produce the lowest mean values of conductivity, use either coating 3 or 4. As type 4 is currently being used, there is probably no need to change.

3-13 Reconsider the experiment in Problem 3-12. Analyze the residuals and draw conclusions about model adequacy.

There is nothing unusual in the normal probability plot. A funnel shape is seen in the plot of residuals versus predicted conductivity indicating a possible non-constant variance.







3-14 An article in the *ACI Materials Journal* (Vol. 84, 1987. pp. 213-216) describes several experiments investigating the rodding of concrete to remove entrapped air. A 3" x 6" cylinder was used, and the number of times this rod was used is the design variable. The resulting compressive strength of the concrete specimen is the response. The data are shown in the following table.

Rodding Level	Compressive Strength		
10	1530	1530	1440
15	1610	1650	1500
20	1560	1730	1530
25	1500	1490	1510

(a) Is there any difference in compressive strength due to the rodding level? Use $\alpha = 0.05$.

There are no differences.

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	28633.33	3	9544.44	1.87	0.2138	not significant
A	*28633.33*	*3*	*9544.44*	*1.87*	*0.2138*	
Residual	40933.33	8	5116.67			
Lack of Fit	*0.000*	*0*				
Pure Error	*40933.33*	*8*	*5116.67*			
Cor Total	69566.67	11				

The "Model F-value" of 1.87 implies the model is not significant relative to the noise. There is a 21.38 % chance that a "Model F-value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

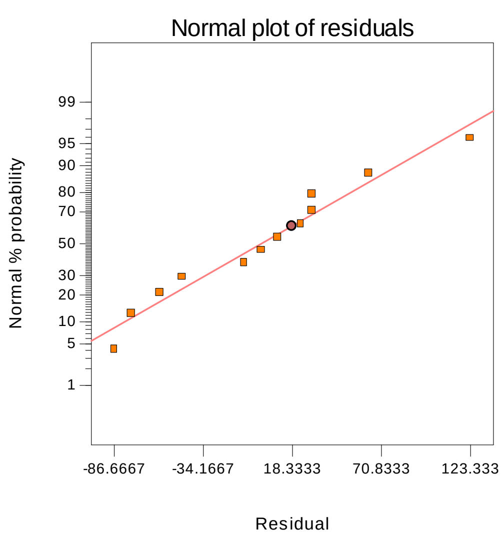
	Estimated Mean	Standard Error
1-10	1500.00	41.30
2-15	1586.67	41.30
3-20	1606.67	41.30
4-25	1500.00	41.30

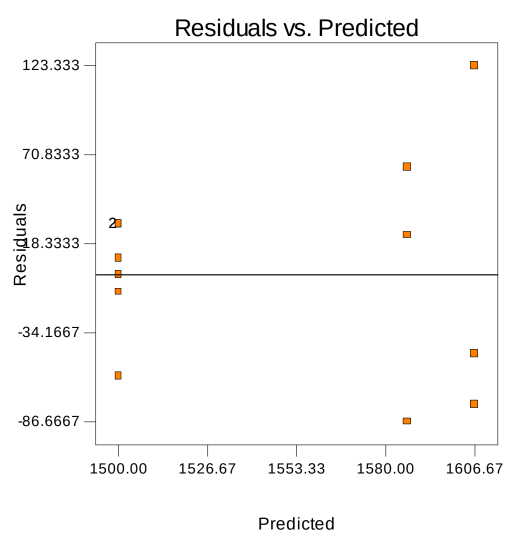
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-86.67	1	58.40	-1.48	0.1761
1 vs 3	-106.67	1	58.40	-1.83	0.1052
1 vs 4	0.000	1	58.40	0.000	1.0000
2 vs 3	-20.00	1	58.40	-0.34	0.7408
2 vs 4	86.67	1	58.40	1.48	0.1761
3 vs 4	106.67	1	58.40	1.83	0.1052

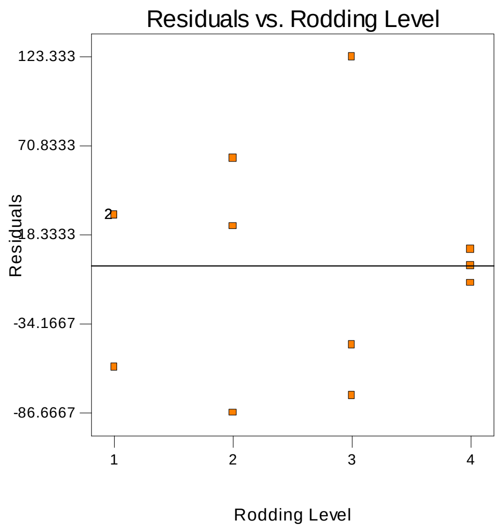
(b) Find the *P*-value for the *F* statistic in part (a). From computer output, *P*=0.2138.

(c) Analyze the residuals from this experiment. What conclusions can you draw about the underlying model assumptions?

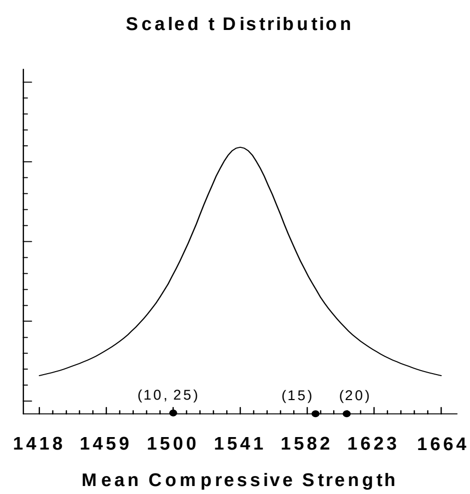
There is nothing unusual about the residual plots.







(d) Construct a graphical display to compare the treatment means as describe in Section 3-5.3.



3-15 An article in *Environment International* (Vol. 18, No. 4, 1992) describes an experiment in which the amount of radon released in showers was investigated. Radon enriched water was used in the experiment and six different orifice diameters were tested in shower heads. The data from the experiment are shown in the following table.

Orifice Dia.	Radon Released (%)			
0.37	80	83	83	85
0.51	75	75	79	79
0.71	74	73	76	77
1.02	67	72	74	74
1.40	62	62	67	69
1.99	60	61	64	66

(a) Does the size of the orifice affect the mean percentage of radon released? Use $\alpha = 0.05$.

Yes. There is at least one treatment mean that is different.

Design Expert Output

Response: Radon Released in %
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1133.38	5	226.68	30.85	< 0.0001	significant
A	*1133.38*	*5*	*226.68*	*30.85*	*< 0.0001*	
Residual	132.25	18	7.35			
Lack of Fit	*0.000*	*0*				
Pure Error	*132.25*	*18*	*7.35*			
Cor Total	1265.63	23				

The Model F-value of 30.85 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

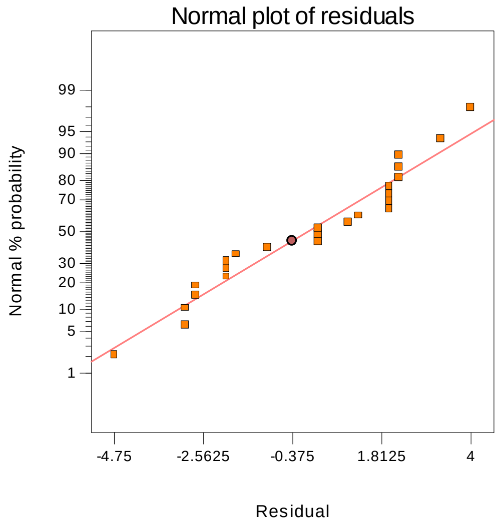
	Estimated Mean	Standard Error
1-0.37	82.75	1.36
2-0.51	77.00	1.36
3-0.71	75.00	1.36
4-1.02	71.75	1.36
5-1.40	65.00	1.36
6-1.99	62.75	1.36

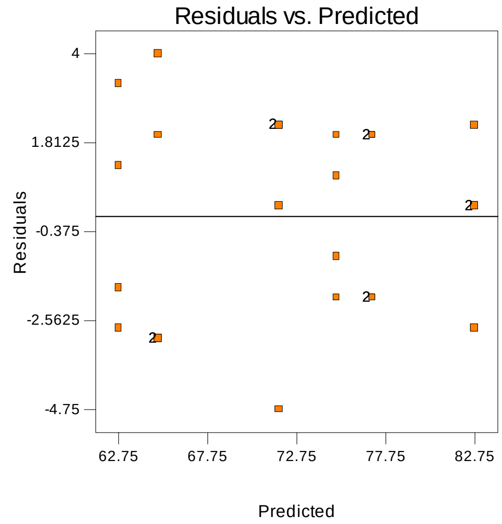
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	5.75	1	1.92	3.00	0.0077
1 vs 3	7.75	1	1.92	4.04	0.0008
1 vs 4	11.00	1	1.92	5.74	< 0.0001
1 vs 5	17.75	1	1.92	9.26	< 0.0001
1 vs 6	20.00	1	1.92	10.43	< 0.0001
2 vs 3	2.00	1	1.92	1.04	0.3105
2 vs 4	5.25	1	1.92	2.74	0.0135
2 vs 5	12.00	1	1.92	6.26	< 0.0001
2 vs 6	14.25	1	1.92	7.43	< 0.0001
3 vs 4	3.25	1	1.92	1.70	0.1072
3 vs 5	10.00	1	1.92	5.22	< 0.0001
3 vs 6	12.25	1	1.92	6.39	< 0.0001
4 vs 5	6.75	1	1.92	3.52	0.0024
4 vs 6	9.00	1	1.92	4.70	0.0002
5 vs 6	2.25	1	1.92	1.17	0.2557

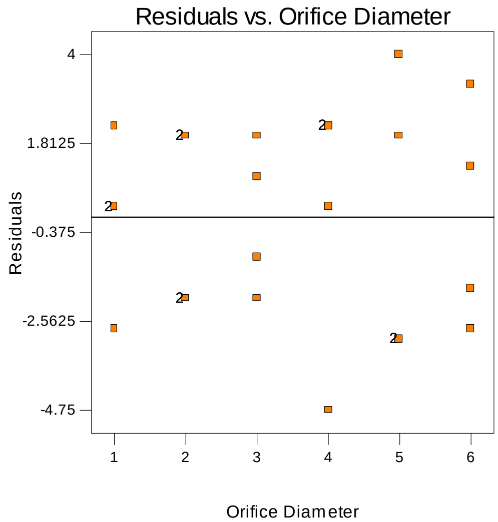
(b) Find the P-value for the F statistic in part (a). P=3.161 x 10^{-8}

(c) Analyze the residuals from this experiment.

There is nothing unusual about the residuals.





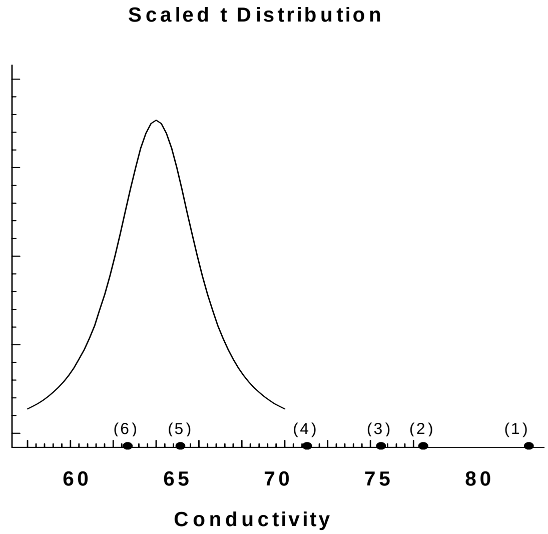


(d) Find a 95 percent confidence interval on the mean percent radon released when the orifice diameter is 1.40.

$$\text{Treatment 5 (Orifice =1.40):} \quad 6 \pm 2.101\sqrt{\frac{7.35}{4}}$$

$$62.152 \leq \mu \leq 67.848$$

(e) Construct a graphical display to compare the treatment means as describe in Section 3-5.3. What conclusions can you draw?



Treatments 5 and 6 as a group differ from the other means; 2, 3, and 4 as a group differ from the other means, 1 differs from the others.

3-16 The response time in milliseconds was determined for three different types of circuits that could be used in an automatic valve shutoff mechanism. The results are shown in the following table.

Circuit Type	Response Time				
1	9	12	10	8	15
2	20	21	23	17	30
3	6	5	8	16	7

(a) Test the hypothesis that the three circuit types have the same response time. Use $\alpha = 0.01$.

From the computer printout, F=16.08, so there is at least one circuit type that is different.

Design Expert Output

Response: Response Time in ms
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	543.60	2	271.80	16.08	0.0004	significant
A	*543.60*	*2*	*271.80*	*16.08*	*0.0004*	
Residual	202.80	12	16.90			
Lack of Fit	*0.000*	*0*				
Pure Error	*202.80*	*12*	*16.90*			
Cor Total	746.40	14				

The Model F-value of 16.08 implies the model is significant. There is only
a 0.04% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	10.80	1.84
2-2	22.20	1.84
3-3	8.40	1.84

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H_0 Coeff=0**	**Prob > |t|**

1 vs 2	-11.40	1	2.60	-4.38	0.0009
1 vs 3	2.40	1	2.60	0.92	0.3742
2 vs 3	13.80	1	2.60	5.31	0.0002

(b) Use Tukey's test to compare pairs of treatment means. Use $\alpha = 0.01$.

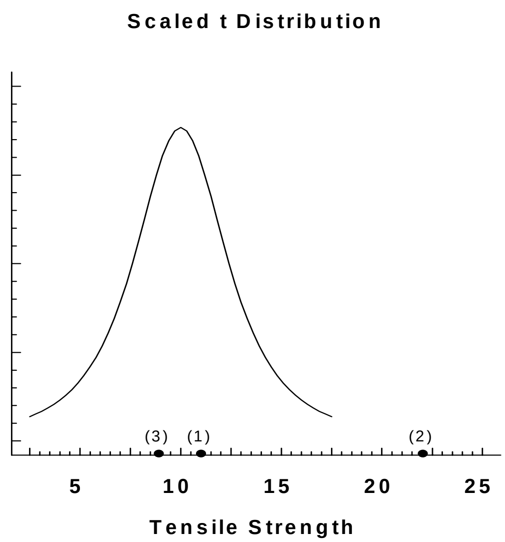
$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{n}} = \sqrt{\frac{1690}{5}} = 1.8385$$
$$q_{0.01,(3,12)} = 5.04$$
$$t_0 = 1.8385(5.04) = 9.266$$
$$\text{1 vs. 2: } |10.8\text{-}22.2| = 11.4 > 9.266$$
$$\text{1 vs. 3: } |10.8\text{-}8.4| = 2.4 < 9.266$$
$$\text{2 vs. 3: } |22.2\text{-}8.4| = 13.8 > 9.266$$

1 and 2 are different. 2 and 3 are different.

Notice that the results indicate that the mean of treatment 2 differs from the means of both treatments 1 and 3, and that the means for treatments 1 and 3 are the same. Notice also that the Fisher LSD procedure (see the computer output) gives the same results.

(c) Use the graphical procedure in Section 3-5.3 to compare the treatment means. What conclusions can you draw? How do they compare with the conclusions from part (a).

The scaled-*t* plot agrees with part (b). In this case, the large difference between the mean of treatment 2 and the other two treatments is very obvious.



(d) Construct a set of orthogonal contrasts, assuming that at the outset of the experiment you suspected the response time of circuit type 2 to be different from the other two.

$$H_0 = \mu_1 - 2\mu_2 + \mu_3 = 0$$
$$H_1 = \mu_1 - 2\mu_2 + \mu_3 \neq 0$$
$$C_1 = y_{1.} - 2y_{2.} + y_{3.}$$
$$C_1 = 54 - 2(111) + 42 = -126$$

$$SS_{C1} = \frac{(-126)^2}{5(6)} = 529.2$$

$$F_{C1} = \frac{529.2}{16.9} = 31.31$$

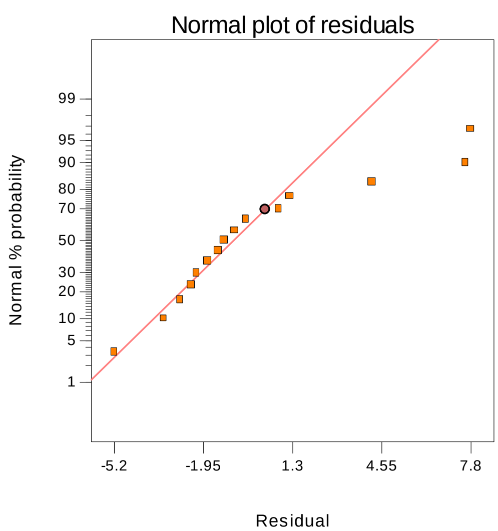
Type 2 differs from the average of type 1 and type 3.

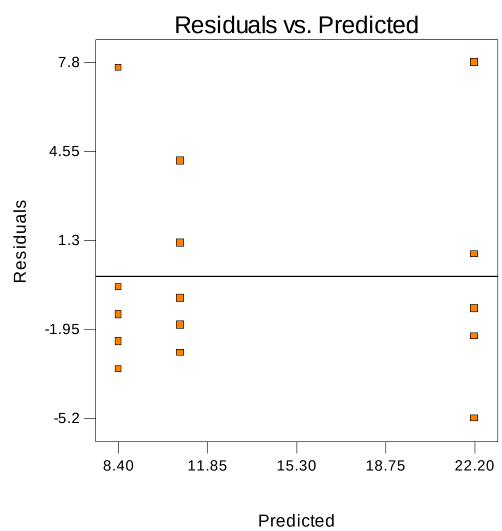
(e) If you were a design engineer and you wished to minimize the response time, which circuit type would you select?

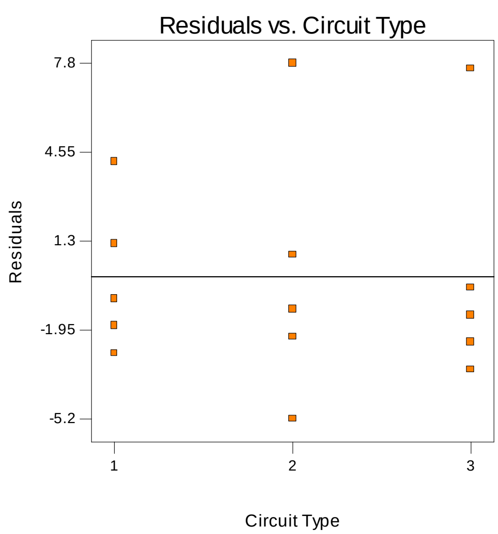
Either type 1 or type 3 as they are not different from each other and have the lowest response time.

(f) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?

The normal probability plot has some points that do not lie along the line in the upper region. This may indicate potential outliers in the data.







3-17 The effective life of insulating fluids at an accelerated load of 35 kV is being studied. Test data have been obtained for four types of fluids. The results were as follows:

Fluid Type	Life (in h) at 35 kV Load					
1	17.6	18.9	16.3	17.4	20.1	21.6
2	16.9	15.3	18.6	17.1	19.5	20.3
3	21.4	23.6	19.4	18.5	20.5	22.3
4	19.3	21.1	16.9	17.5	18.3	19.8

(a) Is there any indication that the fluids differ? Use $\alpha = 0.05$.

At $\alpha = 0.05$ there are no difference, but at since the *P***-**value is just slightly above 0.05, there is probably a difference in means.

Design Expert Output

Response: Life in in h
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	30.17	3	10.06	3.05	0.0525	not significant
A	*30.16*	*3*	*10.05*	*3.05*	*0.0525*	
Residual	65.99	20	3.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*65.99*	*20*	*3.30*			
Cor Total	96.16	23				

The Model F-value of 3.05 implies there is a 5.25% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

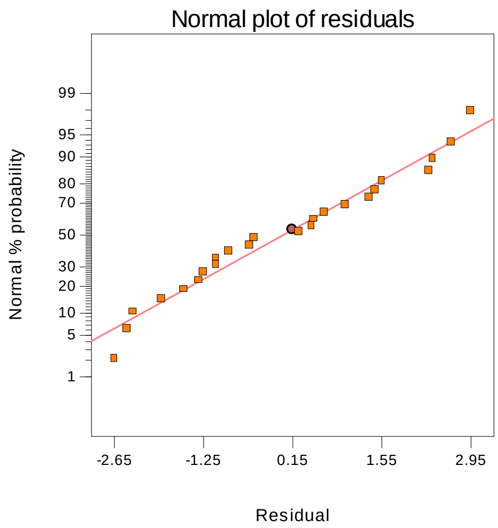
	Estimated Mean	**Standard Error**
1-1	18.65	0.74
2-2	17.95	0.74
3-3	20.95	0.74
4-4	18.82	0.74

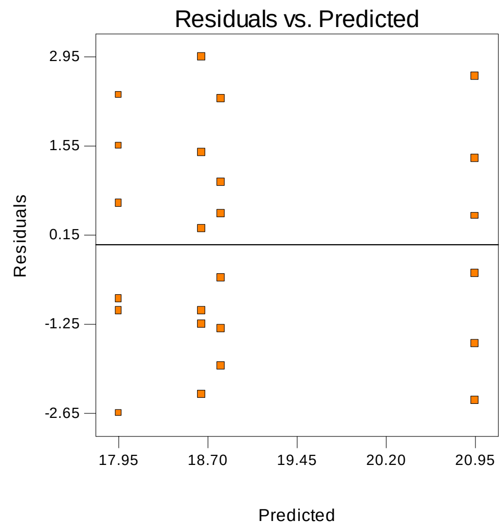
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H_0 Coeff=0**	**Prob > |t|**
1 vs 2	0.70	1	1.05	0.67	0.5121
1 vs 3	-2.30	1	1.05	-2.19	0.0403
1 vs 4	-0.17	1	1.05	-0.16	0.8753
2 vs 3	-3.00	1	1.05	-2.86	0.0097
2 vs 4	-0.87	1	1.05	-0.83	0.4183
3 vs 4	2.13	1	1.05	2.03	0.0554

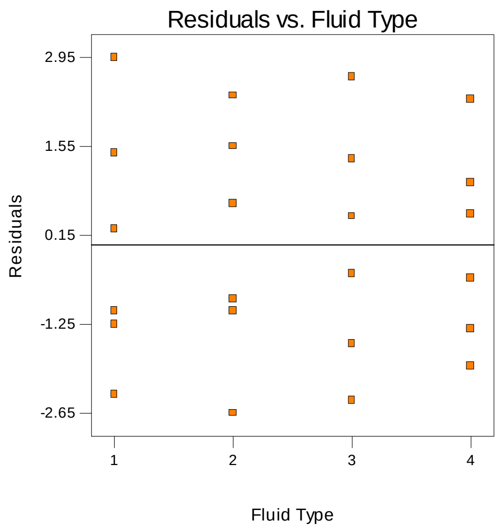
(b) Which fluid would you select, given that the objective is long life?

Treatment 3. The Fisher LSD procedure in the computer output indicates that the fluid 3 is different from the others, and it's average life also exceeds the average lives of the other three fluids.

(c) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?
There is nothing unusual in the residual plots.







3-18 Four different designs for a digital computer circuit are being studied in order to compare the amount of noise present. The following data have been obtained:

Circuit Design			Noise Observed		
1	19	20	19	30	8
2	80	61	73	56	80
3	47	26	25	35	50
4	95	46	83	78	97

(a) Is the amount of noise present the same for all four designs? Use $\alpha = 0.05$.

No, at least one treatment mean is different.

Design Expert Output

Response: Noise
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	12042.00	3	4014.00	21.78	< 0.0001	significant
A	*12042.00*	*3*	*4014.00*	*21.78*	*< 0.0001*	
Residual	2948.80	16	184.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*2948.80*	*16*	*184.30*			
Cor Total	14990.80	19				

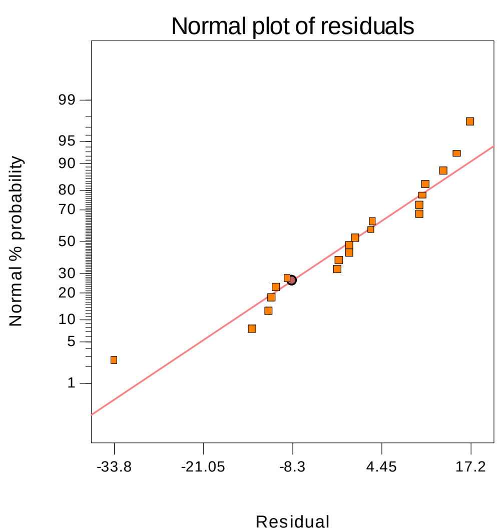
The Model F-value of 21.78 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

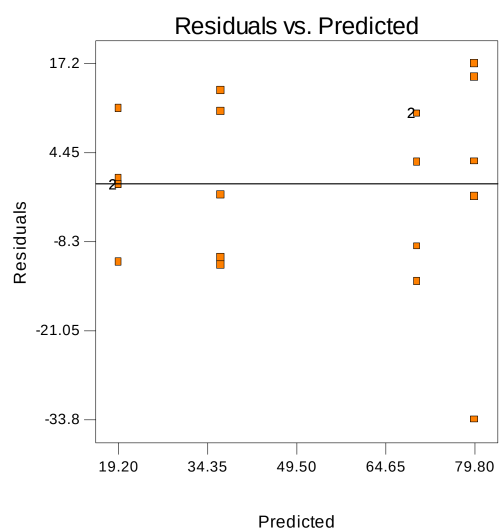
Treatment Means (Adjusted, If Necessary)

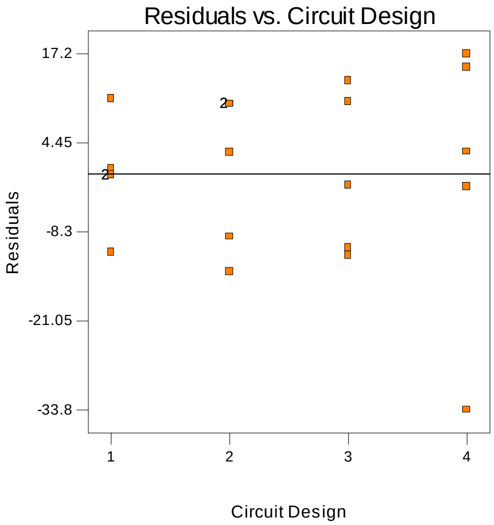
	Estimated Mean	Standard Error
1-1	19.20	6.07
2-2	70.00	6.07
3-3	36.60	6.07
4-4	79.80	6.07

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	-50.80	1	8.59	-5.92	< 0.0001
1 vs 3	-17.40	1	8.59	-2.03	0.0597
1 vs 4	-60.60	1	8.59	-7.06	< 0.0001
2 vs 3	33.40	1	8.59	3.89	0.0013
2 vs 4	-9.80	1	8.59	-1.14	0.2705
3 vs 4	-43.20	1	8.59	-5.03	0.0001

(b) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied? There is nothing too unusual about the residual plots, although there is a mild outlier present.







(c) Which circuit design would you select for use? Low noise is best.

From the Design Expert Output, the Fisher LSD procedure comparing the difference in means identifies Type 1 as having lower noise than Types 2 and 4. Although the LSD procedure comparing Types 1 and 3 has a *P*-value greater than 0.05, it is less than 0.10. Unless there are other reasons for choosing Type 3, Type 1 would be selected.

3-19 Four chemists are asked to determine the percentage of methyl alcohol in a certain chemical compound. Each chemist makes three determinations, and the results are the following:

Chemist	Percentage of Methyl Alcohol		
1	84.99	84.04	84.38
2	85.15	85.13	84.88
3	84.72	84.48	85.16
4	84.20	84.10	84.55

(a) Do chemists differ significantly? Use $\alpha = 0.05$.

There is no significant difference at the 5% level, but chemists differ significantly at the 10% level.

Design Expert Output

Response: Methyl Alcohol in %
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.04	3	0.35	3.25	0.0813	not significant
A	*1.04*	*3*	*0.35*	*3.25*	*0.0813*	
Residual	0.86	8	0.11			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.86*	*8*	*0.11*			
Cor Total	1.90	11				

The Model F-value of 3.25 implies there is a 8.13% chance that a "Model F-Value" this large could occur due to noise.

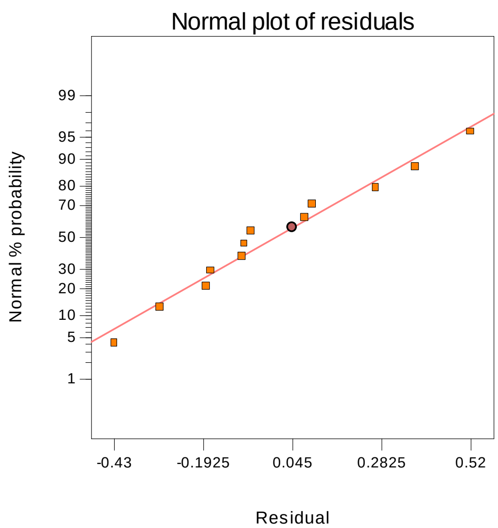
Treatment Means (Adjusted, If Necessary)
Estimated Standard

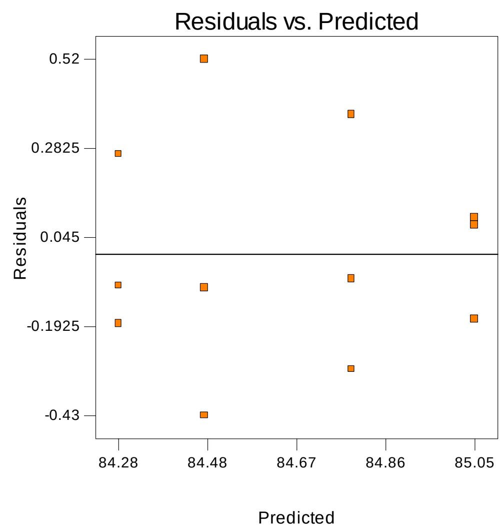
	Mean	**Error**
1-1	84.47	0.19
2-2	85.05	0.19
3-3	84.79	0.19
4-4	84.28	0.19

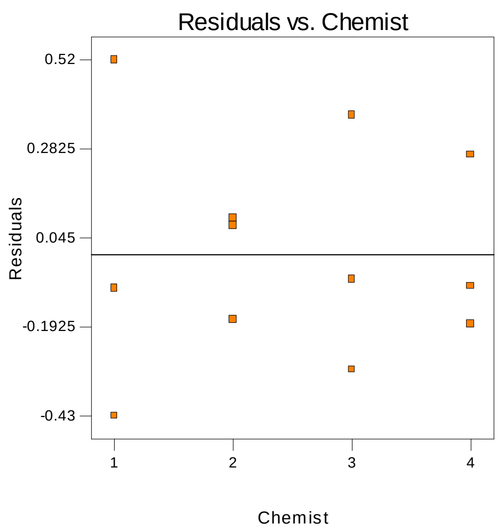
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-0.58	1	0.27	-2.18	0.0607
1 vs 3	-0.32	1	0.27	-1.18	0.2703
1 vs 4	0.19	1	0.27	0.70	0.5049
2 vs 3	0.27	1	0.27	1.00	0.3479
2 vs 4	0.77	1	0.27	2.88	0.0205
3 vs 4	0.50	1	0.27	1.88	0.0966

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.







(c) If chemist 2 is a new employee, construct a meaningful set of orthogonal contrasts that might have been useful at the start of the experiment.

Chemists	Total	C1	C2	C3
1	253.41	1	-2	0
2	255.16	-3	0	0
3	254.36	1	1	-1
4	252.85	1	1	1
	Contrast Totals:	-4.86	0.39	-1.51

$$SS_{C1} = \frac{(-4.86)^2}{3(12)} = 0.656 \quad F_{C1} = \frac{0.656}{0.10727} = 6.115*$$

$$SS_{C2} = \frac{(0.39)^2}{3(6)} = 0.008 \quad F_{C2} = \frac{0.008}{0.10727} = 0.075$$

$$SS_{C3} = \frac{(-1.51)^2}{3(2)} = 0.380 \quad F_{C3} = \frac{0.380}{0.10727} = 3.54$$

Only contrast 1 is significant at 5%.

3-20 Three brands of batteries are under study. It is s suspected that the lives (in weeks) of the three brands are different. Five batteries of each brand are tested with the following results:

Weeks of Life

Brand 1	Brand 2	Brand 3
100	76	108
96	80	100
92	75	96
96	84	98
92	82	100

(a) Are the lives of these brands of batteries different?

Yes, at least one of the brands is different.

Design Expert Output

Response: Life in Weeks

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1196.13	2	598.07	38.34	< 0.0001	significant
A	*1196.13*	*2*	*598.07*	*38.34*	*< 0.0001*	
Residual	187.20	12	15.60			
Lack of Fit	*0.000*	*0*				
Pure Error	*187.20*	*12*	*15.60*			
Cor Total	1383.33	14				

The Model F-value of 38.34 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

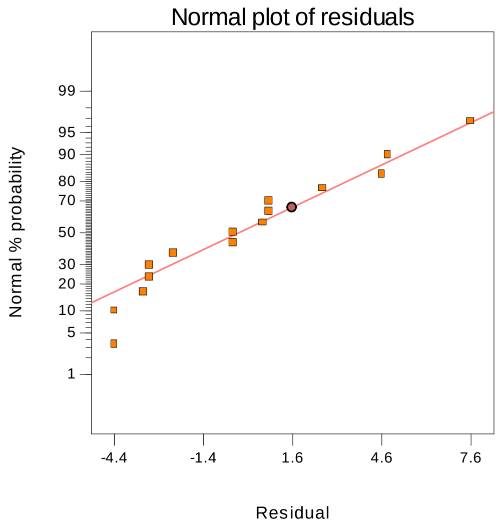
Treatment Means (Adjusted, If Necessary)

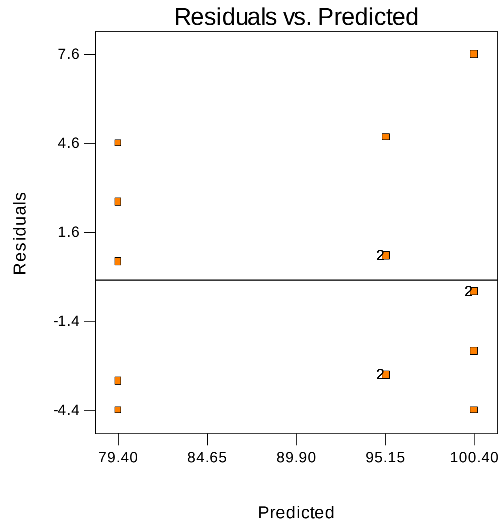
	Estimated Mean	**Standard Error**
1-1	95.20	1.77
2-2	79.40	1.77
3-3	100.40	1.77

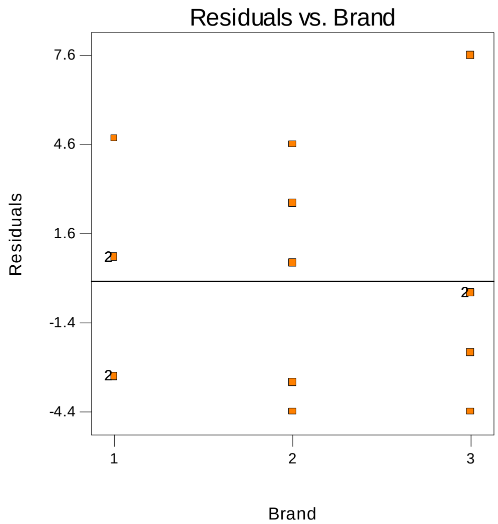
Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	15.80	1	2.50	6.33	< 0.0001
1 vs 3	-5.20	1	2.50	-2.08	0.0594
2 vs 3	-21.00	1	2.50	-8.41	< 0.0001

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residuals.







(c) Construct a 95 percent interval estimate on the mean life of battery brand 2. Construct a 99 percent interval estimate on the mean difference between the lives of battery brands 2 and 3.

$$\bar{y}_{i.} \pm t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Brand 2:}\quad 79.4 \pm 2.179\sqrt{\frac{15.60}{5}}$$
$$79.40 \pm 3.849$$
$$75.551 \le \mu_2 \le 83.249$$

$$\text{Brand 2 - Brand 3:}\quad \bar{y}_{i.} - \bar{y}_{j.} \pm t_{\alpha/2,N-a}\sqrt{\frac{2MS_E}{n}}$$
$$79.4 - 100.4 \pm 3.055\sqrt{\frac{2(15.60)}{5}}$$
$$-28.631 \le \mu_2 - \mu_3 \le -13.369$$

(d) Which brand would you select for use? If the manufacturer will replace without charge any battery that fails in less than 85 weeks, what percentage would the company expect to replace?

Chose brand 3 for longest life. Mean life of this brand in 100.4 weeks, and the variance of life is estimated by 15.60 (*MSE*). Assuming normality, then the probability of failure before 85 weeks is:

$$\Phi\left(\frac{85-100.4}{\sqrt{15.60}}\right) = \Phi(-3.90) = 0.00005$$

That is, about 5 out of 100,000 batteries will fail before 85 week.

3-21 Four catalysts that may affect the concentration of one component in a three component liquid mixture are being investigated. The following concentrations are obtained:

Catalyst 1	2	3	4
58.2	56.3	50.1	52.9
57.2	54.5	54.2	49.9
58.4	57.0	55.4	50.0
55.8	55.3		51.7
54.9			

(a) Do the four catalysts have the same effect on concentration?

No, their means are different.

Design Expert Output

Response: Concentration
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	85.68	3	28.56	9.92	0.0014	significant
A	*85.68*	*3*	*28.56*	*9.92*	*0.0014*	
Residual	34.56	12	2.88			
Lack of Fit	*0.000*	*0*				
Pure Error	*34.56*	*12*	*2.88*			
Cor Total	120.24	15				

The Model F-value of 9.92 implies the model is significant. There is only
a 0.14% chance that a "Model F-Value" this large could occur due to noise.

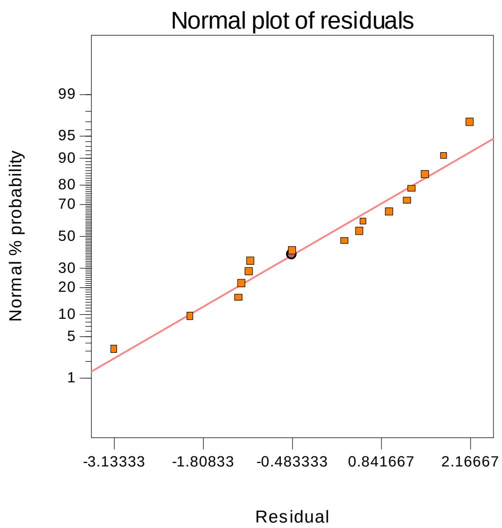
Treatment Means (Adjusted, If Necessary)
Estimated Standard

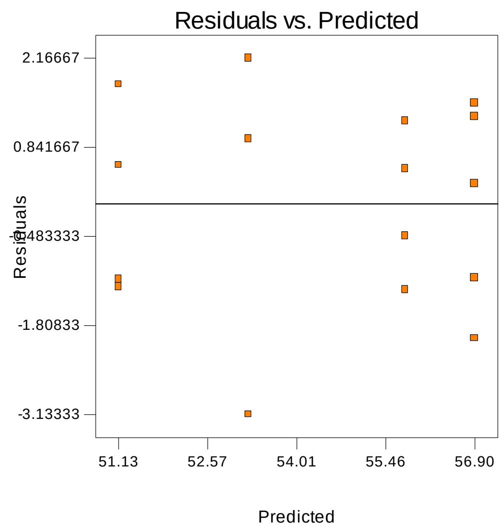
	Mean	Error
1-1	56.90	0.76
2-2	55.77	0.85
3-3	53.23	0.98
4-4	51.13	0.85

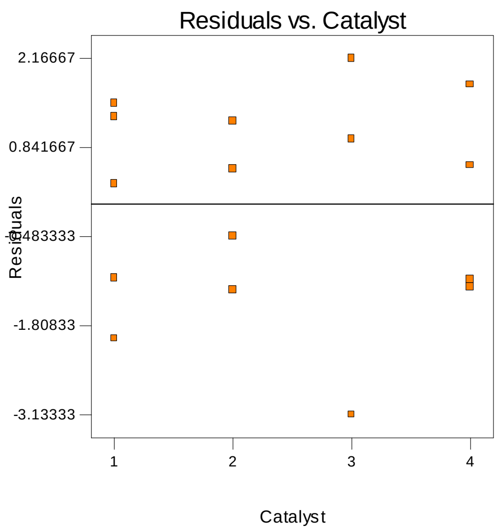
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	1.13	1	1.14	0.99	0.3426
1 vs 3	3.67	1	1.24	2.96	0.0120
1 vs 4	5.77	1	1.14	5.07	0.0003
2 vs 3	2.54	1	1.30	1.96	0.0735
2 vs 4	4.65	1	1.20	3.87	0.0022
3 vs 4	2.11	1	1.30	1.63	0.1298

(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.







(c) Construct a 99 percent confidence interval estimate of the mean response for catalyst 1.

$$\bar{y}_{i.} \pm t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}$$

$$\text{Catalyst 1:}\quad 56.9 \pm 3.055\sqrt{\frac{2.88}{5}}$$

$$56.9 \pm 2.3186$$

$$54.5814 \le \mu_1 \le 59.2186$$

3-22 An experiment was performed to investigate the effectiveness of five insulating materials. Four samples of each material were tested at an elevated voltage level to accelerate the time to failure. The failure times (in minutes) is shown below.

Material	Failure Time (minutes)			
1	110	157	194	178
2	1	2	4	18
3	880	1256	5276	4355
4	495	7040	5307	10050
5	7	5	29	2

(a) Do all five materials have the same effect on mean failure time?

No, at least one material is different.

Design Expert Output

Response: Failure Timein Minutes

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.032E+008	4	2.580E+007	6.19	0.0038	significant
A	*1.032E+008*	*4*	*2.580E+007*	*6.19*	*0.0038*	
Residual	6.251E+007	15	4.167E+006			
Lack of Fit	*0.000*	*0*				
Pure Error	*6.251E+007*	*15*	*4.167E+006*			
Cor Total	1.657E+008	19				

The Model F-value of 6.19 implies the model is significant. There is only a 0.38% chance that a "Model F-Value" this large could occur due to noise.

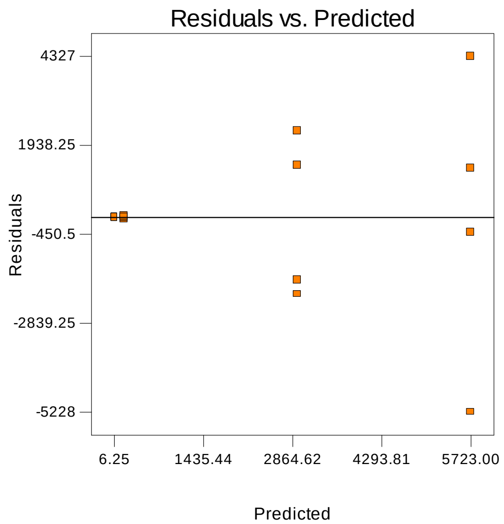
Treatment Means (Adjusted, If Necessary)

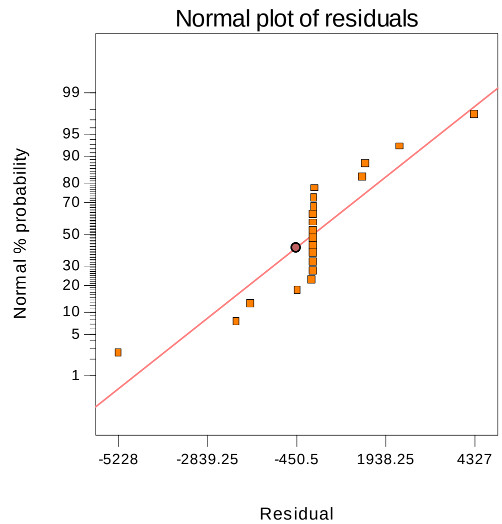
	Estimated Mean	Standard Error
1-1	159.75	1020.67
2-2	6.25	1020.67
3-3	2941.75	1020.67
4-4	5723.00	1020.67
5-5	10.75	1020.67

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	153.50	1	1443.44	0.11	0.9167
1 vs 3	-2782.00	1	1443.44	-1.93	0.0731
1 vs 4	-5563.25	1	1443.44	-3.85	0.0016
1 vs 5	149.00	1	1443.44	0.10	0.9192
2 vs 3	-2935.50	1	1443.44	-2.03	0.0601
2 vs 4	-5716.75	1	1443.44	-3.96	0.0013
2 vs 5	-4.50	1	1443.44	-3.118E-003	0.9976
3 vs 4	-2781.25	1	1443.44	-1.93	0.0732
3 vs 5	2931.00	1	1443.44	2.03	0.0604

4 vs 5	5712.25	1	1443.44	3.96	0.0013

(b) Plot the residuals versus the predicted response. Construct a normal probability plot of the residuals. What information do these plots convey?





The plot of residuals versus predicted has a strong outward-opening funnel shape, which indicates the variance of the original observations is not constant. The normal probability plot also indicates that the normality assumption is not valid. A data transformation is recommended.

(c) Based on your answer to part (b) conduct another analysis of the failure time data and draw appropriate conclusions.

A natural log transformation was applied to the failure time data. The analysis in the log scale identifies that there exists at least one difference in treatment means.

Design Expert Output

Response: Failure Timein Minutes Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	165.06	4	41.26	37.66	< 0.0001	significant
A	*165.06*	*4*	*41.26*	*37.66*	*< 0.0001*	
Residual	16.44	15	1.10			
Lack of Fit	*0.000*	*0*				
Pure Error	*16.44*	*15*	*1.10*			
Cor Total	181.49	19				

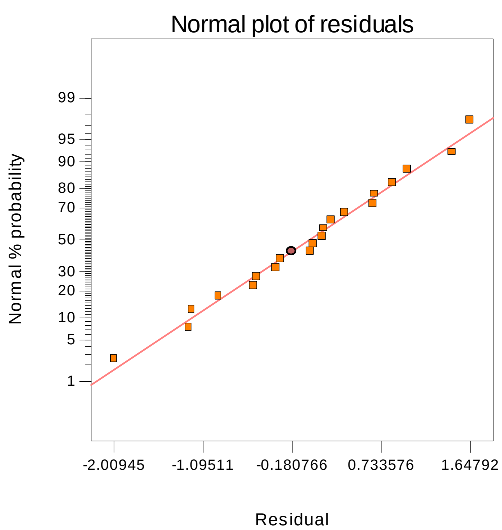
The Model F-value of 37.66 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

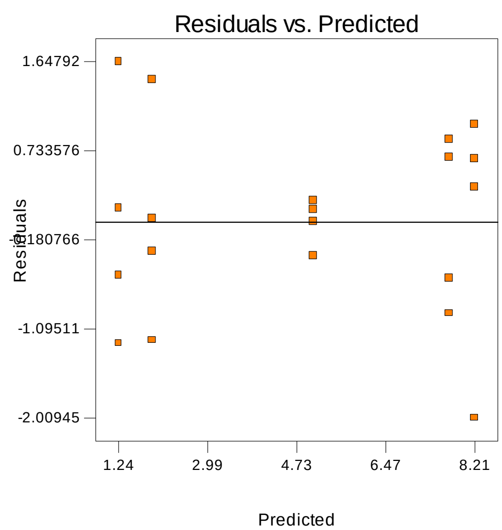
Treatment Means (Adjusted, If Necessary)

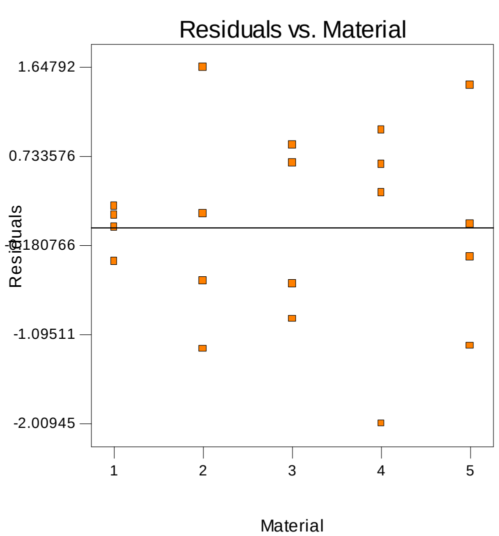
	Estimated Mean	**Standard Error**
1-1	5.05	0.52
2-2	1.24	0.52
3-3	7.72	0.52
4-4	8.21	0.52
5-5	1.90	0.52

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	3.81	1	0.74	5.15	0.0001
1 vs 3	-2.66	1	0.74	-3.60	0.0026
1 vs 4	-3.16	1	0.74	-4.27	0.0007
1 vs 5	3.15	1	0.74	4.25	0.0007
2 vs 3	-6.47	1	0.74	-8.75	< 0.0001
2 vs 4	-6.97	1	0.74	-9.42	< 0.0001
2 vs 5	-0.66	1	0.74	-0.89	0.3856
3 vs 4	-0.50	1	0.74	-0.67	0.5116
3 vs 5	5.81	1	0.74	7.85	< 0.0001
4 vs 5	6.31	1	0.74	8.52	< 0.0001

There is nothing unusual about the residual plots when the natural log transformation is applied.







3-23 A semiconductor manufacturer has developed three different methods for reducing particle counts on wafers. All three methods are tested on five wafers and the after-treatment particle counts obtained. The data are shown below.

Method	Count				
1	31	10	21	4	1
2	62	40	24	30	35
3	58	27	120	97	68

(a) Do all methods have the same effect on mean particle count?

No, at least one method has a different effect on mean particle count.

Design Expert Output

Response: Count

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	8963.73	2	4481.87	7.91	0.0064	significant
A	*8963.73*	*2*	*4481.87*	*7.91*	*0.0064*	
Residual	6796.00	12	566.33			
Lack of Fit	*0.000*	*0*				
Pure Error	*6796.00*	*12*	*566.33*			
Cor Total	15759.73	14				

The Model F-value of 7.91 implies the model is significant. There is only a 0.64% chance that a "Model F-Value" this large could occur due to noise.

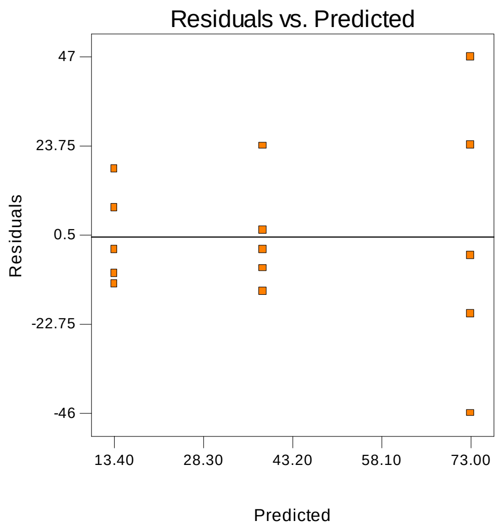
Treatment Means (Adjusted, If Necessary)

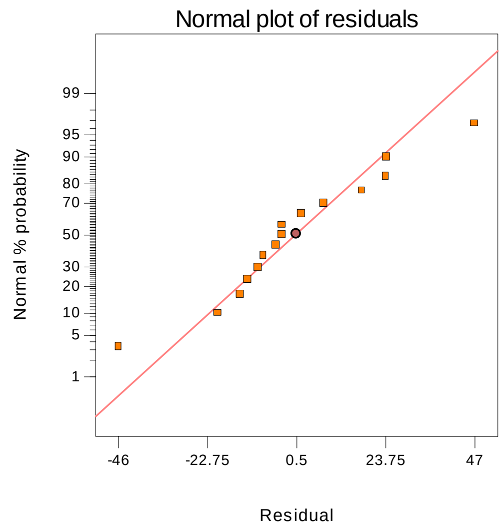
	Estimated Mean	**Standard Error**
1-1	13.40	10.64
2-2	38.20	10.64
3-3	73.00	10.64

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-24.80	1	15.05	-1.65	0.1253
1 vs 3	-59.60	1	15.05	-3.96	0.0019
2 vs 3	-34.80	1	15.05	-2.31	0.0393

(b) Plot the residuals versus the predicted response. Construct a normal probability plot of the residuals. Are there potential concerns about the validity of the assumptions?

The plot of residuals versus predicted appears to be funnel shaped. This indicates the variance of the original observations is not constant. The residuals plotted in the normal probability plot do not fall along a straight line, which suggests that the normality assumption is not valid. A data transformation is recommended.





(c) Based on your answer to part (b) conduct another analysis of the particle count data and draw appropriate conclusions.

For count data, a square root transformation is often very effective in resolving problems with inequality of variance. The analysis of variance for the transformed response is shown below. The difference between methods is much more apparent after applying the square root transformation.

Design Expert Output

Response: Count Transform: Square root Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	63.90	2	31.95	9.84	0.0030	significant
A	*63.90*	*2*	*31.95*	*9.84*	*0.0030*	
Residual	38.96	12	3.25			
Lack of Fit	*0.000*	*0*				
Pure Error	*38.96*	*12*	*3.25*			
Cor Total	102.86	14				

The Model F-value of 9.84 implies the model is significant. There is only a 0.30% chance that a "Model F-Value" this large could occur due to noise.

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	3.26	0.81
2-2	6.10	0.81
3-3	8.31	0.81

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H_0 Coeff=0**	**Prob > |t|**
1 vs 2	-2.84	1	1.14	-2.49	0.0285
1 vs 3	-5.04	1	1.14	-4.42	0.0008
2 vs 3	-2.21	1	1.14	-1.94	0.0767

3-24 Consider testing the equality of the means of two normal populations, where the variances are unknown but are assumed to be equal. The appropriate test procedure is the pooled *t* test. Show that the pooled *t* test is equivalent to the single factor analysis of variance.

$$t_0 = \frac{\bar{y}_{1.} - \bar{y}_{2.}}{S_p\sqrt{\frac{2}{n}}} \sim t_{2n-2} \quad \text{assuming } n_1 = n_2 = n$$

$$S_p = \frac{\sum_{j=1}^{n}\left(y_{1j} - \bar{y}_{1.}\right)^2 + \sum_{j=1}^{n}\left(y_{2j} - \bar{y}_{2.}\right)^2}{2n-2} = \frac{\sum_{i=1}^{2}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{1.}\right)^2}{2n-2} \equiv MS_E \quad \text{for a=2}$$

Furthermore, $\left(\bar{y}_{1.} - \bar{y}_{2.}\right)^2\left(\frac{n}{2}\right) = \sum_{i=1}^{2}\frac{y_{i.}^2}{n} - \frac{y_{..}^2}{2n}$, which is exactly the same as $SS_{Treatments}$ in a one-way classification with a=2. Thus we have shown that $t_0^2 = \frac{SS_{Treatments}}{MS_E}$. In general, we know that $t_u^2 = F_{1,u}$ so that $t_0^2 \sim F_{1,2n-2}$. Thus the square of the test statistic from the pooled *t*-test is the same test statistic that results from a single-factor analysis of variance with a=2.

3-25 Show that the variance of the linear combination $\sum_{i=1}^{a} c_i y_{i.}$ is $\sigma^2\sum_{i=1}^{a} n_i c_i^2$.

$$V\left[\sum_{i=1}^{a} c_i y_{i.}\right] = \sum_{i=1}^{a} V\left(c_i y_{i.}\right) = \sum_{i=1}^{a} c_i^2 V\left[\sum_{j=1}^{n_i} y_{ij}\right] = \sum_{i=1}^{a} c_i^2 \sum_{j=1}^{n_i} V\left(y_{ij.}\right),\ V\left(y_{ij}\right) = \sigma^2$$

$$= \sum_{i=1}^{a} c_i^2 n_i \sigma^2$$

3-26 In a fixed effects experiment, suppose that there are *n* observations for each of four treatments. Let Q_1^2, Q_2^2, Q_3^2 be single-degree-of-freedom components for the orthogonal contrasts. Prove that $SS_{Treatments} = Q_1^2 + Q_2^2 + Q_3^2$.

$$\begin{aligned} C_1 &= 3y_{1.} - y_{2.} - y_{3.} - y_{4.} & SS_{C1} &= Q_1^2 \\ C_2 &= 2y_{2.} - y_{3.} - y_{4.} & SS_{C2} &= Q_2^2 \\ C_3 &= y_{3.} - y_{4.} & SS_{C3} &= Q_3^2 \end{aligned}$$

$$Q_1^2 = \frac{(3y_{1.} - y_{2.} - y_{3.} - y_{4.})^2}{12n}$$

$$Q_2^2 = \frac{(2y_{2.} - y_{3.} - y_{4.})^2}{6n}$$

$$Q_3^2 = \frac{(y_{3.} - y_{4.})^2}{2n}$$

$$Q_1^2 + Q_2^2 + Q_3^2 = \frac{9\sum_{i=1}^{4} y_{i.}^2 - 6\left(\sum\sum_{i<j} y_{i.} y_{j.}\right)}{12n} \quad \text{and since}$$

$$\sum_{i<j}\sum y_{i.}y_{j.} = \frac{1}{2}\left(y_{..}^2 - \sum_{i=1}^{4} y_{i.}^2\right), \text{ we have } Q_1^2 + Q_2^2 + Q_3^2 = \frac{12\sum_{i=1}^{4} y_{i.}^2 - 3y_{..}^2}{12n} = \sum_{i=1}^{4}\frac{y_{i.}^2}{n} - \frac{y_{..}^2}{4n} = SS_{Treatments}$$

for a=4.

3-27 Use Bartlett's test to determine if the assumption of equal variances is satisfied in Problem 3-14. Use α = 0.05. Did you reach the same conclusion regarding the equality of variance by examining the residual plots?

$$\chi_0^2 = 2.3026\frac{q}{c}, \text{ where}$$

$$q = (N-a)log_{10} S_p^2 - \sum_{i=1}^{a}(n_i - 1)log_{10} S_i^2$$

$$c = 1 + \frac{1}{3(a-1)}\left(\sum_{i=1}^{a}(n_i-1)^{-1} - (N-a)^{-1}\right)$$

$$S_p^2 = \frac{\sum_{i=1}^{a}(n_i-1)S_i^2}{N-a}$$

$$S_1^2 = 11.2 \qquad S_p^2 = \frac{(5-1)11.2 + (5-1)14.8 + (5-1)20.8}{15-3}$$

$$S_2^2 = 14.8$$

$$S_3^2 = 20.8 \qquad S_p^2 = \frac{(5-1)11.2 + (5-1)14.8 + (5-1)20.8}{15-3} = 15.6$$

$$c = 1 + \frac{1}{3(3-1)}\left(\sum_{i=1}^{a}(5-1)^{-1} - (15-3)^{-1}\right)$$

$$c = 1 + \frac{1}{3(3-1)}\left(\frac{3}{4} + \frac{1}{12}\right) = 1.1389$$

$$q = (N-a)log_{10} S_p^2 - \sum_{i=1}^{a}(n_i-1)log_{10} S_i^2$$

$$q = (15-3)log_{10} 15.6 - 4(log_{10} 11.2 + log_{10} 14.8 + log_{10} 20.8)$$

$$q = 14.3175 - 14.150 = 0.1675$$

$$\chi_0^2 = 2.3026\frac{q}{c} = 2.3026\frac{0.1675}{1.1389} = 0.3386 \qquad \chi_{0.05,4}^2 = 9.49$$

Cannot reject null hypothesis; conclude that the variance are equal. This agrees with the residual plots in Problem 3-16.

3-28 Use the modified Levene test to determine if the assumption of equal variances is satisfied on Problem 3-20. Use α = 0.05. Did you reach the same conclusion regarding the equality of variances by examining the residual plots?

The absolute value of Battery Life – brand median is:

$$\left|y_{ij} - \tilde{y}_i\right|$$

Brand 1	Brand 2	Brand 3
4	4	8
0	0	0
4	5	4
0	4	2
4	2	0

The analysis of variance indicates that there is not a difference between the different brands and therefore the assumption of equal variances is satisfired.

Design Expert Output

Response: Mod Levine
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	0.93	2	0.47	0.070	0.9328
A	*0.93*	*2*	*0.47*	*0.070*	*0.9328*
Pure Error	80.00	12	6.67		
Cor Total	80.93	14			

3-29 Refer to Problem 3-16. If we wish to detect a maximum difference in mean response times of 10 milliseconds with a probability of at least 0.90, what sample size should be used? How would you obtain a preliminary estimate of σ^2?

$$\Phi^2 = \frac{nD^2}{2a\sigma^2}\text{, use } MS_E \text{ from Problem 3-10 to estimate } \sigma^2.$$

$$\Phi^2 = \frac{n(10)^2}{2(3)(16.9)} = 0.986n$$

Letting $\alpha = 0.05$, P(accept) = 0.1 , $\upsilon_1 = a - 1 = 2$

Trial and Error yields:

n	υ_2	Φ	P(accept)
5	12	2.22	0.17
6	15	2.43	0.09
7	18	2.62	0.04

Choose n ≥ 6, therefore N ≥ 18

Notice that we have used an estimate of the variance obtained from the present experiment. This indicates that we probably didn't use a large enough sample (*n* was 5 in problem 3-10) to satisfy the criteria specified in this problem. However, the sample size *was* adequate to detect differences in one of the circuit types.

When we have no prior estimate of variability, sometimes we will generate sample sizes for a range of possible variances to see what effect this has on the size of the experiment. Often a knowledgeable expert will be able to bound the variability in the response, by statements such as "the standard deviation is going to be *at least*…" or "the standard deviation shouldn't be larger than…".

3-30 Refer to Problem 3-20.

(a) If we wish to detect a maximum difference in mean battery life of 0.5 percent with a probability of at least 0.90, what sample size should be used? Discuss how you would obtain a preliminary estimate of σ^2 for answering this question.

Use the MS_E from Problem 3-14.

$$\Phi^2 = \frac{nD^2}{2a\sigma^2} \qquad \Phi^2 = \frac{n(0.005\times 91.6667)^2}{2(3)(15.60)} = 0.002244n$$

Letting $\alpha = 0.05$, P(accept) = 0.1 , $\upsilon_1 = a-1 = 2$

Trial and Error yields:

n	υ_2	Φ	*P(accept)*
40	117	1.895	0.18
45	132	2.132	0.10
50	147	2.369	0.05

Choose n ≥ 45, therefore N ≥ 135

See the discussion from the previous problem about the estimate of variance.

(b) If the difference between brands is great enough so that the standard deviation of an observation is increased by 25 percent, what sample size should be used if we wish to detect this with a probability of at least 0.90?

$$\upsilon_1 = a-1 = 2 \qquad \upsilon_2 = N-a = 15-3 = 12 \qquad \alpha = 0.05 \qquad P(\,accept\,) \le 0.1$$

$$\lambda = \sqrt{1+n\left[(1+0.01P)^2-1\right]} = \sqrt{1+n\left[(1+0.01(25))^2-1\right]} = \sqrt{1+0.5625n}$$

Trial and Error yields:

n	υ_2	λ	*P(accept)*
40	117	4.84	0.13
45	132	5.13	0.11
50	147	5.40	0.10

Choose n ≥ 50, therefore N ≥ 150

3-31 Consider the experiment in Problem 3-20. If we wish to construct a 95 percent confidence interval on the difference in two mean battery lives that has an accuracy of ±2 weeks, how many batteries of each brand must be tested?

$$\alpha = 0.05 \qquad MS_E = 15.6$$

$$width = t_{0.025,N-a}\sqrt{\frac{2MS_E}{n}}$$

Trial and Error yields:

n	υ_2	t	width

5	12	2.179	5.44
10	27	2.05	3.62
31	90	1.99	1.996
32	93	1.99	1.96

Choose n ≥ 31, therefore N ≥ 93

3-32 Suppose that four normal populations have means of μ_1=50, μ_2=60, μ_3=50, and μ_4=60. How many observations should be taken from each population so that the probability or rejecting the null hypothesis of equal population means is at least 0.90? Assume that α=0.05 and that a reasonable estimate of the error variance is σ^2=25.

$$\mu_i = \mu + \tau_i, i = 1,2,3,4$$

$$\mu = \frac{\sum_{i=1}^{4}\mu_i}{4} = \frac{220}{4} = 55$$

$$\tau_1 = -5, \tau_2 = 5, \tau_3 = -5, \tau_4 = 5$$

$$\sum_{i=1}^{4}\tau_i^2 = 100$$

$$\Phi^2 = \frac{n\sum\tau_i^2}{a\sigma^2} = \frac{100n}{4(25)} = n$$

$$\Phi = \sqrt{n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	1-β
4	2.00	12	0.18	0.82
5	2.24	16	0.08	0.92

Therefore, select n=5

3-33 Refer to Problem 3-32.

(a) How would your answer change if a reasonable estimate of the experimental error variance were σ^2 = 36?

$$\Phi^2 = \frac{n\sum\tau_i^2}{a\sigma^2} = \frac{100n}{4(36)} = 0.6944n$$

$$\Phi = \sqrt{0.6944n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	1-β
5	1.863	16	0.24	0.76
6	2.041	20	0.15	0.85
7	2.205	24	0.09	0.91

Therefore, select n=7

(b) How would your answer change if a reasonable estimate of the experimental error variance were $\sigma^2 = 49$?

$$\Phi^2 = \frac{n\sum \tau_i^2}{a\sigma^2} = \frac{100n}{4(49)} = 0.5102n$$
$$\Phi = \sqrt{0.5102n}$$

$\upsilon_1 = 3, \upsilon_2 = 4(n-1), \alpha = 0.05$, From the O.C. curves we can construct the following:

n	Φ	υ_2	β	1-β
7	1.890	24	0.16	0.84
8	2.020	28	0.11	0.89
9	2.142	32	0.09	0.91

Therefore, select n=9

(c) Can you draw any conclusions about the sensitivity of your answer in the particular situation about how your estimate of σ affects the decision about sample size?

As our estimate of variability increases the sample size must increase to ensure the same power of the test.

(d) Can you make any recommendations about how we should use this general approach to choosing *n* in practice?

When we have no prior estimate of variability, sometimes we will generate sample sizes for a range of possible variances to see what effect this has on the size of the experiment. Often a knowledgeable expert will be able to bound the variability in the response, by statements such as “the standard deviation is going to be *at least*…” or “the standard deviation shouldn’t be larger than…”.

3-34 Refer to the aluminum smelting experiment described in Section 3-8. Verify that ratio control methods do not affect average cell voltage. Construct a normal probability plot of residuals. Plot the residuals versus the predicted values. Is there an indication that any underlying assumptions are violated?

Design Expert Output

Response: Cell Average
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2.746E-003	3	9.153E-004	0.20	0.8922	not significant
A	*2.746E-003*	*3*	*9.153E-004*	*0.20*	*0.8922*	
Residual	0.090	20	4.481E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.090*	*20*	*4.481E-003*			
Cor Total	0.092	23				

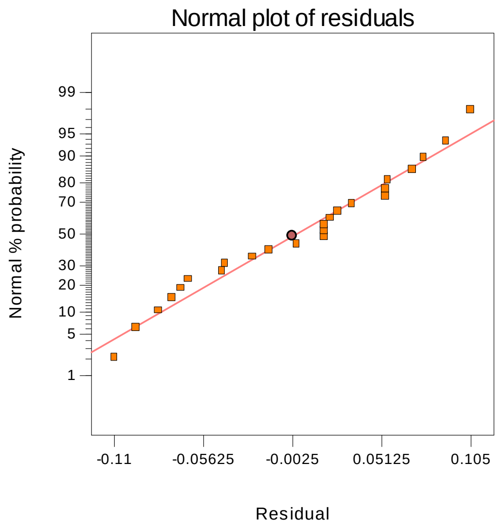
The "Model F-value" of 0.20 implies the model is not significant relative to the noise. There is a 89.22 % chance that a "Model F-value" this large could occur due to noise.

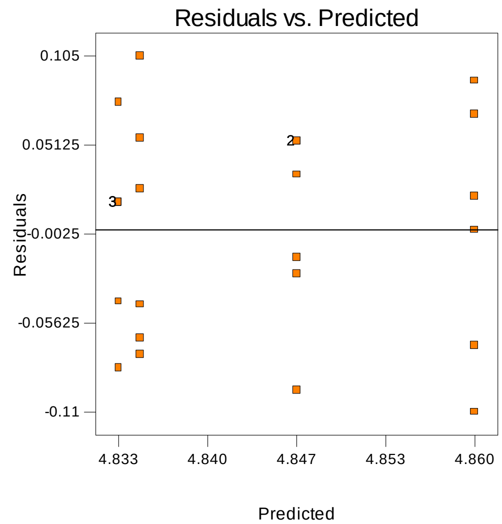
Treatment Means (Adjusted, If Necessary)

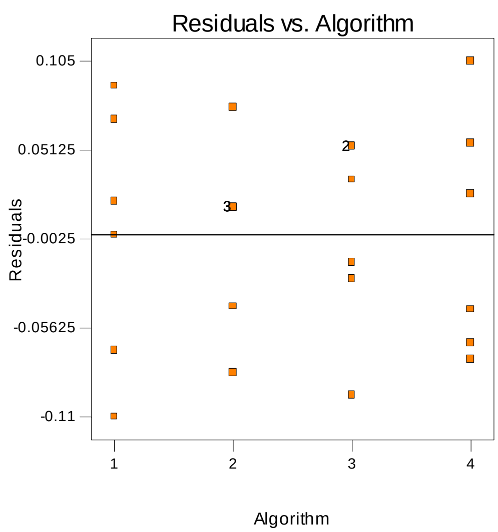
	Estimated Mean	**Standard Error**
1-1	4.86	0.027
2-2	4.83	0.027
3-3	4.85	0.027
4-4	4.84	0.027

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.027	1	0.039	0.69	0.4981
1 vs 3	0.013	1	0.039	0.35	0.7337
1 vs 4	0.025	1	0.039	0.65	0.5251
2 vs 3	-0.013	1	0.039	-0.35	0.7337
2 vs 4	-1.667E-003	1	0.039	-0.043	0.9660
3 vs 4	0.012	1	0.039	0.30	0.7659

The following residual plots are satisfactory.







3-35 Refer to the aluminum smelting experiment in Section 3-8. Verify the ANOVA for pot noise summarized in Table 3-13. Examine the usual residual plots and comment on the experimental validity.

Design Expert Output

Response: Cell StDev Transform: Natural log Constant: 0.000
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	6.17	3	2.06	21.96	< 0.0001	significant
A	*6.17*	*3*	*2.06*	*21.96*	*< 0.0001*	
Residual	1.87	20	0.094			
Lack of Fit	*0.000*	*0*				
Pure Error	*1.87*	*20*	*0.094*			
Cor Total	8.04	23				

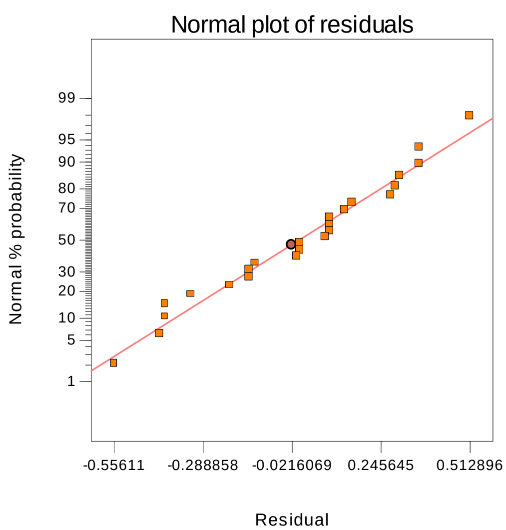
The Model F-value of 21.96 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

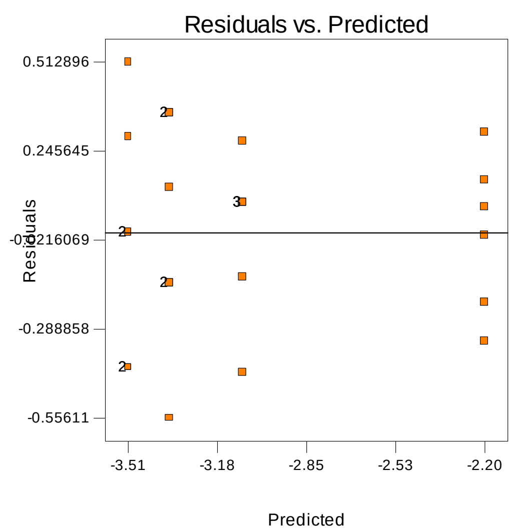
Treatment Means (Adjusted, If Necessary)

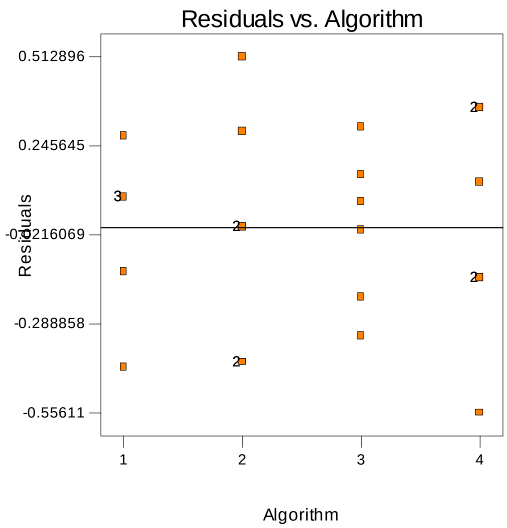
	Estimated Mean	Standard Error
1-1	-3.09	0.12
2-2	-3.51	0.12
3-3	-2.20	0.12
4-4	-3.36	0.12

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	0.42	1	0.18	2.38	0.0272
1 vs 3	-0.89	1	0.18	-5.03	< 0.0001
1 vs 4	0.27	1	0.18	1.52	0.1445
2 vs 3	-1.31	1	0.18	-7.41	< 0.0001
2 vs 4	-0.15	1	0.18	-0.86	0.3975
3 vs 4	1.16	1	0.18	6.55	< 0.0001

The following residual plots identify the residuals to be normally distributed, randomly distributed through the range of prediction, and uniformly distributed across the different algorithms. This validates the assumptions for the experiment.







3-36 Four different feed rates were investigated in an experiment on a CNC machine producing a component part used in an aircraft auxiliary power unit. The manufacturing engineer in charge of the experiment knows that a critical part dimension of interest may be affected by the feed rate. However, prior experience has indicated that only dispersion effects are likely to be present. That is, changing the feed rate does not affect the average dimension, but it could affect dimensional variability. The engineer makes five production runs at each feed rate and obtains the standard deviation of the critical dimension (in 10^{-3} mm). The data are shown below. Assume that all runs were made in random order.

Feed Rate		Production	Run		
(in/min)	1	2	3	4	5
10	0.09	0.10	0.13	0.08	0.07
12	0.06	0.09	0.12	0.07	0.12
14	0.11	0.08	0.08	0.05	0.06
16	0.19	0.13	0.15	0.20	0.11

(a) Does feed rate have any effect on the standard deviation of this critical dimension?

Because the residual plots were not acceptable for the non-transformed data, a square root transformation was applied to the standard deviations of the critical dimension. Based on the computer output below, the feed rate has an effect on the standard deviation of the critical dimension.

Design Expert Output

Response: Run StDev Transform: Square root Constant: 0.000
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.040	3	0.013	7.05	0.0031	significant
A	*0.040*	*3*	*0.013*	*7.05*	*0.0031*	
Residual	0.030	16	1.903E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.030*	*16*	*1.903E-003*			
Cor Total	0.071	19				

The Model F-value of 7.05 implies the model is significant. There is only a 0.31% chance that a "Model F-Value" this large could occur due to noise.

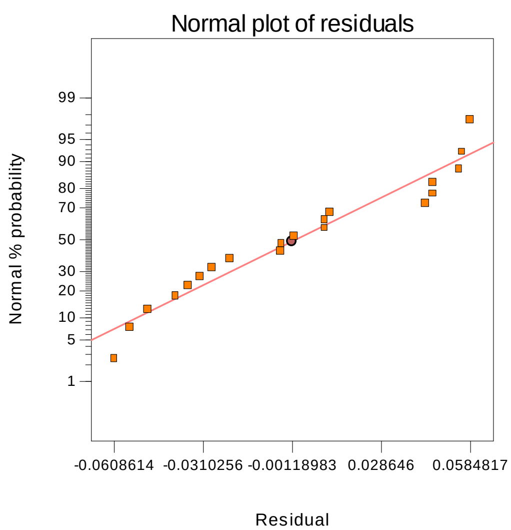
Treatment Means (Adjusted, If Necessary)

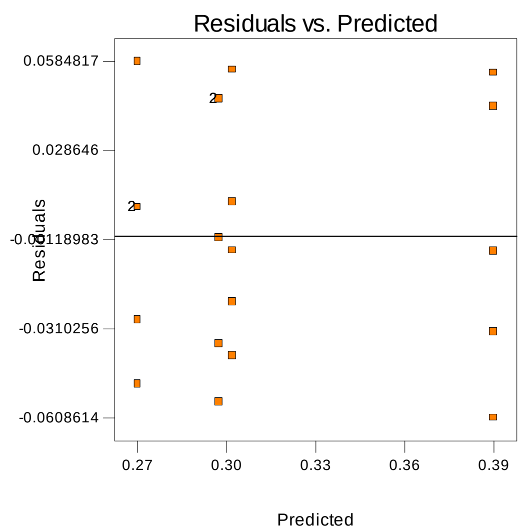
	Estimated Mean	Standard Error
1-10	0.30	0.020
2-12	0.30	0.020
3-14	0.27	0.020
4-16	0.39	0.020

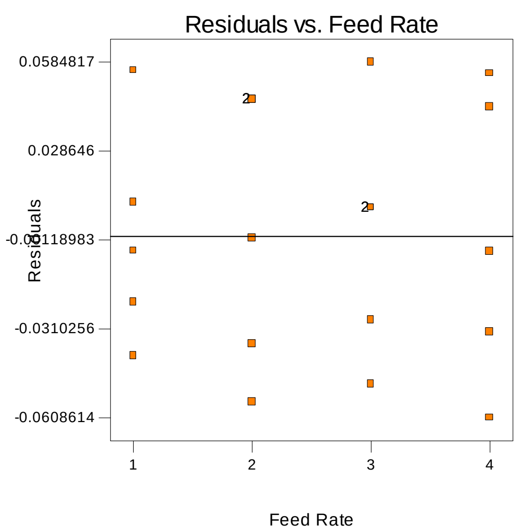
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	4.371E-003	1	0.028	0.16	0.8761
1 vs 3	0.032	1	0.028	1.15	0.2680
1 vs 4	-0.088	1	0.028	-3.18	0.0058
2 vs 3	0.027	1	0.028	0.99	0.3373
2 vs 4	-0.092	1	0.028	-3.34	0.0042
3 vs 4	-0.12	1	0.028	-4.33	0.0005

(b) Use the residuals from this experiment of investigate model adequacy. Are there any problems with experimental validity?

The residual plots are satisfactory.







3-37 Consider the data shown in Problem 3-16.

(a) Write out the least squares normal equations for this problem, and solve them for $\hat{\mu}$ and $\hat{\tau}_i$, using the usual constraint $\left(\sum_{i=1}^{3} \hat{\tau}_i = 0\right)$. Estimate $\tau_1 - \tau_2$.

$$\begin{array}{lllll} 15\hat{\mu} & +5\hat{\tau}_1 & +5\hat{\tau}_2 & +5\hat{\tau}_3 & =207 \\ 5\hat{\mu} & +5\hat{\tau}_1 & & & =54 \\ 5\hat{\mu} & & +5\hat{\tau}_2 & & =111 \\ 15\hat{\mu} & & & +5\hat{\tau}_3 & =42 \end{array}$$

Imposing $\sum_{i=1}^{3} \hat{\tau}_i = 0$, therefore $\hat{\mu} = 13.80$, $\hat{\tau}_1 = -3.00$, $\hat{\tau}_2 = 8.40$, $\hat{\tau}_3 = -5.40$

$$\hat{\tau}_1 - \hat{\tau}_2 = -3.00 - 8.40 = -11.40$$

(b) Solve the equations in (a) using the constraint $\hat{\tau}_3 = 0$. Are the estimators $\hat{\tau}_i$ and $\hat{\mu}$ the same as you found in (a)? Why? Now estimate $\tau_1 - \tau_2$ and compare your answer with that for (a). What statement can you make about estimating contrasts in the τ_i?

Imposing the constraint, $\hat{\tau}_3 = 0$ we get the following solution to the normal equations: $\hat{\mu} = 8.40$, $\hat{\tau}_1 = 2.40$, $\hat{\tau}_2 = 13.8$, and $\hat{\tau}_3 = 0$. These estimators are not the same as in part (a). However, $\hat{\tau}_1 - \hat{\tau}_2 = 2.40 - 13.80 = -11.40$, is the same as in part (a). The contrasts are estimable.

(c) Estimate $\mu + \tau_1$, $2\tau_1 - \tau_2 - \tau_3$ and $\mu + \tau_1 + \tau_2$ using the two solutions to the normal equations. Compare the results obtained in each case.

	Contrast	Estimated from Part (a)	Estimated from Part (b)
1	$\mu + \tau_1$	10.80	10.80
2	$2\tau_1 - \tau_2 - \tau_3$	-9.00	-9.00
3	$\mu + \tau_1 + \tau_2$	19.20	24.60

Contrasts 1 and 2 are estimable, 3 is not estimable.

3-38 Apply the general regression significance test to the experiment in Example 3-1. Show that the procedure yields the same results as the usual analysis of variance.

From Table 3-3:

$y_{..} = 12355$

from Example 3-1, we have:

$\hat{\mu} = 617.75 \quad \hat{\tau}_1 = -66.55 \quad \hat{\tau}_2 = -30.35$
$\hat{\tau}_3 = 7.65 \quad \hat{\tau}_4 = 89.25$

$\sum_{i=1}^{4}\sum_{j=1}^{5} y_{ij}^2 = 7,704,511$, with 20 degrees of freedom.

$$
\begin{aligned}
R(\mu,\tau) &= \hat{\mu}y_{..} + \sum_{i=1}^{5}\hat{\tau}y_{i.} \\
&= (617.75)(12355)+(-66.55)(2756)+(-30.35)(2937)+(7.65)(3127)+(89.25)(3535) \\
&= 7,632,301.25 + 66,870.55 = 7,699,172.80
\end{aligned}
$$

with 4 degrees of freedom.

$$SS_E = \sum_{i=1}^{4}\sum_{j=1}^{5} y_{ij}^2 - R(\mu,\tau) = 7,704,511 - 7,699,172.80 = 5339.2$$

with 20-4 degrees of freedom.

This is identical to the SS_E found in Example 3-1.

The reduced model:

$R(\mu) = \hat{\mu}y_{..} = (617.75)(12355) = 7,632,301.25$, with 1 degree of freedom.

$R(\tau|\mu) = R(\mu,\tau) - R(\mu) = 7,699,172.80 - 7,632,301.25 = 66,870.55$, with 4-1=3 degrees of freedom.

Note: $R(\tau|\mu) = SS_{Treatment}$ from Example 3-1.

Finally,

$$F_0 = \frac{\frac{R(\tau|\mu)}{3}}{\frac{SS_E}{16}} = \frac{\frac{66,870.55}{3}}{\frac{5339.2}{16}} = \frac{22290.8}{333.7} = 66.8$$

which is the same as computed in Example 3-1.

3-39 Use the Kruskal-Wallis test for the experiment in Problem 3-17. Are the results comparable to those found by the usual analysis of variance?

From Design Expert Output of Problem 3-17

Response: Life in in h
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	30.17	3	10.06	3.05	0.0525	not significant
A	*30.16*	*3*	*10.05*	*3.05*	*0.0525*	
Residual	65.99	20	3.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*65.99*	*20*	*3.30*			
Cor Total	96.16	23				

$$H = \frac{12}{N(N+1)}\left[\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i}\right] - 3(N+1) = \frac{12}{24(24+1)}[4040.5] - 3(24+1) = 5.81$$

$$\chi^2_{0.05,3} = 7.81$$

Accept the null hypothesis; the treatments are not different. This agrees with the analysis of variance.

3-40 Use the Kruskal-Wallis test for the experiment in Problem 3-18. Compare conclusions obtained with those from the usual analysis of variance?

From Design Expert Output of Problem 3-12

Response: Noise
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	12042.00	3	4014.00	21.78	< 0.0001	significant
A	*12042.00*	*3*	*4014.00*	*21.78*	*< 0.0001*	
Residual	2948.80	16	184.30			
Lack of Fit	*0.000*	*0*				
Pure Error	*2948.80*	*16*	*184.30*			
Cor Total	14990.80	19				

$$H = \frac{12}{N(N+1)}\left[\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i}\right] - 3(N+1) = \frac{12}{20(20+1)}[2691.6] - 3(20+1) = 13.90$$

$$\chi^2_{0.05,4} = 12.84$$

Reject the null hypothesis because the treatments are different. This agrees with the analysis of variance.

3-41 Consider the experiment in Example 3-1. Suppose that the largest observation on etch rate is incorrectly recorded as 250A/min. What effect does this have on the usual analysis of variance? What effect does it have on the Kruskal-Wallis test?

The incorrect observation reduces the analysis of variance F_0 from 66.8 to 0.50. It does change the value of the Kruskal-Wallis test.

Minitab Output

One-way ANOVA: Etch Rate 2 versus Power

```
Analysis of Variance for Etch Rat
Source     DF        SS        MS        F        P
Power       3     15927      5309     0.50    0.685
Error      16    168739     10546
Total      19    184666
```

Chapter 4
Randomized Blocks, Latin Squares, and Related Designs
Solutions

4-1 A chemist wishes to test the effect of four chemical agents on the strength of a particular type of cloth. Because there might be variability from one bolt to another, the chemist decides to use a randomized block design, with the bolts of cloth considered as blocks. She selects five bolts and applies all four chemicals in random order to each bolt. The resulting tensile strengths follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw appropriate conclusions.

	Bolt				
Chemical	1	2	3	4	5
1	73	68	74	71	67
2	73	67	75	72	70
3	75	68	78	73	68
4	73	71	75	75	69

Design Expert Output

Response: Strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	157.00	4	39.25			
Model	12.95	3	4.32	2.38	0.1211	not significant
A	*12.95*	*3*	*4.32*	*2.38*	*0.1211*	
Residual	21.80	12	1.82			
Cor Total	191.75	19				

The "Model F-value" of 2.38 implies the model is not significant relative to the noise. There is a 12.11 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	1.35	R-Squared	0.3727
Mean	71.75	Adj R-Squared	0.2158
C.V.	1.88	Pred R-Squared	-0.7426
PRESS	60.56	Adeq Precision	10.558

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	70.60	0.60
2-2	71.40	0.60
3-3	72.40	0.60
4-4	72.60	0.60

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-0.80	1	0.85	-0.94	0.3665
1 vs 3	-1.80	1	0.85	-2.11	0.0564
1 vs 4	-2.00	1	0.85	-2.35	0.0370
2 vs 3	-1.00	1	0.85	-1.17	0.2635
2 vs 4	-1.20	1	0.85	-1.41	0.1846
3 vs 4	-0.20	1	0.85	-0.23	0.8185

There is no difference among the chemical types at $\alpha = 0.05$ level.

4-2 Three different washing solutions are being compared to study their effectiveness in retarding bacteria growth in five-gallon milk containers. The analysis is done in a laboratory, and only three trials can be run on any day. Because days could represent a potential source of variability, the experimenter decides to use a randomized block design. Observations are taken for four days, and the data are shown here. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Days			
Solution	1	2	3	4
1	13	22	18	39
2	16	24	17	44
3	5	4	1	22

Design Expert Output

Response: Growth

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	1106.92	3	368.97			
Model	703.50	2	351.75	40.72	0.0003	significant
A	*703.50*	*2*	*351.75*	*40.72*	*0.0003*	
Residual	51.83	6	8.64			
Cor Total	1862.25	11				

The Model F-value of 40.72 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.94	R-Squared	0.9314
Mean	18.75	Adj R-Squared	0.9085
C.V.	15.68	Pred R-Squared	0.7255
PRESS	207.33	Adeq Precision	19.687

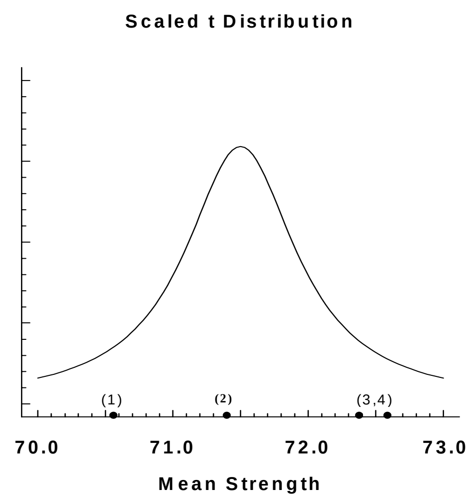
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	23.00	1.47
2-2	25.25	1.47
3-3	8.00	1.47

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-2.25	1	2.08	-1.08	0.3206
1 vs 3	15.00	1	2.08	7.22	0.0004
2 vs 3	17.25	1	2.08	8.30	0.0002

There is a difference between the means of the three solutions. The Fisher LSD procedure indicates that solution 3 is significantly different than the other two.

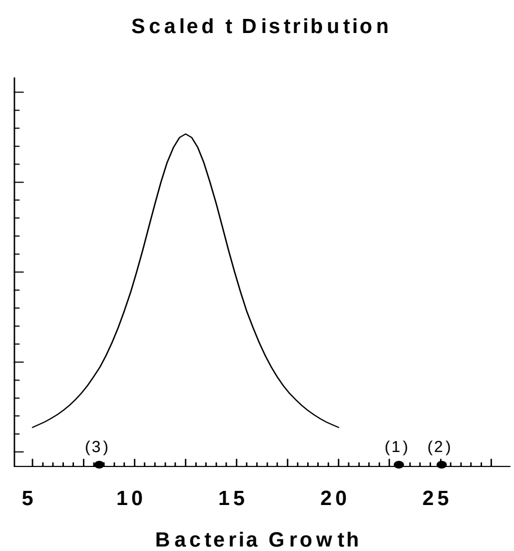
4-3 Plot the mean tensile strengths observed for each chemical type in Problem 4-1 and compare them to a scaled *t* distribution. What conclusions would you draw from the display?



$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{1.82}{5}} = 0.603$$

There is no obvious difference between the means. This is the same conclusion given by the analysis of variance.

4-4 Plot the average bacteria counts for each solution in Problem 4-2 and compare them to an appropriately scaled *t* distribution. What conclusions can you draw?



$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{8.64}{4}} = 1.47$$

There is no difference in mean bacteria growth between solutions 1 and 2. However, solution 3 produces significantly lower mean bacteria growth. This is the same conclusion reached from the Fisher LSD procedure in Problem 4-4.

4-5 Consider the hardness testing experiment described in Section 4-1. Suppose that the experiment was conducted as described and the following Rockwell C-scale data (coded by subtracting 40 units) obtained:

	Coupon			
Tip	1	2	3	4
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

(a) Analyize the data from this experiment.

There is a difference between the means of the four tips.

Design Expert Output

Response: Hardness

ANOVA for Selected Factorial Model

Analysis of variance table [Terms added sequentially (first to last)]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Bock	0.82	3	0.27			
Model	0.38	3	0.13	14.44	0.0009	significant
A	0.38	3	0.13	14.44	0.0009	
Residual	0.080	9	8.889E-003			
Cor Total	1.29	15				

The Model F-value of 14.44 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.094	R-Squared	0.8280
Mean	9.63	Adj R-Squared	0.7706
C.V.	0.98	Pred R-Squared	0.4563
PRESS	0.25	Adeq Precision	15.635

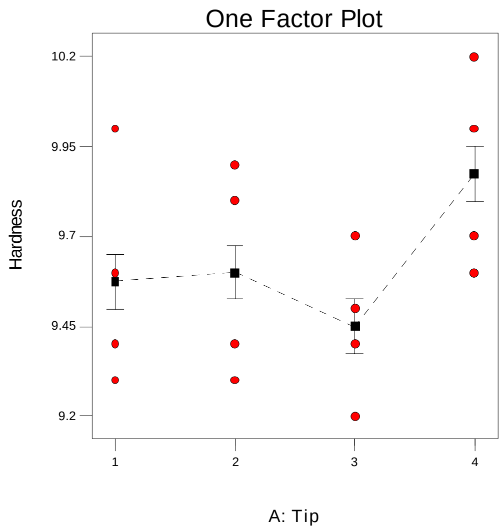
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	9.57	0.047
2-2	9.60	0.047
3-3	9.45	0.047
4-4	9.88	0.047

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	-0.025	1	0.067	-0.38	0.7163
1 vs 3	0.13	1	0.067	1.87	0.0935
1 vs 4	-0.30	1	0.067	-4.50	0.0015
2 vs 3	0.15	1	0.067	2.25	0.0510
2 vs 4	-0.27	1	0.067	-4.12	0.0026
3 vs 4	-0.43	1	0.067	-6.37	0.0001

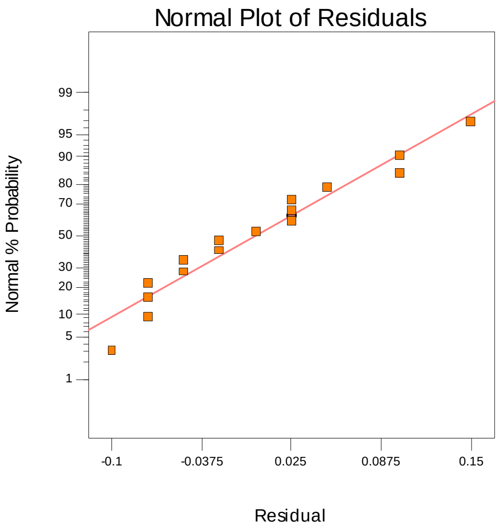
(b) Use the Fisher LSD method to make comparisons among the four tips to determine specifically which tips differ in mean hardness readings.

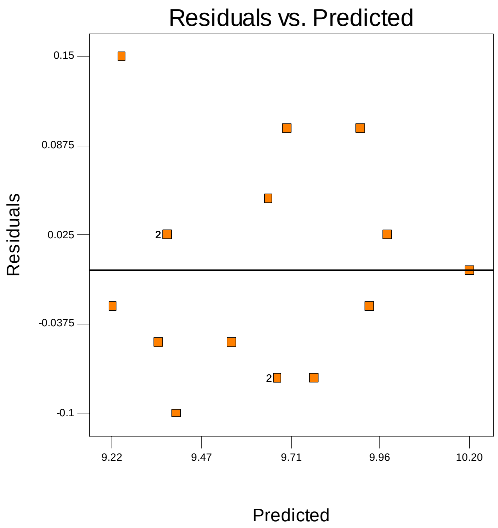
Based on the LSD bars in the Design Expert plot below, the mean of tip 4 differs from the means of tips 1, 2, and 3. The LSD metod identifies a marginal difference between the means of tips 2 and 3.

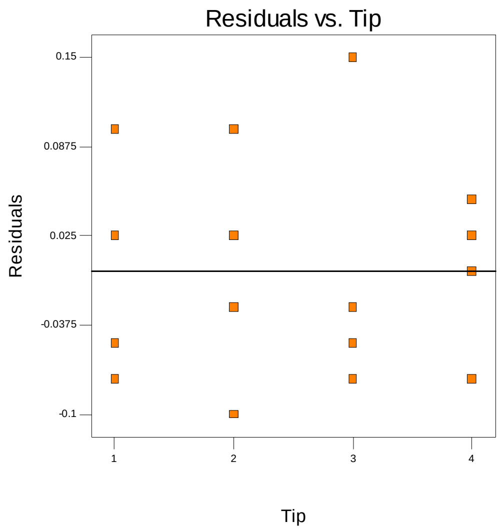


(c) Analyze the residuals from this experiment.

The residual plots below do not identify any violations to the assumptions.







4-6 A soncumer products company relies on direct mail marketing pieces as a major component of its advertising campaigns. The company yhas three different designs for a new brochure and want to evaluate their effectiveness, as there are substantial differences in costs between the three designs. The company decides to test the three designs by mailing 5,000 samples of each to potential customers in four different regions of the country. Since there are known regional differences in the customer base, regions are considered as blocks. The number of responses to each mailing is shown below.

	Region			
Design	NE	NW	SE	SW
1	250	350	219	375
2	400	525	390	580
3	275	340	200	310

(a) Analyize the data from this experiment.

The residuals of the analsysis below identify concerns with the normality and equality of variance assumptions. As a result, a squreroot transformation was applied as shown in the second analsysis table. The residuals of both analysis are presented for comparison in part (c) of this problem. The analysis concludes that there is a difference between the mean number of responses for the three designs.

Design Expert Output

Response: Number of responses
ANOVA for Selected Factorial Model
Analysis of variance table [Terms added sequentially (first to last)]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	49035.67	3	16345.22			
Model	90755.17	2	45377.58	50.15	0.0002	significant
A	*90755.17*	*2*	*45377.58*	*50.15*	*0.0002*	
Residual	5428.83	6	904.81			
Cor Total	1.452E+005	11				

The Model F-value of 50.15 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	30.08	R-Squared	0.9436
Mean	351.17	Adj R-Squared	0.9247
C.V.	8.57	Pred R-Squared	0.7742

PRESS	21715.33	Adeq Precision	16.197

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	298.50	15.04
2-2	473.75	15.04
3-3	281.25	15.04

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-175.25	1	21.27	-8.24	0.0002
1 vs 3	17.25	1	21.27	0.81	0.4483
2 vs 3	192.50	1	21.27	9.05	0.0001

Design Expert Output for Transformed Data

Response: Number of responses Transform: Square root Constant: 0

ANOVA for Selected Factorial Model

Analysis of variance table [Terms added sequentially (first to last)]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	35.89	3	11.96			
Model	60.73	2	30.37	60.47	0.0001	significant
A	*60.73*	*2*	*30.37*	*60.47*	*0.0001*	
Residual	3.01	6	0.50			
Cor Total	99.64	11				

The Model F-value of 60.47 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.71	R-Squared	0.9527
Mean	18.52	Adj R-Squared	0.9370
C.V.	3.83	Pred R-Squared	0.8109
PRESS	12.05	Adeq Precision	18.191

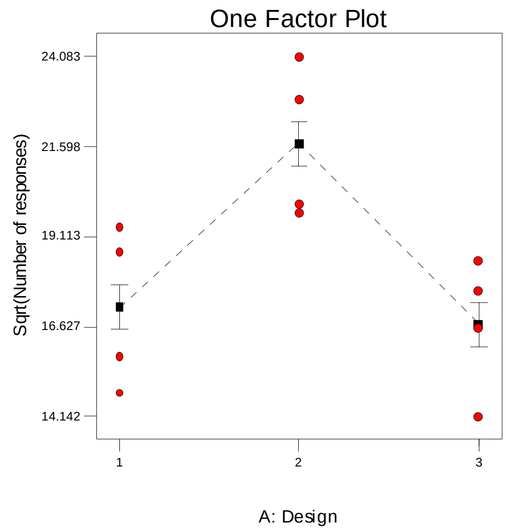
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	17.17	0.35
2-2	21.69	0.35
3-3	16.69	0.35

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	-4.52	1	0.50	-9.01	0.0001
1 vs 3	0.48	1	0.50	0.95	0.3769
2 vs 3	4.99	1	0.50	9.96	< 0.0001

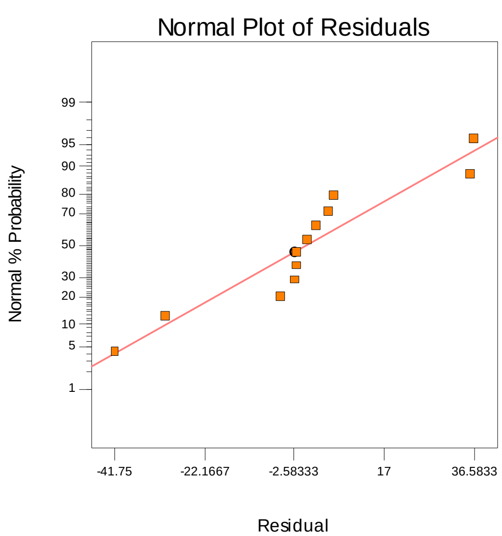
(b) Use the Fisher LSD method to make comparisons among the three designs to determine specifically which designs differ in mean response rate.

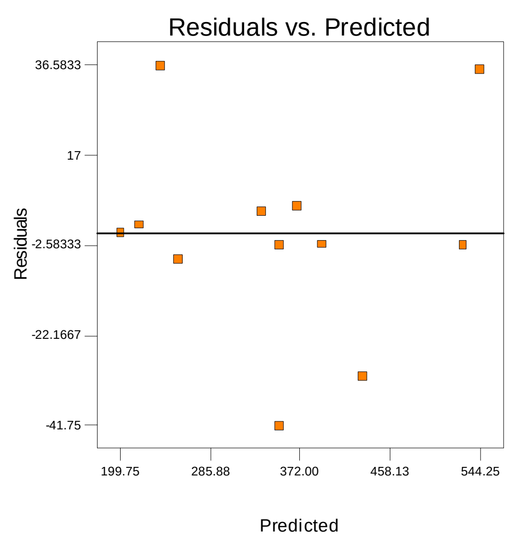
Based on the LSD bars in the Design Expert plot below, designs 1 and 3 do not differ; however, design 2 is different than designs 1 and 3.

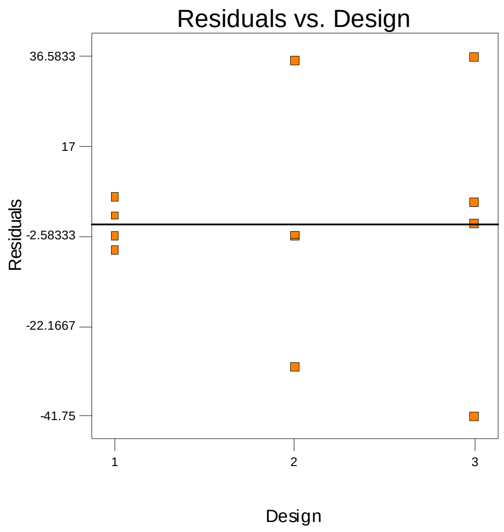


(c) Analyze the residuals from this experiment.

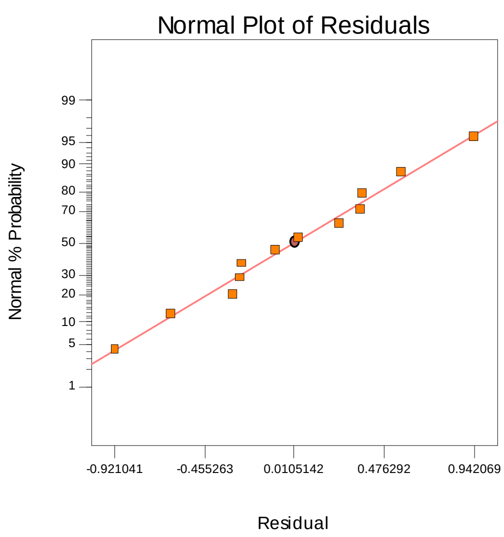
The first set of residual plots presented below represent the untransformed data. Concerns with the normality as well as inequality of variance are presented. The second set of residual plots represent transformed data and do not identify significant violations to the assumptions. The residuals vs. design identify a slight inequality; however, not a strong violation and an improvement to the non-transformed data.

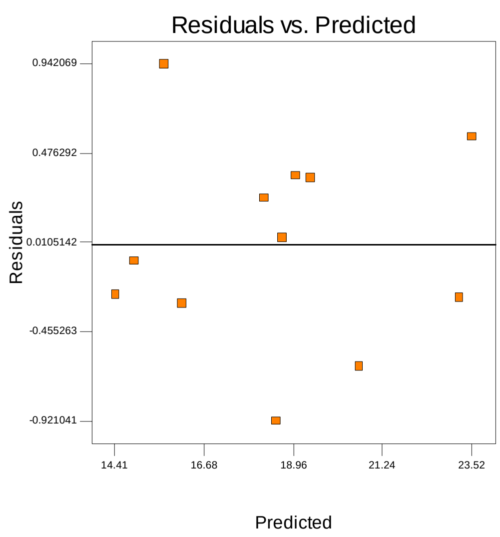


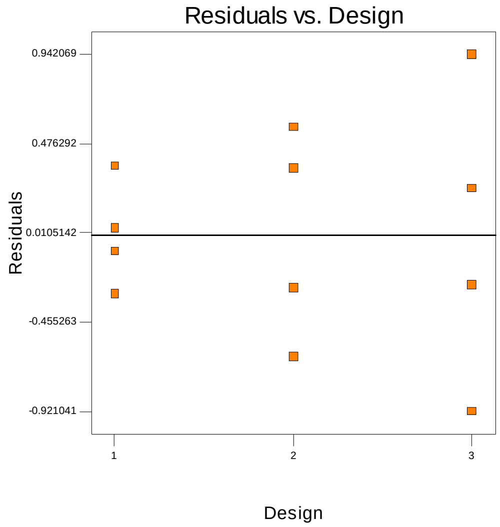




The following are the square root transformed data residual plots.







4-7 The effect of three different lubricating oils on fuel economy is diesel truck engines is being studied. Fuel economy is measured using brake-specific fuel consumption after the engine has been running for 15 minutes. Five different truck engines are available for the study, and the experimenters conduct the following randomized complete block design.

	Truck				
Oil	1	2	3	4	5
1	0.500	0.634	0.487	0.329	0.512
2	0.535	0.675	0.520	0.435	0.540
3	0.513	0.595	0.488	0.400	0.510

(a) Analyize the data from this experiment.

The residuals of the analsysis below identify concerns with the normality and equality of variance assumptions. As a result, a squreroot transformation was applied as shown in the second analsysis table. The residuals of both analysis are presented for comparison in part (c) of this problem. The analysis concludes that there is a difference between the mean number of responses for the three designs.

Design Expert Output

Response: Fuel consumption
ANOVA for Selected Factorial Model
Analysis of variance table [Terms added sequentially (first to last)]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.092	4	0.023			
Model	6.706E-003	2	3.353E-003	6.35	0.0223	significant
A	*6.706E-003*	*2*	*3.353E-003*	*6.35*	*0.0223*	
Residual	4.222E-003	8	5.278E-004			
Cor Total	0.10	14				

The Model F-value of 6.35 implies the model is significant. There is only
a 2.23% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.023	R-Squared	0.6136
Mean	0.51	Adj R-Squared	0.5170
C.V.	4.49	Pred R-Squared	-0.3583
PRESS	0.015	Adeq Precision	18.814

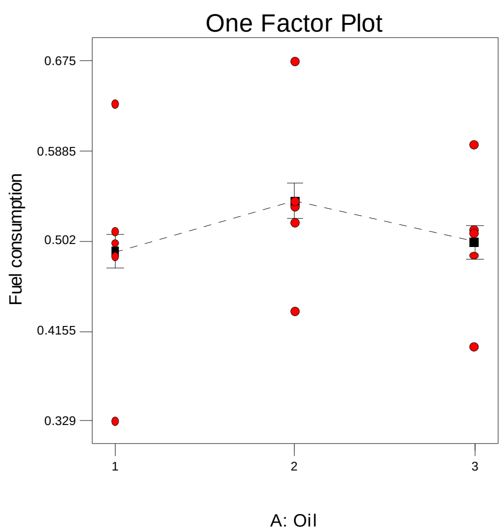
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-1	0.49	0.010
2-2	0.54	0.010
3-3	0.50	0.010

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	-0.049	1	0.015	-3.34	0.0102
1 vs 3	-8.800E-003	1	0.015	-0.61	0.5615
2 vs 3	0.040	1	0.015	2.74	0.0255

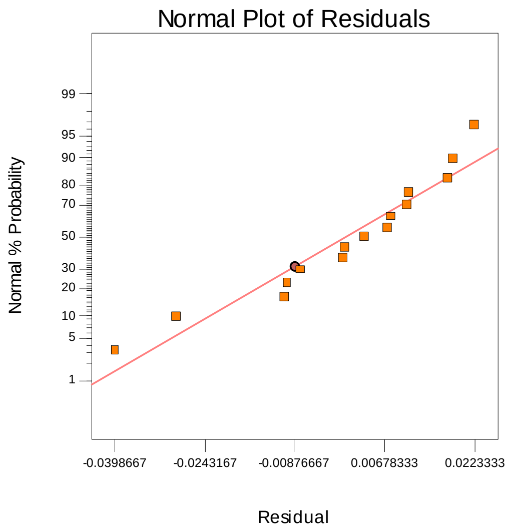
(b) Use the Fisher LSD method to make comparisons among the three lubricating oils to determine specifically which oils differ in break-specific fuel consumption.

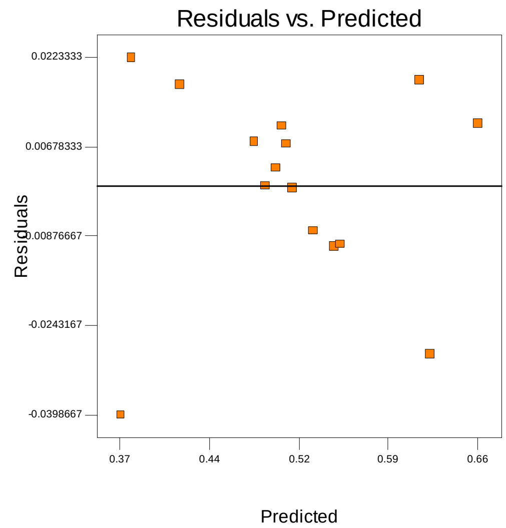
Based on the LSD bars in the Design Expert plot below, the means for break-specific fuel consumption for oils 1 and 3 do not differ; however, oil 2 is different than oils 1 and 3.

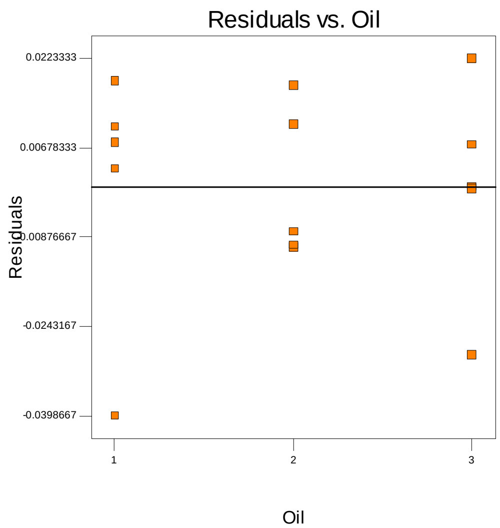


(c) Analyze the residuals from this experiment.

The residual plots below do not identify any violations to the assumptions.







4-8 An article in the *Fire Safety Journal* ("The Effect of Nozzle Design on the Stability and Performance of Turbulent Water Jets," Vol. 4, August 1981) describes an experiment in which a shape factor was determined for several different nozzle designs at six levels of efflux velocity. Interest focused on potential differences between nozzle designs, with velocity considered as a nuisance variable. The data are shown below:

	Jet Efflux Velocity (m/s)					
Nozzle Design	11.73	14.37	16.59	20.43	23.46	28.74
1	0.78	0.80	0.81	0.75	0.77	0.78
2	0.85	0.85	0.92	0.86	0.81	0.83
3	0.93	0.92	0.95	0.89	0.89	0.83
4	1.14	0.97	0.98	0.88	0.86	0.83
5	0.97	0.86	0.78	0.76	0.76	0.75

(a) Does nozzle design affect the shape factor? Compare nozzles with a scatter plot and with an analysis of variance, using $\alpha = 0.05$.

Design Expert Output

Response: Shape
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.063	5	0.013			
Model	0.10	4	0.026	8.92	0.0003	significant
A	*0.10*	*4*	*0.026*	*8.92*	*0.0003*	
Residual	0.057	20	2.865E-003			
Cor Total	0.22	29				

The Model F-value of 8.92 implies the model is significant. There is only a 0.03% chance that a "Model F-Value" this large could occur due to noise.

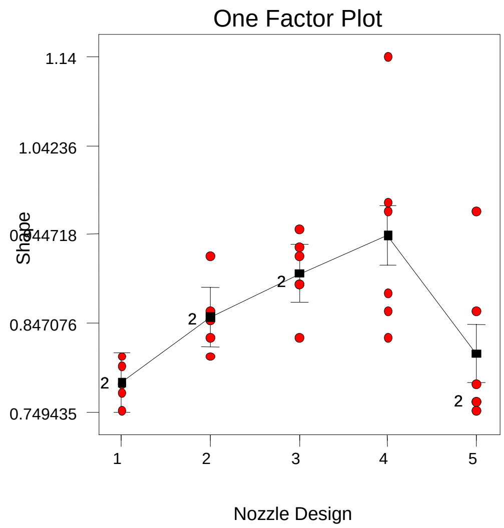
Std. Dev.	0.054	R-Squared	0.6407
Mean	0.86	Adj R-Squared	0.5688
C.V.	6.23	Pred R-Squared	0.1916
PRESS	0.13	Adeq Precision	9.438

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	0.78	0.022
2-2	0.85	0.022
3-3	0.90	0.022
4-4	0.94	0.022
5-5	0.81	0.022

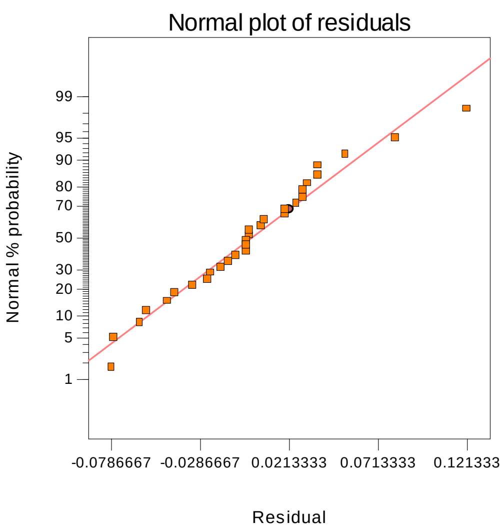
Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H_0 Coeff=0**	**Prob > |t|**
1 vs 2	-0.072	1	0.031	-2.32	0.0311
1 vs 3	-0.12	1	0.031	-3.88	0.0009
1 vs 4	-0.16	1	0.031	-5.23	< 0.0001
1 vs 5	-0.032	1	0.031	-1.02	0.3177
2 vs 3	-0.048	1	0.031	-1.56	0.1335
2 vs 4	-0.090	1	0.031	-2.91	0.0086
2 vs 5	0.040	1	0.031	1.29	0.2103
3 vs 4	-0.042	1	0.031	-1.35	0.1926
3 vs 5	0.088	1	0.031	2.86	0.0097
4 vs 5	0.13	1	0.031	4.21	0.0004

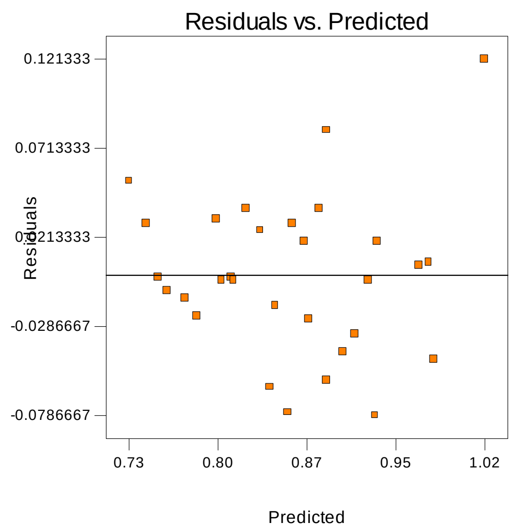
Nozzle design has a significant effect on shape factor.

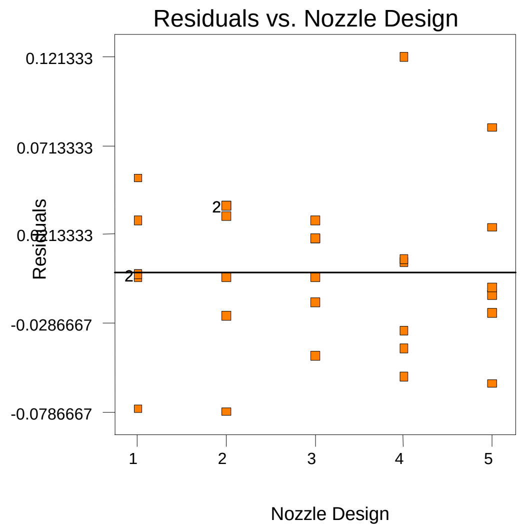


(b) Analyze the residual from this experiment.

The plots shown below do not give any indication of serious problems. Thre is some indication of a mild outlier on the normal probability plot and on the plot of residuals versus the predicted velocity.







(c) Which nozzle designs are different with respect to shape factor? Draw a graph of average shape factor for each nozzle type and compare this to a scaled *t* distribution. Compare the conclusions that you draw from this plot to those from Duncan's multiple range test.

$$S_{\bar{y}_{i.}} = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{0.002865}{6}} = 0.021852$$

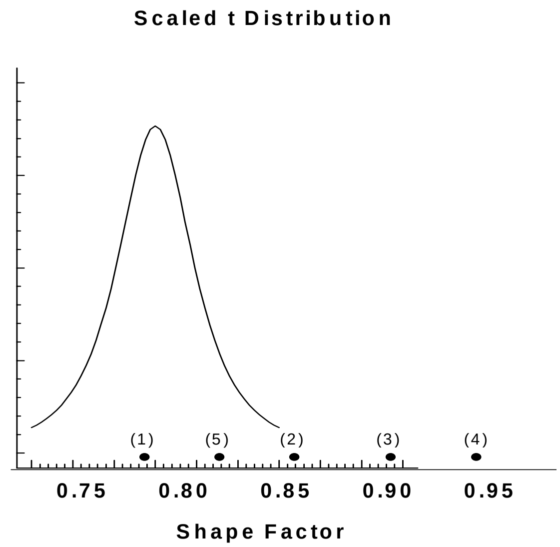
$R_2 = r_{0.05}(2,20)\ S_{\bar{y}_{i.}} = (2.95)(0.021852) = 0.06446$

$R_3 = r_{0.05}(3,20)\ S_{\bar{y}_{i.}} = (3.10)(0.021852) = 0.06774$

$R_4 = r_{0.05}(4,20)\ S_{\bar{y}_{i.}} = (3.18)(0.021852) = 0.06949$

$R_5 = r_{0.05}(5,20)\ S_{\bar{y}_{i.}} = (3.25)(0.021852) = 0.07102$

	Mean Difference		*R*	
1 vs 4	0.16167	>	0.07102	different
1 vs 3	0.12000	>	0.06949	different
1 vs 2	0.07167	>	0.06774	different
1 vs 5	0.03167	<	0.06446	
5 vs 4	0.13000	>	0.06949	different
5 vs 3	0.08833	>	0.06774	different
5 vs 2	0.04000	<	0.06446	
2 vs 4	0.09000	>	0.06774	different
2 vs 3	0.04833	<	0.06446	
3 vs 4	0.04167	<	0.06446	



4-9 Consider the ratio control algorithm experiment described in Chapter 3, Section 3-8. The experiment was actually conducted as a randomized block design, where six time periods were selected as the blocks, and all four ratio control algorithms were tested in each time period. The average cell voltage and the standard deviation of voltage (shown in parentheses) for each cell as follows:

Ratio Control	Time Period					
Algorithms	1	2	3	4	5	6
1	4.93 (0.05)	4.86 (0.04)	4.75 (0.05)	4.95 (0.06)	4.79 (0.03)	4.88 (0.05)
2	4.85 (0.04)	4.91 (0.02)	4.79 (0.03)	4.85 (0.05)	4.75 (0.03)	4.85 (0.02)
3	4.83 (0.09)	4.88 (0.13)	4.90 (0.11)	4.75 (0.15)	4.82 (0.08)	4.90 (0.12)
4	4.89 (0.03)	4.77 (0.04)	4.94 (0.05)	4.86 (0.05)	4.79 (0.03)	4.76 (0.02)

(a) Analyze the average cell voltage data. (Use α = 0.05.) Does the choice of ratio control algorithm affect the cell voltage?

Design Expert Output

Response: Average
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.017	5	3.487E-003			
Model	2.746E-003	3	9.153E-004	0.19	0.9014	not significant
A	*2.746E-003*	*3*	*9.153E-004*	*0.19*	*0.9014*	
Residual	0.072	15	4.812E-003			
Cor Total	0.092	23				

The "Model F-value" of 0.19 implies the model is not significant relative to the noise. There is a 90.14 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	0.069	R-Squared	0.0366
Mean	4.84	Adj R-Squared	-0.1560
C.V.	1.43	Pred R-Squared	-1.4662
PRESS	0.18	Adeq Precision	2.688

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-1	4.86	0.028

2-2	4.83	0.028
3-3	4.85	0.028
4-4	4.84	0.028

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.027	1	0.040	0.67	0.5156
1 vs 3	0.013	1	0.040	0.33	0.7438
1 vs 4	0.025	1	0.040	0.62	0.5419
2 vs 3	-0.013	1	0.040	-0.33	0.7438
2 vs 4	-1.667E-003	1	0.040	-0.042	0.9674
3 vs 4	0.012	1	0.040	0.29	0.7748

The ratio control algorithm does not affect the mean cell voltage.

(b) Perform an appropriate analysis of the standard deviation of voltage. (Recall that this is called "pot noise.") Does the choice of ratio control algorithm affect the pot noise?

Design Expert Output

Response: StDev Transform: Natural log Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	0.94	5	0.19			
Model	6.17	3	2.06	33.26	< 0.0001	significant
A	*6.17*	*3*	*2.06*	*33.26*	*< 0.0001*	
Residual	0.93	15	0.062			
Cor Total	8.04	23				

The Model F-value of 33.26 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.25	R-Squared	0.8693
Mean	-3.04	Adj R-Squared	0.8432
C.V.	-8.18	Pred R-Squared	0.6654
PRESS	2.37	Adeq Precision	12.446

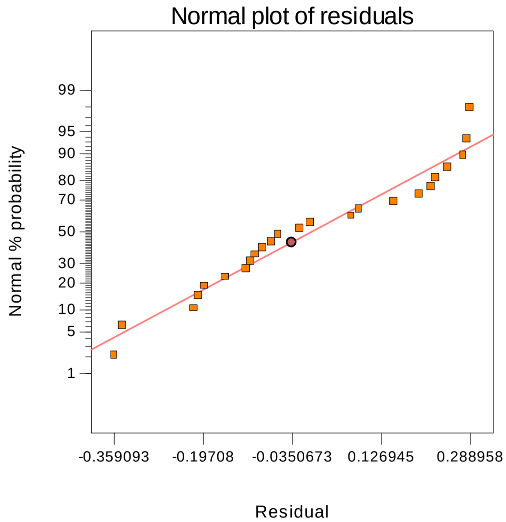
Treatment Means (Adjusted, If Necessary)

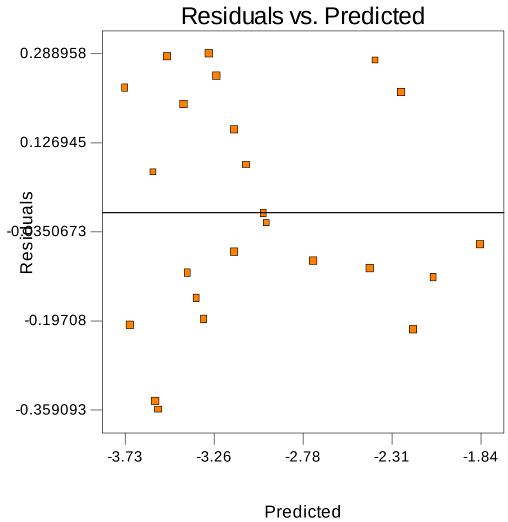
	Estimated Mean	Standard Error
1-1	-3.09	0.10
2-2	-3.51	0.10
3-3	-2.20	0.10
4-4	-3.36	0.10

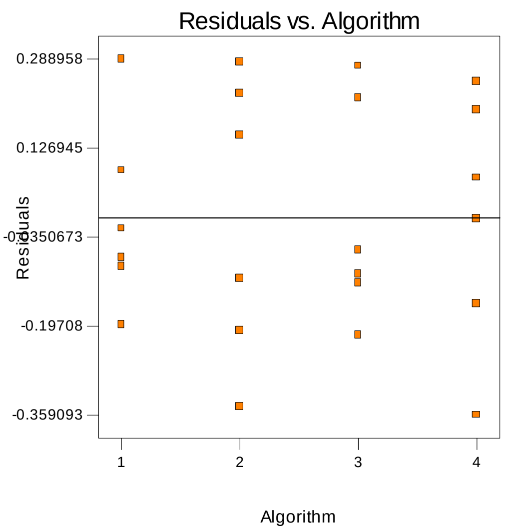
Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	0.42	1	0.14	2.93	0.0103
1 vs 3	-0.89	1	0.14	-6.19	< 0.0001
1 vs 4	0.27	1	0.14	1.87	0.0813
2 vs 3	-1.31	1	0.14	-9.12	< 0.0001
2 vs 4	-0.15	1	0.14	-1.06	0.3042
3 vs 4	1.16	1	0.14	8.06	< 0.0001

A natural log transformation was applied to the pot noise data. The ratio control algorithm does affect the pot noise.

(c) Conduct any residual analyses that seem appropriate.







The normal probability plot shows slight deviations from normality; however, still acceptable.

(d) Which ratio control algorithm would you select if your objective is to reduce both the average cell voltage and the pot noise?

Since the ratio control algorithm has little effect on average cell voltage, select the algorithm that minimizes pot noise, that is algorithm #2.

4-10 An aluminum master alloy manufacturer produces grain refiners in ingot form. This company produces the product in four furnaces. Each furnace is known to have its own unique operating characteristics, so any experiment run in the foundry that involves more than one furnace will consider furnace a nuisance variable. The process engineers suspect that stirring rate impacts the grain size of the product. Each furnace can be run at four different stirring rates. A randomized block design is run for a particular refiner and the resulting grain size data is shown below.

	Furnace			
Stirring Rate	1	2	3	4

5	8	4	5	6
10	14	5	6	9
15	14	6	9	2
20	17	9	3	6

(a) Is there any evidence that stirring rate impacts grain size?

Design Expert Output

Response: Grain Size
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	165.19	3	55.06			
Model	22.19	3	7.40	0.85	0.4995	not significant
A	*22.19*	*3*	*7.40*	*0.85*	*0.4995*	
Residual	78.06	9	8.67			
Cor Total	265.44	15				

The "Model F-value" of 0.85 implies the model is not significant relative to the noise. There is a 49.95 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	2.95	R-Squared	0.2213
Mean	7.69	Adj R-Squared	-0.0382
C.V.	38.31	Pred R-Squared	-1.4610
PRESS	246.72	Adeq Precision	5.390

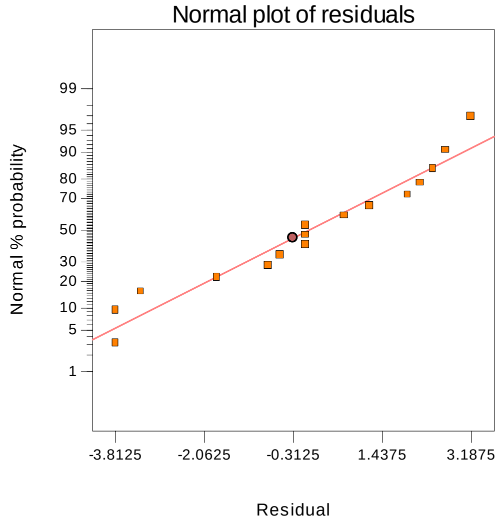
Treatment Means (Adjusted, If Necessary)

	Estimated Mean	**Standard Error**
1-5	5.75	1.47
2-10	8.50	1.47
3-15	7.75	1.47
4-20	8.75	1.47

Treatment	**Mean Difference**	**DF**	**Standard Error**	**t for H0 Coeff=0**	**Prob > |t|**
1 vs 2	-2.75	1	2.08	-1.32	0.2193
1 vs 3	-2.00	1	2.08	-0.96	0.3620
1 vs 4	-3.00	1	2.08	-1.44	0.1836
2 vs 3	0.75	1	2.08	0.36	0.7270
2 vs 4	-0.25	1	2.08	-0.12	0.9071
3 vs 4	-1.00	1	2.08	-0.48	0.6425

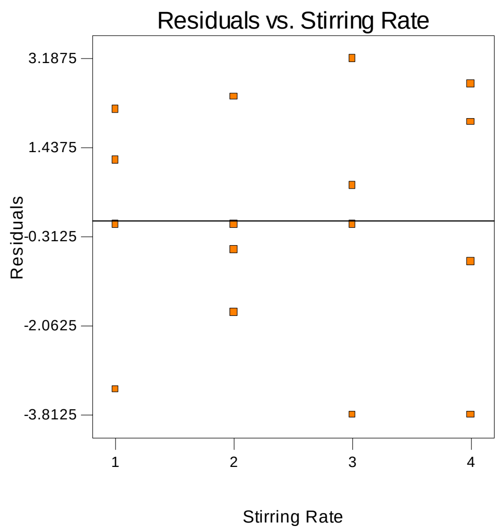
The analysis of variance shown above indicates that there is no difference in mean grain size due to the different stirring rates.

(b) Graph the residuals from this experiment on a normal probability plot. Interpret this plot.



The plot indicates that normality assumption is valid.

(c) Plot the residuals versus furnace and stirring rate. Does this plot convey any useful information?



The variance is consistent at different stirring rates. Not only does this validate the assumption of uniform variance, it also identifies that the different stirring rates do not affect variance.

(d) What should the process engineers recommend concerning the choice of stirring rate and furnace for this particular grain refiner if small grain size is desirable?

There really is no effect due to the stirring rate.

4-11 Analyze the data in Problem 4-2 using the general regression significance test.

$$
\begin{array}{lllllllllll}
\mu: & 12\hat{\mu} & +4\hat{\tau}_1 & +4\hat{\tau}_2 & +4\hat{\tau}_3 & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =225 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =92 \\
\tau_2: & 4\hat{\mu} & & +4\hat{\tau}_2 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =101
\end{array}
$$

$\tau_3:$	$4\hat{\mu}$			$+4\hat{\tau}_3$	$+\hat{\beta}_1$	$+\hat{\beta}_2$	$+\hat{\beta}_3$	$+\hat{\beta}_4$	=32
$\beta_1:$	$3\hat{\mu}$	$+\hat{\tau}_1$	$+\hat{\tau}_2$	$+\hat{\tau}_3$	$+3\hat{\beta}_1$				=34
$\beta_2:$	$3\hat{\mu}$	$+\hat{\tau}_1$	$+\hat{\tau}_2$	$+\hat{\tau}_3$		$+3\hat{\beta}_2$			=50
$\beta_3:$	$3\hat{\mu}$	$+\hat{\tau}_1$	$+\hat{\tau}_2$	$+\hat{\tau}_3$			$+3\hat{\beta}_3$		=36
$\beta_4:$	$3\hat{\mu}$	$+\hat{\tau}_1$	$+\hat{\tau}_2$	$+\hat{\tau}_3$				$+3\hat{\beta}_4$	=105

Applying the constraints $\sum\hat{\tau}_i = \sum\hat{\beta}_j = 0$, we obtain:

$$\hat{\mu}=\frac{225}{12},\ \hat{\tau}_1=\frac{51}{12},\ \hat{\tau}_2=\frac{78}{12},\ \hat{\tau}_3=\frac{-129}{12},\ \hat{\beta}_1=\frac{-89}{12},\ \hat{\beta}_2=\frac{-25}{12},\ \hat{\beta}_3=\frac{-81}{12},\ \hat{\beta}_4=\frac{195}{12}$$

$$R(\mu,\tau,\beta)=\left(\frac{225}{12}\right)(225)+\left(\frac{51}{12}\right)(92)+\left(\frac{78}{12}\right)(101)+\left(\frac{-129}{12}\right)(32)+\left(\frac{-89}{12}\right)(34)+\left(\frac{-25}{12}\right)(50)+\left(\frac{-81}{12}\right)(36)+\left(\frac{195}{12}\right)(105)=6029.17$$

$$\sum\sum y_{ij}^2=6081,\ SS_E=\sum\sum y_{ij}^2-R(\mu,\tau,\beta)=6081-6029.17=51.83$$

Model Restricted to $\tau_i=0$:

$\mu:$	$12\hat{\mu}$	$+3\hat{\beta}_1$	$+3\hat{\beta}_2$	$+3\hat{\beta}_3$	$+3\hat{\beta}_4$	=225
$\beta_1:$	$3\hat{\mu}$	$+3\hat{\beta}_1$				=34
$\beta_2:$	$3\hat{\mu}$		$+3\hat{\beta}_2$			=50
$\beta_3:$	$3\hat{\mu}$			$+3\hat{\beta}_3$		=36
$\beta_4:$	$3\hat{\mu}$				$+3\hat{\beta}_4$	=105

Applying the constraint $\sum\hat{\beta}_j=0$, we obtain:

$$\hat{\mu}=\frac{225}{12},\ \hat{\beta}_1=-89/12,\ \hat{\beta}_2=\frac{-25}{12},\ \hat{\beta}_3=\frac{-81}{12},\ \hat{\beta}_4=\frac{195}{12}.\ \text{Now:}$$

$$R(\mu,\beta)=\left(\frac{225}{12}\right)(225)+\left(\frac{-89}{12}\right)(34)+\left(\frac{-25}{12}\right)(50)+\left(\frac{-81}{12}\right)(36)+\left(\frac{195}{12}\right)(105)=5325.67$$

$$R(\tau|\mu,\beta)=R(\mu,\tau,\beta)-R(\mu,\beta)=6029.17-5325.67=703.50=SS_{Treatments}$$

Model Restricted to $\beta_j=0$:

$\mu:$	$12\hat{\mu}$	$+4\hat{\tau}_1$	$+4\hat{\tau}_2$	$+4\hat{\tau}_3$	=225
$\tau_1:$	$4\hat{\mu}$	$+4\hat{\tau}_1$			=92
$\tau_2:$	$4\hat{\mu}$		$+4\hat{\tau}_2$		=101
$\tau_3:$	$4\hat{\mu}$			$+4\hat{\tau}_3$	=32

Applying the constraint $\sum\hat{\tau}_i=0$, we obtain:

$$\hat{\mu}=\frac{225}{12},\ \hat{\tau}_1=\frac{51}{12},\ \hat{\tau}_2=\frac{78}{12},\ \hat{\tau}_3=\frac{-129}{12}$$

$$R(\mu,\tau)=\left(\frac{225}{12}\right)(225)+\left(\frac{51}{12}\right)(92)+\left(\frac{78}{12}\right)(101)+\left(\frac{-129}{12}\right)(32)=4922.25$$
$$R(\beta|\mu,\tau)=R(\mu,\tau,\beta)-R(\mu,\tau)=6029.17-4922.25=1106.92=SS_{Blocks}$$

4-12 Assuming that chemical types and bolts are fixed, estimate the model parameters τ_i and β_j in Problem 4-1.

Using Equations 4-14, Applying the constraints, we obtain:

$$\hat{\mu}=\frac{35}{20},\ \hat{\tau}_1=\frac{-23}{20},\ \hat{\tau}_2=\frac{-7}{20},\ \hat{\tau}_3=\frac{13}{20},\ \hat{\tau}_4=\frac{17}{20},\ \hat{\beta}_1=\frac{35}{20},\ \hat{\beta}_2=\frac{-65}{20},\ \hat{\beta}_3=\frac{75}{20},\ \hat{\beta}_4=\frac{20}{20},\ \hat{\beta}_5=\frac{-65}{20}$$

4-13 Draw an operating characteristic curve for the design in Problem 4-2. Does this test seem to be sensitive to small differences in treatment effects?

Assuming that solution type is a fixed factor, we use the OC curve in appendix V. Calculate

$$\Phi^2=\frac{b\sum\tau_i^2}{a\sigma^2}=\frac{4\sum\tau_i^2}{3(8.69)}$$

using MS_E to estimate σ^2. We have:

$$\upsilon_1=a-1=2 \qquad \upsilon_2=(a-1)(b-1)=(2)(3)=6\ .$$

If $\sum\hat{\tau}^2_i=\sigma^2=MS_E$, then:

$$\Phi=\sqrt{\frac{4}{3(1)}}=1.15 \text{ and } \beta\cong 0.70$$

If $\sum\hat{\tau}_i=2\sigma^2=2MS_E$, then:

$$\Phi=\sqrt{\frac{4}{3(2)}}=1.63 \text{ and } \beta\cong 0.55\text{, etc.}$$

This test is not very sensitive to small differences.

4-14 Suppose that the observation for chemical type 2 and bolt 3 is missing in Problem 4-1. Analyze the problem by estimating the missing value. Perform the exact analysis and compare the results.

$$y_{23}\text{ is missing.}\quad \hat{y}_{23}=\frac{ay'_{2.}+by'_{.3}-y'_{..}}{(a-1)(b-1)}=\frac{4(282)+5(227)-1360}{(4)(3)}=75.25$$

Thus, $y_{2.}$=357.25, $y_{.3}$=3022.25, and $y_{..}$=1435.25

Source	SS	DF	MS	F_0
Chemicals	12.7844	3	4.2615	2.154

Bolts	158.8875	4	
Error	21.7625	11	1.9784
Total	193.4344	18	

$F_{0.10,3,11}$=2.66, Chemicals are not significant.

4-12 ***Two missing values in a randomized block.*** Suppose that in Problem 4-1 the observations for chemical type 2 and bolt 3 and chemical type 4 and bolt 4 are missing.

(a) Analyze the design by iteratively estimating the missing values as described in Section 4-1.3.

$$\hat{y}_{23} = \frac{4y'_{2.} + 5y'_{.3} - y'_{..}}{12} \text{ and } \hat{y}_{44} = \frac{4y'_{4.} + 5y'_{.4} - y'_{..}}{12}$$

Data is coded *y*-70. As an initial guess, set y_{23}^{0} equal to the average of the observations available for chemical 2. Thus, $y_{23}^{0} = \frac{2}{4} = 0.5$. Then ,

$$\hat{y}_{44}^{0} = \frac{4(8)+5(6)-25.5}{12} = 3.04$$
$$\hat{y}_{23}^{1} = \frac{4(2)+5(17)-28.04}{12} = 5.41$$
$$\hat{y}_{44}^{1} = \frac{4(8)+5(6)-30.41}{12} = 2.63$$
$$\hat{y}_{44}^{2} = \frac{4(2)+5(17)-27.63}{12} = 5.44$$
$$\hat{y}_{44}^{2} = \frac{4(8)+5(6)-30.44}{12} = 2.63$$
$$\therefore \hat{y}_{23} = 5.44 \quad \hat{y}_{44} = 2.63$$

Design Expert Output

ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	156.83	4	39.21			
Model	9.59	3	3.20	2.08	0.1560	not significant
A	*9.59*	*3*	*3.20*	*2.08*	*0.1560*	
Residual	18.41	12	1.53			
Cor Total	184.83	19				

(b) Differentiate SS_E with respect to the two missing values, equate the results to zero, and solve for estimates of the missing values. Analyze the design using these two estimates of the missing values.

$$SS_E = \sum\sum y_{ij}^2 - \tfrac{1}{5}\sum y_{i.}^2 - \tfrac{1}{4}\sum y_{.j}^2 + \tfrac{1}{20}\sum y_{..}^2$$
$$SS_E = 0.6y_{23}^2 + 0.6y_{44}^2 - 6.8y_{23} - 3.7y_{44} + 0.1y_{23}y_{44} + R$$

From $\frac{\partial SS_E}{\partial y_{23}} = \frac{\partial SS_E}{\partial y_{44}} = 0$, we obtain:

$$\begin{aligned} 1.2\hat{y}_{23} + 0.1\hat{y}_{44} &= 6.8 \\ 0.1\hat{y}_{23} + 1.2\hat{y}_{44} &= 3.7 \end{aligned} \quad \Rightarrow \hat{y}_{23} = 5.45\,,\ \hat{y}_{44} = 2.63$$

These quantities are almost identical to those found in part (a). The analysis of variance using these new data does not differ substantially from part (a).

(c) Derive general formulas for estimating two missing values when the observations are in *different* blocks.

$$SS_E = y_{iu}^2 + y_{kv}^2 - \frac{\left(y'_{i.} + y'_{iu}\right)^2 + \left(y'_{k.} + y_{kv}\right)^2}{b} - \frac{\left(y'_{.u} + y_{iu}\right)^2 + \left(y'_{.v} + y_{kv}\right)^2}{a} + \frac{\left(y'_{..} + y_{iu} + y\right)_{kv}^{\ 2}}{ab}$$

From $\frac{\partial SS_E}{\partial y_{23}} = \frac{\partial SS_E}{\partial y_{44}} = 0$, we obtain:

$$\hat{y}_{iu}\left[\frac{(a-1)(b-1)}{ab}\right] = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{ab} - \frac{\hat{y}_{kv}}{ab}$$
$$\hat{y}_{kv}\left[\frac{(a-1)(b-1)}{ab}\right] = \frac{ay'_{k.} + by'_{.v} - y'_{..}}{ab} - \frac{\hat{y}_{iu}}{ab}$$

whose simultaneous solution is:

$$\hat{y}_{iu} = \frac{y'_{i.}\,a\left[1-(a-1)^2(b-1)^2 - ab\right] + y'_{.u}\,b\left[1-(a-1)^2(b-1)^2 - ab\right] - y'_{..}\left[1-ab(a-1)^2(b-1)^2\right]}{(a-1)(b-1)\left[1-(a-1)^2(b-1)^2\right]} + \frac{ab\left[ay'_{k.} + by'_{.v} - y'_{..}\right]}{\left[1-(a-1)^2(b-1)^2\right]}$$

$$\hat{y}_{kv} = \frac{ay'_{i.} + by'_{.u} - y'_{..} - (b-1)(a-1)\left[ay'_{k.} + by'_{.v} - y'_{..}\right]}{\left[1-(a-1)^2(b-1)^2\right]}$$

(d) Derive general formulas for estimating two missing values when the observations are in the *same* block. Suppose that two observations y_{ij} and y_{kj} are missing, $i \neq k$ (same block j).

$$SS_E = y_{ij}^2 + y_{kj}^2 - \frac{\left(y'_{i.} + y_{ij}\right)^2 + \left(y'_{k.} + y_{kj}\right)^2}{b} - \frac{\left(y'_{.j} + y_{ij} + y_{kj}\right)^2}{a} + \frac{\left(y'_{..} + y_{ij} + y_{kj}\right)^2}{ab}$$

From $\frac{\partial SS_E}{\partial y_{23}} = \frac{\partial SS_E}{\partial y_{44}} = 0$, we obtain

$$\hat{y}_{ij} = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \hat{y}_{kj}(a-1)(b-1)^2$$
$$\hat{y}_{kj} = \frac{ay'_{k.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \hat{y}_{ij}(a-1)(b-1)^2$$

whose simultaneous solution is:

$$\hat{y}_{ij} = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} + \frac{(b-1)\left[ay'_{k.} + by'_{.j} - y'_{..} + (a-1)(b-1)^2\left(ay'_{i.} + by'_{.j} - y'_{..}\right)\right]}{\left[1-(a-1)^2(b-1)\right]^2}$$

$$\hat{y}_{kj} = \frac{ay'_{k.} + by'_{.j} - y'_{..} - (b-1)^2(a-1)\left[ay'_{i.} + by'_{.j} - y'_{..}\right]}{(a-1)(b-1)\left[1-(a-1)^2(b-1)^4\right]}$$

4-17 An industrial engineer is conducting an experiment on eye focus time. He is interested in the effect of the distance of the object from the eye on the focus time. Four different distances are of interest. He has five subjects available for the experiment. Because there may be differences among individuals, he decides to conduct the experiment in a randomized block design. The data obtained follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw appropriate conclusions.

	Subject				
Distance (ft)	1	2	3	4	5
4	10	6	6	6	6
6	7	6	6	1	6
8	5	3	3	2	5
10	6	4	4	2	3

Design Expert Output

Response: Focus Time
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	36.30	4	9.07			
Model	32.95	3	10.98	8.61	0.0025	significant
A	*32.95*	*3*	*10.98*	*8.61*	*0.0025*	
Residual	15.30	12	1.27			
Cor Total	84.55	19				

The Model F-value of 8.61 implies the model is significant. There is only a 0.25% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.13	R-Squared	0.6829
Mean	4.85	Adj R-Squared	0.6036
C.V.	23.28	Pred R-Squared	0.1192
PRESS	42.50	Adeq Precision	10.432

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1-4	6.80	0.50
2-6	5.20	0.50
3-8	3.60	0.50
4-10	3.80	0.50

Treatment	Mean Difference	DF	Standard Error	t for H0 Coeff=0	Prob > |t|
1 vs 2	1.60	1	0.71	2.24	0.0448
1 vs 3	3.20	1	0.71	4.48	0.0008
1 vs 4	3.00	1	0.71	4.20	0.0012
2 vs 3	1.60	1	0.71	2.24	0.0448
2 vs 4	1.40	1	0.71	1.96	0.0736
3 vs 4	-0.20	1	0.71	-0.28	0.7842

Distance has a statistically significant effect on mean focus time.

4-18 The effect of five different ingredients (*A, B, C, D, E*) on reaction time of a chemical process is being studied. Each batch of new material is only large enough to permit five runs to be made. Furthermore, each runs requires approximately 1 1/2 hours, so only five runs can be made in one day. The experimenter decides to run the experiment as a Latin square so that day and batch effects can be systematically controlled. She obtains the data that follow. Analyze the data from this experiment (use α = 0.05) and draw conclusions.

	Day				
Batch	1	2	3	4	5
1	*A*=8	*B*=7	*D*=1	*C*=7	*E*=3
2	*C*=11	*E*=2	*A*=7	*D*=3	*B*=8
3	*B*=4	*A*=9	*C*=10	*E*=1	*D*=5
4	*D*=6	*C*=8	*E*=6	*B*=6	*A*=10
5	*E*=4	*D*=2	*B*=3	*A*=8	*C*=8

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Batch    random      5 1 2 3 4 5
Day      random      5 1 2 3 4 5
Catalyst  fixed      5 A B C D E

Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Catalyst    4    141.440    141.440     35.360   11.31  0.000
Batch       4     15.440     15.440      3.860    1.23  0.348
Day         4     12.240     12.240      3.060    0.98  0.455
Error      12     37.520     37.520      3.127
Total      24    206.640
```

4-19 An industrial engineer is investigating the effect of four assembly methods (*A, B, C, D*) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment (α = 0.05) draw appropriate conclusions.

Order of	Operator			
Assembly	1	2	3	4
1	*C*=10	*D*=14	*A*=7	*B*=8
2	*B*=7	*C*=18	*D*=11	*A*=8
3	*A*=5	*B*=10	*C*=11	*D*=9
4	*D*=10	*A*=10	*B*=12	*C*=14

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Order    random      4 1 2 3 4
Operator random      4 1 2 3 4
Method    fixed      4 A B C D


Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
```

```
Method       3     72.500     72.500     24.167   13.81  0.004
Order        3     18.500     18.500      6.167    3.52  0.089
Operator     3     51.500     51.500     17.167    9.81  0.010
Error        6     10.500     10.500      1.750
Total       15    153.000
```

4-20 Suppose that in Problem 4-18 the observation from batch 3 on day 4 is missing. Estimate the missing value from Equation 4-24, and perform the analysis using this value.

y_{354} is missing. $\hat{y}_{354} = \dfrac{p\left[y'_{i..} + y'_{.j.} + y'_{..k}\right] - 2y'_{...}}{(p-2)(p-1)} = \dfrac{5[28+15+24]-2(146)}{(3)(4)} = 3.58$

Minitab Output

General Linear Model

```
Factor      Type Levels Values
Batch     random      5 1 2 3 4 5
Day       random      5 1 2 3 4 5
Catalyst   fixed      5 A B C D E

Analysis of Variance for Time, using Adjusted SS for Tests

Source      DF     Seq SS     Adj SS     Adj MS       F      P
Catalyst     4    128.676    128.676     32.169   11.25  0.000
Batch        4     16.092     16.092      4.023    1.41  0.290
Day          4      8.764      8.764      2.191    0.77  0.567
Error       12     34.317     34.317      2.860
Total       24    187.849
```

4-21 Consider a *p x p* Latin square with rows (α_i), columns (β_k), and treatments (τ_j) fixed. Obtain least squares estimates of the model parameters α_i, β_k, τ_j.

$$\mu : p^2\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{...}$$

$$\alpha_i : p\hat{\mu} + p\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{i..} \quad , \ i = 1,2,...,p$$

$$\tau_j : p\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\hat{\tau}_j + p\sum_{k=1}^{p}\hat{\beta}_k = y_{.j.}, \ j = 1,2,...,p$$

$$\beta_k : p\hat{\mu} + p\sum_{i=1}^{p}\hat{\alpha}_i + p\sum_{j=1}^{p}\hat{\tau}_j + p\hat{\beta}_k = y_{..k}, \ k = 1,2,...,p$$

There are 3*p*+1 equations in 3*p*+1 unknowns. The rank of the system is 3*p*-2. Three side conditions are necessary. The usual conditions imposed are: $\sum_{i=1}^{p}\hat{\alpha}_i = \sum_{j=1}^{p}\hat{\tau}_j = \sum_{k=1}^{p}\hat{\beta}_k = 0$. The solution is then:

$$\hat{\mu} = \frac{y_{...}}{p^2} = \bar{y}_{...}$$

$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}, i = 1, 2, ..., p$$

$$\hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...},\ j = 1,2,...,p$$
$$\hat{\beta}_k = \bar{y}_{i..} - \bar{y}_{...},\ k = 1,2,...,p$$

4-22 Derive the missing value formula (Equation 4-24) for the Latin square design.

$$SS_E = \sum\sum\sum y_{ijk}^2 - \sum\frac{y_{i..}^2}{p} - \sum\frac{y_{.j.}^2}{p} - \sum\frac{y_{..k}^2}{p} + 2\left(\frac{y_{...}^2}{p^2}\right)$$

Let y_{ijk} be missing. Then

$$SS_E = y_{ijk}^2 - \frac{\left(y'_{i..} + y_{ijk}\right)^2}{p} - \frac{\left(y'_{.j.} + y_{ijk}\right)^2}{p} - \frac{\left(y'_{..k} + y_{ijk}\right)^2}{p} + \frac{2\left(y'_{...} + y_{ijk}\right)}{p^2} + R$$

where R is all terms without y_{ijk}.. From $\frac{\partial SS_E}{\partial y_{ijk}} = 0$, we obtain:

$$y_{ijk}\frac{(p-1)(p-2)}{p^2} = \frac{p\left(y'_{i..} + y'_{.j.} + y'_{..k}\right) - 2y'_{...}}{p^2},\ \text{or}\ y_{ijk} = \frac{p\left(y'_{i..} + y'_{.j.} + y'_{..k}\right) - 2y'_{...}}{(p-1)(p-2)}$$

4-23 ***Designs involving several Latin squares***. [See Cochran and Cox (1957), John (1971).] The *p x p* Latin square contains only *p* observations for each treatment. To obtain more replications the experimenter may use several squares, say *n*. It is immaterial whether the squares used are the same are different. The appropriate model is

$$y_{ijkh} = \mu + \rho_h + \alpha_{i(h)} + \tau_j + \beta_{k(h)} + (\tau\rho)_{jh} + \varepsilon_{ijkh} \quad \begin{cases} i = 1,2,...,p \\ j = 1,2,...,p \\ k = 1,2,...,p \\ h = 1,2,...,n \end{cases}$$

where y_{ijkh} is the observation on treatment *j* in row *i* and column *k* of the *h*th square. Note that $\alpha_{i(h)}$ and $\beta_{k(h)}$ are row and column effects in the *h*th square, and ρ_h is the effect of the *h*th square, and $(\tau\rho)_{jh}$ is the interaction between treatments and squares.

(a) Set up the normal equations for this model, and solve for estimates of the model parameters. Assume that appropriate side conditions on the parameters are $\sum_h \hat{\rho}_h = 0$, $\sum_i \hat{\alpha}_{i(h)} = 0$, and $\sum_k \hat{\beta}_{k(h)} = 0$ for each *h*, $\sum_j \hat{\tau}_j = 0$, $\sum_j (\hat{\tau}\rho)_{jh} = 0$ for each *h*, and $\sum_h (\hat{\tau}\rho)_{jh} = 0$ for each *j*.

$$\hat{\mu} = \bar{y}_{....}$$
$$\hat{\rho}_h = \bar{y}_{...h} - \bar{y}_{....}$$
$$\hat{\tau}_j = \bar{y}_{.j..} - \bar{y}_{....}$$
$$\hat{\alpha}_{i(h)} = \bar{y}_{i..h} - \bar{y}_{...h}$$
$$\hat{\beta}_{k(h)} = \bar{y}_{..kh} - \bar{y}_{...h}$$
$$\left(\hat{\tau\rho}\right)_{jh} = \bar{y}_{.j.h} - \bar{y}_{.j..} - \bar{y}_{...h} + \bar{y}_{....}$$

(b) Write down the analysis of variance table for this design.

Source	SS	DF
Treatments	$\sum \frac{y_{.j..}^2}{np} - \frac{y_{....}^2}{np^2}$	p-1
Squares	$\sum \frac{y_{...h}^2}{p^2} - \frac{y_{....}^2}{np^2}$	n-1
Treatment x Squares	$\sum \frac{y_{.j.h}^2}{p} - \frac{y_{....}^2}{np^2} - SS_{Treatments} - SS_{Squares}$	(p-1)(n-1)
Rows	$\sum \frac{y_{i..h}^2}{p} - \frac{y_{...h}^2}{np^2}$	n(p-1)
Columns	$\sum \frac{y_{..kh}^2}{p} - \frac{y_{...h}^2}{np^2}$	n(p-1)
Error	subtraction	n(p-1)(p-2)
Total	$\sum\sum\sum\sum y_{ijkh}^2 - \frac{y_{....}^2}{np^2}$	np^2-1

4-24 Discuss how the operating characteristics curves in the Appendix may be used with the Latin square design.

For the fixed effects model use:

$$\Phi^2 = \frac{\sum p\tau_j^2}{p\sigma^2} = \sum \frac{\tau_j^2}{\sigma^2}, \ \upsilon_1 = p-1 \quad \upsilon_2 = (p-2)(p-1)$$

For the random effects model use:

$$\lambda = \sqrt{1 + \frac{p\sigma_\tau^2}{\sigma^2}}, \ \upsilon_1 = p-1 \quad \upsilon_2 = (p-2)(p-1)$$

4-25 Suppose that in Problem 4-14 the data taken on day 5 were incorrectly analyzed and had to be discarded. Develop an appropriate analysis for the remaining data.

Two methods of analysis exist: (1) Use the general regression significance test, or (2) recognize that the design is a Youden square. The data can be analyzed as a balanced incomplete block design with $a = b = 5$, $r = k = 4$ and $\lambda = 3$. Using either approach will yield the same analysis of variance.

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Catalyst  fixed      5 A B C D E
Batch    random      5 1 2 3 4 5
Day      random      4 1 2 3 4

Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS        F      P
Catalyst    4    119.800    120.167     30.042     7.48  0.008
Batch       4     11.667     11.667      2.917     0.73  0.598
Day         3      6.950      6.950      2.317     0.58  0.646
Error       8     32.133     32.133      4.017
Total      19    170.550
```

4-26 The yield of a chemical process was measured using five batches of raw material, five acid concentrations, five standing times, (*A*, *B*, *C*, *D*, *E*) and five catalyst concentrations (α, β, γ, δ, ε). The Graeco-Latin square that follows was used. Analyze the data from this experiment (use α = 0.05) and draw conclusions.

	Acid Concentration				
Batch	1	2	3	4	5
1	$A\alpha$=26	$B\beta$=16	$C\gamma$=19	$D\delta$=16	$E\varepsilon$=13
2	$B\gamma$=18	$C\delta$=21	$D\varepsilon$=18	$E\alpha$=11	$A\beta$=21
3	$C\varepsilon$=20	$D\alpha$=12	$E\beta$=16	$A\gamma$=25	$B\delta$=13
4	$D\beta$=15	$E\gamma$=15	$A\delta$=22	$B\varepsilon$=14	$C\alpha$=17
5	$E\delta$=10	$A\varepsilon$=24	$B\alpha$=17	$C\beta$=17	$D\gamma$=14

General Linear Model

```
Factor     Type Levels Values
Time      fixed      5 A B C D E
Catalyst random      5 a b c d e
Batch    random      5 1 2 3 4 5
Acid     random      5 1 2 3 4 5

Analysis of Variance for Yield, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS        F      P
Time        4    342.800    342.800     85.700    14.65  0.001
Catalyst    4     12.000     12.000      3.000     0.51  0.729
Batch       4     10.000     10.000      2.500     0.43  0.785
Acid        4     24.400     24.400      6.100     1.04  0.443
Error       8     46.800     46.800      5.850
Total      24    436.000
```

4-27 Suppose that in Problem 4-19 the engineer suspects that the workplaces used by the four operators may represent an additional source of variation. A fourth factor, workplace (α, β, γ, δ) may be introduced and another experiment conducted, yielding the Graeco-Latin square that follows. Analyze the data from this experiment (use α = 0.05) and draw conclusions.

Order of	Operator			
Assembly	1	2	3	4
1	$C\beta$=11	$B\gamma$=10	$D\delta$=14	$A\alpha$=8
2	$B\alpha$=8	$C\delta$=12	$A\gamma$=10	$D\beta$=12
3	$A\delta$=9	$D\alpha$=11	$B\beta$=7	$C\gamma$=15

4	$D\gamma$=9	$A\beta$=8	$C\alpha$=18	$B\delta$=6

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Method    fixed      4 A B C D
Order    random      4 1 2 3 4
Operator random      4 1 2 3 4
Workplac random      4 a b c d

Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Method      3     95.500     95.500     31.833    3.47  0.167
Order       3      0.500      0.500      0.167    0.02  0.996
Operator    3     19.000     19.000      6.333    0.69  0.616
Workplac    3      7.500      7.500      2.500    0.27  0.843
Error       3     27.500     27.500      9.167
Total      15    150.000
```

However, there are only three degrees of freedom for error, so the test is not very sensitive.

4-28 Construct a 5 x 5 hypersquare for studying the effects of five factors. Exhibit the analysis of variance table for this design.

Three 5 x 5 orthogonal Latin Squares are:

ABCDE	$\alpha\beta\gamma\delta\varepsilon$	12345
BCDEA	$\gamma\delta\varepsilon\alpha\beta$	45123
CDEAB	$\varepsilon\alpha\beta\gamma\delta$	23451
DEABC	$\beta\gamma\delta\varepsilon\alpha$	51234
EABCD	$\delta\varepsilon\alpha\beta\gamma$	34512

Let rows = factor 1, columns = factor 2, Latin letters = factor 3, Greek letters = factor 4 and numbers = factor 5. The analysis of variance table is:

Source	DF
Rows	4
Columns	4
Latin Letters	4
Greek Letters	4
Numbers	4
Error	4
Total	24

4-29 Consider the data in Problems 4-19 and 4-27. Suppressing the Greek letters in 4-27, analyze the data using the method developed in Problem 4-23.

Square 1 - Operator

Batch	1	2	3	4	Row Total
1	*C*=10	*D*=14	*A*=7	*B*=8	(39)
2	*B*=7	*C*=18	*D*=11	*A*=8	(44)
3	*A*=5	*B*=10	*C*=11	*D*=9	(35)
4	*D*=10	*A*=10	*B*=12	*C*=14	(46)
	(32)	(52)	(41)	(36)	164=$y_{...1}$

Square 2 - Operator

Batch	1	2	3	4	Row Total
1	*C*=11	*B*=10	*D*=14	*A*=8	(43)
2	*B*=8	*C*=12	*A*=10	*D*=12	(42)
3	*A*=9	*D*=11	*B*=7	*C*=15	(42)
4	*D*=9	*A*=8	*C*=18	*B*=6	(41)
	(37)	(41)	(49)	(41)	168=$y_{...2}$

Assembly Methods	Totals
A	$y_{.1..}$=65
B	$y_{.2..}$=68
C	$y_{.3..}$=109
D	$y_{.4..}$=90

Source	*SS*	*DF*	*MS*	F_0
Assembly Methods	159.25	3	53.08	14.00*
Squares	0.50	1	0.50	
A x *S*	8.75	3	2.92	0.77
Assembly Order (Rows)	19.00	6	3.17	
Operators (columns)	70.50	6	11.75	
Error	45.50	12	3.79	
Total	303.50	31		

Significant at 1%.

4-30 Consider the randomized block design with one missing value in Problem 4-15. Analyze this data by using the exact analysis of the missing value problem discussed in Section 4-1.4. Compare your results to the approximate analysis of these data given in Table 4-15.

$$
\begin{array}{llllllllllll}
\mu: & 15\hat{\mu} & +4\hat{\tau}_1 & +3\hat{\tau}_2 & +4\hat{\tau}_3 & +4\hat{\tau}_4 & +4\hat{\beta}_1 & +4\hat{\beta}_2 & +3\hat{\beta}_3 & +4\hat{\beta}_4 & =17 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =3 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & +\hat{\beta}_1 & +\hat{\beta}_2 & & +\hat{\beta}_4 & =1 \\
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =-2 \\
\tau_4: & 4\hat{\mu} & & & & +4\hat{\tau}_4 & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =15 \\
\beta_1: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & +4\hat{\beta}_1 & & & & =-4 \\
\beta_2: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & +3\hat{\beta}_2 & & & =-3 \\
\beta_3: & 3\hat{\mu} & +\hat{\tau}_1 & & +\hat{\tau}_3 & +\hat{\tau}_4 & & & +4\hat{\beta}_3 & & =6 \\
\beta_4: & 4\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & & & +4\hat{\beta}_4 & =19
\end{array}
$$

Applying the constraints $\sum \hat{\tau}_i = \sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = \frac{41}{36},\ \hat{\tau}_1 = \frac{-14}{36},\ \hat{\tau}_2 = \frac{-24}{36},\ \hat{\tau}_3 = \frac{-59}{36},\ \hat{\tau}_4 = \frac{94}{36},\ \hat{\beta}_1 = \frac{-77}{36},\ \hat{\beta}_2 = \frac{-68}{36},\ \hat{\beta}_3 = \frac{24}{36},\ \hat{\beta}_4 = \frac{121}{36}$$

$$R(\mu,\tau,\beta)=\hat{\mu}y_{..}+\sum_{i=1}^{4}\hat{\tau}_i y_{i.}+\sum_{j=1}^{4}\hat{\beta}_j y_{.j}=138.78$$

With 7 degrees of freedom.

$$\sum\sum y_{ij}^2=145.00\,,\ SS_E=\sum\sum y_{ij}^2-R(\mu,\tau,\beta)=145.00-138.78=6.22$$

which is identical to SS_E obtained in the approximate analysis. In general, the SS_E in the exact and approximate analyses will be the same.

To test H_o: $\tau_i=0$ the reduced model is $y_{ij}=\mu+\beta_j+\varepsilon_{ij}$. The normal equations used are:

$$\begin{array}{lcccccc}
\mu: & 15\hat{\mu} & +4\hat{\beta}_1 & +4\hat{\beta}_2 & +3\hat{\beta}_3 & +4\hat{\beta}_4 & =17 \\
\beta_1: & 4\hat{\mu} & +4\hat{\beta}_1 & & & & =-4 \\
\beta_2: & 4\hat{\mu} & & +4\hat{\beta}_2 & & & =-3 \\
\beta_3: & 3\hat{\mu} & & & +3\hat{\beta}_3 & & =6 \\
\beta_4: & 4\hat{\mu} & & & & +4\hat{\beta}_4 & =18
\end{array}$$

Applying the constraint $\sum\hat{\beta}_j=0$, we obtain:

$$\hat{\mu}=\frac{19}{16}\ ,\ \hat{\beta}_1=\frac{-35}{16}\,,\ \hat{\beta}_2=\frac{-31}{16}\,,\ \hat{\beta}_3=\frac{13}{16}\,,\ \hat{\beta}_4=\frac{53}{16}\,.\ \text{Now}\ \ R(\mu,\beta)=\hat{\mu}y_{..}+\sum_{j=1}^{4}\hat{\beta}_j y_{.j}=99.25$$

with 4 degrees of freedom.

$$R(\tau|\mu,\beta)=R(\mu,\tau,\beta)-R(\mu,\beta)=138.78-99.25=39.53=SS_{Treatments}$$

with 7-4=3 degrees of freedom. $R(\tau|\mu,\beta)$ is used to test H_o: $\tau_i=0$.

The sum of squares for blocks is found from the reduced model $y_{ij}=\mu+\tau_i+\varepsilon_{ij}$. The normal equations used are:

Model Restricted to $\beta_j=0$:

$$\begin{array}{lcccccc}
\mu: & 15\hat{\mu} & +4\hat{\tau}_1 & +3\hat{\tau}_2 & +4\hat{\tau}_3 & +4\hat{\tau}_4 & =17 \\
\tau_1: & 4\hat{\mu} & +4\hat{\tau}_1 & & & & =3 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & =1 \\
\tau_3: & 4\hat{\mu} & & & +4\hat{\tau}_3 & & =-2 \\
\tau_4: & 4\hat{\mu} & & & & +4\hat{\tau}_4 & =15
\end{array}$$

Applying the constraint $\sum\hat{\tau}_i=0$, we obtain:

$$\hat{\mu}=\frac{13}{12}\,,\ \hat{\tau}_1=\frac{-4}{12}\,,\ \hat{\tau}_2=\frac{-9}{12}\,,\ \hat{\tau}_3=\frac{-19}{12}\,,\ \hat{\tau}_4=\frac{32}{12}$$

$$R(\mu,\tau)=\hat{\mu}y_{..}+\sum_{i=1}^{4}\hat{\tau}_i y_{i.}=59.83$$

with 4 degrees of freedom.

$$R(\beta|\mu,\tau)=R(\mu,\tau,\beta)-R(\mu,\tau)=138.78-59.83=78.95=SS_{Blocks}$$

with 7-4=3 degrees of freedom.

Source	*DF*	*SS*(exact)	*SS*(approximate)
Tips	3	39.53	39.98
Blocks	3	78.95	79.53
Error	8	6.22	6.22
Total	14	125.74	125.73

Note that for the exact analysis, $SS_T \neq SS_{Tips}+SS_{Blocks}+SS_E$.

4-31 An engineer is studying the mileage performance characteristics of five types of gasoline additives. In the road test he wishes to use cars as blocks; however, because of a time constraint, he must use an incomplete block design. He runs the balanced design with the five blocks that follow. Analyze the data from this experiment (use α = 0.05) and draw conclusions.

	Car				
Additive	1	2	3	4	5
1		17	14	13	12
2	14	14		13	10
3	14		13	14	9
4	13	11	11	12	
5	11	12	10		8

There are several computer software packages that can analyze the incomplete block designs discussed in this chapter. The Minitab General Linear Model procedure is a widely available package with this capability. The output from this routine for Problem 4-27 follows. The adjusted sums of squares are the appropriate sums of squares to use for testing the difference between the means of the gasoline additives.

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Additive  fixed      5 1 2 3 4 5
Car      random      5 1 2 3 4 5

Analysis of Variance for Mileage, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Additive    4    31.7000    35.7333     8.9333    9.81  0.001
Car         4    35.2333    35.2333     8.8083    9.67  0.001
Error      11    10.0167    10.0167     0.9106
Total      19    76.9500
```

4-32 Construct a set of orthogonal contrasts for the data in Problem 4-31. Compute the sum of squares for each contrast.

One possible set of orthogonal contrasts is:

$$H_0 : \mu_4 + \mu_5 = \mu_1 + \mu_2 \qquad (1)$$
$$H_0 : \mu_1 = \mu_2 \qquad (2)$$
$$H_0 : \mu_4 = \mu_5 \qquad (3)$$
$$H_0 : 4\mu_3 = \mu_4 + \mu_5 + \mu_1 + \mu_2 \qquad (4)$$

The sums of squares and *F*-tests are:

Brand ->	1	2	3	4	5			
Q_i	33/4	11/4	-3/4	-14/4	-27/4	$\sum c_i Q_i$	*SS*	F_0
(1)	-1	-1	0	1	1	-85/4	30.10	39.09
(2)	1	-1	0	0	0	-22/4	4.03	5.23
(3)	0	0	0	-1	1	-13/4	1.41	1.83
(4)	-1	-1	4	-1	-1	-15/4	0.19	0.25

Contrasts (1) and (2) are significant at the 1% and 5% levels, respectively.

4-33 Seven different hardwood concentrations are being studied to determine their effect on the strength of the paper produced. However the pilot plant can only produce three runs each day. As days may differ, the analyst uses the balanced incomplete block design that follows. Analyze this experiment (use $\alpha = 0.05$) and draw conclusions.

Hardwood	Days						
Concentration (%)	1	2	3	4	5	6	7
2	114				120		117
4	126	120				119	
6		137	114				134
8	141		129	149			
10		145		150	143		
12			120		118	123	
14				136		130	127

There are several computer software packages that can analyze the incomplete block designs discussed in this chapter. The Minitab General Linear Model procedure is a widely available package with this capability. The output from this routine for Problem 4-33 follows. The adjusted sums of squares are the appropriate sums of squares to use for testing the difference between the means of the hardwood concentrations.

Minitab Output

General Linear Model

```
Factor     Type Levels Values
Concentr  fixed      7  2  4  6  8 10 12 14
Days     random      7 1 2 3 4 5 6 7

Analysis of Variance for Strength, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Concentr    6    2037.62    1317.43     219.57   10.42  0.002
Days        6     394.10     394.10      65.68    3.12  0.070
Error       8     168.57     168.57      21.07
Total      20    2600.29
```

4-34 Analyze the data in Example 4-6 using the general regression significance test.

$$
\begin{array}{lllllllllll}
\mu: & 12\hat{\mu} & +3\hat{\tau}_1 & +3\hat{\tau}_2 & +3\hat{\tau}_3 & +3\hat{\tau}_4 & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =870 \\
\tau_1: & 3\hat{\mu} & +3\hat{\tau}_1 & & & & +\hat{\beta}_1 & & +\hat{\beta}_3 & +\hat{\beta}_4 & =218 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & & +\hat{\beta}_2 & +\hat{\beta}_3 & +\hat{\beta}_4 & =214 \\
\tau_3: & 3\hat{\mu} & & & +3\hat{\tau}_3 & & +\hat{\beta}_1 & +\hat{\beta}_2 & +\hat{\beta}_3 & & =216 \\
\tau_4: & 3\hat{\mu} & & & & +3\hat{\tau}_4 & +\hat{\beta}_1 & +\hat{\beta}_2 & & +\hat{\beta}_4 & =222 \\
\beta_1: & 3\hat{\mu} & +\hat{\tau}_1 & & +\hat{\tau}_3 & +\hat{\tau}_4 & +3\hat{\beta}_1 & & & & =221 \\
\beta_2: & 3\hat{\mu} & & +\hat{\tau}_2 & +\hat{\tau}_3 & +\hat{\tau}_4 & & +3\hat{\beta}_2 & & & =207 \\
\beta_3: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & +\hat{\tau}_3 & & & & +3\hat{\beta}_3 & & =224 \\
\beta_4: & 3\hat{\mu} & +\hat{\tau}_1 & +\hat{\tau}_2 & & +\hat{\tau}_4 & & & & +3\hat{\beta}_4 & =218
\end{array}
$$

Applying the constraints $\sum \hat{\tau}_i = \sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = 870/12,\ \hat{\tau}_1 = -9/8,\ \hat{\tau}_2 = -7/8,\ \hat{\tau}_3 = -4/8,\ \hat{\tau}_4 = 20/8,$$
$$\hat{\beta}_1 = 7/8,\ \hat{\beta}_2 = -31/8,\ \hat{\beta}_3 = 24/8,\ \hat{\beta}_4 = 0/8$$
$$R(\mu,\tau,\beta) = \hat{\mu}y_{..} + \sum_{i=1}^{4}\hat{\tau}_i y_{i.} + \sum_{j=1}^{4}\hat{\beta}_j y_{.j} = 63{,}152.75$$

with 7 degrees of freedom.

$$\sum\sum y_{ij}^2 = 63{,}156.00$$
$$SS_E = \sum\sum y_{ij}^2 - R(\mu,\tau,\beta) = 63156.00 - 63152.75 = 3.25.$$

To test H_o: $\tau_i = 0$ the reduced model is $y_{ij} = \mu + \beta_j + \varepsilon_{ij}$. The normal equations used are:

$$
\begin{array}{lllllll}
\mu: & 12\hat{\mu} & +3\hat{\beta}_1 & +3\hat{\beta}_2 & +3\hat{\beta}_3 & +3\hat{\beta}_4 & =870 \\
\beta_1: & 3\hat{\mu} & +3\hat{\beta}_1 & & & & =221 \\
\beta_2: & 3\hat{\mu} & & +3\hat{\beta}_2 & & & =207 \\
\beta_3: & 3\hat{\mu} & & & +3\hat{\beta}_3 & & =224 \\
\beta_4: & 3\hat{\mu} & & & & +3\hat{\beta}_4 & =218
\end{array}
$$

Applying the constraint $\sum \hat{\beta}_j = 0$, we obtain:

$$\hat{\mu} = \frac{870}{12},\ \hat{\beta}_1 = \frac{7}{6},\ \hat{\beta}_2 = \frac{-21}{6},\ \hat{\beta}_3 = \frac{13}{6},\ \hat{\beta}_4 = \frac{1}{6}$$
$$R(\mu,\beta) = \hat{\mu}y_{..} + \sum_{j=1}^{4}\hat{\beta}_j y_{.j} = 63{,}130.00$$

with 4 degrees of freedom.

$$R(\tau|\mu,\beta) = R(\mu,\tau,\beta) - R(\mu,\beta) = 63152.75 - 63130.00 = 22.75 = SS_{Treatments}$$

with 7 – 4 = 3 degrees of freedom. $R(\tau|\mu,\beta)$ is used to test H_o: $\tau_i = 0$.

The sum of squares for blocks is found from the reduced model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$. The normal equations used are:

Model Restricted to $\beta_j = 0$:

$$\begin{array}{lllllll}
\mu: & 12\hat{\mu} & +3\hat{\tau}_1 & +3\hat{\tau}_2 & +3\hat{\tau}_3 & +3\hat{\tau}_4 & =870 \\
\tau_1: & 3\hat{\mu} & +3\hat{\tau}_1 & & & & =218 \\
\tau_2: & 3\hat{\mu} & & +3\hat{\tau}_2 & & & =214 \\
\tau_3: & 3\hat{\mu} & & & +3\hat{\tau}_3 & & =216 \\
\tau_4: & 3\hat{\mu} & & & & +3\hat{\tau}_4 & =222
\end{array}$$

The sum of squares for blocks is found as in Example 4-6. We may use the method shown above to find an adjusted sum of squares for blocks from the reduced model, $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$.

4-35 Prove that $\dfrac{k\sum_{i=1}^{a} Q_i^2}{(\lambda a)}$ is the adjusted sum of squares for treatments in a BIBD.

We may use the general regression significance test to derive the computational formula for the adjusted treatment sum of squares. We will need the following:

$$\hat{\tau}_i = \frac{kQ_i}{(\lambda a)},\ kQ_i = ky_{i.} - \sum_{i=1}^{b} n_{ij} y_{.j}$$

$$R(\mu,\tau,\beta) = \hat{\mu} y_{..} + \sum_{i=1}^{a} \hat{\tau}_i y_{i.} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j}$$

and the sum of squares we need is:

$$R(\tau|\mu,\beta) = \hat{\mu} y_{..} + \sum_{i=1}^{a} \hat{\tau}_i y_{i.} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j} - \sum_{j=1}^{b} \frac{y_{.j}^2}{k}$$

The normal equation for β is, from equation (4-35),

$$\beta : k\hat{\mu} + \sum_{i=1}^{a} n_{ij}\hat{\tau}_i + k\hat{\beta}_j = y_{.j}$$

and from this we have:

$$ky_{.j}\hat{\beta}_j = y_{.j}^2 - ky_{.j}\hat{\mu} - y_{.j}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i$$

therefore,

$$R(\tau|\mu,\beta) = \hat{\mu}y_{..} + \sum_{i=1}^{a}\hat{\tau}_i y_{i.} + \sum_{j=1}^{b}\left[\frac{y_{.j}^2}{k} - \frac{k\hat{\mu}y_{.j}}{k} - \frac{y_{.j}\sum_{i=1}^{a} n_{ij}\hat{\tau}_i}{k} - \frac{y_{.j}^2}{k}\right]$$

$$R(\tau|\mu,\beta) = \sum_{i=1}^{a}\hat{\tau}_i\left(y_{i.} - \frac{1}{k}\sum_{i=1}^{a} n_{ij}y_{.j}\right) = \sum_{i=1}^{a} Q_i\left(\frac{kQ}{\lambda a}\right) = k\sum_{i=1}^{a}\left(\frac{Q_i^2}{\lambda a}\right) \equiv SS_{Treatments(adjusted)}$$

4-36 An experimenter wishes to compare four treatments in blocks of two runs. Find a BIBD for this experiment with six blocks.

Treatment	Block 1	Block 2	Block 3	Block 4	Block 5	Block 6
1	X	X	X			
2	X			X	X	
3		X		X		X
4			X		X	X

Note that the design is formed by taking all combinations of the 4 treatments 2 at a time. The parameters of the design are $\lambda = 1$, $a = 4$, $b = 6$, $k = 3$, and $r = 2$

4-37 An experimenter wishes to compare eight treatments in blocks of four runs. Find a BIBD with 14 blocks and $\lambda = 3$.

The design has parameters $a = 8$, $b = 14$, $\lambda = 3$, $r = 2$ and $k = 4$. It may be generated from a 2^3 factorial design confounded in two blocks of four observations each, with each main effect and interaction successively confounded (7 replications) forming the 14 blocks. The design is discussed by John (1971, pg. 222) and Cochran and Cox (1957, pg. 473). The design follows:

Blocks	1=(*I*)	2=*a*	3=*b*	4=*ab*	5=*c*	6=*ac*	7=*bc*	8=*abc*
1	X		X		X		X	
2		X		X		X		X
3	X		X			X		X
4		X		X	X		X	
5	X	X			X	X		
6			X	X			X	X
7	X	X					X	X
8			X	X	X	X		
9	X	X	X	X				
10					X	X	X	X
11	X			X		X	X	
12		X	X		X			X
13	X			X	X			X
14		X	X			X	X	

4-38 Perform the interblock analysis for the design in Problem 4-31.

The interblock analysis for Problem 4-31 uses $\hat{\sigma}^2 = 0.77$ and $\hat{\sigma}_\beta^2 = 2.14$. A summary of the interblock, intrablock and combined estimates is:

Parameter	Intrablock	Interblock	Combined
τ_1	2.20	-1.80	2.18
τ_2	0.73	0.20	0.73
τ_3	-0.20	-5.80	-0.23
τ_4	-0.93	9.20	-0.88
τ_5	-1.80	-1.80	-1.80

4-39 Perform the interblock analysis for the design in Problem 4-33. The interblock analysis for problem 4-33 uses $\hat{\sigma}^2 = 21.07$ and $\sigma_\beta^2 = \frac{\left[MS_{Blocks(adj)} - MS_E\right](b-1)}{a(r-1)} = \frac{\left[65.68 - 21.07\right](6)}{7(2)} = 19.12$. A summary of the interblock, intrablock, and combined estimates is give below

Parameter	Intrablock	Interblock	Combined
τ_1	-12.43	-11.79	-12.38
τ_2	-8.57	-4.29	-7.92
τ_3	2.57	-8.79	1.76
τ_4	10.71	9.21	10.61
τ_5	13.71	21.21	14.67
τ_6	-5.14	-22.29	-6.36
τ_7	-0.86	10.71	-0.03

4-40 Verify that a BIBD with the parameters a = 8, r = 8, k = 4, and b = 16 does not exist. These conditions imply that $\lambda = \frac{r(k-1)}{a-1} = \frac{8(3)}{7} = \frac{24}{7}$, which is not an integer, so a balanced design with these parameters cannot exist.

4-41 Show that the variance of the intra block estimators $\{\hat{\tau}_i\}$ is $\frac{k((a-1))\sigma^2}{(\lambda a^2)}$.

Note that $\hat{\tau}_i = \frac{kQ_i}{(\lambda a)}$, and $Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j}$, and $kQ_i = ky_{i.} - \sum_{j=1}^{b} n_{ij} y_{.j} = (k-1)y_{i.} - \left(\sum_{j=1}^{b} n_{ij} y_{.j} - y_{i.}\right)$

$y_{i.}$ contains r observations, and the quantity in the parenthesis is the sum of $r(k$-1) observations, not including treatment i. Therefore,

$$V(kQ_i) = k^2 V(Q_i) = r(k-1)^2\sigma^2 + r(k-1)\sigma^2$$

or

$$V(Q_i)=\frac{1}{k^2}\left[r(k-1)\sigma^2\{(k-1)+1\}\right]=\frac{r(k-1)\sigma^2}{k}$$

To find $V(\hat{\tau}_i)$, note that:

$$V(\hat{\tau}_i)=\left(\frac{k}{\lambda a}\right)^2 V(Q)_i=\left(\frac{k}{\lambda a}\right)^2\frac{r(k-1)}{k}\sigma^2=\frac{kr(k-1)}{(\lambda a)^2}\sigma^2$$

However, since $\lambda(a-1)=r(k-1)$, we have:

$$V(\hat{\tau}_i)=\frac{k(a-1)}{\lambda a^2}\sigma^2$$

Furthermore, the $\{\hat{\tau}_i\}$ are not independent, this is required to show that $V(\hat{\tau}_i-\hat{\tau}_j)=\frac{2k}{\lambda a}\sigma^2$

4-42 ***Extended incomplete block designs.*** Occasionally the block size obeys the relationship $a < k < 2a$. An extended incomplete block design consists of a single replicate or each treatment in each block along with an incomplete block design with $k^* = k-a$. In the balanced case, the incomplete block design will have parameters $k^* = k-a$, $r^* = r-b$, and λ^*. Write out the statistical analysis. (Hint: In the extended incomplete block design, we have $\lambda = 2r-b+\lambda^*$.)

As an example of an extended incomplete block design, suppose we have a=5 treatments, b=5 blocks and k=9. A design could be found by running all five treatments in each block, plus a block from the balanced incomplete block design with $k^* = k-a$=9-5=4 and λ^*=3. The design is:

Block	Complete Treatment	Incomplete Treatment
1	1,2,3,4,5	2,3,4,5
2	1,2,3,4,5	1,2,4,5
3	1,2,3,4,5	1,3,4,5
4	1,2,3,4,5	1,2,3,4
5	1,2,3,4,5	1,2,3,5

Note that r=9, since the augmenting incomplete block design has r*=4, and $r = r^* + b$ = 4+5=9, and $\lambda = 2r-b+\lambda^*$=18-5+3=16. Since some treatments are repeated in each block it is possible to compute an error sum of squares between repeat observations. The difference between this and the residual sum of squares is due to interaction. The analysis of variance table is shown below:

Source	*SS*	*DF*
Treatments (adjusted)	$k\sum\frac{Q_i^2}{a\lambda}$	a-1
Blocks	$\sum\frac{y_{.j}^2}{k}-\frac{y_{..}^2}{N}$	b-1
Interaction	Subtraction	(a-1)(b-1)
Error	[SS between repeat observations]	b(k-a)

Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$	N-1

Chapter 5
Introduction to Factorial Designs
Solutions

5-1 The yield of a chemical process is being studied. The two most important variables are thought to be the pressure and the temperature. Three levels of each factor are selected, and a factorial experiment with two replicates is performed. The yield data follow:

	Pressure		
Temperature	200	215	230
150	90.4	90.7	90.2
	90.2	90.6	90.4
160	90.1	90.5	89.9
	90.3	90.6	90.1
170	90.5	90.8	90.4
	90.7	90.9	90.1

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Both pressure (A) and temperature (B) are significant, the interaction is not.

Design Expert Output

Response:Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

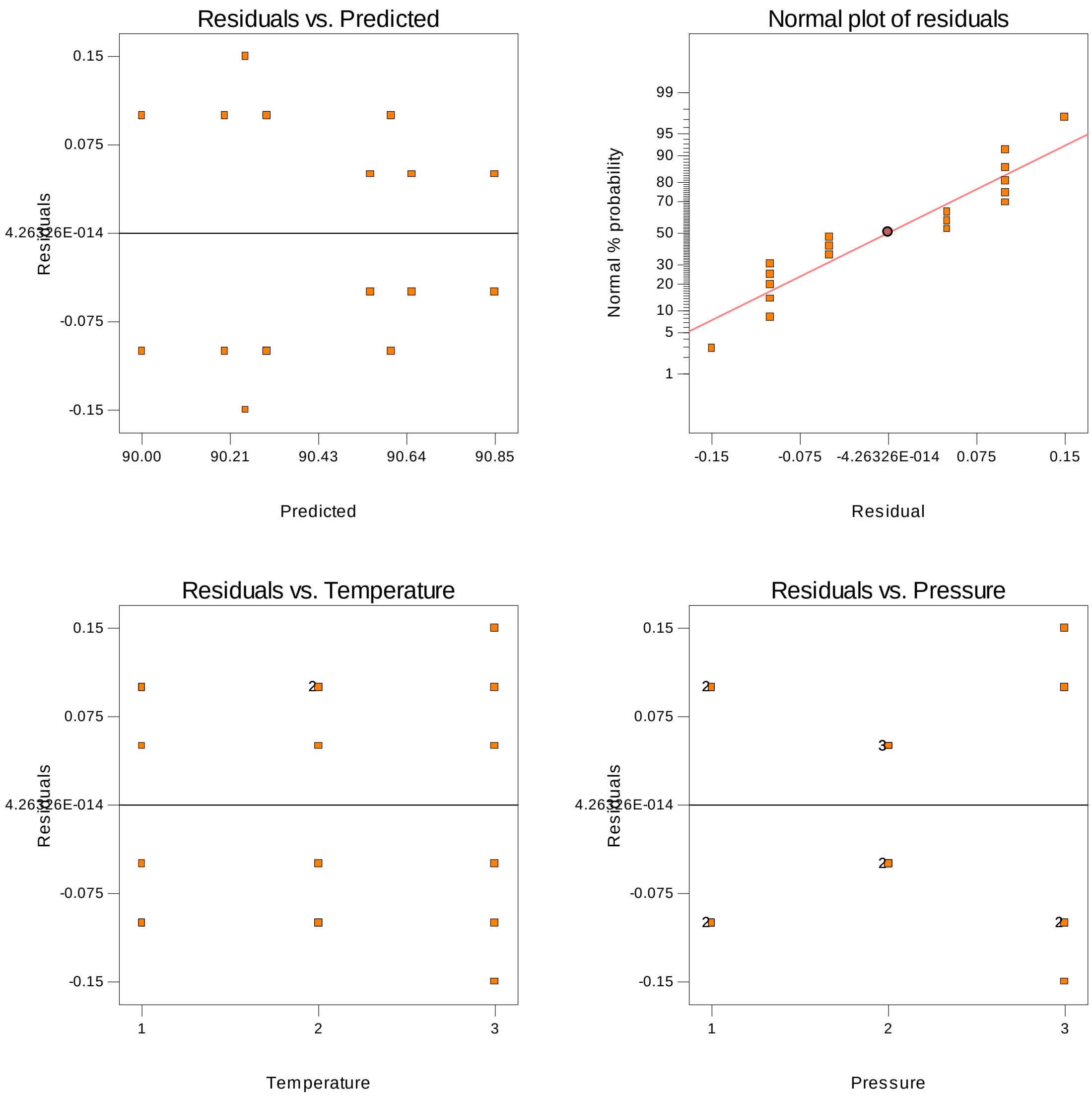
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.14	8	0.14	8.00	0.0026	significant
A	*0.77*	*2*	*0.38*	*21.59*	*0.0004*	
B	*0.30*	*2*	*0.15*	*8.47*	*0.0085*	
AB	*0.069*	*4*	*0.017*	*0.97*	*0.4700*	
Residual	0.16	9	0.018			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.16*	*9*	*0.018*			
Cor Total	1.30	17				

The Model F-value of 8.00 implies the model is significant. There is only a 0.26% chance that a "Model F-Value" this large could occur due to noise.

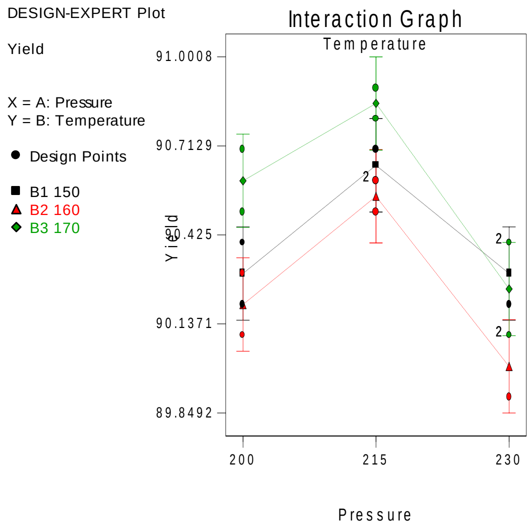
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residual plots show no serious deviations from the assumptions.



(c) Under what conditions would you operate this process?



Set pressure at 215 and Temperature at the high level, 170 degrees C, as this gives the highest yield.

The standard analysis of variance treats all design factors as if they were qualitative. In this case, both factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since both factors in this problem are quantitative and have three levels, we can fit linear and quadratic effects of both temperature and pressure, exactly as in Example 5-5 in the text. The Design-Expert output, including the response surface plots, now follows.

Design Expert Output

Response:Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.13	5	0.23	16.18	< 0.0001	significant
A	*0.10*	*1*	*0.10*	*7.22*	*0.0198*	
B	*0.067*	*1*	*0.067*	*4.83*	*0.0483*	
A2	*0.67*	*1*	*0.67*	*47.74*	*< 0.0001*	
B2	*0.23*	*1*	*0.23*	*16.72*	*0.0015*	
AB	*0.061*	*1*	*0.061*	*4.38*	*0.0582*	
Residual	0.17	12	0.014			
Lack of Fit	*7.639E-003*	*3*	*2.546E-003*	*0.14*	*0.9314*	*not significant*
Pure Error	*0.16*	*9*	*0.018*			
Cor Total	1.30	17				

The Model F-value of 16.18 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, A^2, B^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy),
model reduction may improve your model.

Std. Dev.	0.12	R-Squared	0.8708
Mean	90.41	Adj R-Squared	0.8170
C.V.	0.13	Pred R-Squared	0.6794
PRESS	0.42	Adeq Precision	11.968

	Coefficient		Standard	95% CI	95% CI

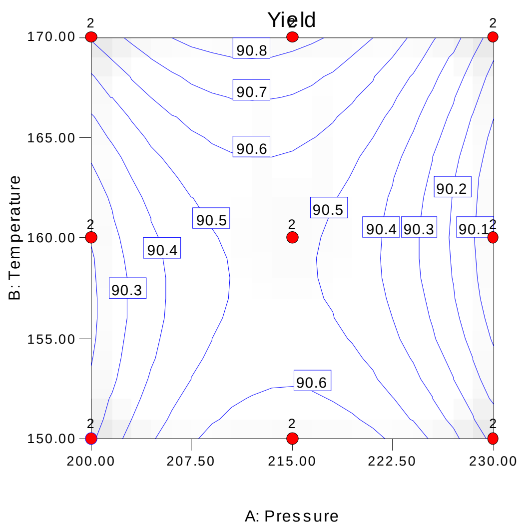
Factor	Estimate	DF	Error	Low	High	VIF
Intercept	90.52	1	0.062	90.39	90.66	
A-Pressure	-0.092	1	0.034	-0.17	-0.017	1.00
B-Temperature	0.075	1	0.034	6.594E-004	0.15	1.00
A^2	-0.41	1	0.059	-0.54	-0.28	1.00
B^2	0.24	1	0.059	0.11	0.37	1.00
AB	-0.087	1	0.042	-0.18	3.548E-003	1.00

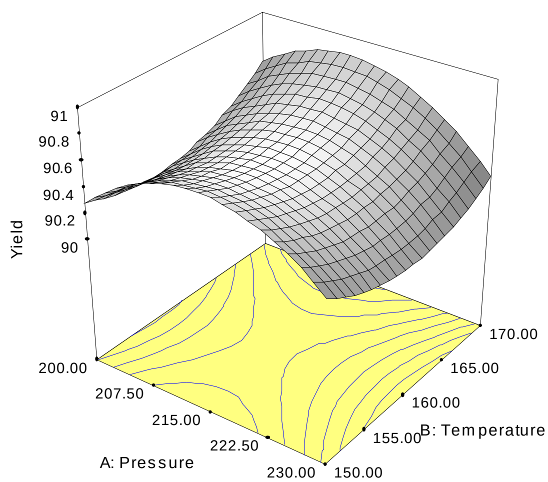
Final Equation in Terms of Coded Factors:

Yield =
+90.52
-0.092 * A
+0.075 * B
-0.41 * A^2
+0.24 * B^2
-0.087 * A * B

Final Equation in Terms of Actual Factors:

Yield =
+48.54630
+0.86759 * Pressure
-0.64042 * Temperature
-1.81481E-003 * Pressure^2
+2.41667E-003 * Temperature^2
-5.83333E-004 * Pressure * Temperature





5-2 An engineer suspects that the surface finish of a metal part is influenced by the feed rate and the depth of cut. She selects three feed rates and four depths of cut. She then conducts a factorial experiment and obtains the following data:

	Depth of Cut (in)			
Feed Rate (in/min)	0.15	0.18	0.20	0.25
	74	79	82	99
0.20	64	68	88	104
	60	73	92	96
	92	98	99	104

0.25	86	104	108	110
	88	88	95	99
	99	104	108	114
0.30	98	99	110	111
	102	95	99	107

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

The depth (A) and feed rate (B) are significant, as is the interaction (AB).

Design Expert Output

Response: Surface Finish
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

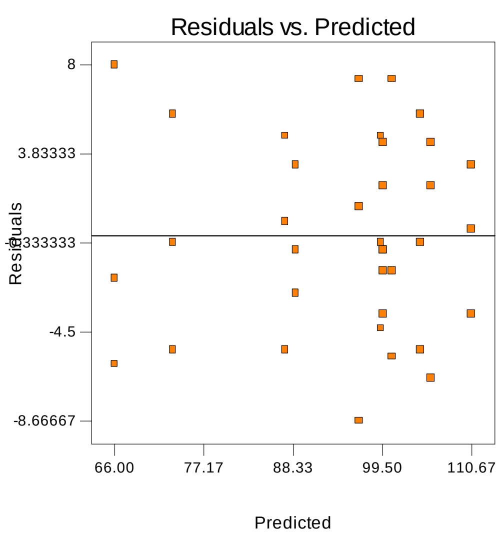
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5842.67	11	531.15	18.49	< 0.0001	significant
A	*2125.11*	*3*	*708.37*	*24.66*	*< 0.0001*	
B	*3160.50*	*2*	*1580.25*	*55.02*	*< 0.0001*	
AB	*557.06*	*6*	*92.84*	*3.23*	*0.0180*	
Residual	689.33	24	28.72			
Lack of Fit	*0.000*	*0*				
Pure Error	*689.33*	*24*	*28.72*			
Cor Total	6532.00	35				

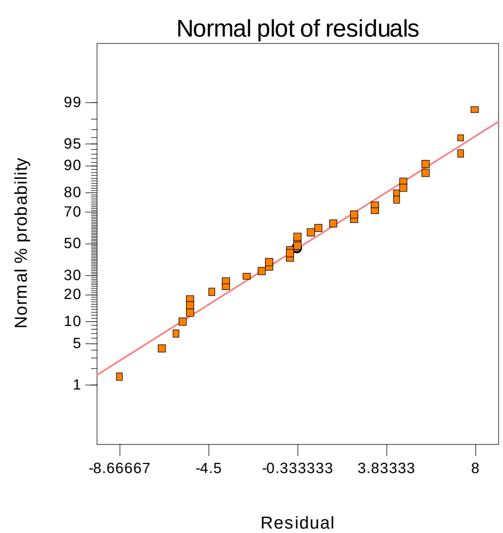
The Model F-value of 18.49 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

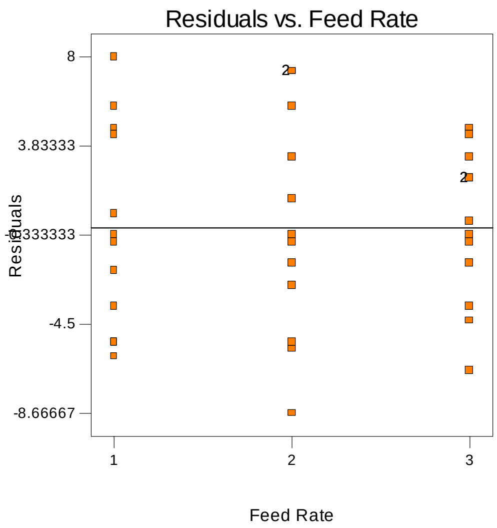
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, AB are significant model terms.

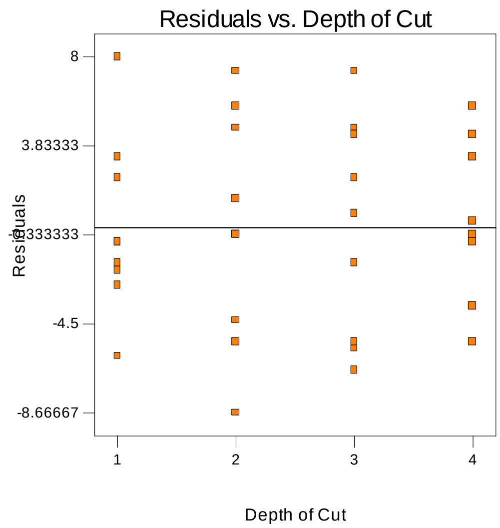
(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residual plots shown indicate nothing unusual.



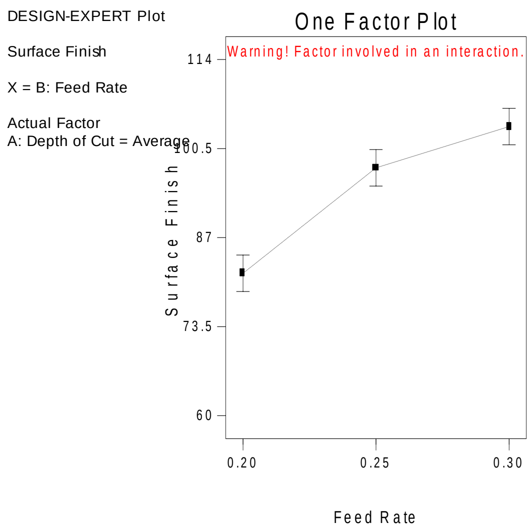






(c) Obtain point estimates of the mean surface finish at each feed rate.

Feed Rate	Average
0.20	81.58
0.25	97.58
0.30	103.83



(d) Find *P*-values for the tests in part (a).

The *P*-values are given in the computer output in part (a).

5-3 For the data in Problem 5-2, compute a 95 percent interval estimate of the mean difference in response for feed rates of 0.20 and 0.25 in/min.

We wish to find a confidence interval on $\mu_1 - \mu_2$, where μ_1 is the mean surface finish for 0.20 in/min and μ_2 is the mean surface finish for 0.25 in/min.

$$\bar{y}_{1..} - \bar{y}_{2..} - t_{\alpha/2,ab(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_1 - \mu_2 \le \bar{y}_{1..} - \bar{y}_{2..} + t_{\alpha/2,ab(n-1))}\sqrt{\frac{2MS_E}{n}}$$

$$(81.5833 - 97.5833) \pm (2.064)\sqrt{\frac{2(28.7222)}{3}} = -16 \pm 9.032$$

Therefore, the 95% confidence interval for $\mu_1 - \mu_2$ is -16.000 ± 9.032.

5-4 An article in *Industrial Quality Control* (1956, pp. 5-8) describes an experiment to investigate the effect of the type of glass and the type of phosphor on the brightness of a television tube. The response variable is the current necessary (in microamps) to obtain a specified brightness level. The data are as follows:

Glass	Phosphor Type		
Type	1	2	3
	280	300	290
1	290	310	285
	285	295	290
	230	260	220
2	235	240	225
	240	235	230

(a) Is there any indication that either factor influences brightness? Use $\alpha = 0.05$.

Both factors, phosphor type (*A*) and Glass type (*B*) influence brightness.

Design Expert Output

Response: Current in microamps
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	15516.67	5	3103.33	58.80	< 0.0001	significant
A	*933.33*	*2*	*466.67*	*8.84*	*0.0044*	
B	*14450.00*	*1*	*14450.00*	*273.79*	*< 0.0001*	
AB	*133.33*	*2*	*66.67*	*1.26*	*0.3178*	
Residual	633.33	12	52.78			
Lack of Fit	*0.000*	*0*				
Pure Error	*633.33*	*12*	*52.78*			
Cor Total	16150.00	17				

The Model F-value of 58.80 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

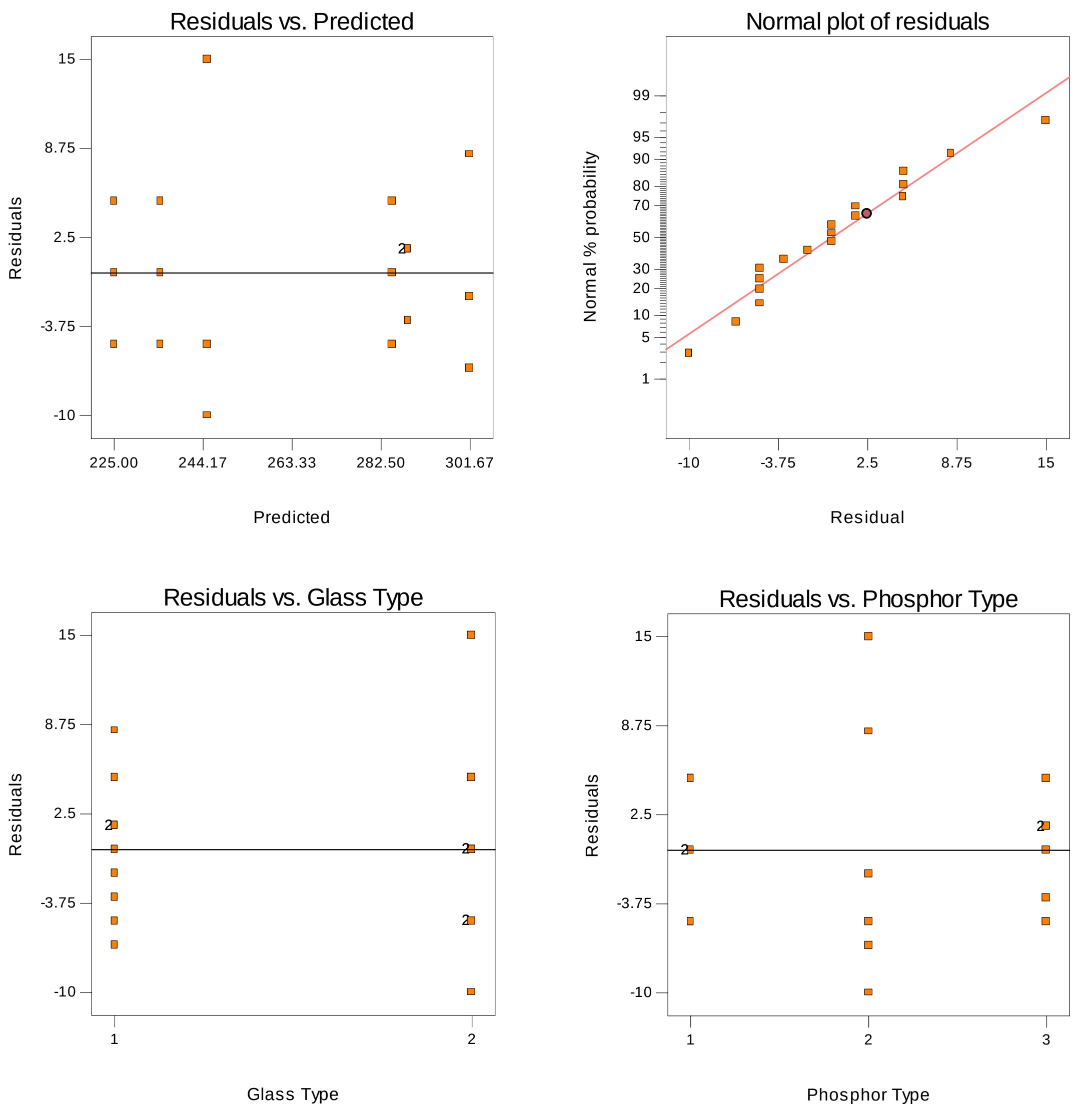
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

(b) Do the two factors interact? Use $\alpha = 0.05$.

There is no interaction effect.

(c) Analyze the residuals from this experiment.

The residual plot of residuals versus phosphor content indicates a very slight inequality of variance. It is not serious enough to be of concern, however.



5-5 Johnson and Leone (*Statistics and Experimental Design in Engineering and the Physical Sciences*, Wiley 1977) describe an experiment to investigate the warping of copper plates. The two factors studies were the temperature and the copper content of the plates. The response variable was a measure of the amount of warping. The data were as follows:

	Copper	Content (%)		
Temperature (°C)	40	60	80	100
50	17,20	16,21	24,22	28,27
75	12,9	18,13	17,12	27,31
100	16,12	18,21	25,23	30,23
125	21,17	23,21	23,22	29,31

(a) Is there any indication that either factor affects the amount of warping? Is there any interaction between the factors? Use $\alpha = 0.05$.

Both factors, copper content (A) and temperature (B) affect warping, the interaction does not.

Design Expert Output

Response: Warping
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

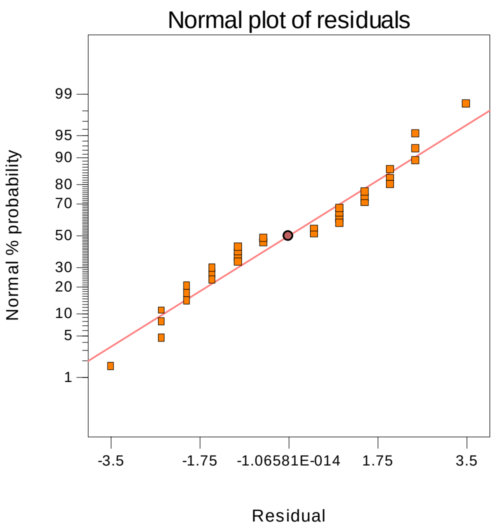
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	968.22	15	64.55	9.52	< 0.0001	significant
A	*698.34*	*3*	*232.78*	*34.33*	*< 0.0001*	
B	*156.09*	*3*	*52.03*	*7.67*	*0.0021*	
AB	*113.78*	*9*	*12.64*	*1.86*	*0.1327*	
Residual	108.50	16	6.78			
Lack of Fit	*0.000*	*0*				
Pure Error	*108.50*	*16*	*6.78*			
Cor Total	1076.72	31				

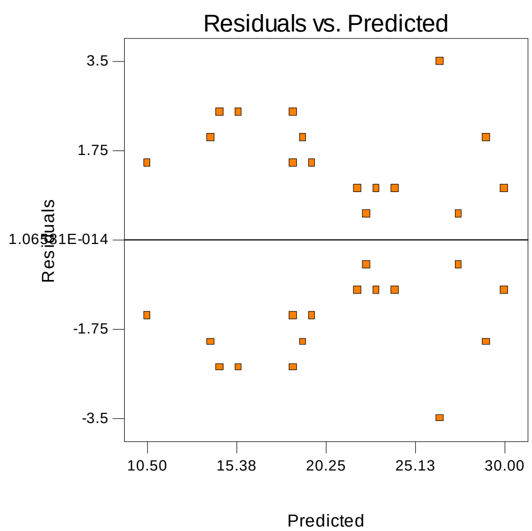
The Model F-value of 9.52 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

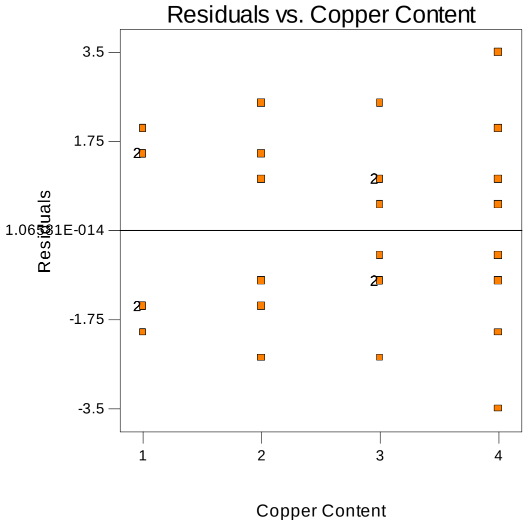
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

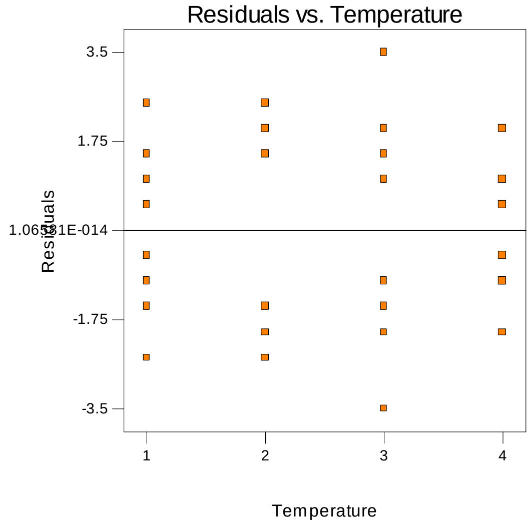
(b) Analyze the residuals from this experiment.

There is nothing unusual about the residual plots.









(c) Plot the average warping at each level of copper content and compare them to an appropriately scaled *t* distribution. Describe the differences in the effects of the different levels of copper content on warping. If low warping is desirable, what level of copper content would you specify?

Design Expert Output

Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**40**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	15.50	1.84	11.60	19.40	3.19	8.74	22.26

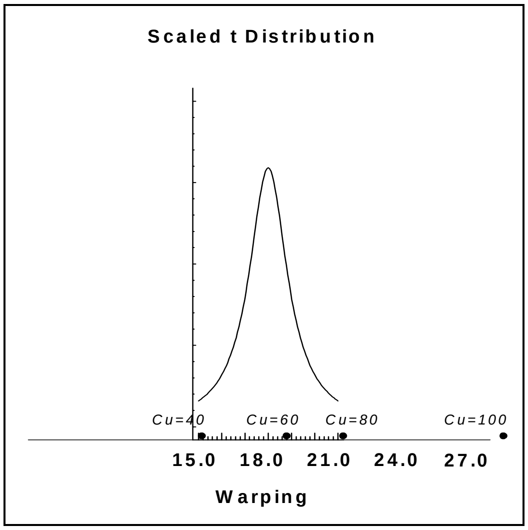
Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**60**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	18.88	1.84	14.97	22.78	3.19	12.11	25.64

Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**80**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	21.00	1.84	17.10	24.90	3.19	14.24	27.76

Factor	**Name**	**Level**	**Low Level**	**High Level**			
A	**Copper Content**	**100**	**40**	**100**			
B	**Temperature**	**Average**	**50**	**125**			
	Prediction	**SE Mean**	**95% CI low**	**95% CI high**	**SE Pred**	**95% PI low**	**95% PI high**
Warping	28.25	1.84	24.35	32.15	3.19	21.49	35.01

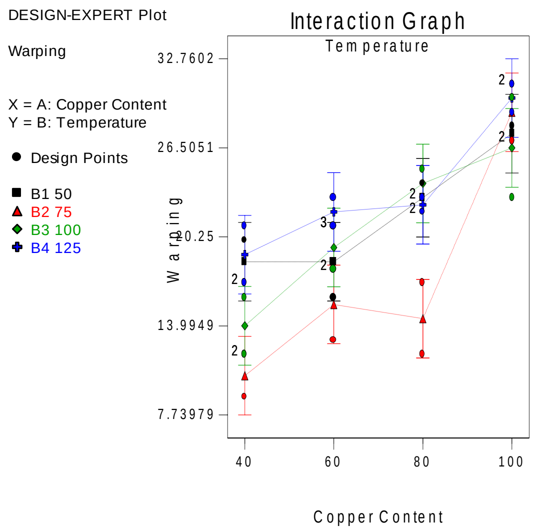
Use a copper content of 40 for the lowest warping.

$$S = \sqrt{\frac{MS_E}{b}} = \sqrt{\frac{6.78125}{8}} = 0.92$$



(d) Suppose that temperature cannot be easily controlled in the environment in which the copper plates are to be used. Does this change your answer for part (c)?

Use a copper of content of 40. This is the same as for part (c).



5-6 The factors that influence the breaking strength of a synthetic fiber are being studied. Four production machines and three operators are chosen and a factorial experiment is run using fiber from the same production batch. The results are as follows:

	Machine			
Operator	1	2	3	4
1	109	110	108	110
	110	115	109	108
2	110	110	111	114
	112	111	109	112
3	116	112	114	120

114	115	119	117

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Only the Operator (A) effect is significant.

Design Expert Output

Response:Stength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

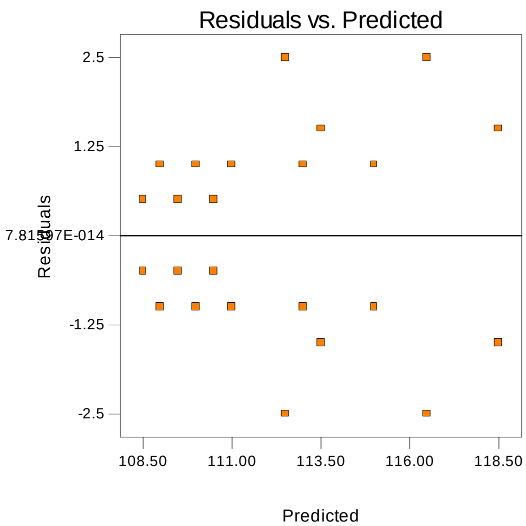
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	217.46	11	19.77	5.21	0.0041	significant
A	*160.33*	*2*	*80.17*	*21.14*	*0.0001*	
B	*12.46*	*3*	*4.15*	*1.10*	*0.3888*	
AB	*44.67*	*6*	*7.44*	*1.96*	*0.1507*	
Residual	45.50	12	3.79			
Lack of Fit	*0.000*	*0*				
Pure Error	*45.50*	*12*	*3.79*			
Cor Total	262.96	23				

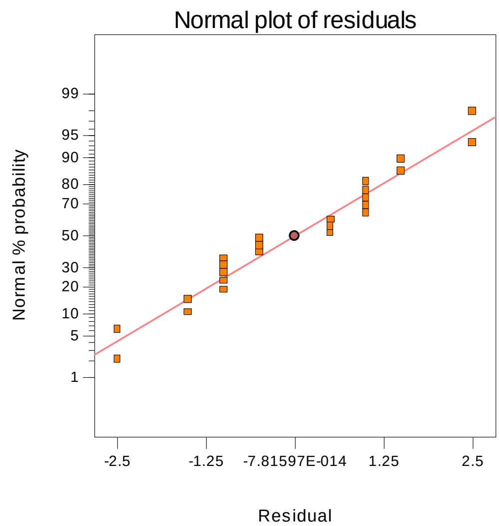
The Model F-value of 5.21 implies the model is significant.
There is only a 0.41% chance that a "Model F-Value" this large could occur due to noise.

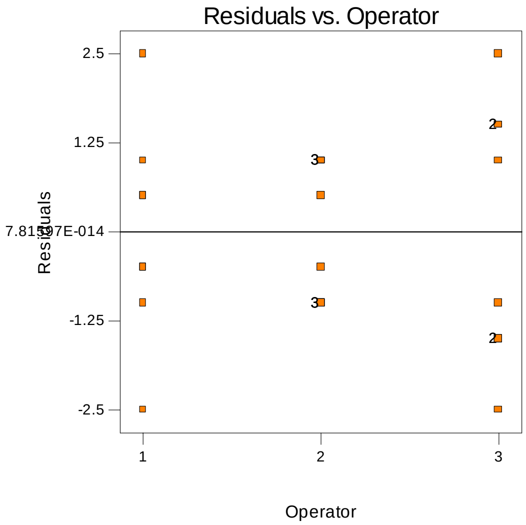
Values of "Prob > F" less than 0.0500 indicate model terms aresignificant.
In this case A are significant model terms.

(b) Prepare appropriate residual plots and comment on the model's adequacy.

The residual plot of residuals versus predicted shows that variance increases very slightly with strength. There is no indication of a severe problem.







5-7 A mechanical engineer is studying the thrust force developed by a drill press. He suspects that the drilling speed and the feed rate of the material are the most important factors. He selects four feed rates and uses a high and low drill speed chosen to represent the extreme operating conditions. He obtains the following results. Analyze the data and draw conclusions. Use $\alpha = 0.05$.

(A)		Feed	Rate (B)	
Drill Speed	0.015	0.030	0.045	0.060
125	2.70	2.45	2.60	2.75
	2.78	2.49	2.72	2.86
200	2.83	2.85	2.86	2.94
	2.86	2.80	2.87	2.88

Design Expert Output

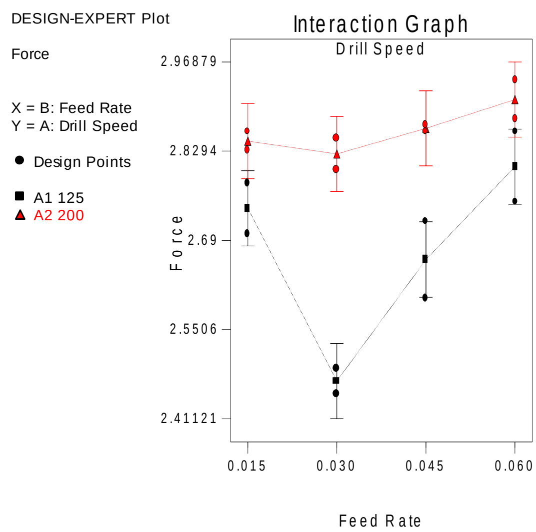
Response: Force
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.28	7	0.040	15.53	0.0005	significant
A	*0.15*	*1*	*0.15*	*57.01*	*< 0.0001*	
B	*0.092*	*3*	*0.031*	*11.86*	*0.0026*	
AB	*0.042*	*3*	*0.014*	*5.37*	*0.0256*	
Residual	0.021	8	2.600E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.021*	*8*	*2.600E-003*			
Cor Total	0.30	15				

The Model F-value of 15.53 implies the model is significant.
There is only a 0.05% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

The factors speed and feed rate, as well as the interaction is important.



The standard analysis of variance treats all design factors as if they were qualitative. In this case, both factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since both factors in this problem are quantitative, we can fit polynomial effects of both speed and feed rate, exactly as in Example 5-5 in the text. The Design-Expert output with only the significant terms retained, including the response surface plots, now follows.

Design Expert Output

Response: Force
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	0.23	3	0.075	11.49	0.0008	significant
A	*0.15*	*1*	*0.15*	*22.70*	*0.0005*	
B	*0.019*	*1*	*0.019*	*2.94*	*0.1119*	
B2	*0.058*	*1*	*0.058*	*8.82*	*0.0117*	
Residual	0.078	12	6.530E-003			
Lack of Fit	*0.058*	*4*	*0.014*	*5.53*	*0.0196*	*significant*
Pure Error	*0.021*	*8*	*2.600E-003*			
Cor Total	0.30	15				

The Model F-value of 11.49 implies the model is significant. There is only a 0.08% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

Std. Dev.	0.081	R-Squared	0.7417
Mean	2.77	Adj R-Squared	0.6772
C.V.	2.92	Pred R-Squared	0.5517
PRESS	0.14	Adeq Precision	9.269

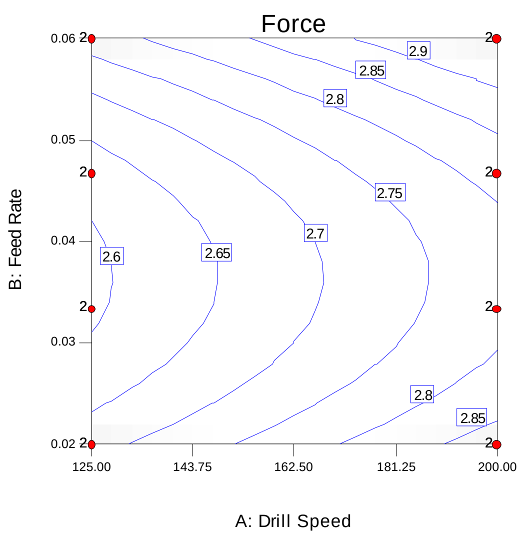
Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	2.69	1	0.032	2.62	2.76	
A-Drill Speed	0.096	1	0.020	0.052	0.14	1.00
B-Feed Rate	0.047	1	0.027	-0.013	0.11	1.00
B2	0.13	1	0.045	0.036	0.23	1.00

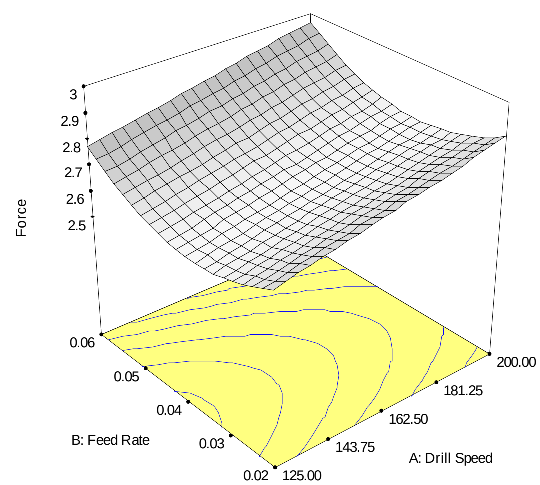
Final Equation in Terms of Coded Factors:

Force	=
+2.69	
+0.096	* A
+0.047	* B
+0.13	* B^2

Final Equation in Terms of Actual Factors:

Force	=
+2.48917	
+3.06667E-003	* Drill Speed
-15.76667	* Feed Rate
+266.66667	* Feed Rate^2





5-8 An experiment is conducted to study the influence of operating temperature and three types of face-plate glass in the light output of an oscilloscope tube. The following data are collected:

	Temperature		
Glass Type	100	125	150
	580	1090	1392
1	568	1087	1380
	570	1085	1386
	550	1070	1328
2	530	1035	1312
	579	1000	1299
	546	1045	867
3	575	1053	904
	599	1066	889

Use α = 0.05 in the analysis. Is there a significant interaction effect? Does glass type or temperature affect the response? What conclusions can you draw? Use the method discussed in the text to partition the temperature effect into its linear and quadratic components. Break the interaction down into appropriate components.

Design Expert Output

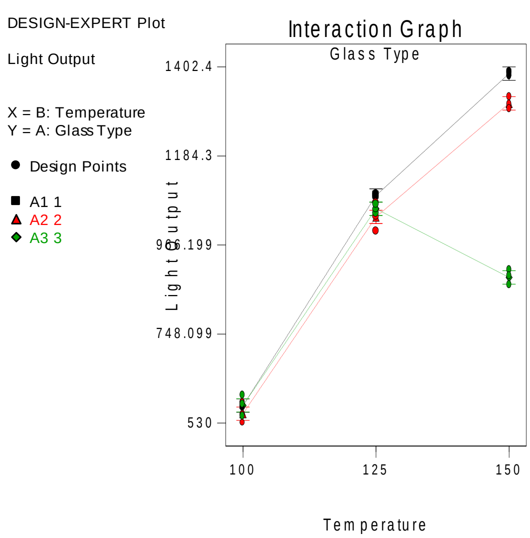
Response: Light Output
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.412E+006	8	3.015E+005	824.77	< 0.0001	significant
A	*1.509E+005*	*2*	*75432.26*	*206.37*	*< 0.0001*	
B	*1.970E+006*	*2*	*9.852E+005*	*2695.26*	*< 0.0001*	
AB	*2.906E+005*	*4*	*72637.93*	*198.73*	*< 0.0001*	
Residual	6579.33	18	365.52			
Lack of Fit	*0.000*	*0*				
Pure Error	*6579.33*	*18*	*365.52*			
Cor Total	2.418E+006	26				

The Model F-value of 824.77 implies the model is significant.
There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

Both factors, Glass Type (A) and Temperature (B) are significant, as well as the interaction (AB). For glass types 1 and 2 the response is fairly linear, for glass type 3, there is a quadratic effect.



Design Expert Output

Response: Light Output
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.412E+006	8	3.015E+005	824.77	< 0.0001	significant
A	*1.509E+005*	*2*	*75432.26*	*206.37*	*< 0.0001*	
B	*1.780E+006*	*1*	*1.780E+006*	*4869.13*	*< 0.0001*	
B2	*1.906E+005*	*1*	*1.906E+005*	*521.39*	*< 0.0001*	
AB	*2.262E+005*	*2*	*1.131E+005*	*309.39*	*< 0.0001*	
AB2	*64373.93*	*2*	*32186.96*	*88.06*	*< 0.0001*	

Pure Error	*6579.33*	*18*	*365.52*
Cor Total	2.418E+006	26	

The Model F-value of 824.77 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, B^2, AB, AB^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

Std. Dev.	19.12	R-Squared	0.9973
Mean	940.19	Adj R-Squared	0.9961
C.V.	2.03	Pred R-Squared	0.9939
PRESS	14803.50	Adeq Precision	75.466

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	1059.00	1	6.37	1045.61	1072.39	
A[1]	28.33	1	9.01	9.40	47.27	
A[2]	-24.00	1	9.01	-42.93	-5.07	
B-Temperature	314.44	1	4.51	304.98	323.91	1.00
B2	-178.22	1	7.81	-194.62	-161.82	1.00
A[1]B	92.22	1	6.37	78.83	105.61	
A[2]B	65.56	1	6.37	52.17	78.94	
A[1]B2	70.22	1	11.04	47.03	93.41	
A[2]B2	76.22	1	11.04	53.03	99.41	

Final Equation in Terms of Coded Factors:

Light Output	=
+1059.00	
+28.33	* A[1]
-24.00	* A[2]
+314.44	* B
-178.22	* B2
+92.22	* A[1]B
+65.56	* A[2]B
+70.22	* A[1]B2
+76.22	* A[2]B2

Final Equation in Terms of Actual Factors:

Glass Type	1
Light Output	=
-3646.00000	
+59.46667	* Temperature
-0.17280	* Temperature2
Glass Type	2
Light Output	=
-3415.00000	
+56.00000	* Temperature
-0.16320	* Temperature2
Glass Type	3
Light Output	=
-7845.33333	
+136.13333	* Temperature
-0.51947	* Temperature2

5-9 Consider the data in Problem 5-1. Use the method described in the text to compute the linear and quadratic effects of pressure.

See the alternative analysis shown in Problem 5-1 part (c).

5-10 Use Duncan's multiple range test to determine which levels of the pressure factor are significantly different for the data in Problem 5-1.

$$\bar{y}_{.3.} = 90.18 \qquad \bar{y}_{.1.} = 90.37 \qquad \bar{y}_{.2.} = 90.68$$

$$S_{\bar{y}_{.j.}} = \sqrt{\frac{MS_E}{an}} = \sqrt{\frac{0.01777}{(3)(2)}} = 0.0543$$

$$r_{0.01}(2,9) = 4.60 \qquad r_{0.01}(3,9) = 4.86$$

$$R_2 = (4.60)(0.0543) = 0.2498 \qquad R_3 = (4.86)(0.0543) = 0.2640$$

2 vs. 3 = 0.50 > 0.2640 (R_3)
2 vs. 1 = 0.31 > 0.2498 (R_2)
1 vs. 3 = 0.19 < 0.2498 (R_2)
Therefore, 2 differs from 1 and 3.

5-11 An experiment was conducted to determine if either firing temperature or furnace position affects the baked density of a carbon anode. The data are shown below.

	Temperature (°C)		
Position	800	825	850
	570	1063	565
1	565	1080	510
	583	1043	590
	528	988	526
2	547	1026	538
	521	1004	532

Suppose we assume that no interaction exists. Write down the statistical model. Conduct the analysis of variance and test hypotheses on the main effects. What conclusions can be drawn? Comment on the model's adequacy.

The model for the two-factor, no interaction model is $y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$. Both factors, furnace position (A) and temperature (B) are significant. The residual plots show nothing unusual.

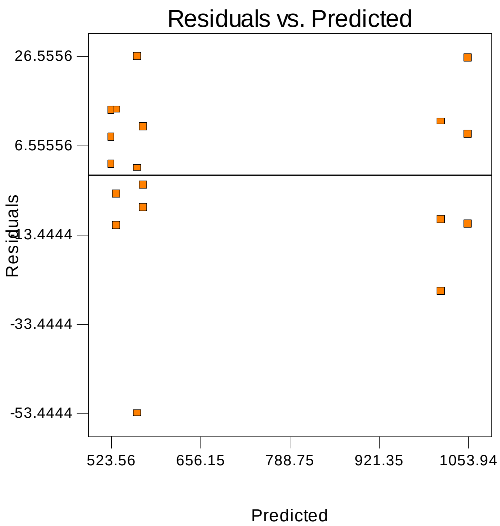
Design Expert Output

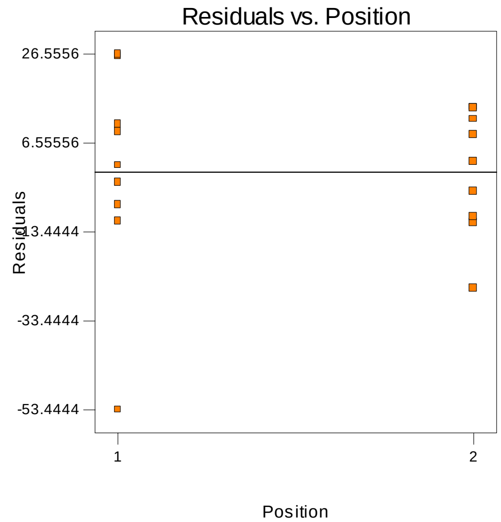
Response: Density
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

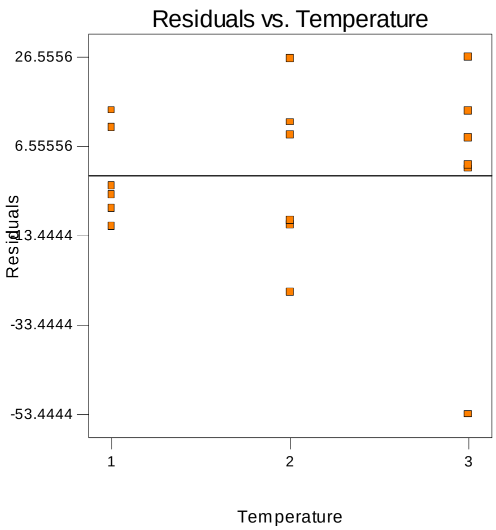
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	9.525E+005	3	3.175E+005	718.24	< 0.0001	significant
A	*7160.06*	*1*	*7160.06*	*16.20*	*0.0013*	
B	*9.453E+005*	*2*	*4.727E+005*	*1069.26*	*< 0.0001*	
Residual	6188.78	14	442.06			
Lack of Fit	*818.11*	*2*	*409.06*	*0.91*	*0.4271*	*not significant*
Pure Error	*5370.67*	*12*	*447.56*			
Cor Total	9.587E+005	17				

The Model F-value of 718.24 implies the model is significant.
There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.







5-12 Derive the expected mean squares for a two-factor analysis of variance with one observation per cell, assuming that both factors are fixed.

	Degrees of Freedom
$E(MS_A)=\sigma^2+b\sum_{i=1}^{a}\frac{\tau_i^2}{(a-1)}$	a-1
$E(MS_B)=\sigma^2+a\sum_{j=1}^{b}\frac{\beta_j^2}{(b-1)}$	b-1
$E(MS_{AB})=\sigma^2+\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{(a-1)(b-1)}{ab-1}$

5-13 Consider the following data from a two-factor factorial experiment. Analyze the data and draw conclusions. Perform a test for nonadditivity. Use $\alpha = 0.05$.

	Column Factor			
Row Factor	1	2	3	4
1	36	39	36	32
2	18	20	22	20
3	30	37	33	34

Design Expert Output

Response: **data**
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	609.42	5	121.88	25.36	0.0006	significant
A	*580.50*	*2*	*290.25*	*60.40*	*0.0001*	
B	*28.92*	*3*	*9.64*	*2.01*	*0.2147*	
Residual	28.83	6	4.81			
Cor Total	638.25	11				

The Model F-value of 25.36 implies the model is significant. There is only a 0.06% chance that a "Model F-Value" this large could occur due to noise.

The row factor (A) is significant.

The test for nonadditivity is as follows:

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_A SS_B}$$

$$SS_N = \frac{\left[4010014 - (357)\left(580.50 + 28.91667 + \frac{357^2}{(4)(3)}\right)\right]^2}{(4)(3)(580.50)(28.91667)}$$

$$SS_N = 3.54051$$

$$SS_{Error} = SS_{Residual} - SS_N = 28.8333 - 3.54051 = 25.29279$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Row	580.50	2	290.25	57.3780
Column	28.91667	3	9.63889	1.9054
Nonadditivity	3.54051	1	3.54051	0.6999
Error	25.29279	5	5.058558	
Total	638.25	11		

5-14 The shear strength of an adhesive is thought to be affected by the application pressure and temperature. A factorial experiment is performed in which both factors are assumed to be fixed. Analyze the data and draw conclusions. Perform a test for nonadditivity.

	Temperature (°F)		
Pressure (lb/in2)	250	260	270
120	9.60	11.28	9.00
130	9.69	10.10	9.57
140	8.43	11.01	9.03
150	9.98	10.44	9.80

Design Expert Output

Response: Strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	5.24	5	1.05	2.92	0.1124	not significant
A	*0.58*	*3*	*0.19*	*0.54*	*0.6727*	
B	*4.66*	*2*	*2.33*	*6.49*	*0.0316*	
Residual	2.15	6	0.36			
Cor Total	7.39	11				

The "Model F-value" of 2.92 implies the model is not significant relative to the noise.
There is a 11.24 % chance that a "Model F-value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B are significant model terms.

Temperature (*B*) is a significant factor.

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_A SS_B}$$

$$SS_N = \frac{\left[415113.777 - (117.93)\left(0.5806917 + 4.65765 + \frac{117.93^2}{(4)(3)}\right)\right]^2}{(4)(3)(0.5806917)(4.65765)}$$

$$SS_N = 0.48948$$

$$SS_{Error} = SS_{Residual} - SS_N = 2.1538833 - 0.48948 = 1.66440$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Row	0.5806917	3	0.1935639	0.5815
Column	4.65765	2	2.328825	6.9960
Nonadditivity	0.48948	1	0.48948	1.4704
Error	1.6644	5	0.33288	
Total	7.392225	11		

5-15 Consider the three-factor model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \varepsilon_{ijk} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,c \end{cases}$$

Notice that there is only one replicate. Assuming the factors are fixed, write down the analysis of variance table, including the expected mean squares. What would you use as the "experimental error" in order to test hypotheses?

Source	Degrees of Freedom	Expected Mean Square
A	*a*-1	$\sigma^2 + bc\sum_{i=1}^{a}\frac{\tau_i^2}{(a-1)}$
B	*b*-1	$\sigma^2 + ac\sum_{j=1}^{b}\frac{\beta_j^2}{(b-1)}$
C	*c*-1	$\sigma^2 + ab\sum_{k=1}^{c}\frac{\gamma_k^2}{(c-1)}$
AB	(*a*-1)(*b*-1)	$\sigma^2 + c\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\tau(\beta)_{ij}^2}{(a-1)(b-1)}$
BC	(*b*-1)(*c*-1)	$\sigma^2 + a\sum_{j=1}^{b}\sum_{k=1}^{c}\frac{(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
Error (*AC* + *ABC*)	*b*(*a*-1)(*c*-1)	σ^2
Total	*abc*-1	

5-16 The percentage of hardwood concentration in raw pulp, the vat pressure, and the cooking time of the pulp are being investigated for their effects on the strength of paper. Three levels of hardwood concentration, three levels of pressure, and two cooking times are selected. A factorial experiment with two replicates is conducted, and the following data are obtained:

Percentage of Hardwood	Cooking Time 3.0 Hours Pressure			Cooking Time 4.0 Hours Pressure		
Concentration	400	500	650	400	500	650
2	196.6	197.7	199.8	198.4	199.6	200.6
	196.0	196.0	199.4	198.6	200.4	200.9
4	198.5	196.0	198.4	197.5	198.7	199.6
	197.2	196.9	197.6	198.1	198.0	199.0
8	197.5	195.6	197.4	197.6	197.0	198.5
	196.6	196.2	198.1	198.4	197.8	199.8

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

Design Expert Output

Response: strength

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	59.73	17	3.51	9.61	< 0.0001	significant
A	*7.76*	*2*	*3.88*	*10.62*	*0.0009*	
B	*20.25*	*1*	*20.25*	*55.40*	*< 0.0001*	
C	*19.37*	*2*	*9.69*	*26.50*	*< 0.0001*	
AB	*2.08*	*2*	*1.04*	*2.85*	*0.0843*	
AC	*6.09*	*4*	*1.52*	*4.17*	*0.0146*	

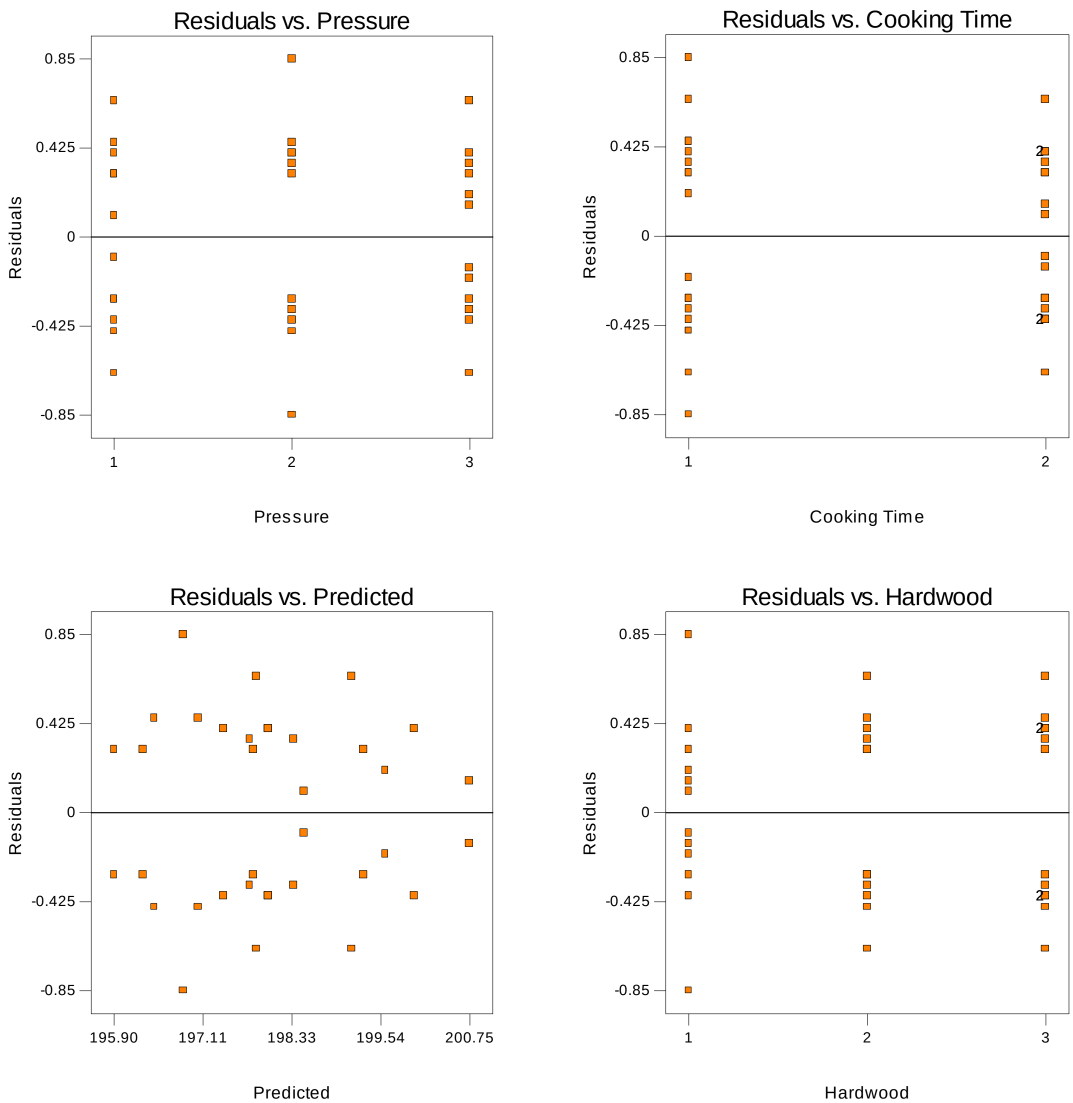
BC	*2.19*	*2*	*1.10*	*3.00*	*0.0750*
ABC	*1.97*	*4*	*0.49*	*1.35*	*0.2903*
Residual	6.58	18	0.37		
Lack of Fit	*0.000*	*0*			
Pure Error	*6.58*	*18*	*0.37*		
Cor Total	66.31	35			

The Model F-value of 9.61 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AC are significant model terms.

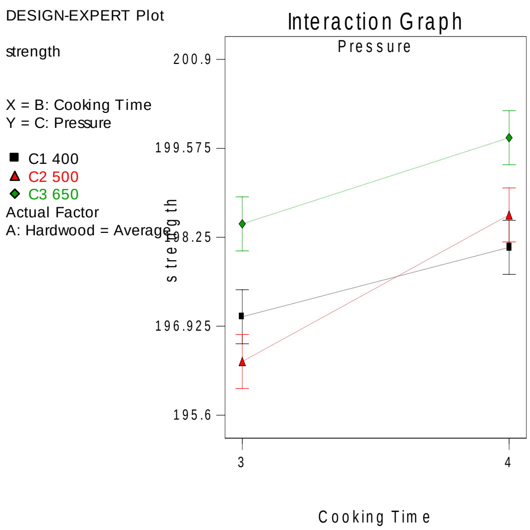
All three main effects, concentration (*A*), pressure (*C*) and time (*B*), as well as the concentration x pressure interaction (*AC*) are significant at the 5% level. The concentration x time (*AB*) and pressure x time interactions (*BC*) are significant at the 10% level.

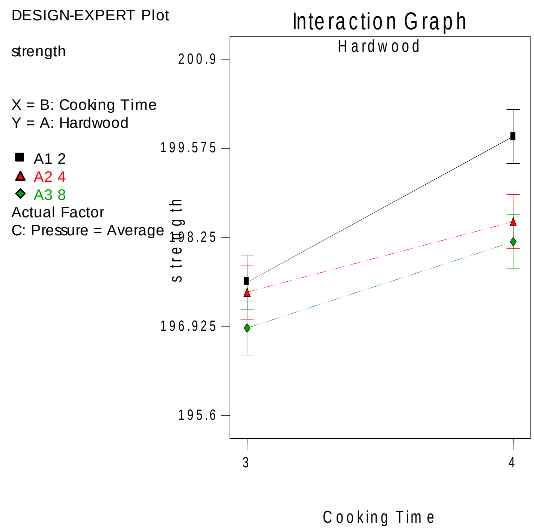
(b) Prepare appropriate residual plots and comment on the model's adequacy.

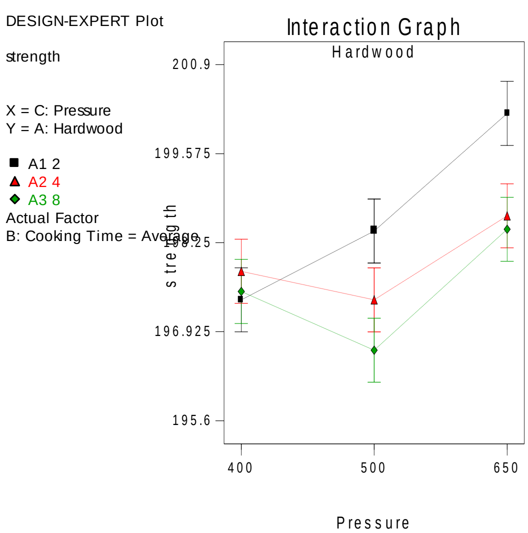


There is nothing unusual about the residual plots.

(c) Under what set of conditions would you run the process? Why?







For the highest strength, run the process with the percentage of hardwood at 2, the pressure at 650, and the time at 4 hours.

The standard analysis of variance treats all design factors as if they were qualitative. In this case, all three factors are quantitative, so some further analysis can be performed. In Section 5-5, we show how response curves and surfaces can be fit to the data from a factorial experiment with at least one quantative factor. Since the factors in this problem are quantitative and two of them have three levels, we can fit a linear term for the two-level factor and linear and quadratic components for the three-level factors. The Minitab output, with the *ABC* interaction removed due to insignificance, now follows. Also included is the Design Expert output; however, if the student choses to use Design Expert, sequential sum of squares must be selected to assure that the sum of squares for the model equals the total of the sum of squares for each factor included in the model.

Minitab Output

General Linear Model: Strength versus

```
Factor                      Type Levels Values

Analysis of Variance for Strength, using Adjusted SS for Tests

Source                      DF      Seq SS     Adj SS     Adj MS       F       P
Hardwood                     1      6.9067     4.9992     4.9992   13.23  0.001
Time                         1     20.2500     1.3198     1.3198    3.49  0.074
Pressure                     1     15.5605     1.5014     1.5014    3.97  0.058
Hardwood*Hardwood            1      0.8571     2.7951     2.7951    7.40  0.012
Pressure*Pressure            1      3.8134     1.8232     1.8232    4.83  0.038
Hardwood*Time                1      0.7779     1.5779     1.5779    4.18  0.053
Hardwood*Pressure            1      2.1179     3.4564     3.4564    9.15  0.006
Time*Pressure                1      0.0190     2.1932     2.1932    5.81  0.024
Hardwood*Hardwood*Time       1      1.3038     1.3038     1.3038    3.45  0.076
Hardwood*Hardwood*
Pressure                     1      2.1885     2.1885     2.1885    5.79  0.025
Hardwood*Pressure*
Pressure                     1      1.6489     1.6489     1.6489    4.36  0.048
Time*Pressure*Pressure       1      2.1760     2.1760     2.1760    5.76  0.025
Error                       23      8.6891     8.6891     0.3778
Total                       35     66.3089

Term                            Coef   SE Coef         T      P
Constant                      236.92     29.38      8.06  0.000
Hardwood                      10.728     2.949      3.64  0.001
Time                         -14.961     8.004     -1.87  0.074
Pressure                     -0.2257    0.1132     -1.99  0.058
Hardwood*Hardwood            -0.6529    0.2400     -2.72  0.012
Pressure*Pressure           0.000234  0.000107      2.20  0.038
Hardwood*Time                -1.1750    0.5749     -2.04  0.053
Hardwood*Pressure          -0.020533  0.006788     -3.02  0.006
Time*Pressure                0.07450   0.03092      2.41  0.024
Hardwood*Hardwood*Time       0.10278   0.05532      1.86  0.076
Hardwood*Hardwood*Pressure  0.000648  0.000269      2.41  0.025
Hardwood*Pressure*Pressure  0.000012  0.000006      2.09  0.048
Time*Pressure*Pressure     -0.000070  0.000029     -2.40  0.025

Unusual Observations for Strength

Obs  Strength       Fit      SE Fit  Residual   St Resid
  6   198.500   197.461       0.364     1.039      2.10R

R denotes an observation with a large standardized residual.
```

Design Expert Output

Response: Strength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	57.62	12	4.80	12.71	< 0.0001	significant
A	*6.91*	*1*	*6.91*	*18.28*	*0.0003*	
B	*20.25*	*1*	*20.25*	*53.60*	*< 0.0001*	
C	*15.56*	*1*	*15.56*	*41.19*	*< 0.0001*	
A2	*0.86*	*1*	*0.86*	*2.27*	*0.1456*	
C2	*3.81*	*1*	*3.81*	*10.09*	*0.0042*	
AB	*0.78*	*1*	*0.78*	*2.06*	*0.1648*	
AC	*2.12*	*1*	*2.12*	*5.61*	*0.0267*	
BC	*0.019*	*1*	*0.019*	*0.050*	*0.8245*	
A2B	*1.30*	*1*	*1.30*	*3.45*	*0.0761*	
A2C	*2.19*	*1*	*2.19*	*5.79*	*0.0245*	
AC2	*1.65*	*1*	*1.65*	*4.36*	*0.0479*	
BC2	*2.18*	*1*	*2.18*	*5.76*	*0.0249*	
Residual	8.69	23	0.38			
Lack of Fit	*2.11*	*5*	*0.42*	*1.15*	*0.3691*	*not significant*
Pure Error	*6.58*	*18*	*0.37*			
Cor Total	66.31	35				

The Model F-value of 12.71 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, C^2, AC, A^2C, AC^2, BC^2 are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy),
model reduction may improve your model.

Std. Dev.	0.61	R-Squared	0.8690
Mean	198.06	Adj R-Squared	0.8006
C.V.	0.31	Pred R-Squared	0.6794
PRESS	21.26	Adeq Precision	15.040

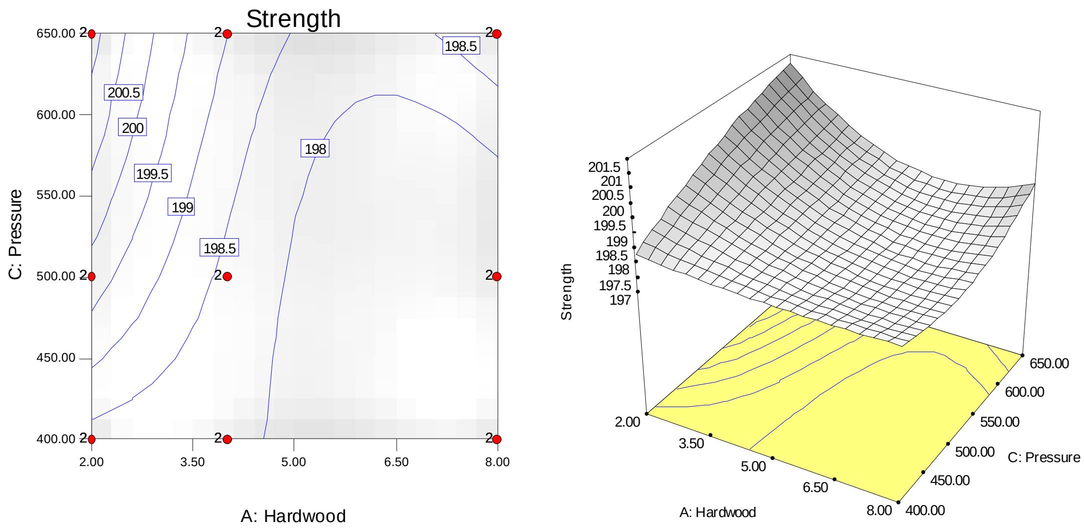
Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	197.21	1	0.26	196.67	197.74	
A-Hardwood	-0.98	1	0.23	-1.45	-0.52	3.36
B-Cooking Time	0.78	1	0.26	0.24	1.31	6.35
C-Pressure	0.19	1	0.25	-0.33	0.71	4.04
A2	0.42	1	0.25	-0.093	0.94	1.04
C2	0.79	1	0.23	0.31	1.26	1.03
AB	-0.22	1	0.13	-0.48	0.039	1.06
AC	-0.46	1	0.15	-0.78	-0.14	1.08
BC	0.062	1	0.13	-0.20	0.32	1.02
A2B	0.46	1	0.25	-0.053	0.98	3.96
A2C	0.73	1	0.30	0.10	1.36	3.97
AC2	0.57	1	0.27	5.625E-003	1.14	3.32
BC2	-0.55	1	0.23	-1.02	-0.075	3.30

Final Equation in Terms of Coded Factors:

Strength	=
+197.21	
-0.98	* A
+0.78	* B
+0.19	* C
+0.42	* A2
+0.79	* C2
-0.22	* A * B
-0.46	* A * C
+0.062	* B * C
+0.46	* A2 * B
+0.73	* A2 * C
+0.57	* A * C2
-0.55	* B * C2

Final Equation in Terms of Actual Factors:

Strength	=
+236.91762	
+10.72773	* Hardwood
-14.96111	* Cooking Time
-0.22569	* Pressure
-0.65287	* Hardwood2
+2.34333E-004	* Pressure2
-1.17500	* Hardwood * Cooking Time
-0.020533	* Hardwood * Pressure
+0.074500	* Cooking Time * Pressure
+0.10278	* Hardwood2 * Cooking Time
+6.48026E-004	* Hardwood2 * Pressure
+1.22143E-005	* Hardwood * Pressure2
-7.00000E-005	* Cooking Time * Pressure2



Cooking Time: B = 4.00

5-17 The quality control department of a fabric finishing plant is studying the effect of several factors on the dyeing of cotton-synthetic cloth used to manufacture men's shirts. Three operators, three cycle times, and two temperatures were selected, and three small specimens of cloth were dyed under each set of conditions. The finished cloth was compared to a standard, and a numerical score was assigned. The results follow. Analyze the data and draw conclusions. Comment on the model's adequacy.

	Temperature					
	300°			350°		
	Operator			Operator		
Cycle Time	1	2	3	1	2	3
	23	27	31	24	38	34
40	24	28	32	23	36	36
	25	26	29	28	35	39
	36	34	33	37	34	34
50	35	38	34	39	38	36
	36	39	35	35	36	31
	28	35	26	26	36	28
60	24	35	27	29	37	26
	27	34	25	25	34	24

All three main effects, and the *AB, AC*, and *ABC* interactions are significant. There is nothing unusual about the residual plots.

Design Expert Output

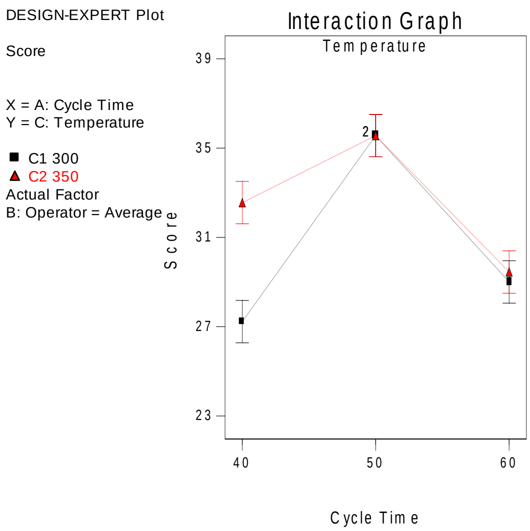
Response: Score
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

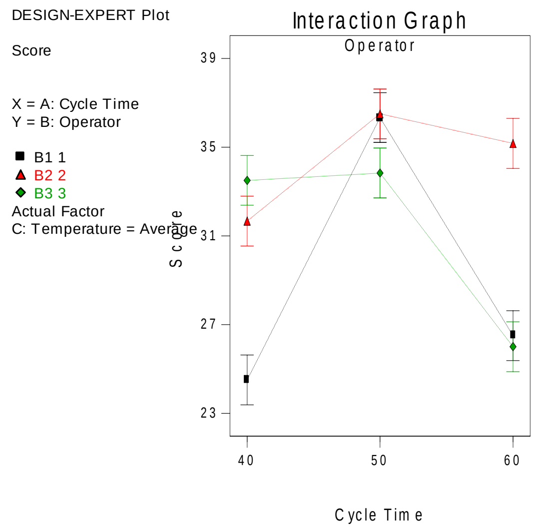
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	1239.33	17	72.90	22.24	< 0.0001	significant
A	*436.00*	*2*	*218.00*	*66.51*	*< 0.0001*	
B	*261.33*	*2*	*130.67*	*39.86*	*< 0.0001*	
C	*50.07*	*1*	*50.07*	*15.28*	*0.0004*	
AB	*355.67*	*4*	*88.92*	*27.13*	*< 0.0001*	

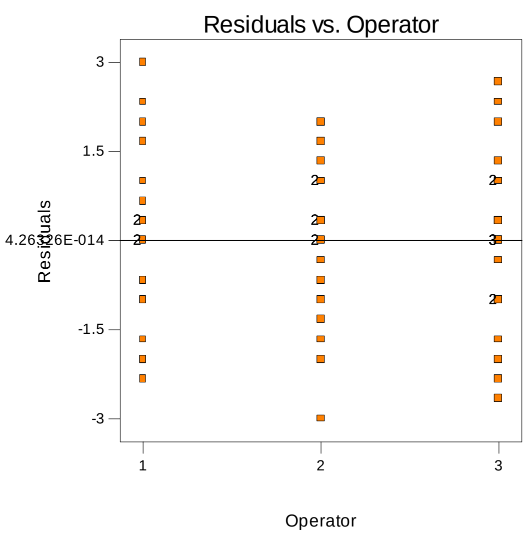
AC	*78.81*	*2*	*39.41*	*12.02*	*0.0001*
BC	*11.26*	*2*	*5.63*	*1.72*	*0.1939*
ABC	*46.19*	*4*	*11.55*	*3.52*	*0.0159*
Residual	118.00	36	3.28		
Lack of Fit	*0.000*	*0*			
Pure Error	*118.00*	*36*	*3.28*		
Cor Total	1357.33	53			

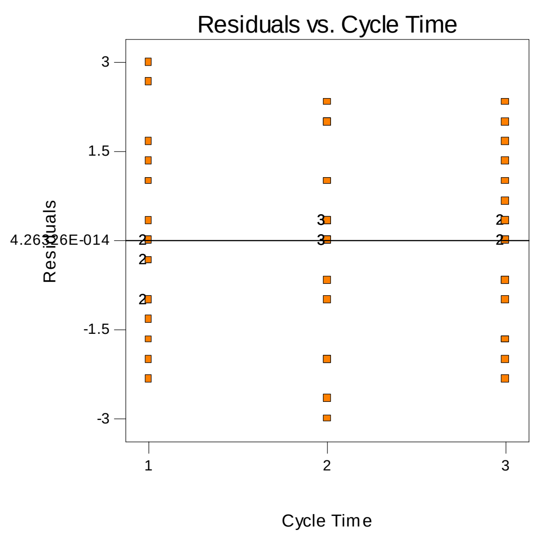
The Model F-value of 22.24 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

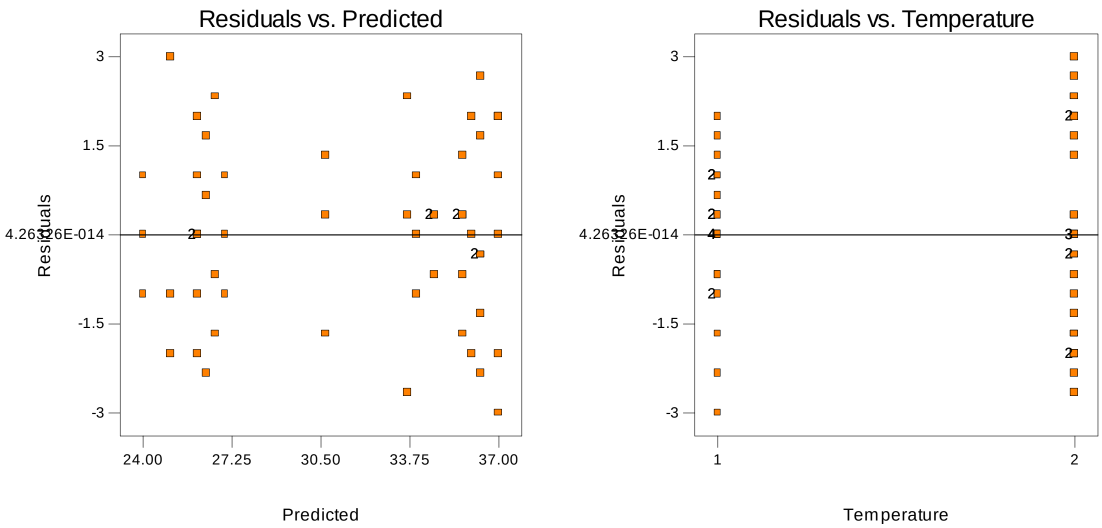
Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, C, AB, AC, ABC are significant model terms.











5-18 In Problem 5-1, suppose that we wish to reject the null hypothesis with a high probability if the difference in the true mean yield at any two pressures is as great as 0.5. If a reasonable prior estimate of the standard deviation of yield is 0.1, how many replicates should be run?

$$\Phi^2 = \frac{naD^2}{2b\sigma^2} = \frac{n(3)(0.5)^2}{2(3)(0.1)^2} = 12.5n$$

n	Φ^2	Φ	$\upsilon_1 = (b-1)$	$\upsilon_2 = ab(n-1)$	β
2	25	5	2	(3)(3)(1)	0.014

2 replications will be enough to detect the given difference.

5-19 The yield of a chemical process is being studied. The two factors of interest are temperature and pressure. Three levels of each factor are selected; however, only 9 runs can be made in one day. The experimenter runs a complete replicate of the design on each day. The data are shown in the following table. Analyze the data assuming that the days are blocks.

	Day 1			Day 2		
	Pressure			Pressure		
Temperature	250	260	270	250	260	270
Low	86.3	84.0	85.8	86.1	85.2	87.3
Medium	88.5	87.3	89.0	89.4	89.9	90.3
High	89.1	90.2	91.3	91.7	93.2	93.7

Design Expert Output

Response: Yield
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	13.01	1	13.01			
Model	109.81	8	13.73	25.84	< 0.0001	significant
A	*5.51*	*2*	*2.75*	*5.18*	*0.0360*	

B	*99.85*	*2*	*49.93*	*93.98*	*< 0.0001*
AB	*4.45*	*4*	*1.11*	*2.10*	*0.1733*
Residual	4.25	8	0.53		
Cor Total	127.07	17			

The Model F-value of 25.84 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

Both main effects, temperature and pressure, are significant.

5-20 Consider the data in Problem 5-5. Analyze the data, assuming that replicates are blocks.

Design Expert Output

Response: Warping
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	11.28	1	11.28			
Model	968.22	15	64.55	9.96	< 0.0001	significant
A	*698.34*	*3*	*232.78*	*35.92*	*< 0.0001*	
B	*156.09*	*3*	*52.03*	*8.03*	*0.0020*	
AB	*113.78*	*9*	*12.64*	*1.95*	*0.1214*	
Residual	97.22	15	6.48			
Cor Total	1076.72	31				

The Model F-value of 9.96 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

Both temperature and copper content are significant. This agrees with the analysis in Problem 5-5.

5-21 Consider the data in Problem 5-6. Analyze the data, assuming that replicates are blocks.

Design-Expert Output

Response: Stength
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Block	1.04	1	1.04			
Model	217.46	11	19.77	4.89	0.0070	significant
A	*160.33*	*2*	*80.17*	*19.84*	*0.0002*	
B	*12.46*	*3*	*4.15*	*1.03*	*0.4179*	
AB	*44.67*	*6*	*7.44*	*1.84*	*0.1799*	
Residual	44.46	11	4.04			
Cor Total	262.96	23				

The Model F-value of 4.89 implies the model is significant. There is only
a 0.70% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A are significant model terms.

Only the operator factor (*A*) is significant. This agrees with the analysis in Problem 5-6.

5-22 An article in the *Journal of Testing and Evaluation* (Vol. 16, no.2, pp. 508-515) investigated the effects of cyclic loading and environmental conditions on fatigue crack growth at a constant 22 MPa stress for a particular material. The data from this experiment are shown below (the response is crack growth rate).

	Environment		
Frequency	Air	H_2O	Salt H_2O
	2.29	2.06	1.90
10	2.47	2.05	1.93
	2.48	2.23	1.75
	2.12	2.03	2.06
	2.65	3.20	3.10
1	2.68	3.18	3.24
	2.06	3.96	3.98
	2.38	3.64	3.24
	2.24	11.00	9.96
0.1	2.71	11.00	10.01
	2.81	9.06	9.36
	2.08	11.30	10.40

(a) Analyze the data from this experiment (use $\alpha = 0.05$).

Design Expert Output

Response: Crack Growth
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	376.11	8	47.01	234.02	< 0.0001	significant
A	*209.89*	*2*	*104.95*	*522.40*	*< 0.0001*	
B	*64.25*	*2*	*32.13*	*159.92*	*< 0.0001*	
AB	*101.97*	*4*	*25.49*	*126.89*	*< 0.0001*	
Residual	5.42	27	0.20			
Lack of Fit	*0.000*	*0*				
Pure Error	*5.42*	*27*	*0.20*			
Cor Total	381.53	35				

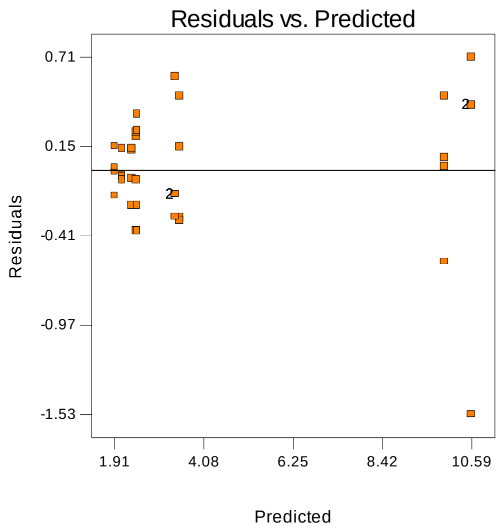
The Model F-value of 234.02 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

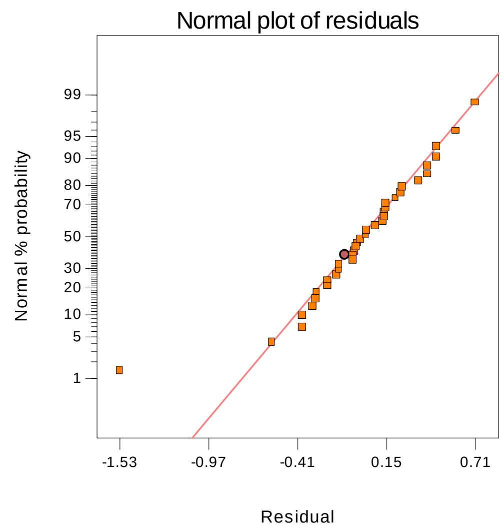
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

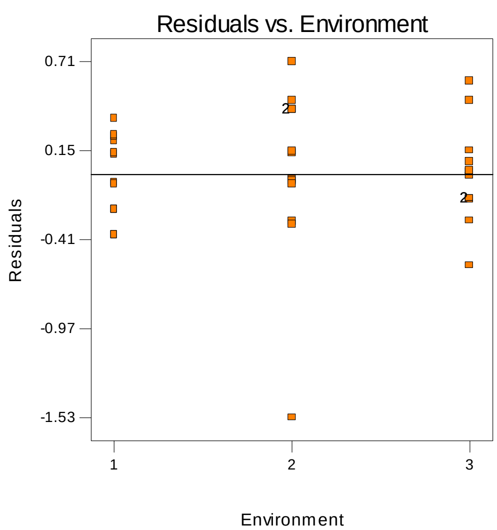
Both frequency and environment, as well as their interaction are significant.

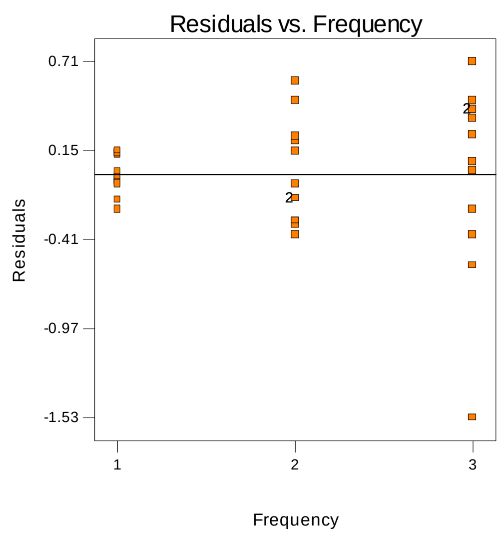
(b) Analyze the residuals.

The residual plots indicate that there may be some problem with inequality of variance. This is particularly noticable on the plot of residuals versus predicted response and the plot of residuals versus frequency.









(c) Repeat the analyses from parts (a) and (b) using ln(y) as the response. Comment on the results.

Design Expert Output

Response: Crack Growth **Transform: Natural log** **Constant: 0.000**

ANOVA for Selected Factorial Model

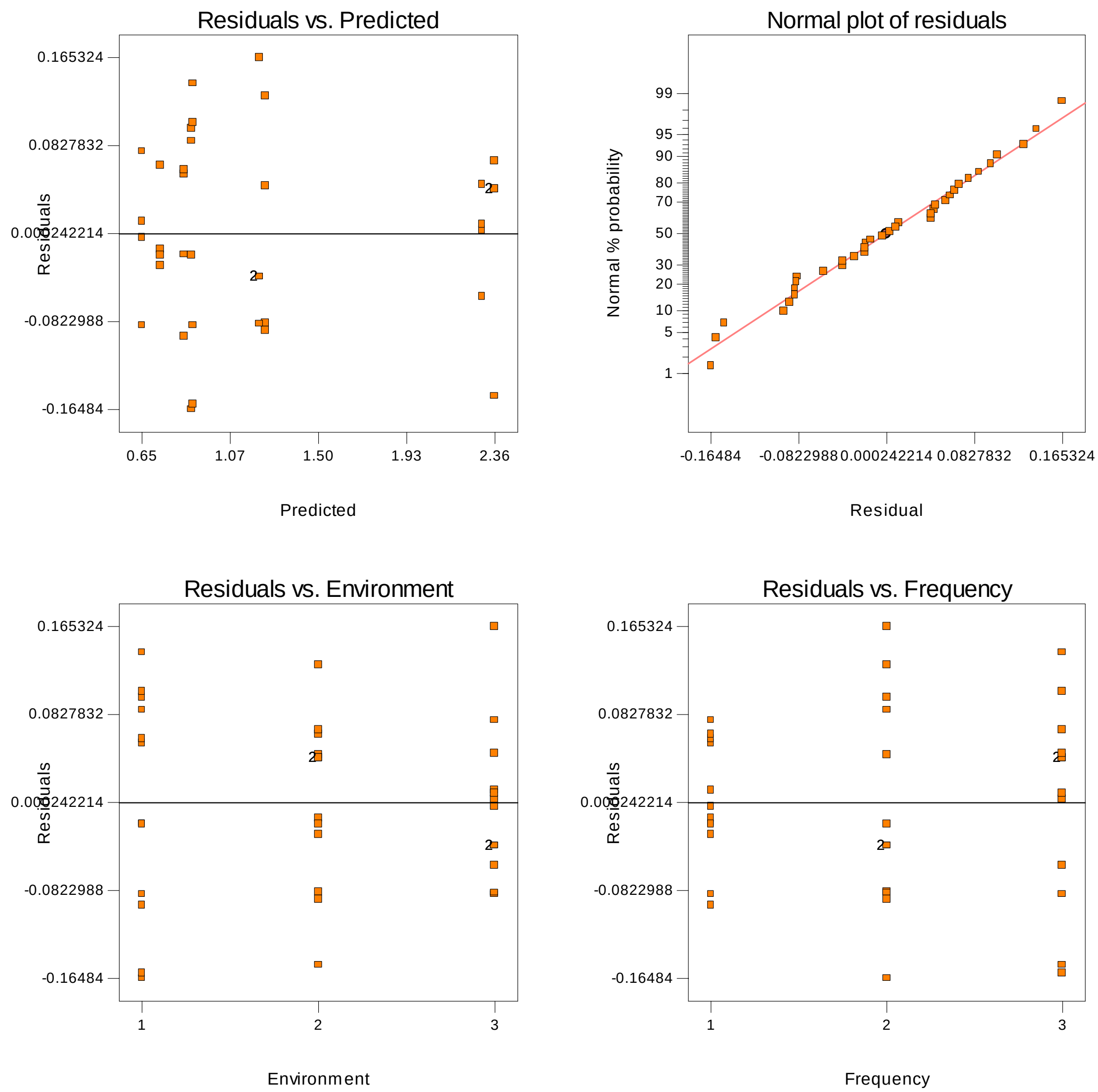
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	13.46	8	1.68	179.57	< 0.0001	significant
A	*7.57*	*2*	*3.79*	*404.09*	*< 0.0001*	
B	*2.36*	*2*	*1.18*	*125.85*	*< 0.0001*	
AB	*3.53*	*4*	*0.88*	*94.17*	*< 0.0001*	
Residual	0.25	27	9.367E-003			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.25*	*27*	*9.367E-003*			
Cor Total	13.71	35				

The Model F-value of 179.57 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

```
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.
```

Both frequency and environment, as well as their interaction are significant. The residual plots of the based on the transformed data look better.



5-23 An article in the *IEEE Transactions on Electron Devices* (Nov. 1986, pp. 1754) describes a study on polysilicon doping. The experiment shown below is a variation of their study. The response variable is base current.

Polysilicon	Anneal Temperature (°C)		
Doping (ions)	900	950	1000
1×10^{20}	4.60	10.15	11.01
	4.40	10.20	10.58
2×10^{20}	3.20	9.38	10.81
	3.50	10.02	10.60

(a) Is there evidence (with $\alpha = 0.05$) indicating that either polysilicon doping level or anneal temperature affect base current?

Design Expert Output

Response: Base Current
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

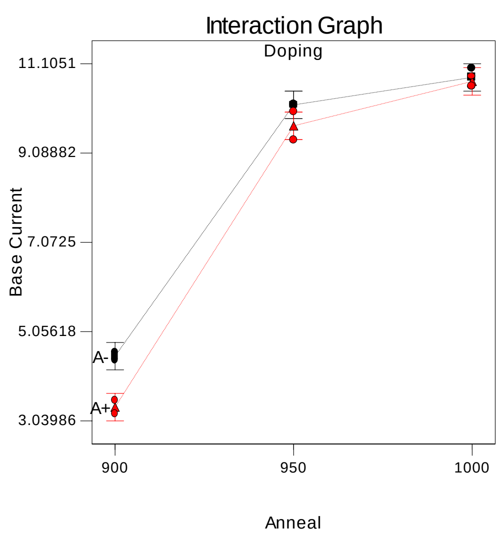
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	112.74	5	22.55	350.91	< 0.0001	significant
A	*0.98*	*1*	*0.98*	*15.26*	*0.0079*	
B	*111.19*	*2*	*55.59*	*865.16*	*< 0.0001*	
AB	*0.58*	*2*	*0.29*	*4.48*	*0.0645*	
Residual	0.39	6	0.064			
Lack of Fit	*0.000*	*0*				
Pure Error	*0.39*	*6*	*0.064*			
Cor Total	113.13	11				

The Model F-value of 350.91 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

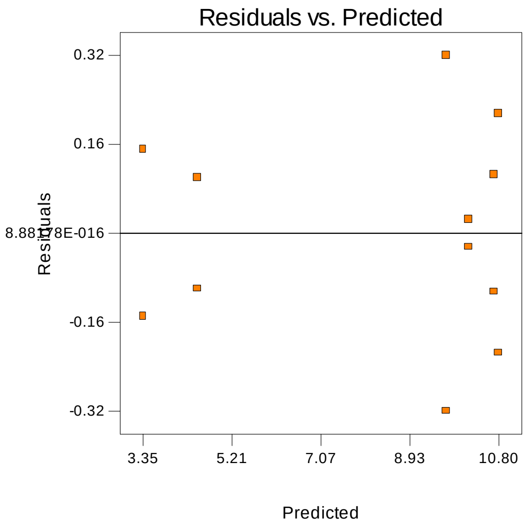
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

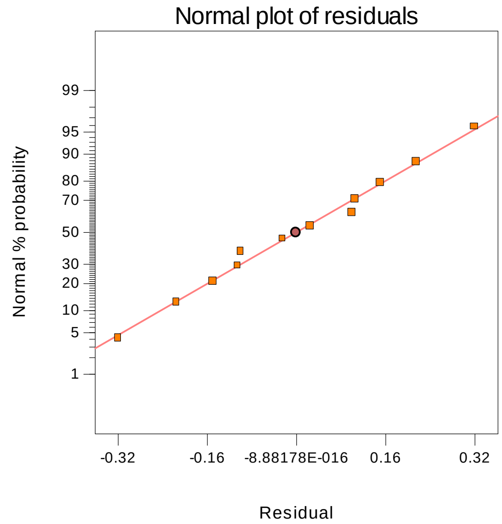
Both factors, doping and anneal are significant. Their interaction is significant at the 10% level.

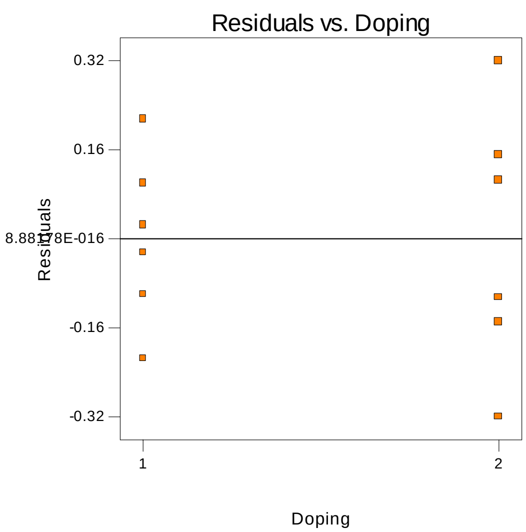
(b) Prepare graphical displays to assist in interpretation of this experiment.

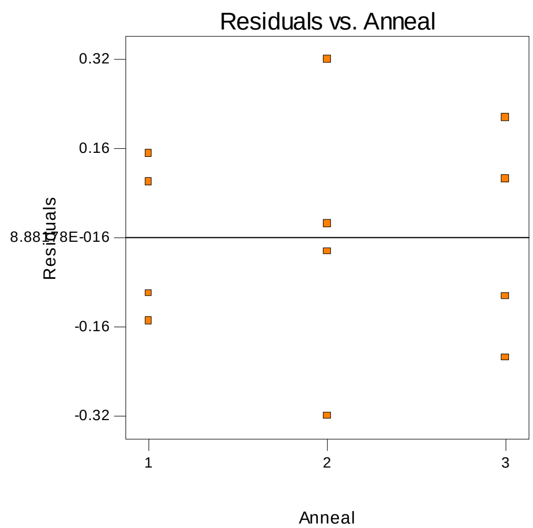


(c) Analyze the residuals and comment on model adequacy.









There is a funnel shape in the plot of residuals versus predicted, indicating some inequality of variance.

(d) Is the model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$ supported by this experiment (x_1 = doping level, x_2 = temperature)? Estimate the parameters in this model and plot the response surface.

Design Expert Output

Response: Base Current
ANOVA for Response Surface Reduced Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	112.73	4	28.18	493.73	< 0.0001	significant
A	*0.98*	*1*	*0.98*	*17.18*	*0.0043*	
B	*93.16*	*1*	*93.16*	*1632.09*	*< 0.0001*	
B^2	*18.03*	*1*	*18.03*	*315.81*	*< 0.0001*	
AB	*0.56*	*1*	*0.56*	*9.84*	*0.0164*	
Residual	0.40	7	0.057			
Lack of Fit	*0.014*	*1*	*0.014*	*0.22*	*0.6569*	*not significant*
Pure Error	*0.39*	*6*	*0.064*			

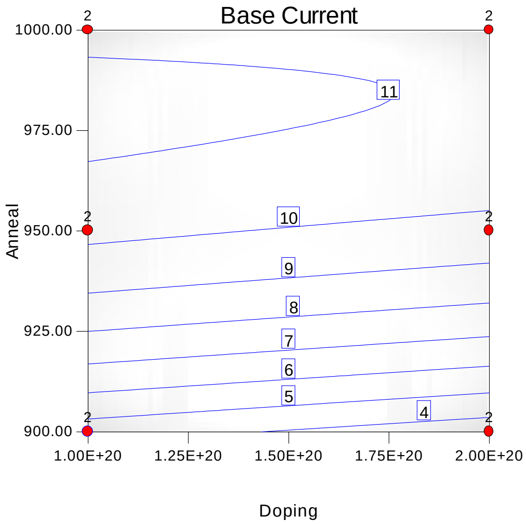
Cor Total 113.13 11

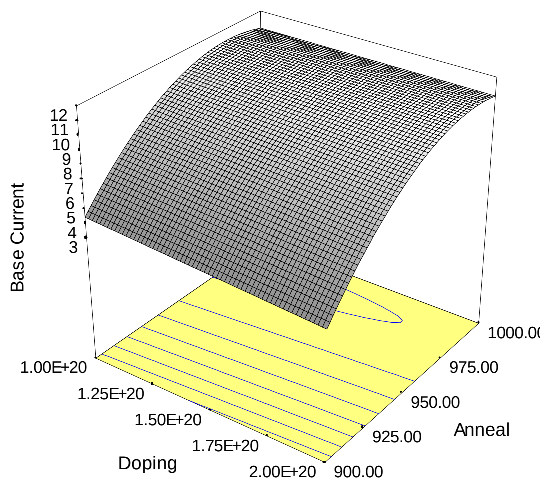
The Model F-value of 493.73 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B, B^2, AB are significant model terms.

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	9.94	1	0.12	9.66	10.22	
A-Doping	-0.29	1	0.069	-0.45	-0.12	1.00
B-Anneal	3.41	1	0.084	3.21	3.61	1.00
B^2	-2.60	1	0.15	-2.95	-2.25	1.00
AB	0.27	1	0.084	0.065	0.46	1.00

All of the coefficients in the assumed model are significant. The quadratic effect is easily observable in the response surface plot.





5-24 An experiment was conducted to study the life (in hours) of two different brands of batteries in three different devices (radio, camera, and portable DVD player). A completely randomized two-factor experiment was conducted, and the following data resulted.

Brand of Battery	Device Radio	Camera	DVD Player
A	8.6	7.9	5.4
	8.2	8.4	5.7
B	9.4	8.5	5.8
	8.8	8.9	5.9

(a) Analyze the data and draw conclusions, using $\alpha = 0.05$.

Both brand of battery (A) and type of device (B) are significant, the interaction is not.

Design Expert Output

Response: **Life**

ANOVA for Selected Factorial Model
Analysis of variance table [Terms added sequentially (first to last)]

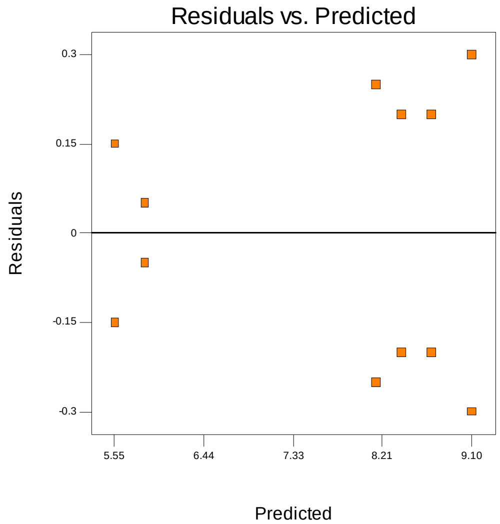
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	23.33	5	4.67	54.36	< 0.0001	significant
A	*0.80*	*1*	*0.80*	*9.33*	*0.0224*	
B	*22.45*	*2*	*11.22*	*130.75*	*< 0.0001*	
AB	*0.082*	*2*	*0.041*	*0.48*	*0.6430*	
Pure Error	*0.52*	*6*	*0.086*			
Cor Total	23.84	11				

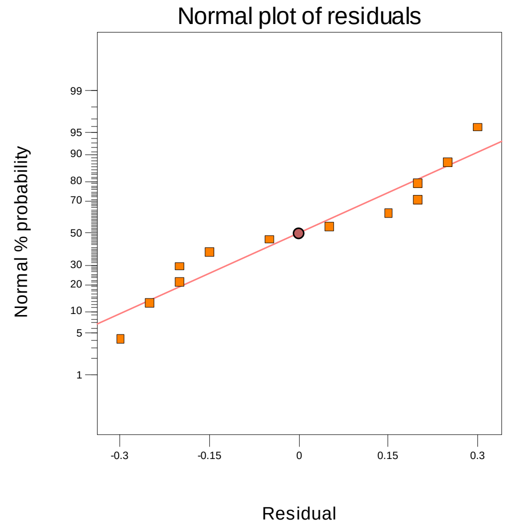
The Model F-value of 54.36 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

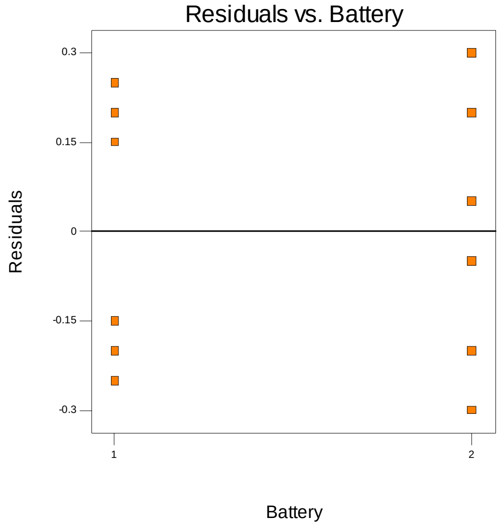
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

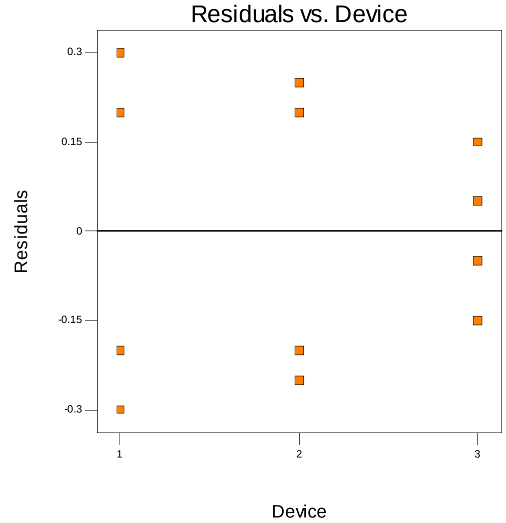
(b) Investigate model adequacy by plotting the residuals.

The residual plots show no serious deviations from the assumptions.



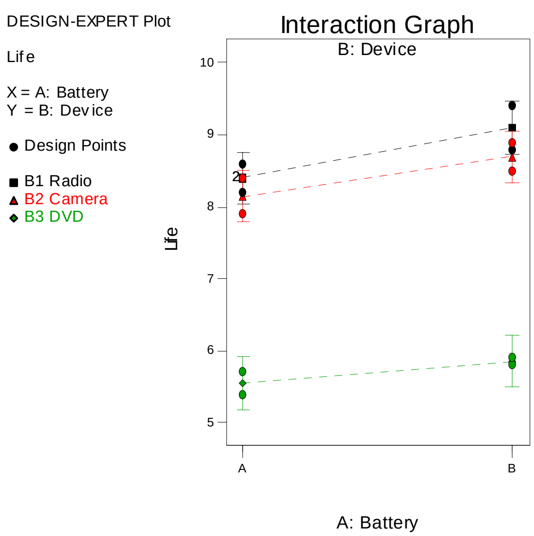






(c) Which brand of batteries would you recommend?

Battery brand B is recommended.



5-25 The author has recently purchased new golf clubs, which he believes witll significantly improve e his game. Below are the scores of three rounds of golf played at three different golf courses with the old and the new clubs.

Clubs	Course		
	Ahwatukee	Karsten	Foothills
Old	90	91	88
	87	93	86
	86	90	90
New	88	90	86
	87	91	85

	85	88	88

(a) Conduct an anlysis of variance. Using $\alpha = 0.05$, what conclusions can you draw?

Although there is a significant difference between the golf courses, there is not a significant difference between the old and new clubs.

Design Expert Output

Response: Score

ANOVA for Selected Factorial Model

Analysis of variance table [Terms added sequentially (first to last)]

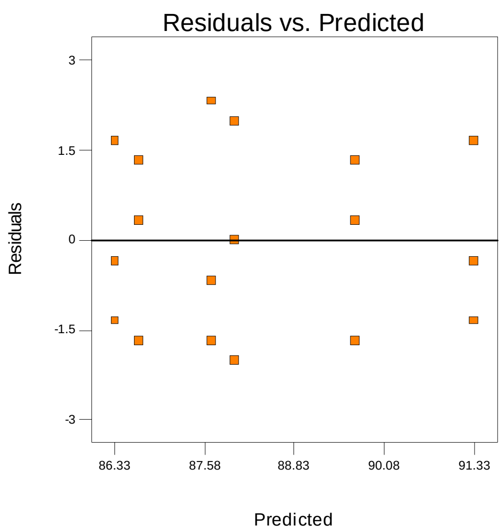
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	54.28	5	10.86	3.69	0.0297	significant
A	*9.39*	*1*	*9.39*	*3.19*	*0.0994*	
B	*44.44*	*2*	*22.22*	*7.55*	*0.0075*	
AB	*0.44*	*2*	*0.22*	*0.075*	*0.9277*	
Pure Error	*35.33*	*12*	*2.94*			
Cor Total	89.61	17				

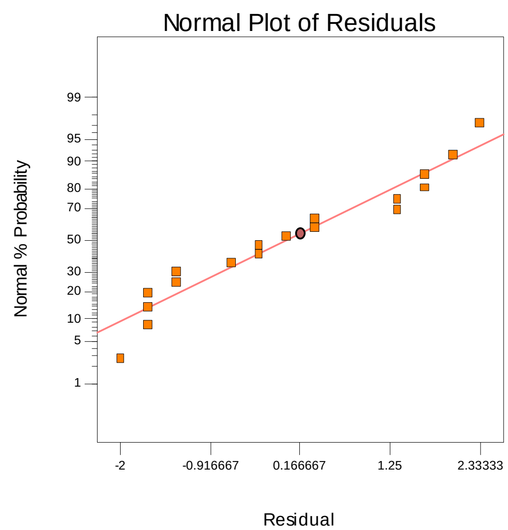
The Model F-value of 3.69 implies the model is significant. There is only a 2.97% chance that a "Model F-Value" this large could occur due to noise.

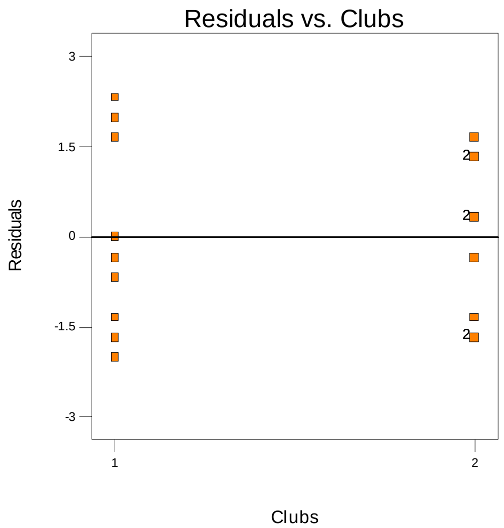
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

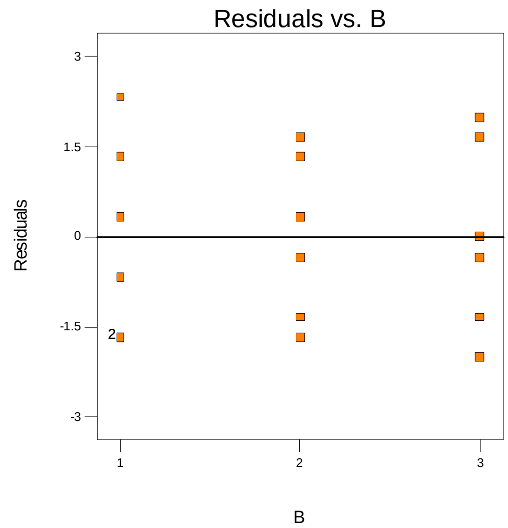
(b) Investigate model adequacy by plotting the residuals.

The residual plots show no deviations from the assumptions.









5-26 A manufacturer of laundry products is investigating the performance of a newly formulated stain remover. The new formulation is compared to the original formulation with respect to its ability to remove a standard tomato-like stain in a test article of cotton cloth using a factorial experiment. The other factors in the experiment are the number of times the test article is washed (1 or 2), and whether or not a detergent booster is used. The response variable is the stain shade after washing (12 is the darkest, 0 is the lightest). The data are shown in the table below.

Formulation	Number of Washings		Number of Washings	
	1	2	1	2
	Booster		Booster	
	Yes	No	Yes	No
New	6	6	3	4
	5	5	2	1
Original	10	11	10	9
	9	11	9	10

(a) Conduct an anlysis of variance. Using $\alpha = 0.05$, what conclusions can you draw?

The formulation, number of washings, and the interaction between these to factors appear to be significant. Continued analysis is required as a result of the residual plots in part (b). Conclusions are presented in part (b).

Design Expert Output

Response: Stain Shade
ANOVA for Selected Factorial Model
Analysis of variance table [Terms added sequentially (first to last)]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	159.44	7	22.78	24.30	< 0.0001	significant
A	*138.06*	*1*	*138.06*	*147.27*	*< 0.0001*	
B	*14.06*	*1*	*14.06*	*15.00*	*0.0047*	
C	*0.56*	*1*	*0.56*	*0.60*	*0.4609*	
AB	*5.06*	*1*	*5.06*	*5.40*	*0.0486*	
AC	*0.56*	*1*	*0.56*	*0.60*	*0.4609*	
BC	*0.56*	*1*	*0.56*	*0.60*	*0.4609*	

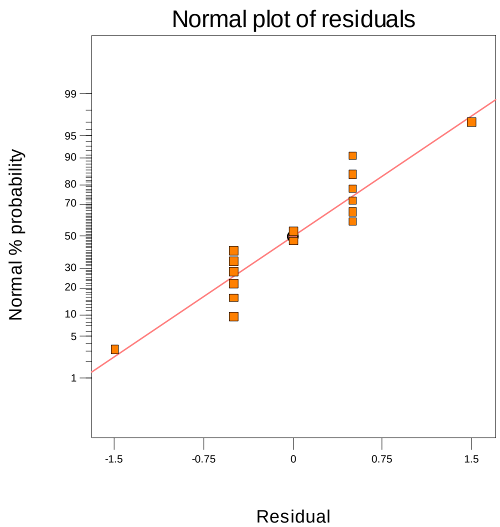
ABC	*0.56*	*1*	*0.56*	*0.60*	*0.4609*
Pure Error	*7.50*	*8*	*0.94*		
Cor Total	166.94	15			

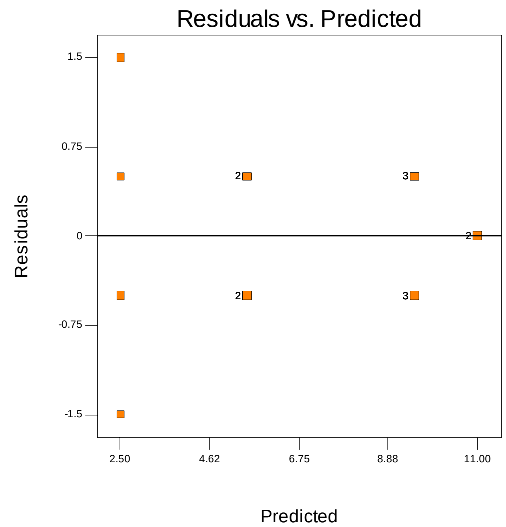
The Model F-value of 24.30 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

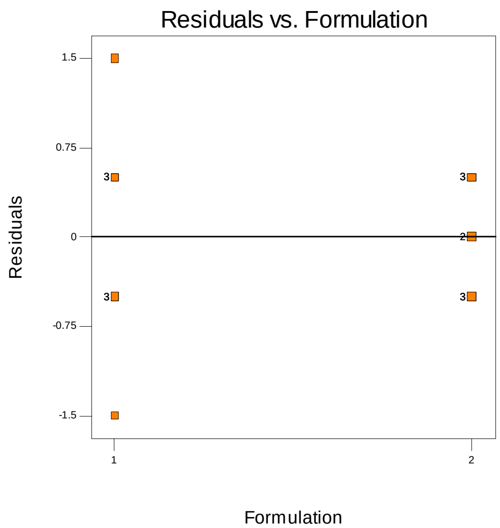
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

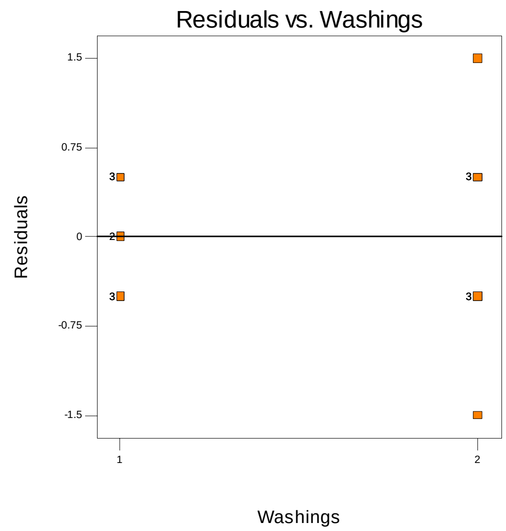
(b) Investigate model adequacy by plotting the residuals.

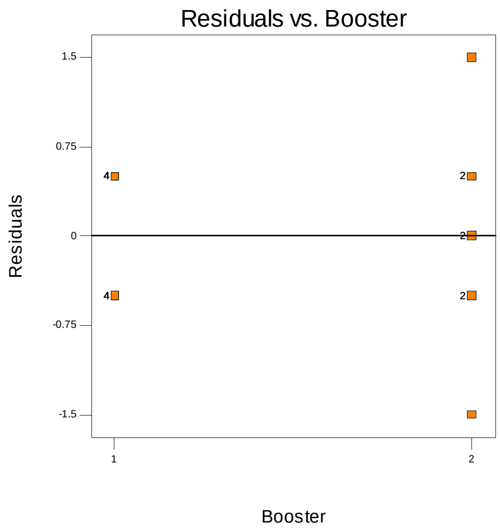
The residual plots shown below identify a violation from our assumptions; nonconstant variance. A power transformation was chosen to correct the violation. λ can be found through trial and error; or the use of a Box-Cox plot that is described in a later chapter. A Box-Cox plot is shown below that identifies a power transformation λ of 1.66.

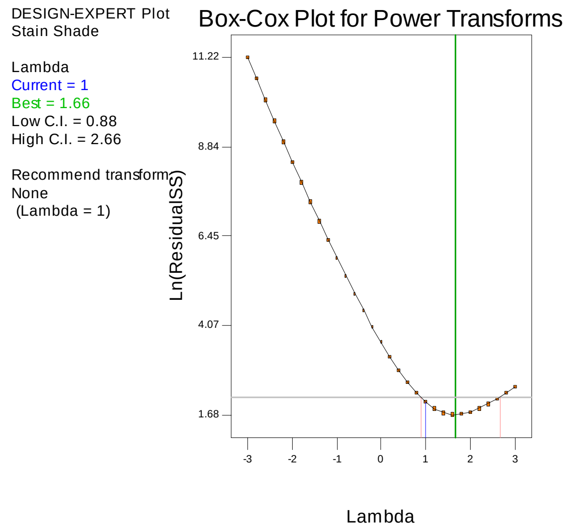












The analysis of variance was performed with the transformed data and is shown below. This time, only the formulation and number of washings appear to be significant; the interaction between these two factors is no longer significant after the data transformation. The residual plots show no deviations from the assumptions. The plot of the effects below identfies the new formulation along with two washings produces the best results. The booster is not significant.

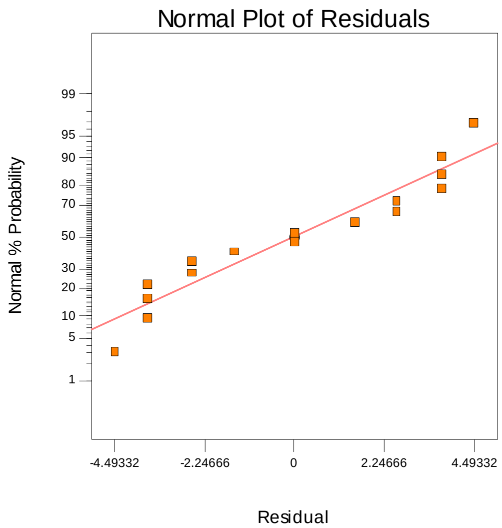
Design Expert Output

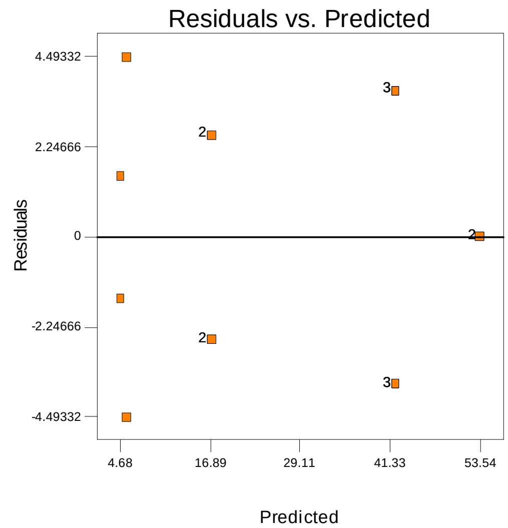
Response: Stain Shade
ANOVA for Selected Factorial Model
Analysis of variance table [Terms added sequentially (first to last)]

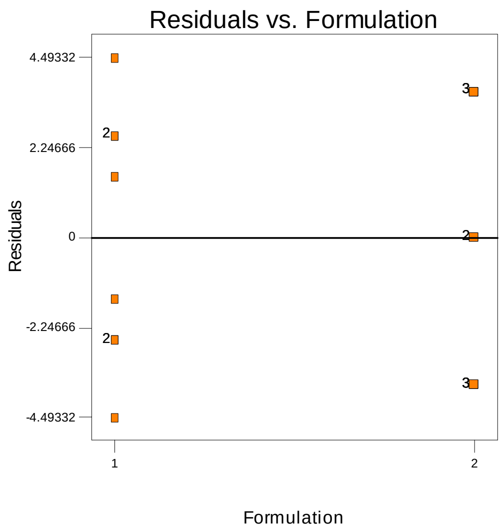
Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	5071.22	7	724.46	38.18	< 0.0001	significant
A	*4587.21*	*1*	*4587.21*	*241.74*	*< 0.0001*	
B	*312.80*	*1*	*312.80*	*16.48*	*0.0036*	
C	*37.94*	*1*	*37.94*	*2.00*	*0.1951*	
AB	*38.24*	*1*	*38.24*	*2.01*	*0.1935*	
AC	*28.55*	*1*	*28.55*	*1.50*	*0.2548*	
BC	*28.55*	*1*	*28.55*	*1.50*	*0.2548*	
ABC	*37.94*	*1*	*37.94*	*2.00*	*0.1951*	
Pure Error	*151.81*	*8*	*18.98*			
Cor Total	5223.03	15				

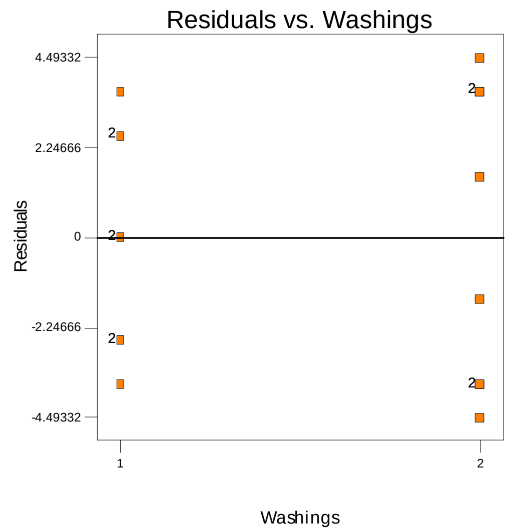
The Model F-value of 38.18 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

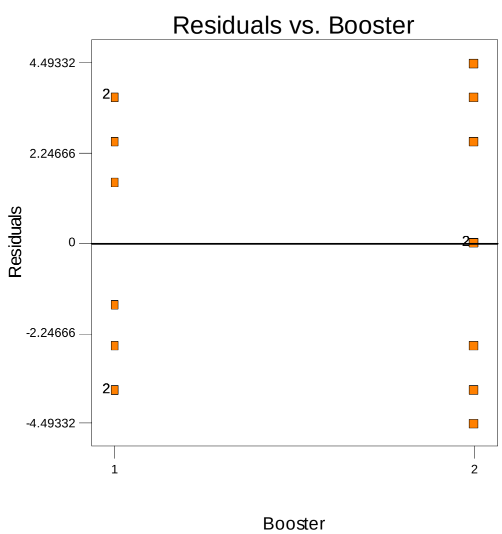
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, are significant model terms.
Values greater than 0.1000 indicate the model terms are not significant.
If there are many insignificant model terms (not counting those required to support hierarchy), model reduction may improve your model.

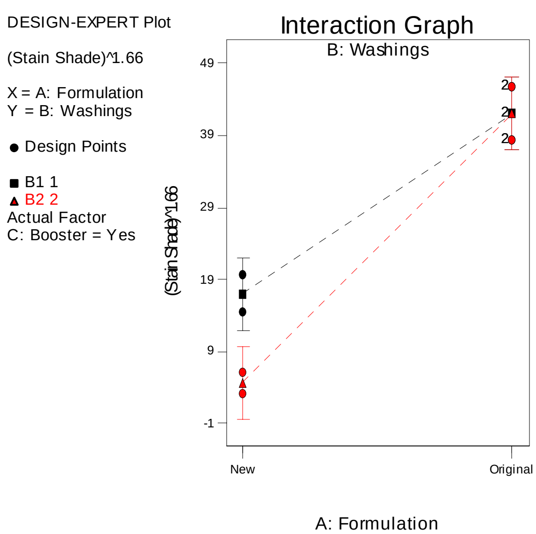

Normal Plot of Residuals
Normal % Probability
99
95
90
80
70
50
30
20
10
5
1
-4.49332
-2.24666
0
2.24666
4.49332
Residual


Residuals vs. Predicted
Residuals
4.49332
2.24666
0
-2.24666
-4.49332
4.68
16.89
29.11
41.33
53.54
Predicted


Residuals vs. Formulation
Residuals
4.49332
2.24666
0
-2.24666
-4.49332
1
2
Formulation


Residuals vs. Washings
Residuals
4.49332
2.24666
0
-2.24666
-4.49332
1
2
Washings


Residuals vs. Booster
Residuals
4.49332
2.24666
0
-2.24666
-4.49332
1
2
Booster


DESIGN-EXPERT Plot
(Stain Shade)^1.66
X = A: Formulation
Y = B: Washings
Design Points
B1 1
B2 2
Actual Factor
C: Booster = Yes
Interaction Graph
B: Washings
(Stain Shade)^1.66
49
39
29
19
9
-1
New
Original
A: Formulation

Chapter 9
Three-Level and Mixed-Level Factorial and Fractional Factorial Design Solutions

9-1 The effects of developer concentration (*A*) and developer time (*B*) on the density of photographic plate film are being studied. Three strengths and three times are used, and four replicates of a 3^2 factorial experiment are run. The data from this experiment follow. Analyze the data using the standard methods for factorial experiments.

	Development Time (minutes)					
Developer Concentration	10		14		18	
10%	0	2	1	3	2	5
	5	4	4	2	4	6
12%	4	6	6	8	9	10
	7	5	7	7	8	5
14%	7	10	10	10	12	10
	8	7	8	7	9	8

Design Expert Output

Response: Data

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	224.22	8	28.03	10.66	< 0.0001	significant
A	*198.22*	*2*	*99.11*	*37.69*	*< 0.0001*	
B	*22.72*	*2*	*11.36*	*4.32*	*0.0236*	
AB	*3.28*	*4*	*0.82*	*0.31*	*0.8677*	
Residual	71.00	27	2.63			
Lack of Fit	*0.000*	*0*				
Pure Error	*71.00*	*27*	*2.63*			
Cor Total	295.22	35				

The Model F-value of 10.66 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case A, B are significant model terms.

Concentration and time are significant. The interaction is not significant. By letting both *A* and *B* be treated as numerical factors, the analysis can be performed as follows:

Design Expert Output

Response: Data

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	221.01	5	44.20	17.87	< 0.0001	significant
A	*192.67*	*1*	*192.67*	*77.88*	*< 0.0001*	
B	*22.04*	*1*	*22.04*	*8.91*	*0.0056*	
A2	*5.56*	*1*	*5.56*	*2.25*	*0.1444*	
B2	*0.68*	*1*	*0.68*	*0.28*	*0.6038*	
AB	*0.062*	*1*	*0.062*	*0.025*	*0.8748*	
Residual	74.22	30	2.47			
Lack of Fit	*3.22*	*3*	*1.07*	*0.41*	*0.7488*	*not significant*
Pure Error	*71.00*	*27*	*2.63*			
Cor Total	295.22	35				

The Model F-value of 17.87 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

9-2 Compute the *I* and *J* components of the two-factor interaction in Problem 9-1.

	B		
	11	10	17
A	22	28	32
	32	35	39

$$AB \text{ Totals} = 77, 78, 71;\quad SS_{AB} = \frac{77^2 + 78^2 + 71^2}{12} - \frac{226^2}{36} = 2.39 = I(AB)$$

$$AB^2 \text{ Totals} = 78, 74, 74;\quad SS_{AB^2} = \frac{78^2 + 74^2 + 74^2}{12} - \frac{226^2}{36} = 0.89 = J(AB)$$

$$SS_{AB} = I(AB) + J(AB) = 3.28$$

9-3 An experiment was performed to study the effect of three different types of 32-ounce bottles (*A*) and three different shelf types (*B*) -- smooth permanent shelves, end-aisle displays with grilled shelves, and beverage coolers -- on the time it takes to stock ten 12-bottle cases on the shelves. Three workers (factor *C*) were employed in this experiment, and two replicates of a 3^3 factorial design were run. The observed time data are shown in the following table. Analyze the data and draw conclusions.

		Replicate I			Replicate 2		
Worker	Bottle Type	Permanent	EndAisle	Cooler	Permanent	EndAisle	Cooler
1	Plastic	3.45	4.14	5.80	3.36	4.19	5.23
	28-mm glass	4.07	4.38	5.48	3.52	4.26	4.85
	38-mm glass	4.20	4.26	5.67	3.68	4.37	5.58
2	Plastic	4.80	5.22	6.21	4.40	4.70	5.88
	28-mm glass	4.52	5.15	6.25	4.44	4.65	6.20
	38-mm glass	4.96	5.17	6.03	4.39	4.75	6.38
3	Plastic	4.08	3.94	5.14	3.65	4.08	4.49
	28-mm glass	4.30	4.53	4.99	4.04	4.08	4.59
	38-mm glass	4.17	4.86	4.85	3.88	4.48	4.90

Design Expert Output

Response: **Time**

ANOVA for Selected Factorial Model

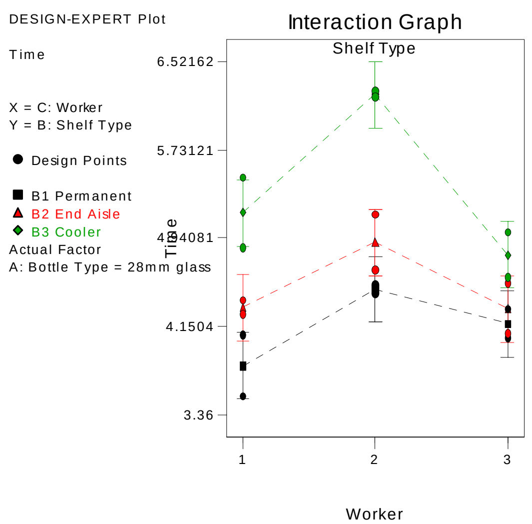
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	28.38	26	1.09	13.06	< 0.0001	significant
A	*0.33*	*2*	*0.16*	*1.95*	*0.1618*	
B	*17.91*	*2*	*8.95*	*107.10*	*< 0.0001*	
C	*7.91*	*2*	*3.96*	*47.33*	*< 0.0001*	
AB	*0.11*	*4*	*0.027*	*0.33*	*0.8583*	
AC	*0.11*	*4*	*0.027*	*0.32*	*0.8638*	
BC	*1.59*	*4*	*0.40*	*4.76*	*0.0049*	
ABC	*0.43*	*8*	*0.053*	*0.64*	*0.7380*	
Residual	2.26	27	0.084			
Lack of Fit	*0.000*	*0*				
Pure Error	*2.26*	*27*	*0.084*			
Cor Total	30.64	53				

The Model F-value of 13.06 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B, C, BC are significant model terms.

Factors *B* and *C*, shelf type and worker, and the *BC* interaction are significant. For the shortest time regardless of worker chose the permanent shelves. This can easily be seen in the interaction plot below.



9-4 A medical researcher is studying the effect of lidocaine on the enzyme level in the heart muscle of beagle dogs. Three different commercial brands of lidocaine (*A*), three dosage levels (*B*), and three dogs (*C*) are used in the experiment, and two replicates of a 3^3 factorial design are run. The observed enzyme levels follow. Analyze the data from this experiment.

		Replicate I			Replicate 2		
		Dog			Dog		
Lidocaine Brand	Dosage Strength	1	2	3	1	2	3
1	1	86	84	85	84	85	86
	2	94	99	98	95	97	90
	3	101	106	98	105	104	103
2	1	85	84	86	80	82	84
	2	95	98	97	93	99	95
	3	108	114	109	110	102	100
3	1	84	83	81	83	80	79
	2	95	97	93	92	96	93
	3	105	100	106	102	111	108

Design Expert Output

Response: Enzyme Level
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	4490.33	26	172.71	16.99	< 0.0001	significant
A	*31.00*	*2*	*15.50*	*1.52*	*0.2359*	
B	*4260.78*	*2*	*2130.39*	*209.55*	*< 0.0001*	
C	*28.00*	*2*	*14.00*	*1.38*	*0.2695*	
AB	*69.56*	*4*	*17.39*	*1.71*	*0.1768*	
AC	*3.33*	*4*	*0.83*	*0.082*	*0.9872*	
BC	*36.89*	*4*	*9.22*	*0.91*	*0.4738*	

ABC	*60.78*	*8*	*7.60*	*0.75*	*0.6502*
Residual	274.50	27	10.17		
Lack of Fit	*0.000*	*0*			
Pure Error	*274.50*	*27*	*10.17*		
Cor Total	4764.83	53			

The Model F-value of 16.99 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant. In this case B are significant model terms.

The dosage is significant.

9-5 Compute the *I* and *J* components of the two-factor interactions for Example 9-1.

	A		
B	134	188	44
	-155	-348	-289
	176	127	288

I totals = 74,75,16 J totals = -128,321,-28

I(AB) = 126.78 J(AB) = 6174.12

SS_{AB} = 6300.90

	A		
C	-190	-58	-211
	399	230	394
	6	-205	-140

I totals = -100,342,-77 J totals = 25,141,-1

I(AC) = 6878.78 J(AC) = 635.12

SS_{AC} = 7513.90

	B		
C	-93	-350	-16
	-155	-133	533
	-104	-309	74

I totals = -152,79,238 J totals =-253,287,131

I(BC) = 4273.00 J(BC) = 8581.34

SS_{BC} = 12854.34

9-6 An experiment is run in a chemical process using a 3^2 factorial design. The design factors are temperature and pressure, and the response variable is yield. The data that result from this experiment are shown below.

	Pressure, psig		
Temperature, °C	100	120	140
80	47.58, 48.77	64.97, 69.22	80.92, 72.60
90	51.86, 82.43	88.47, 84.23	93.95, 88.54
100	71.18, 92.77	96.57, 88.72	76.58, 83.04

(a) Analyze the data from this experiment by conducting an analysis of variance. What conclusions can you draw?

Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3187.13	8	398.39	4.37	0.0205	significant
A	*1096.93*	*2*	*548.47*	*6.02*	*0.0219*	
B	*1503.56*	*2*	*751.78*	*8.25*	*0.0092*	
AB	*586.64*	*4*	*146.66*	*1.61*	*0.2536*	
Pure Error	819.98	9	91.11			
Cor Total	4007.10	17				

The Model F-value of 4.37 implies the model is significant. There is only
a 2.05% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B are significant model terms.

Temperature and pressure are significant. Their interaction is not. An alternate analysis is performed below with *A* and *B* treated as numeric factors:

Design Expert Output

Response: Yield

ANOVA for Selected Factorial Model

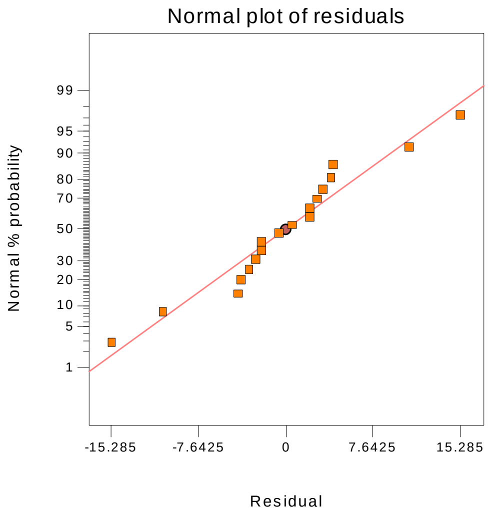
Analysis of variance table [Partial sum of squares]

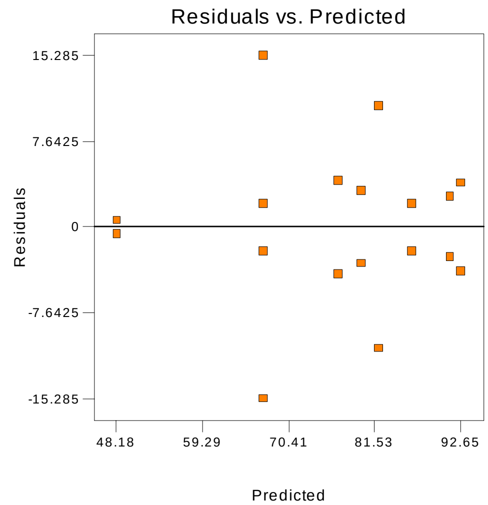
Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3073.27	5	614.65	7.90	0.0017	significant
A	*850.76*	*1*	*850.76*	*10.93*	*0.0063*	
B	*1297.92*	*1*	*1297.92*	*16.68*	*0.0015*	
A2	*246.18*	*1*	*246.18*	*3.16*	*0.1006*	
B2	*205.64*	*1*	*205.64*	*2.64*	*0.1300*	
AB	*472.78*	*1*	*472.78*	*6.08*	*0.0298*	
Residual	933.83	12	77.82			
Lack of Fit	*113.86*	*3*	*37.95*	*0.42*	*0.7454*	*not significant*
Pure Error	*819.98*	*9*	*91.11*			
Cor Total	4007.10	17				

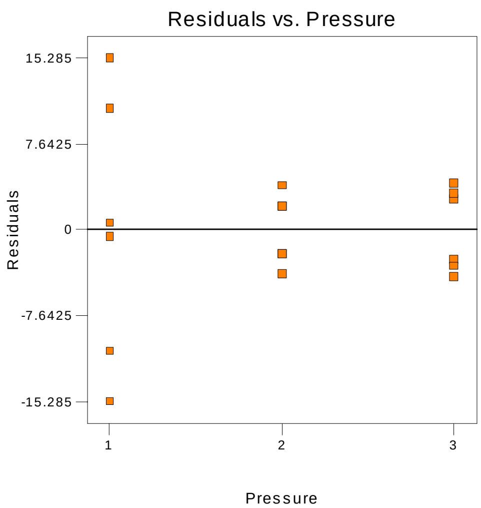
The Model F-value of 7.90 implies the model is significant. There is only
a 0.17% chance that a "Model F-Value" this large could occur due to noise.

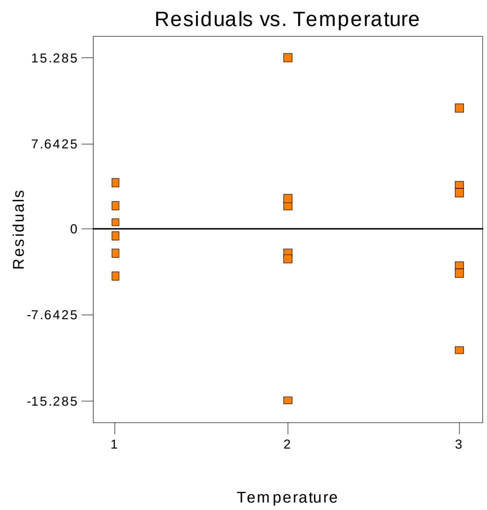
Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case A, B, AB are significant model terms.

(b) Graphically analyze the residuals. Are there any concerns about underlying assumptions or model adequacy?









The plot of residuals versus pressure shows a decreasing funnel shape indicating a non-constant variance.

(c) Verify that if we let the low, medium and high levels of both factors in this experiment take on the levels -1, 0, and +1, then a least squares fit to a second order model for yield is

$$\hat{y} = 86.81 + 10.4x_1 + 8.42x_2 - 7.17x_1^2 - 7.86x_2^2 - 7.69x_1x_2$$

The coefficients can be found in the following table of computer output.

Design Expert Output

Final Equation in Terms of Coded Factors:

Yield =	
+86.81	
+8.42	* A
+10.40	* B
-7.84	* A^2
-7.17	* B^2
-7.69	* A * B

(d) Confirm that the model in part (c) can be written in terms of the natural variables temperature (T) and pressure (P) as

$$\hat{y} = -1335.63 + 18.56T + 8.59P - 0.072T^2 - 0.0196P^2 - 0.0384TP$$

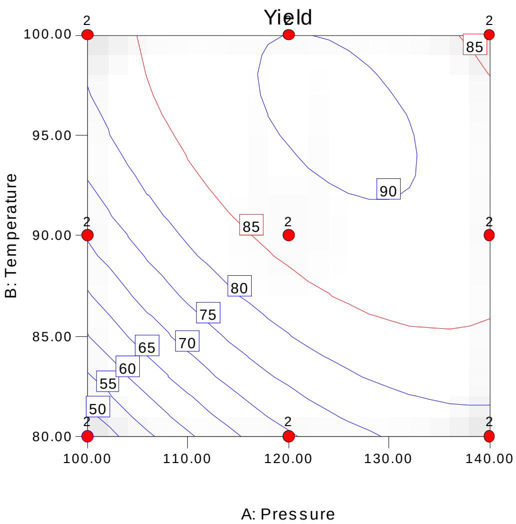
The coefficients can be found in the following table of computer output.

Design Expert Output

Final Equation in Terms of Actual Factors:

Yield	=
-1335.62500	
+8.58737	* Pressure
+18.55850	* Temperature
-0.019612	* Pressure^2
-0.071700	* Temperature^2
-0.038437	* Pressure * Temperature

(e) Construct a contour plot for yield as a function of pressure and temperature. Based on the examination of this plot, where would you recommend running the process.



Run the process in the oval region indicated by the yield of 90.

9-7

(a) Confound a 3^3 design in three blocks using the ABC^2 component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + X_2 + 2X_3$$

Block 1	Block 2	Block 3
000	100	200
112	212	012
210	010	110
120	220	020

022	122	222
202	002	102
221	021	121
101	201	001
011	111	211

The new design is a 180° rotation around the Factor *B* axis.

(b) Confound a 3^3 design in three blocks using the AB^2C component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + 2X_2 + X_3$$

Block 1	Block 2	Block 3
000	210	112
022	202	120
011	221	101
212	100	010
220	122	002
201	111	021
110	012	200
102	020	222
121	001	211

The new design is a 180° rotation around the Factor *C* axis.

(c) Confound a 3^3 design three blocks using the *ABC* component of the three-factor interaction. Compare your results with the design in Figure 9-7.

$$L = X_1 + X_2 + X_3$$

Block 1	Block 2	Block 3
000	112	221
210	022	101
120	202	011
021	100	212
201	010	122
111	220	002
012	121	200
222	001	110
102	211	020

The new design is a 90° rotation around the Factor *C* axis along with switching layer 0 and layer 1 in the *C* axis.

(d) After looking at the designs in parts (a), (b), and (c) and Figure 9-7, what conclusions can you draw?

All four designs are relatively the same. The only differences are rotations and swapping of layers.

9-8 Confound a 3^4 design in three blocks using the AB^2CD component of the four-factor interaction.

$$L = X_1 + 2X_2 + X_3 + X_4$$

Block 1								
0000	1100	0110	0101	2200	0220	0202	1210	1201
0211	1222	2212	2221	0122	2111	1121	1112	2010
2102	0021	2001	2120	1011	2022	0012	1002	1020

Block 2								
1021	1110	1202	0001	0120	0212	1012	1101	1220
0200	0022	0111	2002	2121	2210	0010	0102	0221
1000	1122	1211	2112	2201	2020	2011	2100	2222

Block 3								
2012	2101	2220	1022	1111	1200	2000	2121	2211
1221	1010	1102	0020	0112	0201	1001	1120	1212
2021	2110	2202	0100	0222	0011	0002	0121	0210

9-9 Consider the data from the first replicate of Problem 9-3. Assuming that all 27 observations could not be run on the same day, set up a design for conducting the experiment over three days with AB^2C confounded with blocks. Analyze the data.

	Block 1			Block 2			Block 3	
000	=	3.45	100	=	4.07	200	=	4.20
110	=	4.38	210	=	4.26	010	=	4.14
011	=	5.22	111	=	4.14	211	=	5.17
102	=	4.30	202	=	4.17	002	=	4.08
201	=	4.96	001	=	4.80	101	=	4.52
212	=	4.86	012	=	3.94	112	=	4.53
121	=	6.25	221	=	4.99	021	=	6.21
022	=	5.14	122	=	6.03	222	=	4.85
220	=	5.67	020	=	5.80	120	=	5.48
Totals	=	44.23			43.21			43.18

Design Expert Output

Response: Time
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Block	0.23	2	0.11			
Model	13.17	18	0.73	4.27	0.0404	significant
A	*0.048*	*2*	*0.024*	*0.14*	*0.8723*	
B	*8.92*	*2*	*4.46*	*26.02*	*0.0011*	
C	*1.57*	*2*	*0.78*	*4.57*	*0.0622*	
AB	*1.31*	*4*	*0.33*	*1.91*	*0.2284*	
AC	*0.87*	*4*	*0.22*	*1.27*	*0.3774*	
BC	*0.45*	*4*	*0.11*	*0.66*	*0.6410*	
Residual	1.03	6	0.17			
Cor Total	14.43	26				

The Model F-value of 4.27 implies the model is significant. There is only a 4.04% chance that a "Model F-Value" this large could occur due to noise.

Values of "Prob > F" less than 0.0500 indicate model terms are significant.
In this case B are significant model terms.

9-10 Outline the analysis of variance table for the 3^4 design in nine blocks. Is this a practical design?

Source	DF
A	2
B	2
C	2
D	2
AB	4
AC	4
AD	4
BC	4
BD	4
CD	4
$ABC\ (AB^2C, ABC^2, AB^2C^2)$	6
$ABD\ (ABD, AB^2D, ABD^2)$	6
$ACD\ (ACD, ACD^2, AC^2D^2)$	6
$BCD\ (BCD, BC^2D, BCD^2)$	6
$ABCD$	16
Blocks ($ABC, AB^2C^2, AC^2D, BC^2D^2$)	8
Total	80

Any experiment with 81 runs is large. Instead of having three full levels of each factor, if two levels of each factor could be used, then the overall design would have 16 runs plus some center points. This two-level design could now probably be run in 2 or 4 blocks, with center points in each block. Additional curvature effects could be determined by augmenting the experiment with the axial points of a central composite design and additional enter points. The overall design would be less than 81 runs.

9-11 Consider the data in Problem 9-3. If ABC is confounded in replicate I and ABC^2 is confounded in replicate II, perform the analysis of variance.

$L_1 = X_1 + X_2 + X_3$									$L_2 = X_1 + X_2 + 2X_2$								
Block 1			Block 2			Block 3			Block 1			Block 2			Block 3		
000	=	3.45	001	=	4.80	002	=	4.08	000	=	3.36	100	=	3.52	200	=	3.68
111	=	5.15	112	=	4.53	110	=	4.38	101	=	4.44	201	=	4.39	001	=	4.40
222	=	4.85	220	=	5.67	221	=	6.03	011	=	4.70	111	=	4.65	211	=	4.75
120	=	5.48	121	=	6.25	122	=	4.99	221	=	6.38	021	=	5.88	121	=	6.20
102	=	4.30	100	=	4.07	101	=	4.52	202	=	3.88	002	=	3.65	102	=	4.04
210	=	4.26	211	=	5.17	212	=	4.86	022	=	4.49	122	=	4.59	222	=	4.90
201	=	4.96	202	=	4.17	200	=	4.20	120	=	4.85	220	=	5.58	020	=	5.23
012	=	3.94	010	=	4.14	011	=	5.22	210	=	4.37	010	=	4.19	110	=	4.26
021	=	6.21	022	=	5.14	020	=	5.80	112	=	4.08	212	=	4.48	012	=	4.08

The sums of squares for A, B, C, AB, AC, and BC are calculated as usual. The only sums of squares presenting difficulties with calculations are the four components of the ABC interaction (ABC, ABC^2, AB^2C, and AB^2C^2). ABC is computed using replicate I and ABC^2 is computed using replicate II. AB^2C and AB^2C^2 are computed using data from both replicates.

We will show how to calculate AB^2C and AB^2C^2 from both replicates. Form a two-way table of A x B at each level of C. Find the I(AB) and J(AB) totals for each third of the A x B table.

		A				
C	*B*	0	1	2	I	J
	0	6.81	7.59	7.88	26.70	27.55
0	1	8.33	8.64	8.63	27.25	27.17
	2	11.03	10.33	11.25	26.54	25.77
	0	9.20	8.96	9.35	31.41	31.25
1	1	9.92	9.80	9.92	30.97	31.29

	2	12.09	12.45	12.41	31.72	31.57
	0	7.73	8.34	8.05	26.09	26.29
2	1	8.02	8.61	9.34	27.31	26.11
	2	9.63	9.58	9.75	25.65	26.65

The I and J components for each third of the above table are used to form a new table of diagonal totals.

C	I(*AB*)			J(*AB*)		
0	2.670	27.25	26.54	27.55	27.17	25.77
1	31.41	30.97	31.72	31.25	31.29	31.57
2	26.09	27.31	25.65	26.29	26.11	26.65

I Totals: 85.06,85.26,83.32   I Totals: 85.99,85.03,83.12

J Totals: 85.73,83.60,84.31   J Totals: 83.35,85.06,85.23

Now, $AB^2C^2 = I[C \text{ x } I(AB)] = \dfrac{(85.06)^2+(85.26)^2+(83.32)^2}{18} - \dfrac{(253.64)^2}{54} = 0.1265$

and, $AB^2C = J[C \text{ x } I(AB)] = \dfrac{(85.73)^2+(83.60)^2+(84.31)^2}{18} - \dfrac{(253.64)^2}{54} = 0.1307$

If it were necessary, we could find ABC^2 as ABC^2= I[*C* x J(*AB*)] and *ABC* as J[*C* x J(*AB*)]. However, these components must be computed using the data from the appropriate replicate.

The analysis of variance table:

Source	SS	DF	MS	F_0
Replicates	1.06696	1		
Blocks within Replicates	0.2038	4		
A	0.4104	2	0.2052	5.02
B	17.7514	2	8.8757	217.0
C	7.6631	2	3.8316	93.68
AB	0.1161	4	0.0290	<1
AC	0.1093	4	0.0273	<1
BC	1.6790	4	0.4198	10.26
ABC (rep I)	0.0452	2	0.0226	<1
ABC^2 (rep II)	0.1020	2	0.0510	1.25
AB^2C	0.1307	2	0.0754	1.60
AB^2C^2	0.1265	2	0.0633	1.55
Error	0.8998	22	0.0409	
Total	30.3069	53		

9-12 Consider the data from replicate I in Problem 9-3. Suppose that only a one-third fraction of this design with *I=ABC* is run. Construct the design, determine the alias structure, and analyze the design.

The design is 000, 012, 021, 102, 201, 111, 120, 210, 222.

The alias structure is: $A = BC = AB^2C^2$

$B = AC = AB^2C$

$C = AB = ABC^2$

$AB^2 = AC^2 = BC^2$

		C		
A	*B*	0	1	2
	0	3.45		
0	1			5.48
	2		4.26	
	0			6.21
1	1		5.15	
	2	4.96		
	0		3.94	
2	1	4.30		
	2			4.85

Source	SS	DF
A	2.25	2
B	0.30	2
C	2.81	2
AB^2	0.30	2
Total	5.66	8

9-13 From examining Figure 9-9, what type of design would remain if after completing the first 9 runs, one of the three factors could be dropped?

A full 3^2 factorial design results.

9-14 Construct a 3_{IV}^{4-1} design with $I=ABCD$. Write out the alias structure for this design.

The 27 runs for this design are as follows:

0000	1002	2001
0012	1011	2010
0021	1020	2022
0102	1101	2100
0111	1110	2112
0120	1122	2121
0201	1200	2202
0210	1212	2211
0222	1221	2220

$A = AB^2C^2D^2 = BCD$ $\quad B = AB^2CD = ACD$ $\quad C = ABC^2D = ABD$ $\quad D = ABCD^2 = ABC$
$AB = ABC^2D^2 = CD$ $\quad AB^2 = AC^2D^2 = BC^2D^2$ $\quad AC = AB^2CD^2 = BD$ $\quad AC^2 = AB^2D^2 = BC^2D$
$BC = AB^2C^2D = AD$ $\quad BC^2 = AB^2D = AC^2D$ $\quad BD^2 = AB^2C = ACD^2$ $\quad CD^2 = ABC^2 = ABCD^2$
$AD^2 = AB^2C^2 = BCD^2$

9-15 Verify that the design in Problem 9-14 is a resolution IV design.

The design in Problem 9-14 is a Resolution IV design because no main effect is aliased with a component of a two-factor interaction, but some two-factor interaction components are aliased with each other.

9-16 Construct a 3^{5-2} design with $I=ABC$ and $I=CDE$. Write out the alias structure for this design. What is the resolution of this design?

The complete defining relation for this design is : $I = ABC = CDE = ABC^2DE = ABD^2E^2$
This is a resolution III design. The defining contrasts are $L_1 = X_1 + X_2 + X_3$ and $L_2 = X_3 + X_4 + X_5$.

00000	11120	20111
00012	22111	22222
00022	21021	01210
01200	02111	12000
02100	01222	20120
10202	12012	11111
20101	02120	22201
11102	10210	21012
21200	12021	10222

To find the alias of any effect, multiply the effect by I and I^2. For example, the alias of A is:

$A = AB^2C^2 = ACDE = AB^2CDE = AB^2DE = BC = AC^2D^2E^2 = BC^2DE = BD^2E^2$

9-17 Construct a 3^{9-6} design, and verify that is a resolution III design.

Use the generators $I = AC^2D^2$, $I = AB^2C^2E$, $I = BC^2F^2$, $I = AB^2CG$, $I = ABCH^2$, and $I = ABJ^2$

000000000	021201102	102211001
022110012	212012020	001212210
011220021	100120211	211100110
221111221	122200220	020022222
210221200	010011111	222020101
202001212	201122002	200210122
112222112	002121120	121021010
101002121	111010202	110101022
120112100	220202011	012102201

To find the alias of any effect, multiply the effect by I and I^2. For example, the alias of C is:

$C = C(BC^2F^2) = BF^2$, At least one main effect is aliased with a component of a two-factor interaction.

9-18 Construct a 4×2^3 design confounded in two blocks of 16 observations each. Outline the analysis of variance for this design.

Design is a 4×2^3, with ABC at two levels, and Z at 4 levels. Represent Z with two pseudo-factors D and E as follows:

Factor	Pseudo-	Factors
Z	D	E
Z_1	0	0 = (1)
Z_2	1	0 = d
Z_3	0	1 = e
Z_4	1	1 = de

The 4 x 2^3 is now a 2^5 in the factors *A*, *B*, *C*, *D* and *E*. Confound *ABCDE* with blocks. We have given both the letter notation and the digital notation for the treatment combinations.

Block 1			Block 2		
(1)	=	000	*a*	=	1000
ab	=	1100	*b*	=	0100
ac	=	1010	*c*	=	0010
bc	=	0110	*abc*	=	1110
abcd	=	1111	*bcd*	=	0111
abce	=	1112	*bce*	=	0112
cd	=	0011	*acd*	=	1011
ce	=	0012	*ace*	=	1012
de	=	0003	*ade*	=	1003
abde	=	1103	*bde*	=	0103
bcde	=	0113	*abcde*	=	1113
be	=	0102	*abd*	=	1101
ad	=	1001	*abe*	=	1102
ae	=	1002	*d*	=	0001
acde	=	1013	*e*	=	0002
bd	=	0101	*cde*	=	0013

Source	DF
A	1
B	1
C	1
Z (D+E+DE)	3
AB	1
AC	1
AZ (AD+AE+ADE)	3
BC	1
BZ (BD+BE+BDE)	3
CZ (CD+CE+CDE)	3
ABC	1
ABZ (ABD+ABE+ABDE)	3
ACZ (ACD+ACE+ACDE)	3
BCZ (BCD+BCE+BCDE)	3
ABCZ (ABCD+ABCE)	2
Blocks (or *ABCDE*)	1
Total	31

9-19 Outline the analysis of variance table for a $2^2 3^2$ factorial design. Discuss how this design may be confounded in blocks.

Suppose we have *n* replicates of a $2^2 3^2$ factorial design. *A* and *B* are at 2 levels, and *C* and *D* are at 3 levels.

Source	DF	Components for Confounding
A	1	*A*
B	1	*B*
C	2	*C*
D	2	*D*
AB	1	*AB*
AC	2	*AC*
AD	2	*AD*

BC	2	*BD*
BD	2	CD, CD^2
CD	4	*ABC*
ABC	2	*ABD*
ABD	2	ACD, ACD^2
ACD	4	BCD, BCD^2
BCD	4	$ABCD, ABCD^2$
ABCD	4	
Error	36(n-1)	
Total	36n-1	

Confounding in this series of designs is discussed extensively by Margolin (1967). The possibilities for a single replicate of the $2^2 3^2$ design are:

2 blocks of 18 observations	6 blocks of 6 observations
3 blocks of 12 observations	9 blocks of 4 observations
4 blocks of 9 observations	

For example, one component of the four-factor interaction, say $ABCD^2$, could be selected to confound the design in 3 blocks for 12 observations each, while to confound the design in 2 blocks of 18 observations 3 each we would select the *AB* interaction. Cochran and Cox (1957) and Anderson and McLean (1974) discuss confounding in these designs.

9-20 Starting with a 16-run 2^4 design, show how two three-level factors can be incorporated in this experiment. How many two-level factors can be included if we want some information on two-factor interactions?

Use column *A* and *B* for one three-level factor and columns *C* and *D* for the other. Use the *AC* and *BD* columns for the two, two-level factors. The design will be of resolution V.

9-21 Starting with a 16-run 2^4 design, show how one three-level factor and three two-level factors can be accommodated and still allow the estimation of two-factor interactions.

Use columns *A* and *B* for the three-level factor, and columns *C* and *D* and *ABCD* for the three two-level factors. This design will be of resolution V.

9-22 In Problem 9-26, you met Harry and Judy Peterson-Nedry, two friends of the author who have a winery and vineyard in Newberg, Oregon. That problem described the application of two-level fractional factorial designs to their 1985 Pinor Noir product. In 1987, they wanted to conduct another Pinot Noir experiment. The variables for this experiment were

Variable	Levels
Clone of Pinot Noir	Wadenswil, Pommard
Berry Size	Small, Large
Fermentation temperature	80F, 85F, 90/80F, 90F
Whole Berry	None, 10%
Maceration Time	10 days, 21 days
Yeast Type	Assmanhau, Champagne
Oak Type	Troncais, Allier

Harry and Judy decided to use a 16-run two-level fractional factorial design, treating the four levels of fermentation temperature as two two-level variables. As in Problem 8-26, they used the rankings from a taste-test panel as the response variable. The design and the resulting average ranks are shown below:

Berry	Ferm.	Whole	Macer.	Yeast	Oak	Average

Run	Clone	Size	Temp.		Berry	Time	Type	Type	Rank
1	-	-	-	-	-	-	-	-	4
2	+	-	-	-	-	+	+	+	10
3	-	+	-	-	+	-	+	+	6
4	+	+	-	-	+	+	-	-	9
5	-	-	+	-	+	+	+	-	11
6	+	-	+	-	+	-	-	+	1
7	-	+	+	-	-	+	-	+	15
8	+	+	+	-	-	-	+	-	5
9	-	-	-	+	+	+	-	+	12
10	+	-	-	+	+	-	+	-	2
11	-	+	-	+	-	+	+	-	16
12	+	+	-	+	-	-	-	+	3
13	-	-	+	+	-	-	+	+	8
14	+	-	+	+	-	+	-	-	14
15	-	+	+	+	+	-	-	-	7
16	+	+	+	+	+	+	+	+	13

(a) Describe the aliasing in this design.

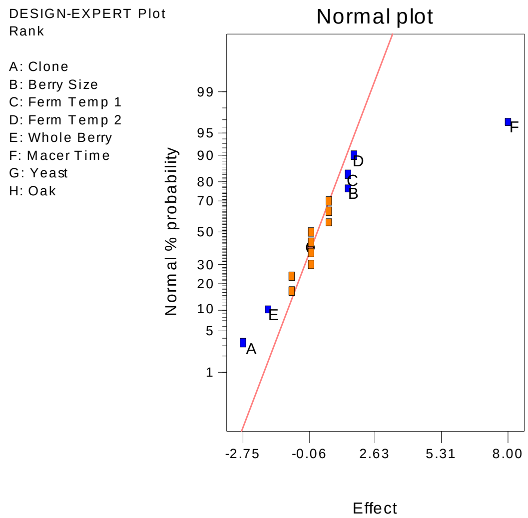
The design is a resolution IV design such that the main effects are aliased with three factor interactions.

Design Expert Output

Term	Aliases
Intercept	ABCG ABDH ABEF ACDF ACEH ADEG AFGH BCDE BCFH BDFG BEGH CDGH CEFG DEFH
A	BCG BDH BEF CDF CEH DEG FGH ABCDE
B	ACG ADH AEF CDE CFH DFG EGH
C	ABG ADF AEH BDE BFH DGH EFG
D	ABH ACF AEG BCE BFG CGH EFH
E	ABF ACH ADG BCD BGH CFG DFH
F	ABE ACD AGH BCH BDG CEG DEH
G	ABC ADE AFH BDF BEH CDH CEF
H	ABD ACE AFG BCF BEG CDG DEF
AB	CG DH EF ACDE ACFH ADFG AEGH BCDF BCEH BDEG BFGH
AC	BG DF EH ABDE ABFH ADGH AEFG BCDH BCEF CDEG CFGH
AD	BH CF EG ABCE ABFG ACGH AEFH BCDG BDEF CDEH DFGH
AE	BF CH DG ABCD ABGH ACFG ADFH BCEG BDEH CDEF EFGH
AF	BE CD GH ABCH ABDG ACEG ADEH BCFG BDFH CEFH DEFG
AG	BC DE FH ABDF ABEH ACDH ACEF BDGH BEFG CDFG CEGH
AH	BD CE FG ABCF ABEG ACDG ADEF BCGH BEFH CDFH DEGH

(b) Analyze the data and draw conclusions.

All of the main effects except Yeast and Oak are significant. The Macer Time is the most significant. None of the interactions were significant.



Design Expert Output

Response: Rank

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	328.75	6	54.79	43.83	< 0.0001	significant
A	*30.25*	*1*	*30.25*	*24.20*	*0.0008*	
B	*9.00*	*1*	*9.00*	*7.20*	*0.0251*	
C	*9.00*	*1*	*9.00*	*7.20*	*0.0251*	
D	*12.25*	*1*	*12.25*	*9.80*	*0.0121*	
E	*12.25*	*1*	*12.25*	*9.80*	*0.0121*	
F	*256.00*	*1*	*256.00*	*204.80*	*< 0.0001*	
Residual	11.25	9	1.25			
Cor Total	340.00	15				

The Model F-value of 43.83 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.12	R-Squared	0.9669
Mean	8.50	Adj R-Squared	0.9449
C.V.	13.15	Pred R-Squared	0.8954
PRESS	35.56	Adeq Precision	19.270

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	8.50	1	0.28	7.87	9.13	
A-Clone	-1.38	1	0.28	-2.01	-0.74	1.00
B-Berry Size	0.75	1	0.28	0.12	1.38	1.00
C-Ferm Temp 1	0.75	1	0.28	0.12	1.38	1.00
D-Ferm Temp 2	0.88	1	0.28	0.24	1.51	1.00
E-Whole Berry	-0.87	1	0.28	-1.51	-0.24	1.00
F-Macer Time	4.00	1	0.28	3.37	4.63	1.00

Final Equation in Terms of Coded Factors:

Rank	=
+8.50	
-1.38	* A
+0.75	* B
+0.75	* C
+0.88	* D
-0.87	* E
+4.00	* F

(c) What comparisons can you make between this experiment and the 1985 Pinot Noir experiment from Problem 8-26?

The experiment from Problem 8-26 indicates that yeast, barrel, whole cluster and the clone x yeast interactions were significant. This experiment indicates that maceration time, whole berry, clone and fermentation temperature are significant.

9-23 An article by W.D. Baten in the 1956 volume of *Industrial Quality Control* described an experiment to study the effect of three factors on the lengths of steel bars. Each bar was subjected to one of two heat treatment processes, and was cut on one of four machines at one of three times during the day (8 am, 11 am, or 3 pm). The coded length data are shown below

(a) Analyze the data from this experiment assuming that the four observations in each cell are replicates.

The Machine effect (C) is significant, the Heat Treat Process (B) is also significant, while the Time of Day (A) is not significant. None of the interactions are significant.

		Machine							
Time of Day	Heat Treat Process	1		2		3		4	
8am	1	6	9	7	9	1	2	6	6
		1	3	5	5	0	4	7	3
	2	4	6	6	5	-1	0	4	5
		0	1	3	4	0	1	5	4
11 am	1	6	3	8	7	3	2	7	9
		1	-1	4	8	1	0	11	6
	2	3	1	6	4	2	0	9	4
		1	-2	1	3	-1	1	6	3
3 pm	1	5	4	10	11	-1	2	10	5
		9	6	6	4	6	1	4	8
	2	6	0	8	7	0	-2	4	3
		3	7	10	0	4	-4	7	0

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	590.33	23	25.67	4.13	< 0.0001	significant
A	*26.27*	*2*	*13.14*	*2.11*	*0.1283*	
B	*42.67*	*1*	*42.67*	*6.86*	*0.0107*	
C	*393.42*	*3*	*131.14*	*21.10*	*< 0.0001*	
AB	*23.77*	*2*	*11.89*	*1.91*	*0.1552*	
AC	*42.15*	*6*	*7.02*	*1.13*	*0.3537*	
BC	*13.08*	*3*	*4.36*	*0.70*	*0.5541*	
ABC	*48.98*	*6*	*8.16*	*1.31*	*0.2623*	
Pure Error	447.50	72	6.22			
Cor Total	1037.83	95				

The Model F-value of 4.13 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.49	R-Squared	0.5688

Mean	3.96	Adj R-Squared	0.4311
C.V.	62.98	Pred R-Squared	0.2334
PRESS	795.56	Adeq Precision	7.020

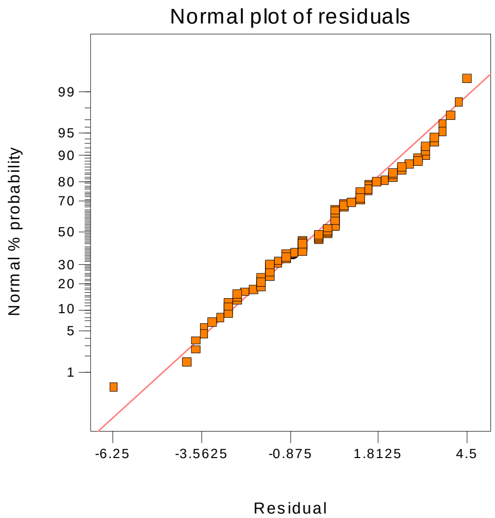
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	3.96	1	0.25	3.45	4.47	
A[1]	0.010	1	0.36	-0.71	0.73	
A[2]	-0.65	1	0.36	-1.36	0.071	
B-Process	-0.67	1	0.25	-1.17	-0.16	1.00
C[1]	-0.54	1	0.44	-1.42	0.34	
C[2]	1.92	1	0.44	1.04	2.80	
C[3]	-3.08	1	0.44	-3.96	-2.20	
A[1]B	0.010	1	0.36	-0.71	0.73	
A[2]B	0.60	1	0.36	-0.11	1.32	
A[1]C[1]	0.32	1	0.62	-0.92	1.57	
A[2]C[1]	-1.27	1	0.62	-2.51	-0.028	
A[1]C[2]	-0.39	1	0.62	-1.63	0.86	
A[2]C[2]	-0.10	1	0.62	-1.35	1.14	
A[1]C[3]	0.24	1	0.62	-1.00	1.48	
A[2]C[3]	0.77	1	0.62	-0.47	2.01	
BC[1]	-0.25	1	0.44	-1.13	0.63	
BC[2]	-0.46	1	0.44	-1.34	0.42	
BC[3]	0.46	1	0.44	-0.42	1.34	
A[1]BC[1]	-0.094	1	0.62	-1.34	1.15	
A[2]BC[1]	-0.44	1	0.62	-1.68	0.80	
A[1]BC[2]	0.11	1	0.62	-1.13	1.36	
A[2]BC[2]	-1.10	1	0.62	-2.35	0.14	
A[1]BC[3]	-0.43	1	0.62	-1.67	0.82	
A[2]BC[3]	0.60	1	0.62	-0.64	1.85	

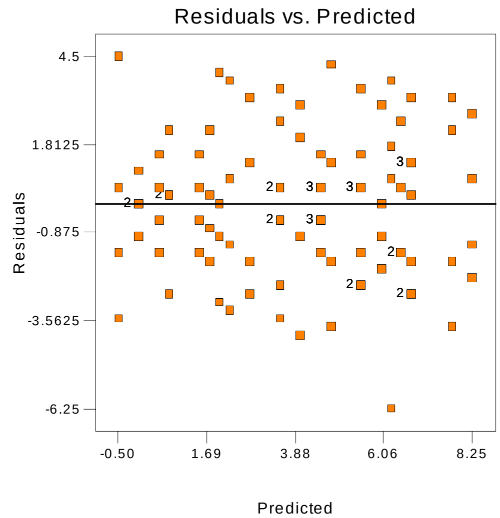
Final Equation in Terms of Coded Factors:

Length	=
+3.96	
+0.010	* A[1]
-0.65	* A[2]
-0.67	* B
-0.54	* C[1]
+1.92	* C[2]
-3.08	* C[3]
+0.010	* A[1]B
+0.60	* A[2]B
+0.32	* A[1]C[1]
-1.27	* A[2]C[1]
-0.39	* A[1]C[2]
-0.10	* A[2]C[2]
+0.24	* A[1]C[3]
+0.77	* A[2]C[3]
-0.25	* BC[1]
-0.46	* BC[2]
+0.46	* BC[3]
-0.094	* A[1]BC[1]
-0.44	* A[2]BC[1]
+0.11	* A[1]BC[2]
-1.10	* A[2]BC[2]
-0.43	* A[1]BC[3]
+0.60	* A[2]BC[3]

(b) Analyze the residuals from this experiment. Is there any indication that there is an outlier in one cell? If you find an outlier, remove it and repeat the analysis from part (a). What are your conclusions?

Standard Order 84, Time of Day at 3:00pm, Heat Treat #2, Machine #2, and length of 0, appears to be an outlier.





The following analysis was performed with the outlier described above removed. As with the original analysis, Machine is significant and Heat Treat Process is also significant, but now Time of Day, factor *A*, is also significant with an *F*-value of 3.05 (the *P*-value is just above 0.05).

Design Expert Output

Response: Length

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	626.58	23	27.24	4.89	< 0.0001	significant
A	*34.03*	*2*	*17.02*	*3.06*	*0.0533*	
B	*33.06*	*1*	*33.06*	*5.94*	*0.0173*	
C	*411.89*	*3*	*137.30*	*24.65*	*< 0.0001*	
AB	*16.41*	*2*	*8.20*	*1.47*	*0.2361*	
AC	*50.19*	*6*	*8.37*	*1.50*	*0.1900*	
BC	*8.38*	*3*	*2.79*	*0.50*	*0.6824*	
ABC	*67.00*	*6*	*11.17*	*2.01*	*0.0762*	
Pure Error	395.42	71	5.57			
Cor Total	1022.00	94				

The Model F-value of 4.89 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.36	R-Squared	0.6131
Mean	4.00	Adj R-Squared	0.4878
C.V.	59.00	Pred R-Squared	0.3100
PRESS	705.17	Adeq Precision	7.447

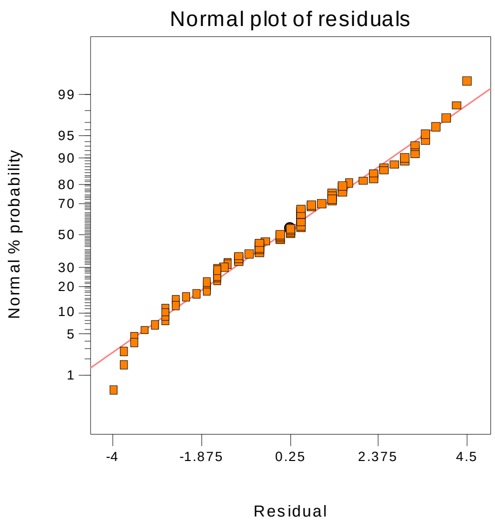
Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	4.05	1	0.24	3.z56	4.53	
A[1]	-0.076	1	0.34	-0.76	0.61	
A[2]	-0.73	1	0.34	-1.41	-0.051	
B-Process	-0.58	1	0.24	-1.06	-0.096	1.00
C[1]	-0.63	1	0.42	-1.46	0.21	
C[2]	2.18	1	0.43	1.33	3.03	
C[3]	-3.17	1	0.42	-4.00	-2.34	
A[1]B	-0.076	1	0.34	-0.76	0.61	
A[2]B	0.52	1	0.34	-0.16	1.20	
A[1]C[1]	0.41	1	0.59	-0.77	1.59	
A[2]C[1]	-1.18	1	0.59	-2.36	-6.278E-003	
A[1]C[2]	-0.65	1	0.60	-1.83	0.54	

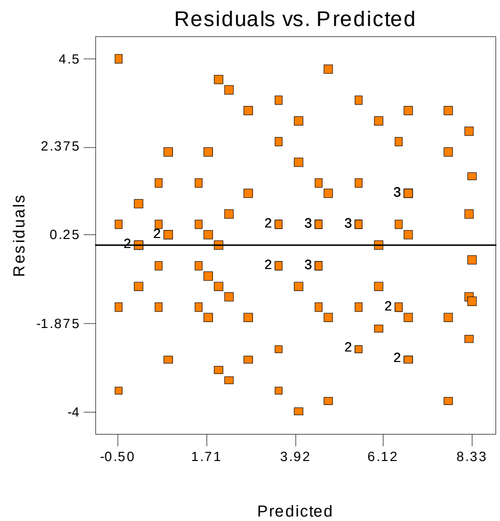
A[2]C[2]	-0.36	1	0.60	-1.55	0.82
A[1]C[3]	0.33	1	0.59	-0.85	1.50
A[2]C[3]	0.86	1	0.59	-0.32	2.04
BC[1]	-0.34	1	0.42	-1.17	0.50
BC[2]	-0.20	1	0.43	-1.05	0.65
BC[3]	0.37	1	0.42	-0.46	1.21
A[1]BC[1]	-6.944E-003	1	0.59	-1.18	1.17
A[2]BC[1]	-0.35	1	0.59	-1.53	0.83
A[1]BC[2]	-0.15	1	0.60	-1.33	1.04
A[2]BC[2]	-1.36	1	0.60	-2.55	-0.18
A[1]BC[3]	-0.34	1	0.59	-1.52	0.84
A[2]BC[3]	0.69	1	0.59	-0.49	1.87

Final Equation in Terms of Coded Factors:

Length	=
+4.05	
-0.076	* A[1]
-0.73	* A[2]
-0.58	* B
-0.63	* C[1]
+2.18	* C[2]
-3.17	* C[3]
-0.076	* A[1]B
+0.52	* A[2]B
+0.41	* A[1]C[1]
-1.18	* A[2]C[1]
-0.65	* A[1]C[2]
-0.36	* A[2]C[2]
+0.33	* A[1]C[3]
+0.86	* A[2]C[3]
-0.34	* BC[1]
-0.20	* BC[2]
+0.37	* BC[3]
-6.944E-003	* A[1]BC[1]
-0.35	* A[2]BC[1]
-0.15	* A[1]BC[2]
-1.36	* A[2]BC[2]
-0.34	* A[1]BC[3]
+0.69	* A[2]BC[3]

The following residual plots are acceptable. Both the normality and constant variance assumptions are satisfied





(c) Suppose that the observations in the cells are the lengths (coded) of bars processed together in heat treating and then cut sequentially (that is, in order) on the three machines. Analyze the data to determine the effects of the three factors on mean length.

The analysis with all effects and interactions included:

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	147.58	23	6.42		
A	*6.57*	*2*	*3.28*		
B	*10.67*	*1*	*10.67*		
C	*98.35*	*3*	*32.78*		
AB	*5.94*	*2*	*2.97*		
AC	*10.54*	*6*	*1.76*		
BC	*3.27*	*3*	*1.09*		
ABC	*12.24*	*6*	*2.04*		
Pure Error	0.000	0			
Cor Total	147.58	23			

The by removing the three factor interaction from the model and applying it to the error, the analysis identifies factor *C* as being significant and *B* as being mildly significant.

Design Expert Output

Response: Length
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	135.34	17	7.96	3.90	0.0502	not significant
A	*6.57*	*2*	*3.28*	*1.61*	*0.2757*	
B	*10.67*	*1*	*10.67*	*5.23*	*0.0623*	
C	*98.35*	*3*	*32.78*	*16.06*	*0.0028*	
AB	*5.94*	*2*	*2.97*	*1.46*	*0.3052*	
AC	*10.54*	*6*	*1.76*	*0.86*	*0.5700*	
BC	*3.27*	*3*	*1.09*	*0.53*	*0.6756*	
Residual	12.24	6	2.04			
Cor Total	147.58	23				

When removing the remaining insignificant factors from the model, *C*, Machine, is the most significant factor while *B*, Heat Treat Process, is moderately significant. Factor *A*, Time of Day, is not significant.

Design Expert Output

Response: Avg
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	109.02	4	27.26	13.43	< 0.0001	significant
B	*10.67*	*1*	*10.67*	*5.26*	*0.0335*	
C	*98.35*	*3*	*32.78*	*16.15*	*< 0.0001*	
Residual	38.56	19	2.03			
Cor Total	147.58	23				

The Model F-value of 13.43 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

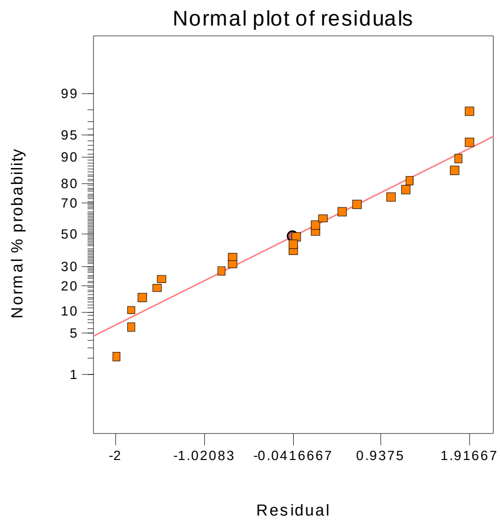
Std. Dev.	1.42	R-Squared	0.7387
Mean	3.96	Adj R-Squared	0.6837
C.V.	35.99	Pred R-Squared	0.5831
PRESS	61.53	Adeq Precision	9.740

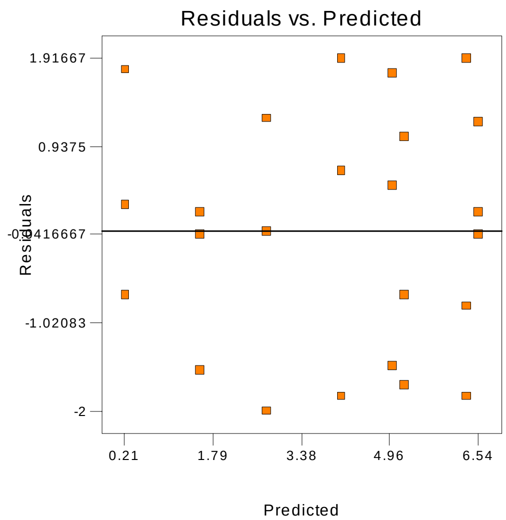
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	3.96	1	0.29	3.35	4.57	
B-Process	-0.67	1	0.29	-1.28	-0.058	1.00
C[1]	-0.54	1	0.50	-1.60	0.51	
C[2]	1.92	1	0.50	0.86	2.97	
C[3]	-3.08	1	0.50	-4.14	-2.03	

Final Equation in Terms of Coded Factors:

Avg =
+3.96
-0.67 * B
-0.54 * C[1]
+1.92 * C[2]
-3.08 * C[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





(d) Calculate the log variance of the observations in each cell. Analyze the average length and the log variance of the length for each of the 12 bars cut at each machine/heat treatment process combination. What conclusions can you draw?

Factor *B*, Heat Treat Process, has an affect on the log variance of the observations while Factor *A*, Time of Day, and Factor *C*, Machine, are not significant at the 5 percent level. However, *A* is significant at the 10 percent level, so Tome of Day has some effect on the variance.

Design Expert Output

Response: Log(Var)
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2.79	11	0.25	2.51	0.0648	not significant
A	*0.58*	*2*	*0.29*	*2.86*	*0.0966*	
B	*0.50*	*1*	*0.50*	*4.89*	*0.0471*	
C	*0.59*	*3*	*0.20*	*1.95*	*0.1757*	
AB	*0.49*	*2*	*0.24*	*2.40*	*0.1324*	
BC	*0.64*	*3*	*0.21*	*2.10*	*0.1538*	

Residual	1.22	12	0.10
Cor Total	4.01	23	

The Model F-value of 2.51 implies there is a 6.48% chance that a "Model F-Value" this large could occur due to noise.

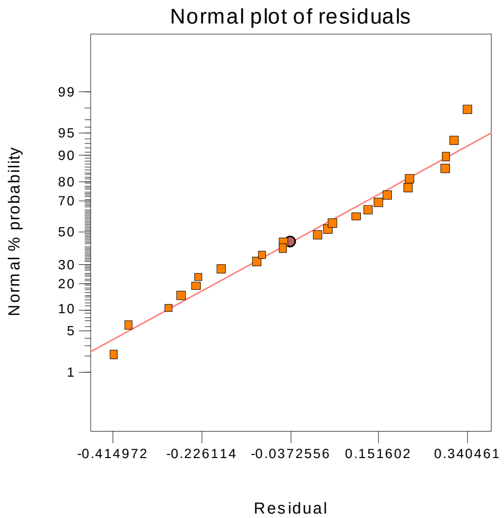
Std. Dev.	0.32	R-Squared	0.6967
Mean	0.65	Adj R-Squared	0.4186
C.V.	49.02	Pred R-Squared	-0.2133
PRESS	4.86	Adeq Precision	5.676

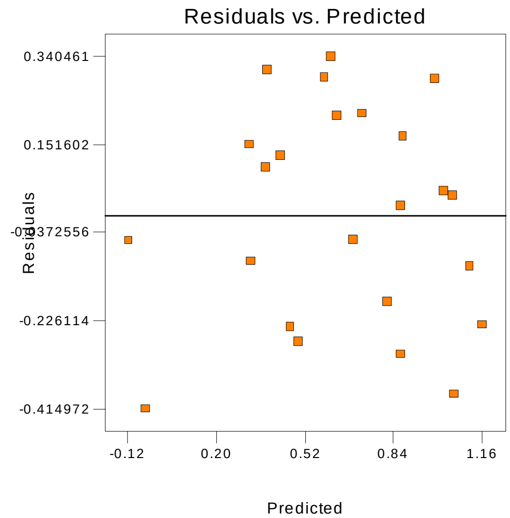
Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	0.65	1	0.065	0.51	0.79	
A[1]	-0.054	1	0.092	-0.25	0.15	
A[2]	-0.16	1	0.092	-0.36	0.043	
B-Process	0.14	1	0.065	2.181E-003	0.29	1.00
C[1]	0.22	1	0.11	-0.025	0.47	
C[2]	0.066	1	0.11	-0.18	0.31	
C[3]	-0.19	1	0.11	-0.44	0.052	
A[1]B	-0.20	1	0.092	-0.40	3.237E-003	
A[2]B	0.14	1	0.092	-0.065	0.34	
BC[1]	-0.18	1	0.11	-0.42	0.068	
BC[2]	-0.15	1	0.11	-0.39	0.098	
BC[3]	0.14	1	0.11	-0.10	0.39	

Final Equation in Terms of Coded Factors:

Log(Var)	=
+0.65	
-0.054	* A[1]
-0.16	* A[2]
+0.14	* B
+0.22	* C[1]
+0.066	* C[2]
-0.19	* C[3]
-0.20	* A[1]B
+0.14	* A[2]B
-0.18	* BC[1]
-0.15	* BC[2]
+0.14	* BC[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





(e) Suppose the time at which a bar is cut really cannot be controlled during routine production. Analyze the average length and the log variance of the length for each of the 12 bars cut at each machine/heat treatment process combination. What conclusions can you draw?

The analysis of the average length is as follows:

Design Expert Output

Response: Avg

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	37.43	7	5.35		
A	3.56	1	3.56		
B	32.78	3	10.93		
AB	1.09	3	0.36		
Pure Error	0.000	0			
Cor Total	37.43	7			

Because the Means Square of the AB interaction is much less than the main effects, it is removed from the model and placed in the error. The average length is strongly affected by Factor *B*, Machine, and moderately affected by Factor *A*, Heat Treat Process. The interaction effect was small and removed from the model.

Design Expert Output

Response: Avg

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	36.34	4	9.09	25.00	0.0122	significant
A	*3.56*	*1*	*3.56*	*9.78*	*0.0522*	
B	*32.78*	*3*	*10.93*	*30.07*	*0.0097*	
Residual	1.09	3	0.36			
Cor Total	37.43	7				

The Model F-value of 25.00 implies the model is significant. There is only a 1.22% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.60	R-Squared	0.9709
Mean	3.96	Adj R-Squared	0.9320

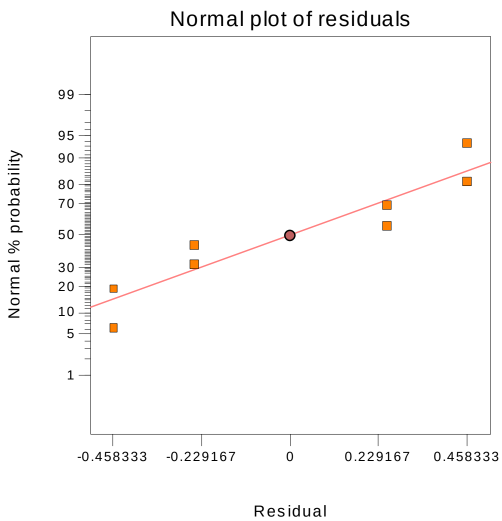
C.V.	15.23		Pred R-Squared	0.7929		
PRESS	7.75		Adeq Precision	13.289		

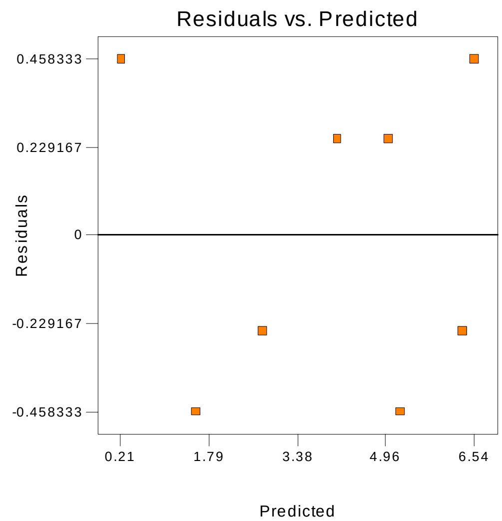
Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	3.96	1	0.21	3.28	4.64	
A-Process	-0.67	1	0.21	-1.34	0.012	1.00
B[1]	-0.54	1	0.37	-1.72	0.63	
B[2]	1.92	1	0.37	0.74	3.09	
B[3]	-3.08	1	0.37	-4.26	-1.91	

Final Equation in Terms of Coded Factors:

Avg =
+3.96
-0.67 * A
-0.54 * B[1]
+1.92 * B[2]
-3.08 * B[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





The Log(Var) is analyzed below:

Design Expert Output

Response: **Log(Var)**
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	0.32	7	0.046		
A	*0.091*	*1*	*0.091*		
B	*0.13*	*3*	*0.044*		
AB	*0.098*	*3*	*0.033*		
Pure Error	0.000	0			
Cor Total	0.32	7			

Because the *AB* interaction has the smallest Mean Square, it was removed from the model and placed in the error. From the following analysis of variance, neither Heat Treat Process, Machine, nor the interaction affect the log variance of the length.

Design Expert Output

Response: Log(Var)
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.22	4	0.056	1.71	0.3441	not significant
A	*0.091*	*1*	*0.091*	*2.80*	*0.1926*	
B	*0.13*	*3*	*0.044*	*1.34*	*0.4071*	
Residual	0.098	3	0.033			
Cor Total	0.32	7				

The "Model F-value" of 1.71 implies the model is not significant relative to the noise. There is a 34.41 % chance that a "Model F-value" this large could occur due to noise.

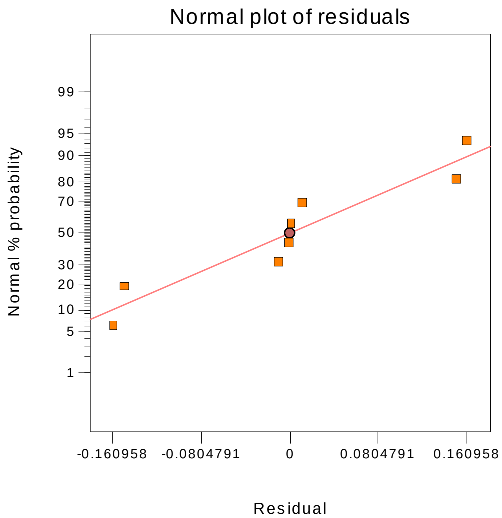
Std. Dev.	0.18	R-Squared	0.6949
Mean	0.79	Adj R-Squared	0.2882
C.V.	22.90	Pred R-Squared	-1.1693
PRESS	0.69	Adeq Precision	3.991

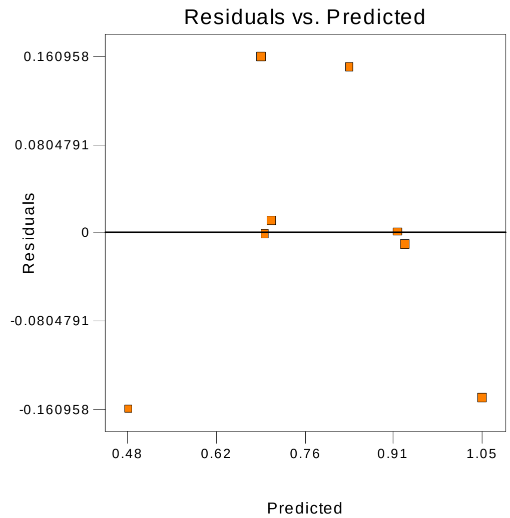
Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	0.79	1	0.064	0.59	0.99	
A-Process	0.11	1	0.064	-0.096	0.31	1.00
B[1]	0.15	1	0.11	-0.20	0.51	
B[2]	0.030	1	0.11	-0.32	0.38	
B[3]	-0.20	1	0.11	-0.55	0.15	

Final Equation in Terms of Coded Factors:

Log(Var) =
+0.79
+0.11 * A
+0.15 * B[1]
+0.030 * B[2]
-0.20 * B[3]

The following residual plots are acceptable. Both the normality and uniformity of variance assumptions are verified.





Chapter 10
Fitting Regression Models
Solutions

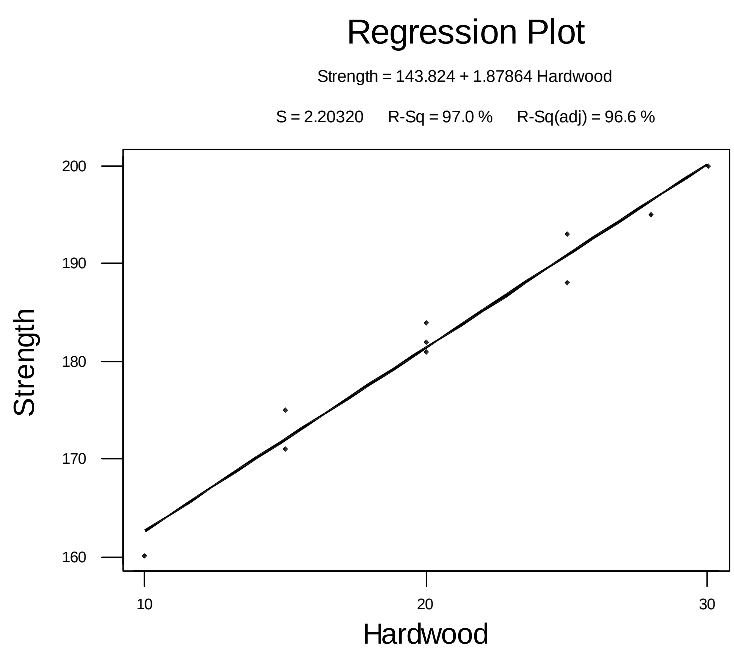
10-1 The tensile strength of a paper product is related to the amount of hardwood in the pulp. Ten samples are produced in the pilot plant, and the data obtained are shown in the following table.

Strength	Percent Hardwood	Strength	Percent Hardwood
160	10	181	20
171	15	188	25
175	15	193	25
182	20	195	28
184	20	200	30

(a) Fit a linear regression model relating strength to percent hardwood.

Minitab Output

```
Regression Analysis: Strength versus Hardwood

The regression equation is
Strength = 144 + 1.88 Hardwood

Predictor        Coef     SE Coef          T        P
Constant      143.824       2.522      57.04    0.000
Hardwood       1.8786      0.1165      16.12    0.000

S = 2.203              R-Sq = 97.0%         R-Sq(adj) = 96.6%
PRESS = 66.2665        R-Sq(pred) = 94.91%
```



(b) Test the model in part (a) for significance of regression.

Minitab Output

```
Analysis of Variance

Source           DF         SS         MS         F        P
Regression        1     1262.1     1262.1    260.00    0.000
Residual Error    8       38.8        4.9
  Lack of Fit     4       13.7        3.4      0.54    0.716
  Pure Error      4       25.2        6.3
Total             9     1300.9

3 rows with no replicates

No evidence of lack of fit (P > 0.1)
```

(c) Find a 95 percent confidence interval on the parameter β_1.

The 95 percent confidence interval is:

$$\hat{\beta}_1 - t_{\alpha/2,n-p} se\left(\hat{\beta}_1\right) \leq \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2,n-p} se\left(\hat{\beta}_1\right)$$
$$1.8786 - 2.3060(0.1165) \leq \beta_1 \leq 1.8786 + 2.3060(0.1165)$$
$$1.6900 \leq \beta_1 \leq 2.1473$$

10-2 A plant distills liquid air to produce oxygen , nitrogen, and argon. The percentage of impurity in the oxygen is thought to be linearly related to the amount of impurities in the air as measured by the "pollution count" in part per million (ppm). A sample of plant operating data is shown below.

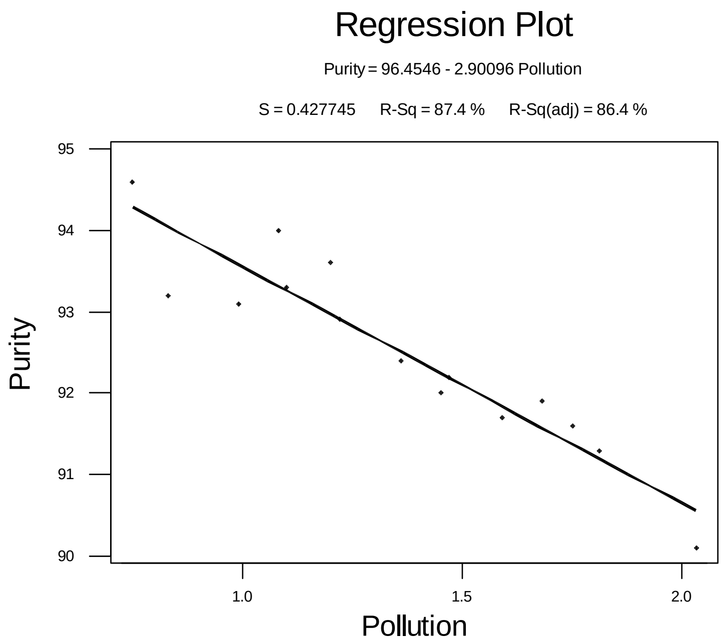
Purity(%)	93.3	92.0	92.4	91.7	94.0	94.6	93.6	93.1	93.2	92.9	92.2	91.3	90.1	91.6	91.9
Pollution count (ppm)	1.10	1.45	1.36	1.59	1.08	0.75	1.20	0.99	0.83	1.22	1.47	1.81	2.03	1.75	1.68

(a) Fit a linear regression model to the data.

Minitab Output

Regression Analysis: Purity versus Pollution

```
The regression equation is
Purity = 96.5 - 2.90 Pollution

Predictor        Coef     SE Coef          T        P
Constant      96.4546      0.4282     225.24    0.000
Pollutio      -2.9010      0.3056      -9.49    0.000

S = 0.4277             R-Sq = 87.4%         R-Sq(adj) = 86.4%
PRESS = 3.43946        R-Sq(pred) = 81.77%
```



(b) Test for significance of regression.

Minitab Output

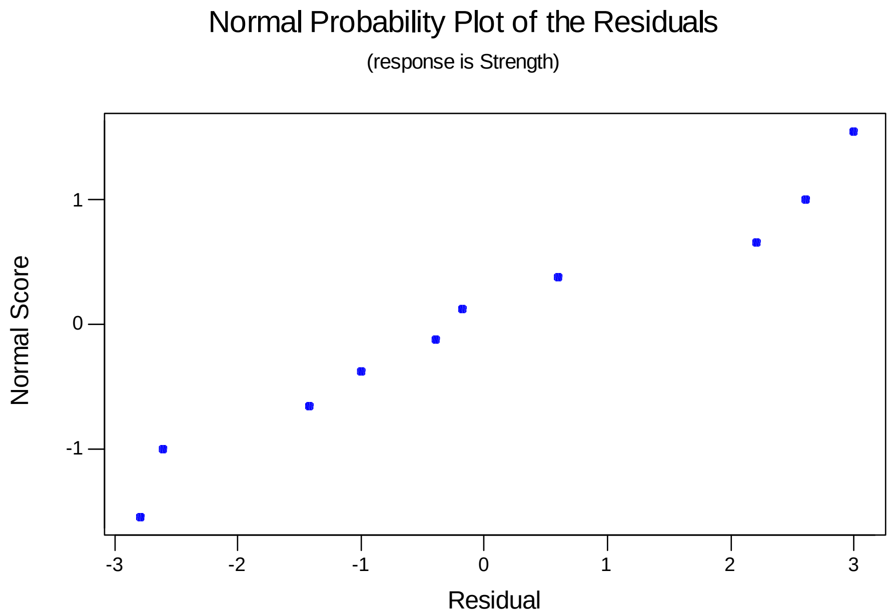
```
Analysis of Variance

Source            DF        SS        MS        F        P
Regression         1    16.491    16.491    90.13    0.000
Residual Error    13     2.379     0.183
Total             14    18.869

No replicates. Cannot do pure error test.

No evidence of lack of fit (P > 0.1)
```

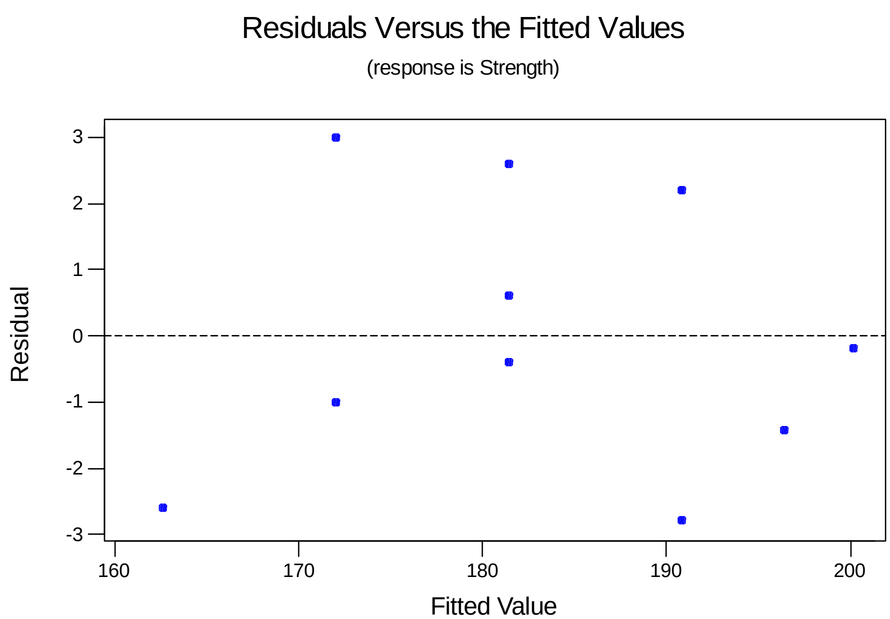
(c) Find a 95 percent confidence interval on β_1.

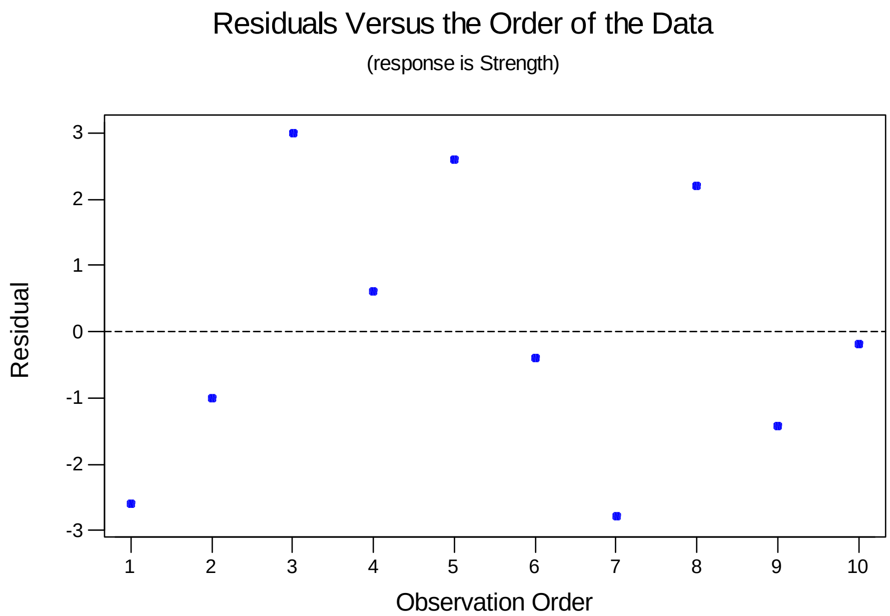
The 95 percent confidence interval is:

$$\hat{\beta}_1 - t_{\alpha/2,n-p}\,se\left(\hat{\beta}_1\right) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2,n-p}\,se\left(\hat{\beta}_1\right)$$
$$-2.9010 - 2.1604(0.3056) \le \beta_1 \le -2.9010 + 2.1604(0.3056)$$
$$-3.5612 \le \beta_1 \le -2.2408$$

10-3 Plot the residuals from Problem 10-1 and comment on model adequacy.

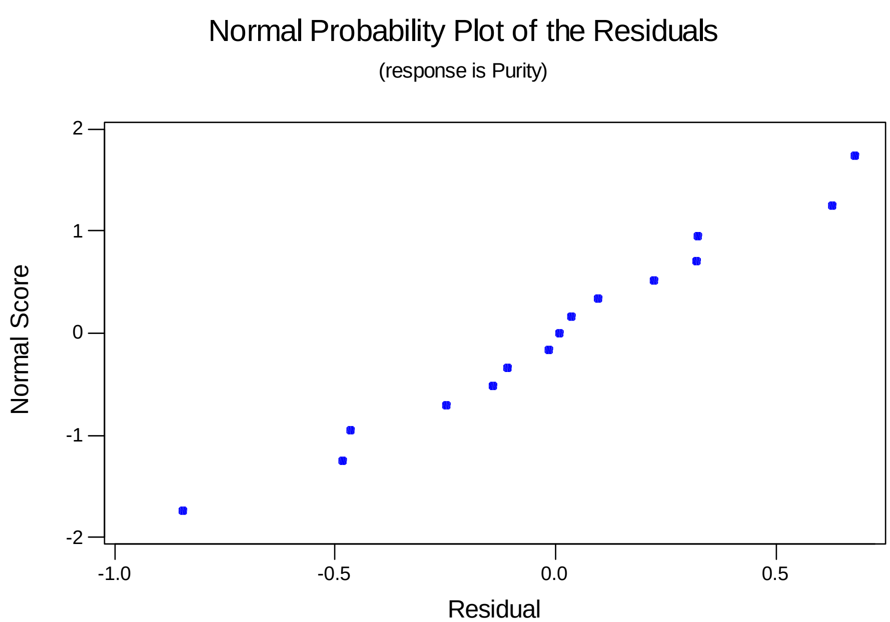

Normal Probability Plot of the Residuals
(response is Strength)
Normal Score
1
0
-1
-3
-2
-1
0
1
2
3
Residual

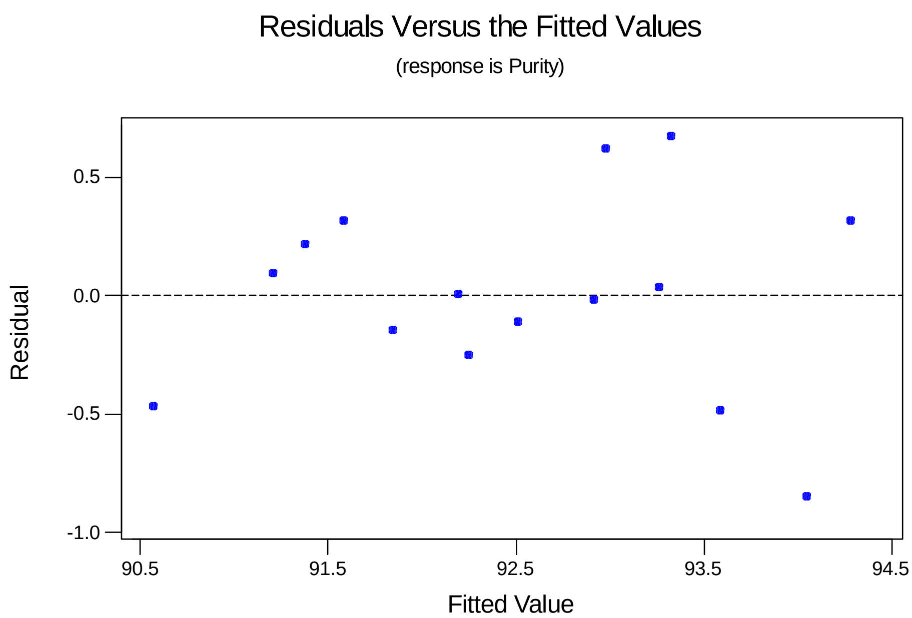

Residuals Versus the Fitted Values
(response is Strength)
Residual
3
2
1
0
-1
-2
-3
160
170
180
190
200
Fitted Value

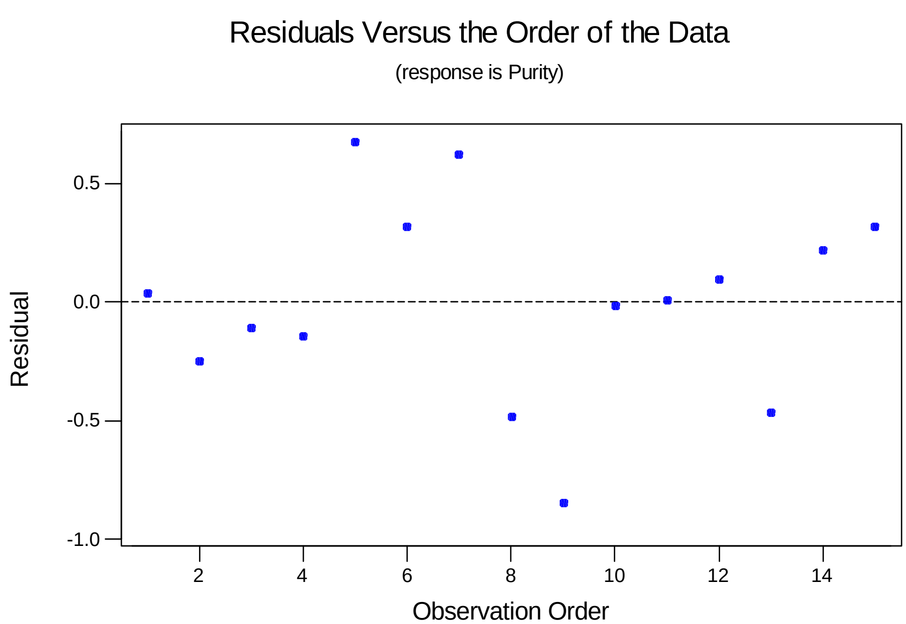


There is nothing unusual about the residual plots. The underlying assumptions have been met.

10-4 Plot the residuals from Problem 10-2 and comment on model adequacy.







There is nothing unusual about the residual plots. The underlying assumptions have been met.

10-5 Using the results of Problem 10-1, test the regression model for lack of fit.

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      1262.1      1262.1    260.00    0.000
Residual Error     8        38.8         4.9
```

```
  Lack of Fit      4       13.7         3.4      0.54    0.716
  Pure Error       4       25.2         6.3
Total              9     1300.9

3 rows with no replicates

No evidence of lack of fit (P > 0.1)
```

10-6 A study was performed on wear of a bearing y and its relationship to x_1 = oil viscosity and x_2 = load. The following data were obtained.

y	x_1	x_2
193	1.6	851
230	15.5	816
172	22.0	1058
91	43.0	1201
113	33.0	1357
125	40.0	1115

(a) Fit a multiple linear regression model to the data.

Minitab Output

Regression Analysis: Wear versus Viscosity, Load

```
The regression equation is
Wear = 351 - 1.27 Viscosity - 0.154 Load

Predictor        Coef     SE Coef          T        P       VIF
Constant       350.99       74.75       4.70    0.018
Viscosit       -1.272       1.169      -1.09    0.356       2.6
Load         -0.15390     0.08953      -1.72    0.184       2.6

S = 25.50              R-Sq = 86.2%         R-Sq(adj) = 77.0%
PRESS = 12696.7        R-Sq(pred) = 10.03%
```

(b) Test for significance of regression.

Minitab Output

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2     12161.6      6080.8      9.35    0.051
Residual Error     3      1950.4       650.1
Total              5     14112.0

No replicates. Cannot do pure error test.

Source       DF      Seq SS
Viscosit      1     10240.4
Load          1      1921.2

* Not enough data for lack of fit test
```

(c) Compute t statistics for each model parameter. What conclusions can you draw?

Minitab Output

Regression Analysis: Wear versus Viscosity, Load

```
The regression equation is
Wear = 351 - 1.27 Viscosity - 0.154 Load

Predictor        Coef     SE Coef          T        P       VIF
Constant       350.99       74.75       4.70    0.018
Viscosit       -1.272       1.169      -1.09    0.356       2.6
```

```
Load         -0.15390     0.08953      -1.72    0.184       2.6

S = 25.50               R-Sq = 86.2%        R-Sq(adj) = 77.0%
PRESS = 12696.7         R-Sq(pred) = 10.03%
```

The *t*-tests are shown in part (a). Notice that overall regression is significant (part(b)), but neither variable has a large *t*-statistic. This could be an indicator that the regressors are nearly linearly dependent.

10-7 The brake horsepower developed by an automobile engine on a dynomometer is thought to be a function of the engine speed in revolutions per minute (rpm), the road octane number of the fuel, and the engine compression. An experiment is run in the laboratory and the data that follow are collected.

Brake Horsepower	rpm	Road Octane Number	Compression
225	2000	90	100
212	1800	94	95
229	2400	88	110
222	1900	91	96
219	1600	86	100
278	2500	96	110
246	3000	94	98
237	3200	90	100
233	2800	88	105
224	3400	86	97
223	1800	90	100
230	2500	89	104

(a) Fit a multiple linear regression model to the data.

Minitab Output

Regression Analysis: Horsepower versus rpm, Octane, Compression

```
The regression equation is
Horsepower = - 266 + 0.0107 rpm + 3.13 Octane + 1.87 Compression

Predictor        Coef     SE Coef          T        P       VIF
Constant      -266.03       92.67      -2.87    0.021
rpm          0.010713    0.004483       2.39    0.044       1.0
Octane         3.1348      0.8444       3.71    0.006       1.0
Compress       1.8674      0.5345       3.49    0.008       1.0

S = 8.812               R-Sq = 80.7%        R-Sq(adj) = 73.4%
PRESS = 2494.05         R-Sq(pred) = 22.33%
```

(b) Test for significance of regression. What conclusions can you draw?

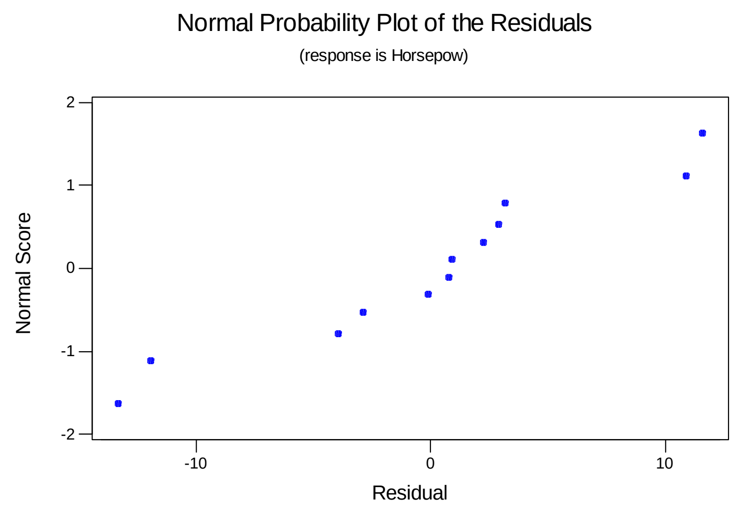
Minitab Output

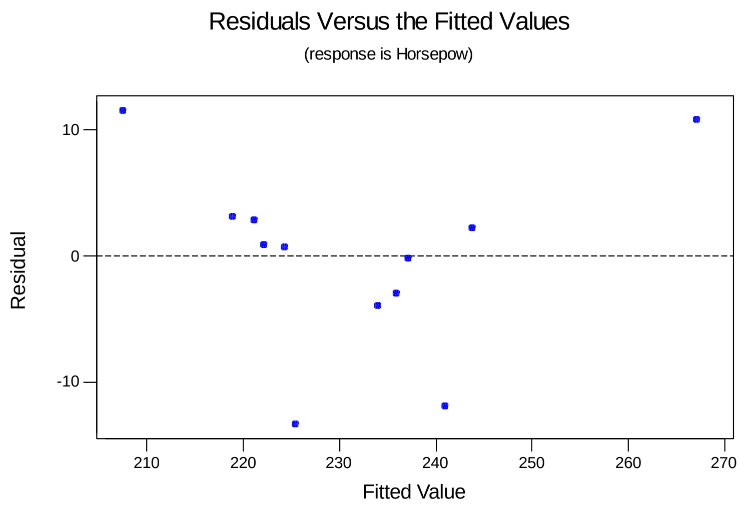
```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3     2589.73      863.24     11.12    0.003
Residual Error     8      621.27       77.66
Total             11     3211.00
r No replicates. Cannot do pure error test.

Source       DF      Seq SS
rpm           1      509.35
Octane        1     1132.56
Compress      1      947.83

Lack of fit test
Possible interactions with variable Octane (P-Value = 0.028)
Possible lack of fit at outer X-values       (P-Value = 0.000)
Overall lack of fit test is significant at P = 0.000
```

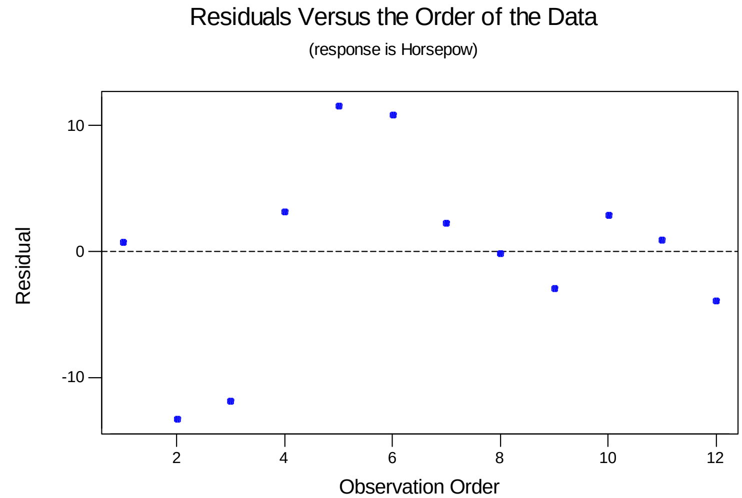
(c) Based on *t* tests, do you need all three regressor variables in the model?

Yes, all of the regressor variables are important.

10-8 Analyze the residuals from the regression model in Problem 10-7. Comment on model adequacy.







The normal probability plot is satisfactory, as is the plot of residuals versus run order (assuming that observation order is run order). The plot of residuals versus predicted response exhibits a slight "bow" shape. This could be an indication of lack of fit. It might be useful to consider adding some ineraction terms to the model.

10-9 The yield of a chemical process is related to the concentration of the reactant and the operating temperature. An experiment has been conducted with the following results.

Yield	Concentration	Temperature
81	1.00	150
89	1.00	180
83	2.00	150
91	2.00	180
79	1.00	150
87	1.00	180
84	2.00	150
90	2.00	180

(a) Suppose we wish to fit a main effects model to this data. Set up the **X'X** matrix using the data exactly as it appears in the table.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1.00 & 1.00 & 2.00 & 2.00 & 1.00 & 1.00 & 2.00 & 2.00 \\ 150 & 180 & 150 & 180 & 150 & 180 & 150 & 180 \end{bmatrix} \begin{bmatrix} 1 & 1.00 & 150 \\ 1 & 1.00 & 180 \\ 1 & 2.00 & 150 \\ 1 & 2.00 & 180 \\ 1 & 1.00 & 150 \\ 1 & 1.00 & 180 \\ 1 & 2.00 & 150 \\ 1 & 2.00 & 180 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 1320 \\ 12 & 20 & 1980 \\ 1320 & 1980 & 219600 \end{bmatrix}$$

(b) Is the matrix you obtained in part (a) diagonal? Discuss your response.

The **X'X** is not diagonal, even though an orthogonal design has been used. The reason is that we have worked with the natural factor levels, not the orthogonally coded variables.

(c) Suppose we write our model in terms of the "usual" coded variables

$$x_1 = \frac{Conc - 1.5}{0.5}\ ,\ x_2 = \frac{Temp - 165}{15}$$

Set up the **X'X** matrix for the model in terms of these coded variables. Is this matrix diagonal? Discuss your response.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

The **X'X** matrix is diagonal because we have used the orthogonally coded variables.

(d) Define a new set of coded variables

$$x_1 = \frac{Conc - 1.0}{1.0}\ ,\ x_2 = \frac{Temp - 150}{30}$$

Set up the **X'X** matrix for the model in terms of this set of coded variables. Is this matrix diagonal? Discuss your response.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 4 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

The **X'X** is not diagonal, even though an orthogonal design has been used. The reason is that we have not used orthogonally coded variables.

(e) Summarize what you have learned from this problem about coding the variables.

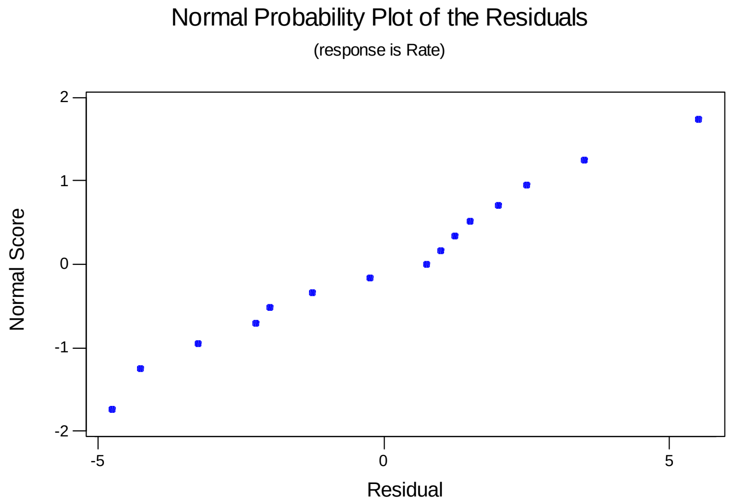
If the design is orthogonal, use the orthogonal coding. This not only makes the analysis somewhat easier, but it also results in model coefficients that are easier to interpret because they are both dimensionless and uncorrelated.

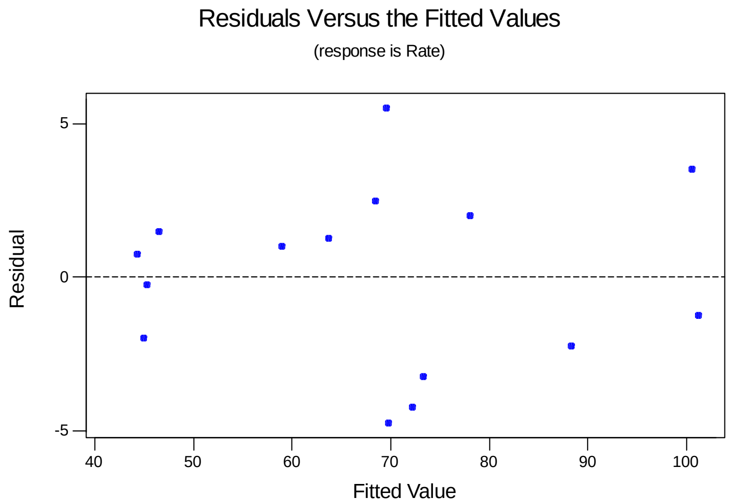
10-10 Consider the 2^4 factorial experiment in Example 6-2. Suppose that the last observation in missing. Reanalyze the data and draw conclusions. How do these conclusions compare with those from the original example?

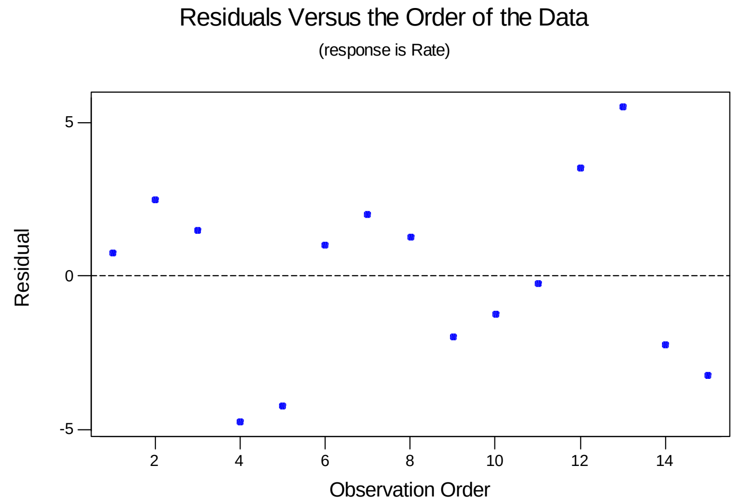
The regression analysis with the one data point missing indicates that the same effects are important.

Minitab Output

```
Regression Analysis: Rate versus A, B, C, D, AB, AC, AD, BC, BD, CD

The regression equation is
Rate = 69.8 + 10.5 A + 1.25 B + 4.63 C + 7.00 D - 0.25 AB - 9.38 AC + 8.00 AD
           + 0.87 BC - 0.50 BD - 0.87 CD

Predictor        Coef     SE Coef          T        P       VIF
Constant       69.750       1.500      46.50    0.000
A              10.500       1.500       7.00    0.002       1.1
B               1.250       1.500       0.83    0.452       1.1
C               4.625       1.500       3.08    0.037       1.1
D               7.000       1.500       4.67    0.010       1.1
AB             -0.250       1.500      -0.17    0.876       1.1
AC             -9.375       1.500      -6.25    0.003       1.1
AD              8.000       1.500       5.33    0.006       1.1
BC              0.875       1.500       0.58    0.591       1.1
BD             -0.500       1.500      -0.33    0.756       1.1
CD             -0.875       1.500      -0.58    0.591       1.1

S = 5.477              R-Sq = 97.6%         R-Sq(adj) = 91.6%
PRESS = 1750.00        R-Sq(pred) = 65.09%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression        10     4893.33      489.33     16.31    0.008
Residual Error     4      120.00       30.00
Total             14     5013.33

No replicates. Cannot do pure error test.

Source       DF      Seq SS
A             1     1414.40
B             1        4.01
C             1      262.86
D             1      758.88
AB            1        0.06
AC            1     1500.63
AD            1      924.50
BC            1       16.07
BD            1        1.72
CD            1       10.21
```


Normal Probability Plot of the Residuals
(response is Rate)
Normal Score
2
1
0
-1
-2
-5
0
5
Residual


Residuals Versus the Fitted Values
(response is Rate)
Residual
5
0
-5
40
50
60
70
80
90
100
Fitted Value



The residual plots are acceptable; therefore, the underlying assumptions are valid.

10-11 Consider the 2^4 factorial experiment in Example 6-2. Suppose that the last two observations are missing. Reanalyze the data and draw conclusions. How do these conclusions compare with those from the original example?

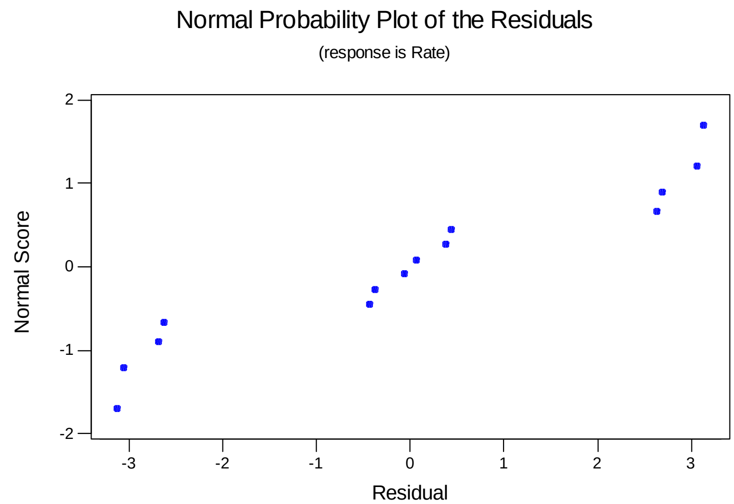
The regression analysis with the one data point missing indicates that the same effects are important.

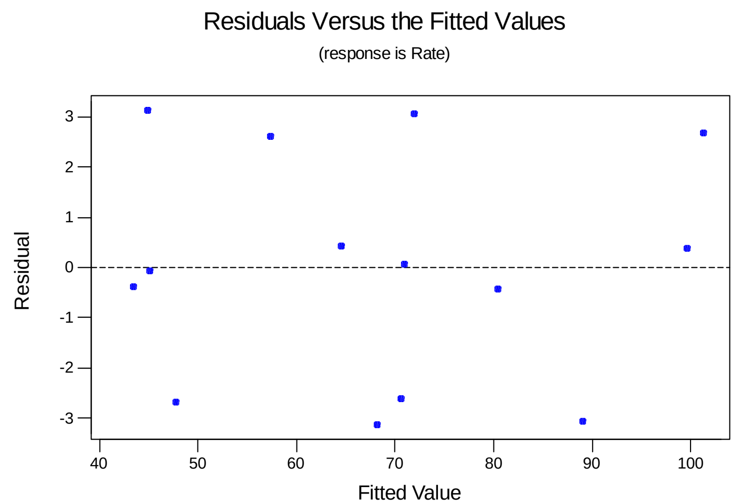
Minitab Output

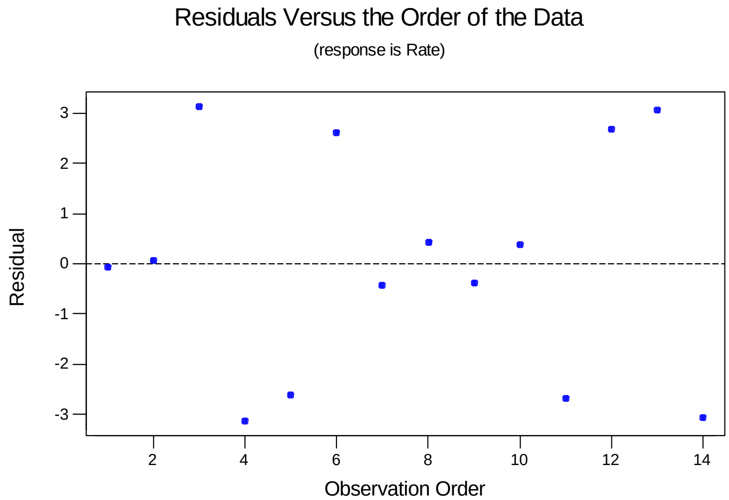
Regression Analysis: Rate versus A, B, C, D, AB, AC, AD, BC, BD, CD

```
The regression equation is
Rate = 71.4 + 10.1 A + 2.87 B + 6.25 C + 8.62 D - 0.66 AB - 9.78 AC + 7.59 AD
           + 2.50 BC + 1.12 BD + 0.75 CD

Predictor        Coef      SE Coef         T        P       VIF
Constant       71.375        1.673     42.66    0.000
A              10.094        1.323      7.63    0.005       1.1
B               2.875        1.673      1.72    0.184       1.7
C               6.250        1.673      3.74    0.033       1.7
D               8.625        1.673      5.15    0.014       1.7
AB             -0.656        1.323     -0.50    0.654       1.1
AC             -9.781        1.323     -7.39    0.005       1.1
AD              7.594        1.323      5.74    0.010       1.1
BC              2.500        1.673      1.49    0.232       1.7
BD              1.125        1.673      0.67    0.549       1.7
CD              0.750        1.673      0.45    0.684       1.7

S = 4.732              R-Sq = 98.7%        R-Sq(adj) = 94.2%
PRESS = 1493.06        R-Sq(pred) = 70.20%

Analysis of Variance

Source             DF        SS        MS        F        P
Regression         10     4943.17    494.32    22.07    0.014
Residual Error      3       67.19     22.40
Total              13     5010.36

No replicates. Cannot do pure error test.

Source        DF      Seq SS
A              1     1543.50
B              1        1.52
C              1      177.63
```

```
D               1       726.01
AB              1         1.17
AC              1      1702.53
AD              1       738.11
BC              1        42.19
BD              1         6.00
CD              1         4.50
```







The residual plots are acceptable; therefore, the underlying assumptions are valid.

10-12 Given the following data, fit the second-order polynomial regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$$

y	x_1	x_2
26	1.0	1.0
24	1.0	1.0
175	1.5	4.0
160	1.5	4.0
163	1.5	4.0
55	0.5	2.0
62	1.5	2.0
100	0.5	3.0
26	1.0	1.5
30	0.5	1.5
70	1.0	2.5
71	0.5	2.5

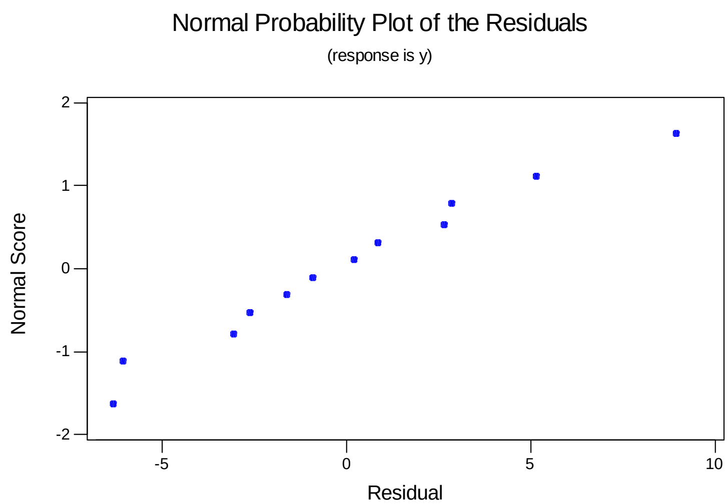
After you have fit the model, test for significance of regression.

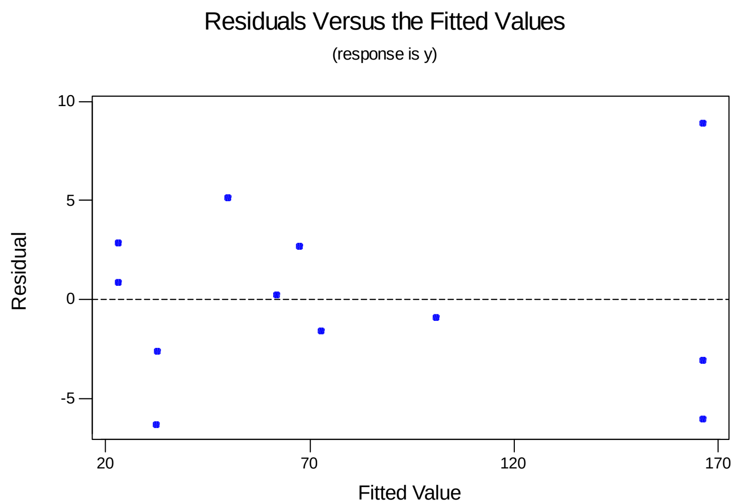
Minitab Output

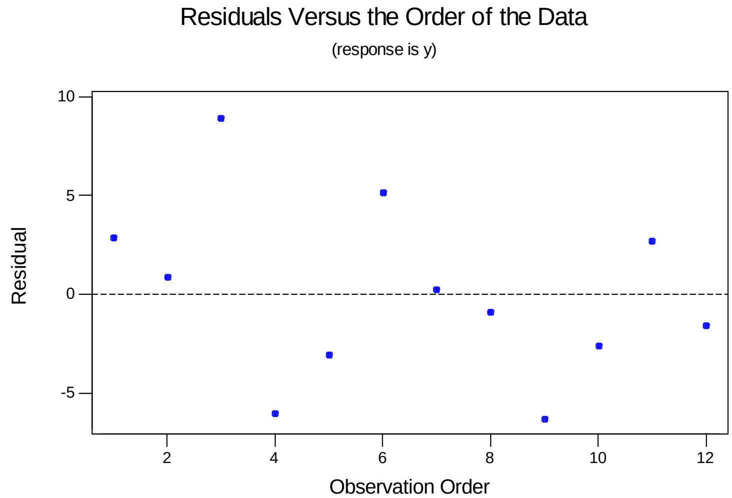
Regression Analysis: y versus x1, x2, x1^2, x2^2, x1x2

```
The regression equation is
y = 24.4 - 38.0 x1 + 0.7 x2 + 35.0 x1^2 + 11.1 x2^2 - 9.99 x1x2

Predictor        Coef     SE Coef          T        P       VIF
Constant        24.41       26.59       0.92    0.394
x1             -38.03       40.45      -0.94    0.383      89.6
x2               0.72       11.69       0.06    0.953      52.1
x1^2            34.98       21.56       1.62    0.156     103.9
x2^2           11.066       3.158       3.50    0.013     104.7
x1x2           -9.986       8.742      -1.14    0.297     105.1

S = 6.042              R-Sq = 99.4%         R-Sq(adj) = 98.9%
PRESS = 1327.71        R-Sq(pred) = 96.24%
r Analysis of Variance

Source              DF          SS          MS          F         P
```

```
Regression          5     35092.6       7018.5    192.23    0.000
Residual Error      6       219.1         36.5
  Lack of Fit       3        91.1         30.4      0.71    0.607
  Pure Error        3       128.0         42.7
Total              11     35311.7

7 rows with no replicates

Source        DF       Seq SS
x1             1      11552.0
x2             1      22950.3
x1^2           1         21.9
x2^2           1        520.8
x1x2           1         47.6
```







10-13

(a) Consider the quadratic regression model from Problem 10-12. Compute *t* statistics for each model parameter and comment on the conclusions that follow from the quantities.

Minitab Output

```
Predictor          Coef      SE Coef          T        P        VIF
Constant          24.41        26.59       0.92    0.394
x1               -38.03        40.45      -0.94    0.383       89.6
x2                 0.72        11.69       0.06    0.953       52.1
x1^2              34.98        21.56       1.62    0.156      103.9
x2^2             11.066        3.158       3.50    0.013      104.7
x1x2             -9.986        8.742      -1.14    0.297      105.1
```

x_2^2 is the only model parameter that is statistically significant with a *t*-value of 3.50. A logical model might also include x_2 to preserve model hierarchy.

(b) Use the extra sum of squares method to evaluate the value of the quadratic terms, x_1^2, x_2^2 and x_1x_2 to the model.

The extra sum of squares due to β_2 is

$$SS_R(\beta_2|\beta_0,\beta_1) = SS_R(\beta_0,\beta_1,\beta_2) - SS_R(\beta_0,\beta_1) = SS_R(\beta_1,\beta_2|\beta_0) - SS_R(\beta_1|\beta_0)$$

$SS_R(\beta_1,\beta_2|\beta_0)$ sum of squares of regression for the model in Problem 10-12 = 35092.6

$$SS_R(\beta_1|\beta_0) = 34502.3$$
$$SS_R(\beta_2|\beta_0,\beta_1) = 35092.6 - 34502.3 = 590.3$$
$$F_0 = \frac{SS_R(\beta_2|\beta_0,\beta_1)/3}{MS_E} = \frac{590.3/3}{36.511} = 5.3892$$

Since $F_{0.05,3,6} = 4.76$, then the addition of the quadratic terms to the model is significant. The P-values indicate that it's probably the term x_2^2 that is responsible for this.

10-14 *Relationship between analysis of variance and regression.* Any analysis of variance model can be expressed in terms of the general linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where the **X** matrix consists of zeros and ones. Show that the single-factor model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, i=1,2,3, j=1,2,3,4 can be written in general linear model form. Then

(a) Write the normal equations $(\mathbf{X}'\mathbf{X})\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ and compare them with the normal equations found for the model in Chapter 3.

The normal equations are $(\mathbf{X}'\mathbf{X})\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$

$$\begin{bmatrix} 12 & 4 & 4 & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \end{bmatrix} = \begin{bmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{bmatrix}$$

which are in agreement with the results of Chapter 3.

(b) Find the rank of $\mathbf{X}'\mathbf{X}$. Can $(\mathbf{X}'\mathbf{X})^{-1}$ be obtained?

$\mathbf{X}'\mathbf{X}$ is a 4 x 4 matrix of rank 3, because the last three columns add to the first column. Thus $(\mathbf{X}'\mathbf{X})^{-1}$ does not exist.

(c) Suppose the first normal equation is deleted and the restriction $\sum_{i=1}^{3} n\hat{\tau}_i = 0$ is added. Can the resulting system of equations be solved? If so, find the solution. Find the regression sum of squares $\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}$, and compare it to the treatment sum of squares in the single-factor model.

Imposing $\sum_{i=1}^{3} n\hat{\tau}_i = 0$ yields the normal equations

$$\begin{bmatrix} 0 & 4 & 4 & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \end{bmatrix} = \begin{bmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{bmatrix}$$

The solution to this set of equations is

$$\hat{\mu} = \frac{y_{..}}{12} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$$

This solution was found be solving the last three equations for $\hat{\tau}_i$, yielding $\hat{\tau}_i = \bar{y}_{i.} - \hat{\mu}$, and then substituting in the first equation to find $\hat{\mu} = \bar{y}_{..}$

The regression sum of squares is

$$SS_R(\boldsymbol{\beta}) = \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} = \bar{y}_{..}y_{..} + \sum_{i=1}^{a}(\bar{y}_{i.} - y_{..})^2 = \frac{y_{..}^2}{an} + \sum_{i=1}^{a}\frac{\bar{y}_{i.}^2}{n} - \frac{y_{..}^2}{an} = \sum_{i=1}^{a}\frac{\bar{y}_{i.}^2}{n}$$

with *a* degrees of freedom. This is the same result found in Chapter 3. For more discussion of the relationship between analysis of variance and regression, see Montgomery and Peck (1992).

10-15 Suppose that we are fitting a straight line and we desire to make the variance of as small as possible. Restricting ourselves to an even number of experimental points, where should we place these points so as to minimize $V(\hat{\beta}_1)$? (Note: Use the design called for in this exercise with great caution because, even though it minimized $V(\hat{\beta}_1)$, it has some undesirable properties; for example, see Myers and Montgomery (1995). Only if you are *very sure* the true functional relationship is linear should you consider using this design.

Since $V(\hat{\beta}_1) = \dfrac{\sigma^2}{S_{xx}}$, we may minimize $V(\hat{\beta}_1)$by making S_{xx} as large as possible. S_{xx} is maximized by spreading out the x_j's as much as possible. The experimenter usually has a "region of interest" for *x*. If *n* is even, *n*/2 of the observations should be run at each end of the "region of interest". If *n* is odd, then run one of the observations in the center of the region and the remaining (*n*-1)/2 at either end.

10-16 ***Weighted least squares.*** Suppose that we are fitting the straight line $y = \beta_0 + \beta_1 x + \varepsilon$, but the variance of the *y*'s now depends on the level of *x*; that is,

$$V(y|x_i) = \sigma^2 = \frac{\sigma^2}{w_i}, i = 1,2,..., n$$

where the w_i are known constants, often called weights. Show that if we choose estimates of the regression coefficients to minimize the weighted sum of squared errors given by $\sum_{i=1}^{n} w_i(y_i - \beta_0 + \beta_1 x_i)^2$, the resulting least squares normal equations are

$$\hat{\beta}_0\sum_{i=1}^{n} w_i + \hat{\beta}_1\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i y_i$$
$$\hat{\beta}_0\sum_{i=1}^{n} w_i x_i + \hat{\beta}_1\sum_{i=1}^{n} w_i x_i^2 = \sum_{i=1}^{n} w_i x_i y_i$$

The least squares normal equations are found:

$$L=\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_1\right)^2 w_i$$
$$\frac{\partial L}{\partial \beta_0}=-2\sum_{i=1}^{n}\left(y_i-\hat{\beta}_0-\hat{\beta}_1x_1\right)w_i=0$$
$$\frac{\partial L}{\partial \beta_1}=-2\sum_{i=1}^{n}\left(y_i-\hat{\beta}_0-\hat{\beta}_1x_1\right)x_1w_i=0$$

which simplify to

$$\hat{\beta}_0\sum_{i=1}^{n}w_i+\hat{\beta}_1\sum_{i=1}^{n}x_1w_i=\sum_{i=1}^{n}w_iy_i$$
$$\hat{\beta}_0\sum_{i=1}^{n}x_1w_i+\hat{\beta}_1\sum_{i=1}^{n}x_1^2w_i=\sum_{i=1}^{n}w_ix_1y_i$$

10-17 Consider the 2_{IV}^{4-1} design discussed in Example 10-5.

(a) Suppose you elect to augment the design with the single run selected in that example. Find the variances and covariances of the regression coefficients in the model (ignoring blocks):

$$y=\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4+\beta_{12}x_1x_2+\beta_{34}x_3x_4+\varepsilon$$

$$\mathbf{X'X}=\begin{bmatrix}1&1&1&1&1&1&1&1&1\\-1&1&-1&1&-1&1&-1&1&-1\\-1&-1&1&1&-1&-1&1&1&-1\\-1&-1&-1&-1&1&1&1&1&-1\\-1&1&1&-1&1&-1&-1&1&1\\1&-1&-1&1&1&-1&-1&1&1\\1&-1&-1&1&1&-1&-1&1&-1\end{bmatrix}\begin{bmatrix}1&-1&-1&-1&-1&1&1\\1&1&-1&-1&1&-1&-1\\1&-1&1&-1&1&-1&-1\\1&1&1&-1&-1&1&1\\1&-1&-1&1&1&1&1\\1&1&-1&1&-1&-1&-1\\1&-1&1&1&-1&-1&-1\\1&1&1&1&1&1&1\\1&-1&-1&-1&1&1&-1\end{bmatrix}=\begin{bmatrix}9&-1&-1&-1&1&1&-1\\-1&9&1&1&-1&-1&1\\-1&1&9&1&-1&-1&1\\-1&1&1&9&-1&-1&1\\1&-1&-1&-1&9&1&-1\\1&-1&-1&-1&1&9&7\\-1&1&1&1&-1&7&9\end{bmatrix}$$

$$(\mathbf{X'X})^{-1}=\begin{bmatrix}0.125&0&0&0&0&-0.0625&0.0625\\0&0.125&0&0&0&0.0625&-0.0625\\0&0&0.125&0&0&0.0625&-0.0625\\0&0&0&0.125&0&0.0625&-0.0625\\0&0&0&0&0.125&-0.0625&0.0625\\-0.0625&0.0625&0.0625&0.0625&0.0625&0.4375&-0.375\\0.0625&-0.0625&-0.0625&-0.0625&0.0625&-0.375&0.4375\end{bmatrix}$$

(b) Are there any other runs in the alternate fraction that would de-alias *AB* from *CD*?

Any other run from the alternate fraction will de-alias *AB* from *CD.*

(c) Suppose you augment the design with four runs suggested in Example 10-5. Find the variance and the covariances of the regression coefficients (ignoring blocks) for the model in part (a).

Choose 4 runs that are one of the quarter fractions not used in the principal half fraction.

$$\mathbf{X'X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{X'X} = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 12 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 12 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 12 & 4 & 0 \\ 0 & 4 & 0 & 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 0.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1071 & 0 & 0 & -0.0179 & -0.0536 & 0.0357 \\ 0 & 0 & 0.0938 & 0.0313 & 0 & 0 & 0 \\ 0 & 0 & 0.0313 & 0.0938 & 0 & 0 & 0 \\ 0 & -0.0179 & 0 & 0 & 0.1071 & -0.0536 & 0.0357 \\ 0 & -0.0536 & 0 & 0 & -0.0536 & 0.2142 & -0.1429 \\ 0 & 0.0357 & 0 & 0 & 0.0357 & -0.1429 & 0.1785 \end{bmatrix}$$

(d) Considering parts (a) and (c), which augmentation strategy would you prefer and why?

If you only have the resources to run one more run, then choose the one-run augmentation. But if resources are not scarce, then augment the design in multiples of two runs, to keep the design orthogonal. Using four runs results in smaller variances of the regression coefficients and a simpler covariance structure.

10-18 Consider the 2^{7-4}_{III}. Suppose after running the experiment, the largest observed effects are *A* + *BD*, *B* + *AD*, and *D* + *AB*. You wish to augment the original design with a group of four runs to de-alias these effects.

(a) Which four runs would you make?

Take the first four runs of the original experiment and change the sign on *A*.

Design Expert Output

			Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
Std	Run	Block	A:x1	B:x2	C:x3	D:x4	E:x5	F:x6	G:x7
1	1	Block 1	-1.00	-1.00	-1.00	1.00	1.00	1.00	-1.00
2	2	Block 1	1.00	-1.00	-1.00	-1.00	-1.00	1.00	1.00
3	3	Block 1	-1.00	1.00	-1.00	-1.00	1.00	-1.00	1.00
4	4	Block 1	1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
5	5	Block 1	-1.00	-1.00	1.00	1.00	-1.00	-1.00	1.00
6	6	Block 1	1.00	-1.00	1.00	-1.00	1.00	-1.00	-1.00
7	7	Block 1	-1.00	1.00	1.00	-1.00	-1.00	1.00	-1.00
8	8	Block 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	9	Block 2	1.00	1.00	1.00	-1.00	-1.00	-1.00	-1.00
10	10	Block 2	1.00	-1.00	-1.00	1.00	-1.00	-1.00	-1.00
11	11	Block 2	-1.00	-1.00	1.00	1.00	-1.00	-1.00	-1.00
12	12	Block 2	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00

Main effects and interactions of interest are:

x1	x2	x4	x1x2	x1x4	x2x4
-1	-1	1	1	-1	-1
1	-1	-1	-1	-1	1
-1	1	-1	-1	1	-1
1	1	1	1	1	1
-1	-1	1	1	-1	-1
1	-1	-1	-1	-1	1
-1	1	-1	-1	1	-1
1	1	1	1	1	1
1	**-1**	**1**	**-1**	**1**	**-1**
-1	**-1**	**-1**	**1**	**1**	**1**
1	**1**	**-1**	**1**	**-1**	**-1**
-1	**1**	**1**	**-1**	**-1**	**1**

(b) Find the variances and covariances of the regression coefficients in the model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_4 x_4 + \beta_{12} x_1 x_2 + \beta_{14} x_1 x_4 + \beta_{24} x_2 x_4 + \varepsilon$$

$$\mathbf{X'X} = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 12 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 12 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 12 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 12 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 0.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1071 & -0.0178 & 0 & 0 & 0.0536 & 0.0714 \\ 0 & -0.0179 & 0.1071 & 0 & 0 & 0.0714 & -0.0536 \\ 0 & 0 & 0 & 0.0938 & 0.0313 & 0 & 0 \\ 0 & 0 & 0 & 0.0313 & 0.0938 & 0 & 0 \\ 0 & -0.0536 & 0.0714 & 0 & 0 & 0.2143 & -0.1607 \\ 0 & 0.0714 & -0.0536 & 0 & 0 & -0.1607 & 0.2143 \end{bmatrix}$$

(c) Is it possible to de-alias these effects with fewer than four additional runs?

It is possible to de-alias these effects in only two runs. By utilizing Design Expert's design augmentation function, the runs 9 and 10 (Block 2) were generated as follows:

Design Expert Output

			Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
Std	Run	Block	A:x1	B:x2	C:x3	D:x4	E:x5	F:x6	G:x7
1	1	Block 1	-1.00	-1.00	-1.00	1.00	1.00	1.00	-1.00
2	2	Block 1	1.00	-1.00	-1.00	-1.00	-1.00	1.00	1.00
3	3	Block 1	-1.00	1.00	-1.00	-1.00	1.00	-1.00	1.00
4	4	Block 1	1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
5	5	Block 1	-1.00	-1.00	1.00	1.00	-1.00	-1.00	1.00
6	6	Block 1	1.00	-1.00	1.00	-1.00	1.00	-1.00	-1.00
7	7	Block 1	-1.00	1.00	1.00	-1.00	-1.00	1.00	-1.00
8	8	Block 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	9	Block 2	-1.00	1.00	-1.00	1.00	-1.00	-1.00	-1.00
10	10	Block 2	1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00

Chapter 11
Response Surface Methods and Designs
Solutions

11-1 A chemical plant produces oxygen by liquefying air and separating it into its component gases by fractional distillation. The purity of the oxygen is a function of the main condenser temperature and the pressure ratio between the upper and lower columns. Current operating conditions are temperature $(\xi_1) =$ -220°C and pressure ratio $(\xi_2) = 1.2$. Using the following data find the path of steepest ascent.

Temperature (x_1)	Pressure Ratio (x_2)	Purity
-225	1.1	82.8
-225	1.3	83.5
-215	1.1	84.7
-215	1.3	85.0
-220	1.2	84.1
-220	1.2	84.5
-220	1.2	83.9
-220	1.2	84.3

Design Expert Output

Response: Purity
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3.14	2	1.57	26.17	0.0050	significant
A	*2.89*	*1*	*2.89*	*48.17*	*0.0023*	
B	*0.25*	*1*	*0.25*	*4.17*	*0.1108*	
Curvature	0.080	1	0.080	1.33	0.3125	not significant
Residual	0.24	4	0.060			
Lack of Fit	*0.040*	*1*	*0.040*	*0.60*	*0.4950*	*not significant*
Pure Error	*0.20*	*3*	*0.067*			
Cor Total	3.46	7				

The Model F-value of 26.17 implies the model is significant. There is only a 0.50% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.24	R-Squared	0.9290
Mean	84.10	Adj R-Squared	0.8935
C.V.	0.29	Pred R-Squared	0.7123
PRESS	1.00	Adeq Precision	12.702

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	84.00	1	0.12	83.66	84.34	
A-Temperature	0.85	1	0.12	0.51	1.19	1.00
B-Pressure Ratio	0.25	1	0.12	-0.090	0.59	1.00
Center Point	0.20	1	0.17	-0.28	0.68	1.00

Final Equation in Terms of Coded Factors:

Purity =
+84.00
+0.85 * A
+0.25 * B

Final Equation in Terms of Actual Factors:

Purity =
+118.40000
+0.17000 * Temperature

```
                    +2.50000    * Pressure Ratio
```

From the computer output use the model $\hat{y} = 84 + 0.85x_1 + 0.25x_2$ as the equation for steepest ascent. Suppose we use a one degree change in temperature as the basic step size. Thus, the path of steepest ascent passes through the point (x_1=0, x_2=0) and has a slope 0.25/0.85. In the coded variables, one degree of temperature is equivalent to a step of Δx_1 = 1/5=0.2. Thus, Δx_2 = (0.25/0.85)0.2=0.059. The path of steepest ascent is:

	Coded	Variables	Natural	Variables
	x_1	x_2	ξ_1	ξ_2
Origin	0	0	-220	1.2
Δ	0.2	0.059	1	0.0059
Origin + Δ	0.2	0.059	-219	1.2059
Origin +5 Δ	1.0	0.295	-215	1.2295
Origin +7 Δ	1.40	0.413	-213	1.2413

11-2 An industrial engineer has developed a computer simulation model of a two-item inventory system. The decision variables are the order quantity and the reorder point for each item. The response to be minimized is the total inventory cost. The simulation model is used to produce the data shown in the following table. Identify the experimental design. Find the path of steepest descent.

Item 1		Item 2		
Order Quantity (x1)	Reorder Point (x2)	Order Quantity (x3)	Reorder Point (x4)	Total Cost
100	25	250	40	625
140	45	250	40	670
140	25	300	40	663
140	25	250	80	654
100	45	300	40	648
100	45	250	80	634
100	25	300	80	692
140	45	300	80	686
120	35	275	60	680
120	35	275	60	674
120	35	275	60	681

The design is a 2^{4-1} fractional factorial with generator *I*=*ABCD*, and three center points.

Design Expert Output

Response: Total Cost
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3880.00	6	646.67	63.26	0.0030	significant
A	*684.50*	*1*	*684.50*	*66.96*	*0.0038*	
C	*1404.50*	*1*	*1404.50*	*137.40*	*0.0013*	
D	*450.00*	*1*	*450.00*	*44.02*	*0.0070*	
AC	*392.00*	*1*	*392.00*	*38.35*	*0.0085*	
AD	*264.50*	*1*	*264.50*	*25.88*	*0.0147*	
CD	*684.50*	*1*	*684.50*	*66.96*	*0.0038*	
Curvature	815.52	1	815.52	79.78	0.0030	significant
Residual	30.67	3	10.22			
Lack of Fit	*2.00*	*1*	*2.00*	*0.14*	*0.7446*	*not significant*
Pure Error	*28.67*	*2*	*14.33*			
Cor Total	4726.18	10				

The Model F-value of 63.26 implies the model is significant. There is only a 0.30% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	3.20	R-Squared	0.9922
Mean	664.27	Adj R-Squared	0.9765
C.V.	0.48	Pred R-Squared	0.9593
PRESS	192.50	Adeq Precision	24.573

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	659.00	1	1.13	655.40	662.60	
A-Item 1 QTY	9.25	1	1.13	5.65	12.85	1.00
C-Item 2 QTY	13.25	1	1.13	9.65	16.85	1.00
D-Item 2 Reorder	7.50	1	1.13	3.90	11.10	1.00
AC	-7.00	1	1.13	-10.60	-3.40	1.00
AD	-5.75	1	1.13	-9.35	-2.15	1.00
CD	9.25	1	1.13	5.65	12.85	1.00
Center Point	19.33	1	2.16	12.44	26.22	1.00

Final Equation in Terms of Coded Factors:

Total Cost	=
+659.00	
+9.25	* A
+13.25	* C
+7.50	* D
-7.00	* A * C
-5.75	* A * D
+9.25	* C * D

Final Equation in Terms of Actual Factors:

Total Cost	=
+175.00000	
+5.17500	* Item 1 QTY
+1.10000	* Item 2 QTY
-2.98750	* Item 2 Reorder
-0.014000	* Item 1 QTY * Item 2 QTY
-0.014375	* Item 1 QTY * Item 2 Reorder
+0.018500	* Item 2 QTY * Item 2 Reorder
+0.019	* Item 2 QTY * Item 2 Reorder

The equation used to compute the path of steepest ascent is $\hat{y} = 659 + 9.25x_1 + 13.25x_3 + 7.50x_4$. Notice that even though the model contains interaction, it is relatively common practice to ignore the interactions in computing the path of steepest ascent. This means that the path constructed is only an approximation to the path that would have been obtained if the interactions were considered, but it's usually close enough to give satisfactory results.

It is helpful to give a general method for finding the path of steepest ascent. Suppose we have a first-order model in *k* variables, say

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i$$

The path of steepest ascent passes through the origin, **x**=**0**, and through the point on a hypersphere of radius, *R* where $\hat{y}$ is a maximum. Thus, the *x*'s must satisfy the constraint

$$\sum_{i=1}^{k} x_i^2 = R^2$$

To find the set of *x*'s that maximize $\hat{y}$ subject to this constraint, we maximize

$$L = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i - \lambda \left[\sum_{i=1}^{k} x_i^2 - R^2 \right]$$

where λ is a LaGrange multiplier. From $\partial L / \partial x_i = \partial L / \partial \lambda = 0$, we find

$$x_i = \frac{\hat{\beta}_i}{2\lambda}$$

It is customary to specify a basic step size in one of the variables, say Δx_j, and then calculate 2 λ as 2 $\lambda = \hat{\beta}_j \,/\, \Delta x_j$. Then this value of 2 λ can be used to generate the remaining coordinates of a point on the path of steepest ascent.

We demonstrate using the data from this problem. Suppose that we use -10 units in ξ_1 as the basic step size. Note that a decrease in ξ_1 is called for, because we are looking for a path of steepest decent. Now -10 units in ξ_1 is equal to -10/20 = -0.5 units change in x_1.

Thus, 2 $\lambda = \hat{\beta}_1 \,/\, \Delta x_1$ = 9.25/(-0.5) = -18.50

Consequently,

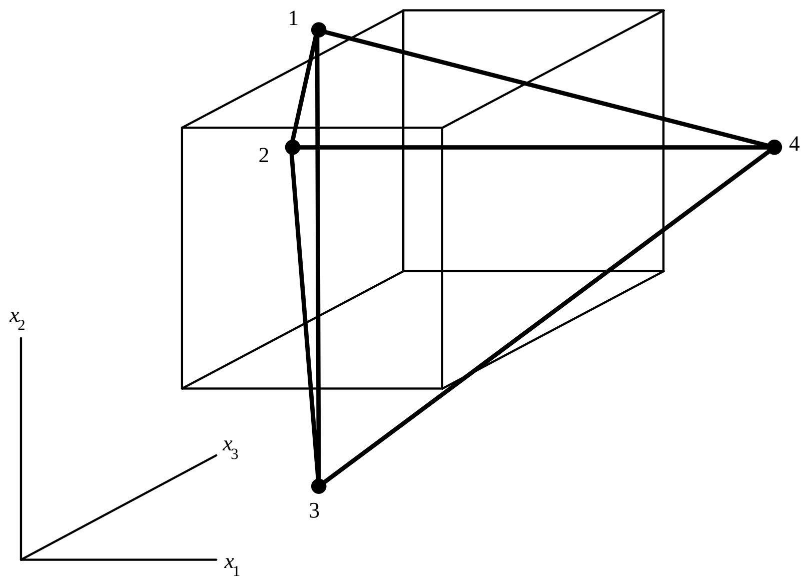
$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} = \frac{13.25}{-18.50} = -0.716$$
$$\Delta x_4 = \frac{\hat{\beta}_4}{2\lambda} = \frac{7.50}{-18.50} = -0.705$$

are the remaining coordinates of points along the path of steepest decent, in terms of the coded variables. The path of steepest decent is shown below:

	Coded	Variables			Natural	Variables		
	x_1	x_2	x_3	x_4	ξ_1	ξ_2	ξ_3	ξ_4
Origin	0	0	0	0	120	35	275	60
Δ	-0.50	0	-0.716	-0.405	-10	0	-17.91	-8.11
Origin + Δ	-0.50	0	-0.716	-0.405	110	35	257.09	51.89
Origin +2 Δ	-1.00	0	-1.432	-0.810	100	35	239.18	43.78

11-3 Verify that the following design is a simplex. Fit the first-order model and find the path of steepest ascent.

Position	x_1	x_2	x_3	y
1	0	$\sqrt{2}$	-1	18.5
2	$-\sqrt{2}$	0	1	19.8
3	0	$-\sqrt{2}$	-1	17.4
4	$\sqrt{2}$	0	1	22.5



The graphical representation of the design identifies a tetrahedron; therefore, the design is a simplex.

Design Expert Output

Response: **y**
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	14.49	3	4.83		
A	*3.64*	*1*	*3.64*		
B	*0.61*	*1*	*0.61*		
C	*10.24*	*1*	*10.24*		
Pure Error	0.000	0			
Cor Total	14.49	3			

Std. Dev.		R-Squared	1.0000
Mean	19.55	Adj R-Squared	
C.V.		Pred R-Squared	N/A
PRESS	N/A	Adeq Precision	0.000

Case(s) with leverage of 1.0000: Pred R-Squared and PRESS statistic not defined

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	19.55	1				
A-x1	1.35	1				1.00
B-x2	0.55	1				1.00
C-x3	1.60	1				1.00

Final Equation in Terms of Coded Factors:

y =
+19.55
+1.35 * A
+0.55 * B
+1.60 * C

Final Equation in Terms of Actual Factors:

y =
+19.55000
+0.95459 * x1
+0.38891 * x2
+1.60000 * x3

The first order model is $\hat{y} = 19.55 + 1.35x_1 + 0.55x_2 + 1.60x_3$.

To find the path of steepest ascent, let the basic step size be $\Delta x_3 = 1$. Then using the results obtained in the previous problem, we obtain

$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} \quad \text{or} \quad 1.0 = \frac{1.60}{2\lambda}$$

which yields $2\lambda = 1.60$. Then the coordinates of points on the path of steepest ascent are defined by

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} = \frac{0.96}{1.60} = 0.60$$
$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{0.24}{1.60} = 0.24$$

Therefore, in the coded variables we have:

	Coded	Variables	
	x_1	x_2	x_3
Origin	0	0	0
Δ	0.60	0.24	1.00
Origin + Δ	0.60	0.24	1.00
Origin +2 Δ	1.20	0.48	2.00

11-4 For the first-order model $\hat{y} = 60 + 1.5x_1 - 0.8x_2 + 2.0x_3$ find the path of steepest ascent. The variables are coded as $-1 \le x_i \le 1$.

Let the basic step size be $\Delta x_3 = 1$. $\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda}$ or $1.0 = \frac{2.0}{2\lambda}$. Then $2\lambda = 2.0$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} = \frac{1.50}{2.0} = 0.75$$
$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{-0.8}{2.0} = -0.40$$

Therefore, in the coded variables we have

	Coded	Variables	
	x_1	x_2	x_3
Origin	0	0	0
Δ	0.75	-0.40	1.00
Origin + Δ	0.75	-0.40	1.00
Origin +2 Δ	1.50	-0.80	2.00

11-5 The region of experimentation for three factors are time ($40 \le T_1 \le 80$ min), temperature ($200 \le T_2 \le 300$ °C), and pressure ($20 \le P \le 50$ psig). A first-order model in coded variables has been fit to yield data from a 2^3 design. The model is

$$\hat{y} = 30 + 5x_1 + 2.5x_2 + 3.5x_3$$

Is the point T_1 = 85, T_2 = 325, P=60 on the path of steepest ascent?

The coded variables are found with the following:

$$x_1 = \frac{T_1 - 60}{20} \qquad x_2 = \frac{T_2 - 250}{50} \qquad x_3 \frac{P_1 - 35}{15}$$

$$\Delta T_1 = 5 \quad \Delta x_1 = \frac{5}{20} = 0.25$$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} \text{ or } 0.25 = \frac{20}{2\lambda} \quad 2\lambda = 20$$

$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{2.5}{20} = 0.125$$

$$\Delta x_3 = \frac{\hat{\beta}_3}{2\lambda} = \frac{3.5}{20} = 0.175$$

	Coded	Variables		Natural	Variables	
	x_1	x_2	x_3	T_1	T_2	P
Origin	0	0	0	60	250	35
Δ	0.25	0.125	0.175	5	6.25	2.625
Origin + Δ	0.25	0.125	0.175	65	256.25	37.625
Origin +5 Δ	1.25	0.625	0.875	85	281.25	48.125

The point T_1=85, T_2=325, and P=60 is not on the path of steepest ascent.

11-6 The region of experimentation for two factors are temperature ($100 \le T \le 300°$ F) and catalyst feed rate ($10 \le C \le 30$ lb/h). A first order model in the usual ± 1 coded variables has been fit to a molecular weight response, yielding the following model.

$$\hat{y} = 2000 + 125x_1 + 40x_2$$

(a) Find the path of steepest ascent.

$$x_1 = \frac{T - 200}{100} \qquad x_2 = \frac{C - 20}{10}$$

$$\Delta T = 100 \quad \Delta x_1 = \frac{100}{100} = 1$$

$$\Delta x_1 = \frac{\hat{\beta}_1}{2\lambda} \text{ or } 1 = \frac{125}{2\lambda} \quad 2\lambda = 125$$

$$\Delta x_2 = \frac{\hat{\beta}_2}{2\lambda} = \frac{40}{125} = 0.32$$

	Coded	Variables	Natural	Variables
	x_1	x_2	T	C
Origin	0	0	200	20
Δ	1	0.32	100	3.2

Origin + Δ	1	0.32	300	23.2
Origin +5 Δ	5	1.60	700	36.0

(a) It is desired to move to a region where molecular weights are above 2500. Based on the information you have from the experiment, in this region, about how may steps along the path of steepest ascent might be required to move to the region of interest?

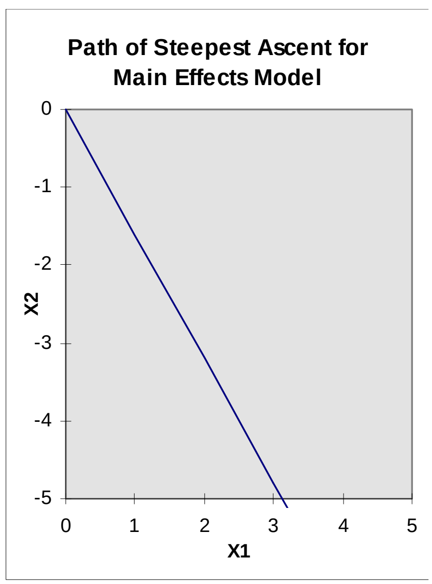
$$\Delta\hat{y} = \Delta x_1 \hat{\beta}_1 + \Delta x_2 \hat{\beta}_2 = (1)(125) + (0.32)(40) = 137.8$$
$$\#\,Steps = \frac{2500 - 2000}{137.8} = 3.63 \rightarrow 4$$

11-7 The path of steepest ascent is usually computed assuming that the model is truly first-order.; that is, there is no interaction. However, even if there is interaction, steepest ascent ignoring the interaction still usually produces good results. To illustrate, suppose that we have fit the model

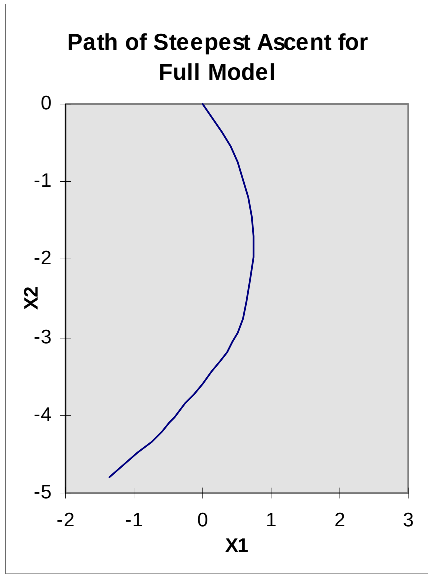
$$\hat{y} = 20 + 5x_1 - 8x_2 + 3x_1 x_2$$

using coded variables ($-1 \le x_1 \le +1$)

(a) Draw the path of steepest ascent that you would obtain if the interaction were ignored.



(b) Draw the path of steepest ascent that you would obtain with the interaction included in the model. Compare this with the path found in part (a).



11-8 The data shown in the following table were collected in an experiment to optimize crystal growth as a function of three variables x_1, x_2, and x_3. Large values of y (yield in grams) are desirable. Fit a second order model and analyze the fitted surface. Under what set of conditions is maximum growth achieved?

x_1	x_2	x_3	y
-1	-1	-1	66
-1	-1	1	70
-1	1	-1	78
-1	1	1	60
1	-1	-1	80
1	-1	1	70
1	1	-1	100
1	1	1	75
-1.682	0	0	100
1.682	0	0	80
0	-1.682	0	68
0	1.682	0	63
0	0	-1.682	65
0	0	1.682	82
0	0	0	113
0	0	0	100
0	0	0	118
0	0	0	88
0	0	0	100
0	0	0	85

Design Expert Output

Response: Yield
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3662.00	9	406.89	2.19	0.1194	not significant
A	*22.08*	*1*	*22.08*	*0.12*	*0.7377*	
B	*25.31*	*1*	*25.31*	*0.14*	*0.7200*	
C	*30.50*	*1*	*30.50*	*0.16*	*0.6941*	
A^2	*204.55*	*1*	*204.55*	*1.10*	*0.3191*	

B^2	*2226.45*	*1*	*2226.45*	*11.96*	*0.0061*	
C^2	*1328.46*	*1*	*1328.46*	*7.14*	*0.0234*	
AB	*66.12*	*1*	*66.12*	*0.36*	*0.5644*	
AC	*55.13*	*1*	*55.13*	*0.30*	*0.5982*	
BC	*171.13*	*1*	*171.13*	*0.92*	*0.3602*	
Residual	1860.95	10	186.09			
Lack of Fit	*1001.61*	*5*	*200.32*	*1.17*	*0.4353*	*not significant*
Pure Error	*859.33*	*5*	*171.87*			
Cor Total	5522.95	19				

The "Model F-value" of 2.19 implies the model is not significant relative to the noise. There is a 11.94 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	13.64	R-Squared	0.6631
Mean	83.05	Adj R-Squared	0.3598
C.V.	16.43	Pred R-Squared	-0.6034
PRESS	8855.23	Adeq Precision	3.882

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	100.67	1	5.56	88.27	113.06	
A-x1	1.27	1	3.69	-6.95	9.50	1.00
B-x2	1.36	1	3.69	-6.86	9.59	1.00
C-x3	-1.49	1	3.69	-9.72	6.73	1.00
A^2	-3.77	1	3.59	-11.77	4.24	1.02
B^2	-12.43	1	3.59	-20.44	-4.42	1.02
C^2	-9.60	1	3.59	-17.61	-1.59	1.02
AB	2.87	1	4.82	-7.87	13.62	1.00
AC	-2.63	1	4.82	-13.37	8.12	1.00
BC	-4.63	1	4.82	-15.37	6.12	1.00

Final Equation in Terms of Coded Factors:

Yield	=
+100.67	
+1.27	* A
+1.36	* B
-1.49	* C
-3.77	* A^2
-12.43	* B^2
-9.60	* C^2
+2.87	* A * B
-2.63	* A * C
-4.63	* B * C

Final Equation in Terms of Actual Factors:

Yield	=
+100.66609	
+1.27146	* x1
+1.36130	* x2
-1.49445	* x3
-3.76749	* $x1^2$
-12.42955	* $x2^2$
-9.60113	* $x3^2$
+2.87500	* x1 * x2
-2.62500	* x1 * x3
-4.62500	* x2 * x3

There are so many nonsignificant terms in this model that we should consider eliminating some of them. A reasonable reduced model is shown below.

Design Expert Output

Response: Yield

ANOVA for Response Surface Reduced Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3143.00	4	785.75	4.95	0.0095	significant
B	25.31	1	25.31	0.16	0.6952	
C	30.50	1	30.50	0.19	0.6673	
B2	2115.31	1	2115.31	13.33	0.0024	
C2	1239.17	1	1239.17	7.81	0.0136	
Residual	2379.95	15	158.66			
Lack of Fit	*1520.62*	*10*	*152.06*	*0.88*	*0.5953*	*not significant*
Pure Error	*859.33*	*5*	*171.87*			
Cor Total	5522.95	19				

The Model F-value of 4.95 implies the model is significant. There is only a 0.95% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	12.60	R-Squared	0.5691
Mean	83.05	Adj R-Squared	0.4542
C.V.	15.17	Pred R-Squared	0.1426
PRESS	4735.52	Adeq Precision	5.778

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	97.58	1	4.36	88.29	106.88	
B-x2	1.36	1	3.41	-5.90	8.63	1.00
C-x3	-1.49	1	3.41	-8.76	5.77	1.00
B2	-12.06	1	3.30	-19.09	-5.02	1.01
C2	-9.23	1	3.30	-16.26	-2.19	1.01

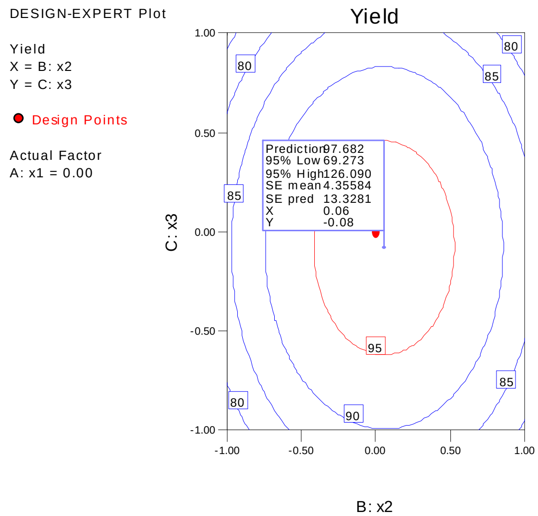
Final Equation in Terms of Coded Factors:

$$\begin{aligned} \text{Yield} &= \\ &+97.58 \\ &+1.36 \quad * B \\ &-1.49 \quad * C \\ &-12.06 \quad * B^2 \\ &-9.23 \quad * C^2 \end{aligned}$$

Final Equation in Terms of Actual Factors:

$$\begin{aligned} \text{Yield} &= \\ &+97.58260 \\ &+1.36130 \quad * x2 \\ &-1.49445 \quad * x3 \\ &-12.05546 \quad * x2^2 \\ &-9.22703 \quad * x3^2 \end{aligned}$$

The contour plot identifies a maximum near the center of the design space.



11-9 The following data were collected by a chemical engineer. The response y is filtration time, x_1 is temperature, and x_2 is pressure. Fit a second-order model.

x_1	x_2	y
-1	-1	54
-1	1	45
1	-1	32
1	1	47
-1.414	0	50
1.414	0	53
0	-1.414	47
0	1.414	51
0	0	41
0	0	39
0	0	44
0	0	42
0	0	40

Design Expert Output

Response: y
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	264.22	4	66.06	2.57	0.1194	not significant
A	13.11	1	13.11	0.51	0.4955	
B	25.72	1	25.72	1.00	0.3467	
A^2	81.39	1	81.39	3.16	0.1132	
AB	144.00	1	144.00	5.60	0.0455	
Residual	205.78	8	25.72			
Lack of Fit	*190.98*	*4*	*47.74*	*12.90*	*0.0148*	*significant*
Pure Error	*14.80*	*4*	*3.70*			
Cor Total	470.00	12				

The "Model F-value" of 2.57 implies the model is not significant relative to the noise. There is a 11.94 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	5.07	R-Squared	0.5622

Mean	45.00	Adj R-Squared	0.3433
C.V.	11.27	Pred R-Squared	-0.5249
PRESS	716.73	Adeq Precision	4.955

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	42.91	1	1.83	38.69	47.14	
A-Temperature	1.28	1	1.79	-2.85	5.42	1.00
B-Pressure	-1.79	1	1.79	-5.93	2.34	1.00
A^2	3.39	1	1.91	-1.01	7.79	1.00
AB	6.00	1	2.54	0.15	11.85	1.00

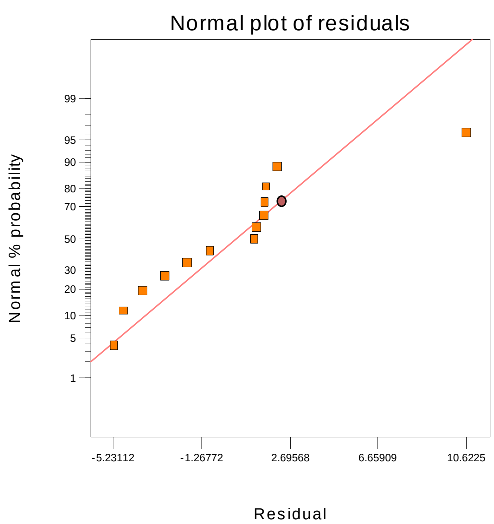
Final Equation in Terms of Coded Factors:

$$\begin{aligned} \text{Time} &= \\ &+42.91 \\ &+1.28 \; * A \\ &-1.79 \; * B \\ &+3.39 \; * A^2 \\ &+6.00 \; * A * B \end{aligned}$$

Final Equation in Terms of Actual Factors:

$$\begin{aligned} \text{Time} &= \\ &+42.91304 \\ &+1.28033 \; * \text{Temperature} \\ &-1.79289 \; * \text{Pressure} \\ &+3.39130 \; * \text{Temperature}^2 \\ &+6.00000 \; * \text{Temperature} * \text{Pressure} \end{aligned}$$

The lack of fit test in the above analysis is significant. Also, the residual plot below identifies an outlier which happens to be standard order number 8.



We chose to remove this run and re-analyze the data.

Design Expert Output

Response: y

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	407.34	4	101.84	30.13	0.0002	significant
A	13.11	1	13.11	3.88	0.0895	

B	132.63	1	132.63	39.25	0.0004	
A^2	155.27	1	155.27	45.95	0.0003	
AB	144.00	1	144.00	42.61	0.0003	
Residual	23.66	7	3.38			
Lack of Fit	*8.86*	*3*	*2.95*	*0.80*	*0.5560*	*not significant*
Pure Error	*14.80*	*4*	*3.70*			
Cor Total	431.00	11				

The Model F-value of 30.13 implies the model is significant. There is only
a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.84	R-Squared	0.9451
Mean	44.50	Adj R-Squared	0.9138
C.V.	4.13	Pred R-Squared	0.8129
PRESS	80.66	Adeq Precision	18.243

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	40.68	1	0.73	38.95	42.40	
A-Temperature	1.28	1	0.65	-0.26	2.82	1.00
B-Pressure	-4.82	1	0.77	-6.64	-3.00	1.02
A^2	4.88	1	0.72	3.18	6.59	1.02
AB	6.00	1	0.92	3.83	8.17	1.00

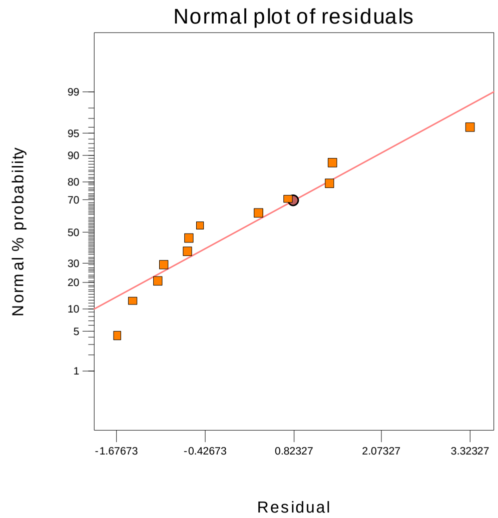
Final Equation in Terms of Coded Factors:

Time =
+40.68
+1.28 * A
-4.82 * B
+4.88 * A^2
+6.00 * A * B

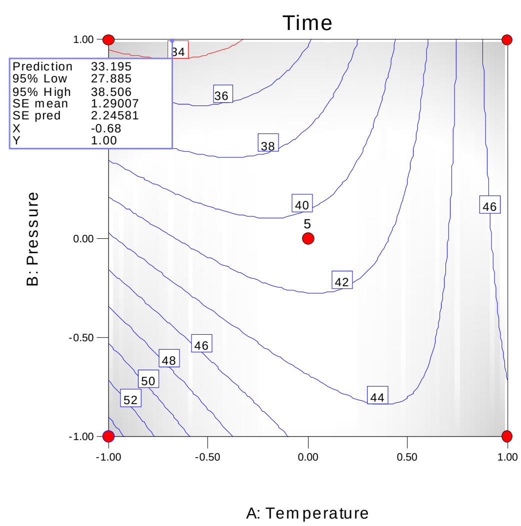
Final Equation in Terms of Actual Factors:

Time =
+40.67673
+1.28033 * Temperature
-4.82374 * Pressure
+4.88218 * Temperature^2
+6.00000 * Temperature * Pressure

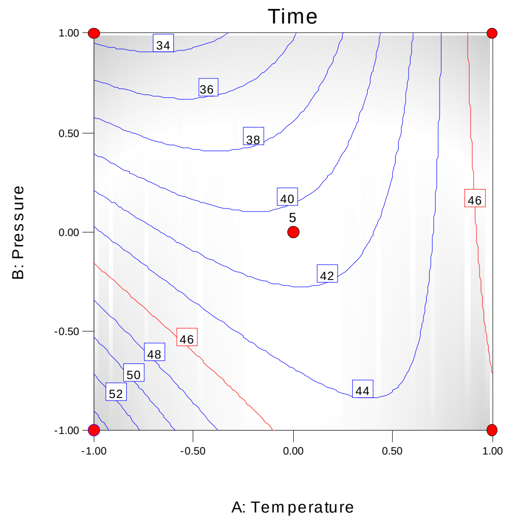
The lack of fit test is satisfactory as well as the following normal plot of residuals:



(a) What operating conditions would you recommend if the objective is to minimize the filtration time?



(b) What operating conditions would you recommend if the objective is to operate the process at a mean filtration time very close to 46?



There are two regions that enable a filtration time of 46. Either will suffice; however, higher temperatures and pressures typically have higher operating costs. We chose the operating conditions at the lower pressure and temperature as shown.

11-10 The hexagon design that follows is used in an experiment that has the objective of fitting a second-order model.

x_1	x_2	y
1	0	68
0.5	$\sqrt{0.75}$	74
-0.5	$\sqrt{0.75}$	65
-1	0	60
-0.5	$-\sqrt{0.75}$	63
0.5	$-\sqrt{0.75}$	70
0	0	58
0	0	60
0	0	57
0	0	55
0	0	69

(a) Fit the second-order model.

Design Expert Output

Response: **y**

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	245.26	5	49.05	1.89	0.2500	not significant
A	*85.33*	*1*	*85.33*	*3.30*	*0.1292*	
B	*9.00*	*1*	*9.00*	*0.35*	*0.5811*	
A^2	*25.20*	*1*	*25.20*	*0.97*	*0.3692*	
B^2	*129.83*	*1*	*129.83*	*5.01*	*0.0753*	
AB	*1.00*	*1*	*1.00*	*0.039*	*0.8519*	
Residual	129.47	5	25.89			

Lack of Fit	*10.67*	*1*	*10.67*	*0.36*	*0.5813*	*not significant*
Pure Error	*118.80*	*4*	*29.70*			
Cor Total	374.73	10				

The "Model F-value" of 1.89 implies the model is not significant relative to the noise. There is a 25.00 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	5.09	R-Squared	0.6545
Mean	63.55	Adj R-Squared	0.3090
C.V.	8.01	Pred R-Squared	-0.5201
PRESS	569.63	Adeq Precision	3.725

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	59.80	1	2.28	53.95	65.65	
A-x1	5.33	1	2.94	-2.22	12.89	1.00
B-x2	1.73	1	2.94	-5.82	9.28	1.00
A^2	4.20	1	4.26	-6.74	15.14	1.00
B^2	9.53	1	4.26	-1.41	20.48	1.00
AB	1.15	1	5.88	-13.95	16.26	1.00

Final Equation in Terms of Coded Factors:

y =
+59.80
+5.33 * A
+1.73 * B
+4.20 * A^2
+9.53 * B^2
+1.15 * A * B

(a) Perform the canonical analysis. What type of surface has been found?

The full quadratic model is used in the following analysis because the reduced model is singular.

Solution

	Variable	**Critical Value**
	X1	-0.627658
	X2	-0.052829
Predicted Value at Solution		58.080492

Eigenvalues and Eigenvectors

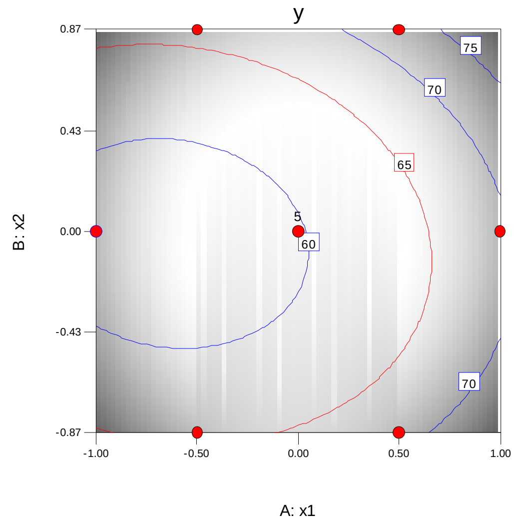
Variable	9.5957	4.1382
X1	0.10640	0.99432
X2	0.99432	-0.10640

Since both eigenvalues are positive, the response is a minimum at the stationary point.

(c) What operating conditions on x_1 and x_2 lead to the stationary point?

The stationary point is (x_1,x_2) = (-0.62766, -0.05283)

(d) Where would you run this process if the objective is to obtain a response that is as close to 65 as possible?



Any value of x_1 and x_2 that give a point on the contour with value of 65 would be satisfactory.

11-11 An experimenter has run a Box-Behnken design and has obtained the results below, where the response variable is the viscosity of a polymer.

Level	Temp.	Agitation Rate	Pressure	x_1	x_2	x_3
High	200	10.0	25	+1	+1	+1
Middle	175	7.5	20	0	0	0
Low	150	5.0	15	-1	-1	-1

Run	x_1	x_2	x_3	y_1
1	-1	-1	0	535
2	1	-1	0	580
3	-1	1	0	596
4	1	1	0	563
5	-1	0	-1	645
6	1	0	-1	458
7	-1	0	1	350
8	1	0	1	600
9	0	-1	-1	595
10	0	1	-1	648
11	0	-1	1	532
12	0	1	1	656
13	0	0	0	653
14	0	0	0	599
15	0	0	0	620

(a) Fit the second-order model.

Design Expert Output

Response: Viscosity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F

Model	89652.58	9	9961.40	9.54	0.0115	significant
A	*703.12*	*1*	*703.12*	*0.67*	*0.4491*	
B	*6105.12*	*1*	*6105.12*	*5.85*	*0.0602*	
C	*5408.00*	*1*	*5408.00*	*5.18*	*0.0719*	
A^2	*20769.23*	*1*	*20769.23*	*19.90*	*0.0066*	
B^2	*1404.00*	*1*	*1404.00*	*1.35*	*0.2985*	
C^2	*4719.00*	*1*	*4719.00*	*4.52*	*0.0868*	
AB	*1521.00*	*1*	*1521.00*	*1.46*	*0.2814*	
AC	*47742.25*	*1*	*47742.25*	*45.74*	*0.0011*	
BC	*1260.25*	*1*	*1260.25*	*1.21*	*0.3219*	
Residual	5218.75	5	1043.75			
Lack of Fit	*3736.75*	*3*	*1245.58*	*1.68*	*0.3941*	*not significant*
Pure Error	*1482.00*	*2*	*741.00*			
Cor Total	94871.33	14				

The Model F-value of 9.54 implies the model is significant. There is only a 1.15% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	32.31	R-Squared	0.9450
Mean	575.33	Adj R-Squared	0.8460
C.V.	5.62	Pred R-Squared	0.3347
PRESS	63122.50	Adeq Precision	10.425

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	624.00	1	18.65	576.05	671.95	
A-Temp	9.37	1	11.42	-19.99	38.74	1.00
B-Agitation Rate	27.62	1	11.42	-1.74	56.99	1.00
C-Pressure	-26.00	1	11.42	-55.36	3.36	1.00
A^2	-75.00	1	16.81	-118.22	-31.78	1.01
B^2	19.50	1	16.81	-23.72	62.72	1.01
C^2	-35.75	1	16.81	-78.97	7.47	1.01
AB	-19.50	1	16.15	-61.02	22.02	1.00
AC	109.25	1	16.15	67.73	150.77	1.00
BC	17.75	1	16.15	-23.77	59.27	1.00

Final Equation in Terms of Coded Factors:

Viscosity	=
+624.00	
+9.37	* A
+27.62	* B
-26.00	* C
-75.00	* A^2
+19.50	* B^2
-35.75	* C^2
-19.50	* A * B
+109.25	* A * C
+17.75	* B * C

Final Equation in Terms of Actual Factors:

Viscosity	=
-629.50000	
+27.23500	* Temp
-9.55000	* Agitation Rate
-111.60000	* Pressure
-0.12000	* Temp^2
+3.12000	* Agitation Rate^2
-1.43000	* Pressure^2
-0.31200	* Temp * Agitation Rate
+0.87400	* Temp * Pressure
+1.42000	* Agitation Rate * Pressure

(b) Perform the canonical analysis. What type of surface has been found?

Solution

Variable	**Critical Value**
X1	2.1849596
X2	-0.871371
X3	2.7586015
Predicted Value at Solution	586.34437

Eigevalues and Eigevectors

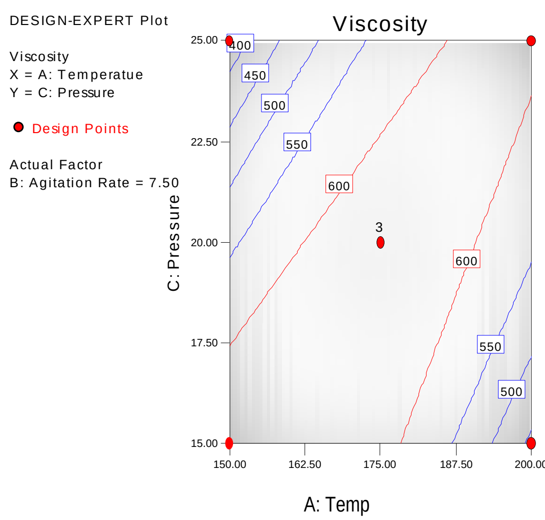
Variable	20.9229	2.5208	-114.694
X1	-0.02739	0.58118	0.81331
X2	0.99129	-0.08907	0.09703
X3	0.12883	0.80888	-0.57368

The system is a saddle point.

(c) What operating conditions on x_1, x_2, and x_3 lead to the stationary point?

The stationary point is (x_1, x_2, x_3) = (2.18496, -0.87167, 2.75860). This is outside the design region. It would be necessary to either examine contour plots or use numerical optimization methods to find desired operating conditions.

(d) What operating conditions would you recommend if it is important to obtain a viscosity that is as close to 600 as possible?



Any point on either of the contours showing a viscosity of 600 is satisfactory.

11-12 Consider the three-variable central composite design shown below. Analyze the data and draw conclusions, assuming that we wish to maximize conversion (y_1) with activity (y_2) between 55 and 60.

Run	Time (min)	Temperature (°C)	Catalyst (%)	Conversion (%) y_1	Activity y_2
1	-1.000	-1.000	-1.000	74.00	53.20

2	1.000	-1.000	-1.000	51.00	62.90
3	-1.000	1.000	-1.000	88.00	53.40
4	1.000	1.000	-1.000	70.00	62.60
5	-1.000	-1.000	1.000	71.00	57.30
6	1.000	-1.000	1.000	90.00	67.90
7	-1.000	1.000	1.000	66.00	59.80
8	1.000	1.000	1.000	97.00	67.80
9	0.000	0.000	0.000	81.00	59.20
10	0.000	0.000	0.000	75.00	60.40
11	0.000	0.000	0.000	76.00	59.10
12	0.000	0.000	0.000	83.00	60.60
13	-1.682	0.000	0.000	76.00	59.10
14	1.682	0.000	0.000	79.00	65.90
15	0.000	-1.682	0.000	85.00	60.00
16	0.000	1.682	0.000	97.00	60.70
17	0.000	0.000	-1.682	55.00	57.40
18	0.000	0.000	1.682	81.00	63.20
19	0.000	0.000	0.000	80.00	60.80
20	0.000	0.000	0.000	91.00	58.90

Quadratic models are developed for the Conversion and Activity response variables as follows:

Design Expert Output

Response: Conversion

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	2555.73	9	283.97	12.76	0.0002	significant
A	*14.44*	*1*	*14.44*	*0.65*	*0.4391*	
B	*222.96*	*1*	*222.96*	*10.02*	*0.0101*	
C	*525.64*	*1*	*525.64*	*23.63*	*0.0007*	
A^2	*48.47*	*1*	*48.47*	*2.18*	*0.1707*	
B^2	*124.48*	*1*	*124.48*	*5.60*	*0.0396*	
C^2	*388.59*	*1*	*388.59*	*17.47*	*0.0019*	
AB	*36.13*	*1*	*36.13*	*1.62*	*0.2314*	
AC	*1035.13*	*1*	*1035.13*	*46.53*	*< 0.0001*	
BC	*120.12*	*1*	*120.12*	*5.40*	*0.0425*	
Residual	222.47	10	22.25			
Lack of Fit	*56.47*	*5*	*11.29*	*0.34*	*0.8692*	*not significant*
Pure Error	*166.00*	*5*	*33.20*			
Cor Total	287.28	19				

The Model F-value of 12.76 implies the model is significant. There is only
a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	4.72	R-Squared	0.9199
Mean	78.30	Adj R-Squared	0.8479
C.V.	6.02	Pred R-Squared	0.7566
PRESS	676.22	Adeq Precision	14.239

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	81.09	1	1.92	76.81	85.38	
A-Time	1.03	1	1.28	-1.82	3.87	1.00
B-Temperature	4.04	1	1.28	1.20	6.88	1.00
C-Catalyst	6.20	1	1.28	3.36	9.05	1.00

A2	-1.83	1	1.24	-4.60	0.93	1.02
B2	2.94	1	1.24	0.17	5.71	1.02
C2	-5.19	1	1.24	-7.96	-2.42	1.02
AB	2.13	1	1.67	-1.59	5.84	1.00
AC	11.38	1	1.67	7.66	15.09	1.00
BC	-3.87	1	1.67	-7.59	-0.16	1.00

Final Equation in Terms of Coded Factors:

Conversion	=
+81.09	
+1.03	* A
+4.04	* B
+6.20	* C
-1.83	* A2
+2.94	* B2
-5.19	* C2
+2.13	* A * B
+11.38	* A * C
-3.87	* B * C

Final Equation in Terms of Actual Factors:

Conversion	=
+81.09128	
+1.02845	* Time
+4.04057	* Temperature
+6.20396	* Catalyst
-1.83398	* Time2
+2.93899	* Temperature2
-5.19274	* Catalyst2
+2.12500	* Time * Temperature
+11.37500	* Time * Catalyst
-3.87500	* Temperature * Catalyst

Design Expert Output

Response: Activity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	256.20	9	28.47	9.16	0.0009	significant
A	*175.35*	*1*	*175.35*	*56.42*	*< 0.0001*	
B	*0.89*	*1*	*0.89*	*0.28*	*0.6052*	
C	*67.91*	*1*	*67.91*	*21.85*	*0.0009*	
A^2	*10.05*	*1*	*10.05*	*3.23*	*0.1024*	
B^2	*0.081*	*1*	*0.081*	*0.026*	*0.8753*	
C^2	*0.047*	*1*	*0.047*	*0.015*	*0.9046*	
AB	*1.20*	*1*	*1.20*	*0.39*	*0.5480*	
AC	*0.011*	*1*	*0.011*	*3.620E-003*	*0.9532*	
BC	*0.78*	*1*	*0.78*	*0.25*	*0.6270*	
Residual	31.08	10	3.11			
Lack of Fit	*27.43*	*5*	*5.49*	*7.51*	*0.0226*	*significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 9.16 implies the model is significant. There is only a 0.09% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.76	R-Squared	0.8918
Mean	60.51	Adj R-Squared	0.7945
C.V.	2.91	Pred R-Squared	0.2536
PRESS	214.43	Adeq Precision	10.911

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	59.85	1	0.72	58.25	61.45	

A-Time	3.58	1	0.48	2.52	4.65	1.00
B-Temperature	0.25	1	0.48	-0.81	1.32	1.00
C-Catalyst	2.23	1	0.48	1.17	3.29	1.00
A^2	0.83	1	0.46	-0.20	1.87	1.02
B^2	0.075	1	0.46	-0.96	1.11	1.02
C^2	0.057	1	0.46	-0.98	1.09	1.02
AB	-0.39	1	0.62	-1.78	1.00	1.00
AC	-0.038	1	0.62	-1.43	1.35	1.00
BC	0.31	1	0.62	-1.08	1.70	1.00

Final Equation in Terms of Coded Factors:

Conversion =
+59.85
+3.58 * A
+0.25 * B
+2.23 * C
+0.83 * A^2
+0.075 * B^2
+0.057 * C^2
-0.39 * A * B
-0.038 * A * C
+0.31 * B * C

Final Equation in Terms of Actual Factors:

Conversion =
+59.84984
+3.58327 * Time
+0.25462 * Temperature
+2.22997 * Catalyst
+0.83491 * Time^2
+0.074772 * Temperature^2
+0.057094 * Catalyst^2
-0.38750 * Time * Temperature
-0.037500 * Time * Catalyst
+0.31250 * Temperature * Catalyst

Because many of the terms are insignificant, the reduced quadratic model is fit as follows:

Design Expert Output

Response: Activity
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	253.20	3	84.40	39.63	< 0.0001	significant
A	*175.35*	*1*	*175.35*	*82.34*	*< 0.0001*	
C	*67.91*	*1*	*67.91*	*31.89*	*< 0.0001*	
A^2	*9.94*	*1*	*9.94*	*4.67*	*0.0463*	
Residual	34.07	16	2.13			
Lack of Fit	*30.42*	*11*	*2.77*	*3.78*	*0.0766*	*not significant*
Pure Error	*3.65*	*5*	*0.73*			
Cor Total	287.28	19				

The Model F-value of 39.63 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	1.46	R-Squared	0.8814
Mean	60.51	Adj R-Squared	0.8591
C.V.	2.41	Pred R-Squared	0.6302
PRESS	106.24	Adeq Precision	20.447

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF

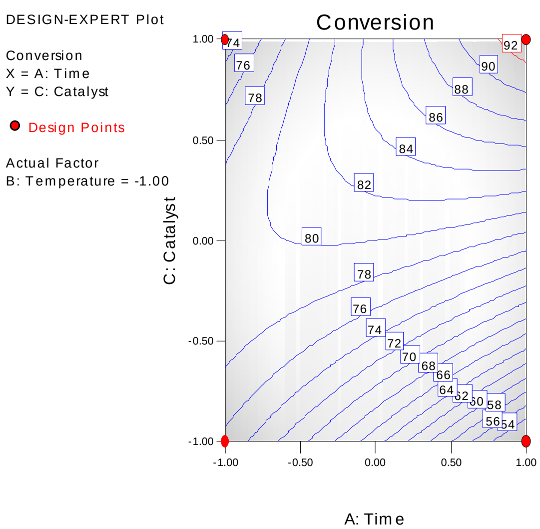
Intercept	59.95	1	0.42	59.06	60.83	
A-Time	3.58	1	0.39	2.75	4.42	1.00
C-Catalyst	2.23	1	0.39	1.39	3.07	1.00
A^2	0.82	1	0.38	0.015	1.63	1.00

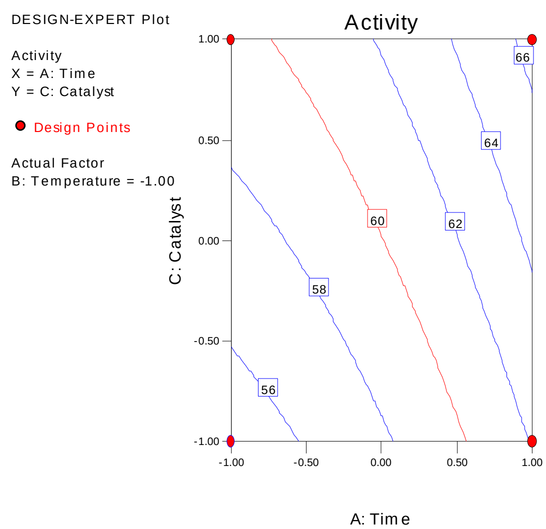
Final Equation in Terms of Coded Factors:

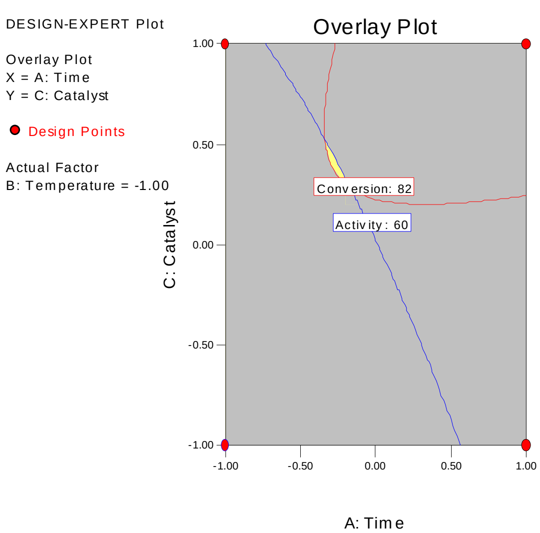
Activity =
+59.95
+3.58 * A
+2.23 * C
+0.82 * A2

Final Equation in Terms of Actual Factors:

Activity =
+59.94802
+3.58327 * Time
+2.22997 * Catalyst
+0.82300 * Time2







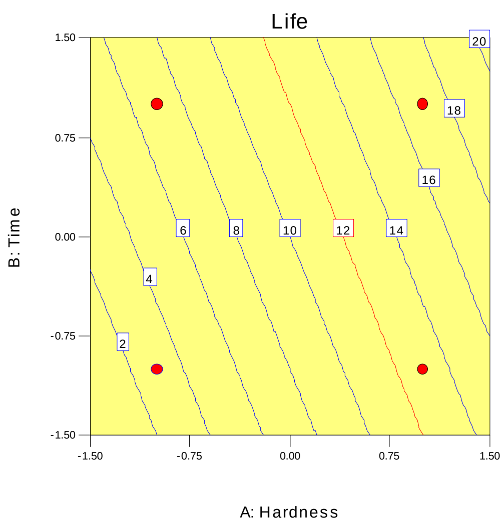
The contour plots visually describe the models while the overlay plots identifies the acceptable region for the process.

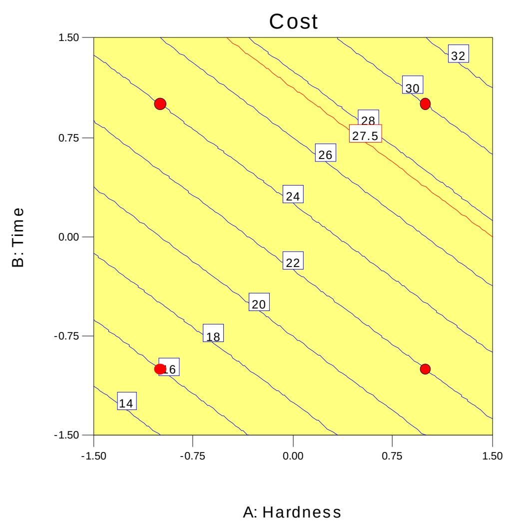
11-13 A manufacturer of cutting tools has developed two empirical equations for tool life in hours (y_1) and for tool cost in dollars (y_2). Both models are linear functions of steel hardness (x_1) and manufacturing time (x_2). The two equations are

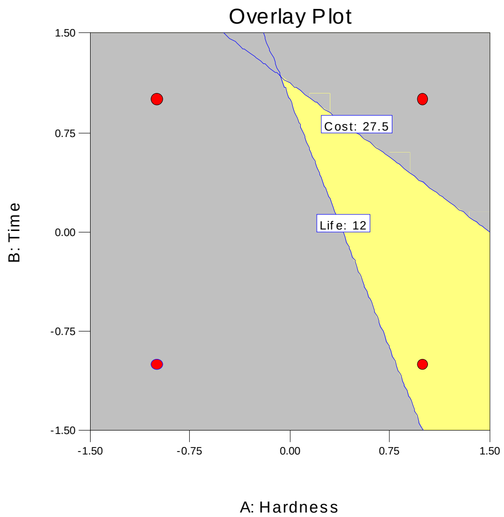
$$\hat{y}_1 = 10 + 5x_1 + 2x_2$$
$$\hat{y}_2 = 23 + 3x_1 + 4x_2$$

and both equations are valid over the range $-1.5 \leq x_1 \leq 1.5$. Unit tool cost must be below \$27.50 and life must exceed 12 hours for the product to be competitive. Is there a feasible set of operating conditions for this process? Where would you recommend that the process be run?

The contour plots below graphically describe the two models. The overlay plot identifies the feasible operating region for the process.







$$10 + 5x_1 + 2x_2 \geq 12$$
$$23 + 3x_1 + 4x_2 \leq 27.50$$

11-14 A central composite design is run in a chemical vapor deposition process, resulting in the experimental data shown below. Four experimental units were processed simultaneously on each run of the design, and the responses are the mean and variance of thickness, computed across the four units.

x_1	x_2	$\bar{y}$	s^2
-1	-1	360.6	6.689
-1	1	445.2	14.230
1	-1	412.1	7.088
1	1	601.7	8.586
1.414	0	518.0	13.130
-1.414	0	411.4	6.644
0	1.414	497.6	7.649
0	-1.414	397.6	11.740
0	0	530.6	7.836
0	0	495.4	9.306
0	0	510.2	7.956
0	0	487.3	9.127

(a) Fit a model to the mean response. Analyze the residuals.

Design Expert Output

Response: Mean Thick
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	47644.26	5	9528.85	16.12	0.0020	significant
A	*22573.36*	*1*	*22573.36*	*38.19*	*0.0008*	
B	*15261.91*	*1*	*15261.91*	*25.82*	*0.0023*	
A^2	*2795.58*	*1*	*2795.58*	*4.73*	*0.0726*	
B^2	*5550.74*	*1*	*5550.74*	*9.39*	*0.0221*	
AB	*2756.25*	*1*	*2756.25*	*4.66*	*0.0741*	
Residual	3546.83	6	591.14			
Lack of Fit	*2462.04*	*3*	*820.68*	*2.27*	*0.2592*	*not significant*
Pure Error	*1084.79*	*3*	*361.60*			

Cor Total	51191.09	11

The Model F-value of 16.12 implies the model is significant. There is only
a 0.20% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	24.31	R-Squared	0.9307
Mean	472.31	Adj R-Squared	0.8730
C.V.	5.15	Pred R-Squared	0.6203
PRESS	19436.37	Adeq Precision	11.261

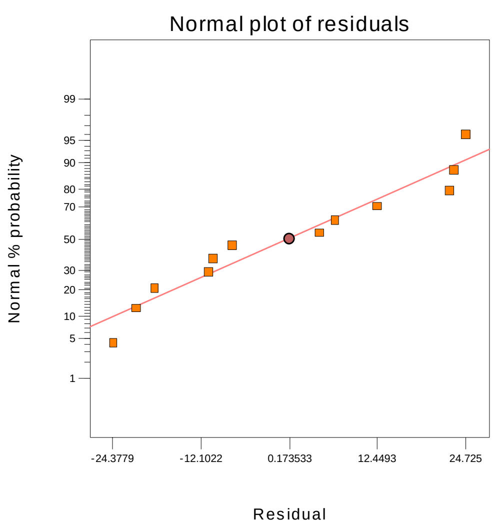
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	505.88	1	12.16	476.13	535.62	
A-x1	53.12	1	8.60	32.09	74.15	1.00
B-x2	43.68	1	8.60	22.64	64.71	1.00
A^2	-20.90	1	9.61	-44.42	2.62	1.04
B^2	-29.45	1	9.61	-52.97	-5.93	1.04
AB	26.25	1	12.16	-3.50	56.00	1.00

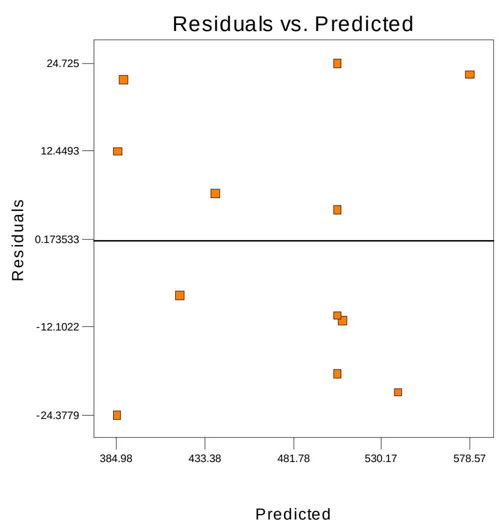
Final Equation in Terms of Coded Factors:

Mean Thick	=
+505.88	
+53.12	* A
+43.68	* B
-20.90	* A^2
-29.45	* B^2
+26.25	* A * B

Final Equation in Terms of Actual Factors:

Mean Thick	=
+505.87500	
+53.11940	* x1
+43.67767	* x2
-20.90000	* $x1^2$
-29.45000	* $x2^2$
+26.25000	* x1 * x2





A modest deviation from normality can be observed in the Normal Plot of Residuals; however, not enough to be concerned.

(b) Fit a model to the variance response. Analyze the residuals.

Design Expert Output

Response: Var Thick
ANOVA for Response Surface 2FI Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	65.80	3	21.93	35.86	< 0.0001	significant
A	*41.46*	*1*	*41.46*	*67.79*	*< 0.0001*	
B	*15.21*	*1*	*15.21*	*24.87*	*0.0011*	
AB	*9.13*	*1*	*9.13*	*14.93*	*0.0048*	
Residual	4.89	8	0.61			
Lack of Fit	*3.13*	*5*	*0.63*	*1.06*	*0.5137*	*not significant*
Pure Error	*1.77*	*3*	*0.59*			
Cor Total	70.69	11				

The Model F-value of 35.86 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.78	R-Squared	0.9308
Mean	9.17	Adj R-Squared	0.9048
C.V.	8.53	Pred R-Squared	0.8920
PRESS	7.64	Adeq Precision	18.572

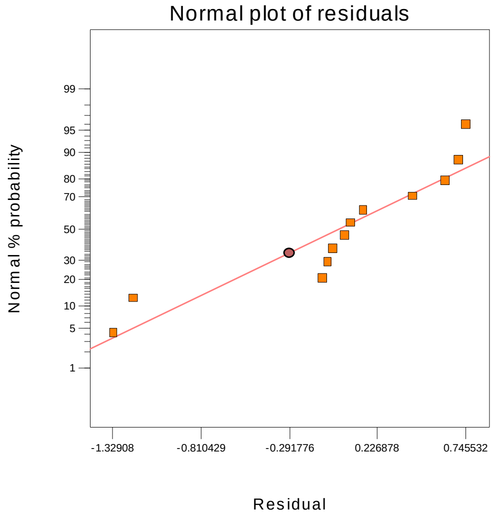
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	9.17	1	0.23	8.64	9.69	
A-x1	2.28	1	0.28	1.64	2.91	1.00
B-x2	-1.38	1	0.28	-2.02	-0.74	1.00
AB	-1.51	1	0.39	-2.41	-0.61	1.00

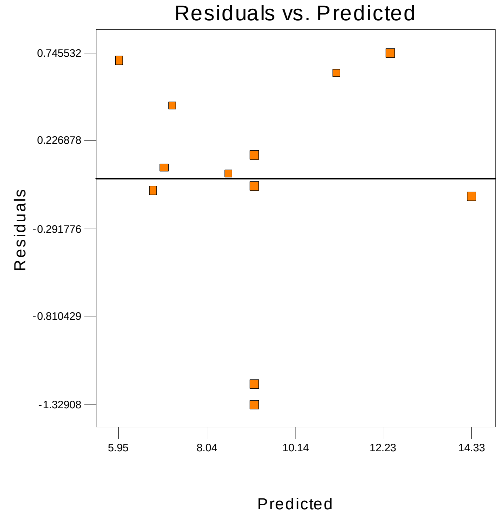
Final Equation in Terms of Coded Factors:

Var Thick =
+9.17
+2.28 * A
-1.38 * B
-1.51 * A * B

Final Equation in Terms of Actual Factors:

Var Thick =
+9.16508
+2.27645 * x1
-1.37882 * x2
-1.51075 * x1 * x2





The residual plots are not acceptable. A transformation should be considered. If not successful at correcting the residual plots, further investigation into the two apparently unusual points should be made.

(c) Fit a model to the $\ln(s^2)$. Is this model superior to the one you found in part (b)?

Design Expert Output

Response: Var Thick Transform: Natural log Constant: 0

ANOVA for Response Surface 2FI Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	0.67	3	0.22	36.94	< 0.0001	significant
A	*0.46*	*1*	*0.46*	*74.99*	*< 0.0001*	
B	*0.14*	*1*	*0.14*	*22.80*	*0.0014*	
AB	*0.079*	*1*	*0.079*	*13.04*	*0.0069*	
Residual	0.049	8	6.081E-003			
Lack of Fit	*0.024*	*5*	*4.887E-003*	*0.61*	*0.7093*	*not significant*
Pure Error	*0.024*	*3*	*8.071E-003*			
Cor Total	0.72	11				

The Model F-value of 36.94 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.078	R-Squared	0.9327
Mean	2.18	Adj R-Squared	0.9074
C.V.	3.57	Pred R-Squared	0.8797
PRESS	0.087	Adeq Precision	18.854

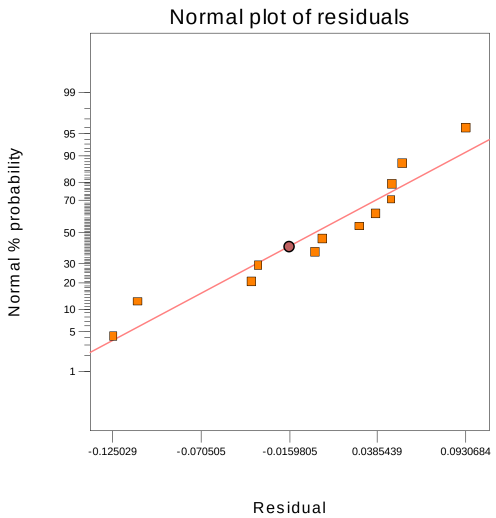
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	2.18	1	0.023	2.13	2.24	
A-x1	0.24	1	0.028	0.18	0.30	1.00
B-x2	-0.13	1	0.028	-0.20	-0.068	1.00
AB	-0.14	1	0.039	-0.23	-0.051	1.00

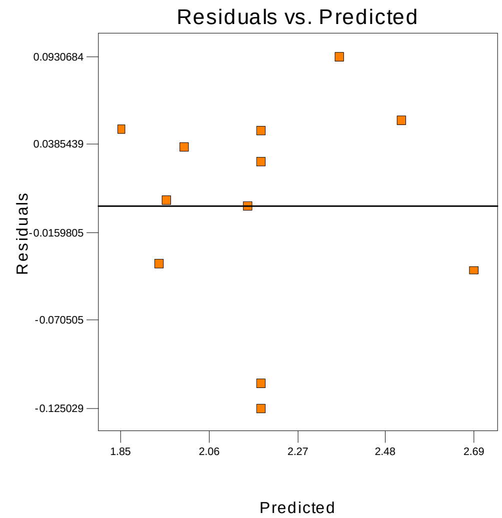
Final Equation in Terms of Coded Factors:

Ln(Var Thick) =
+2.18
+0.24 * A
-0.13 * B
-0.14 * A * B

Final Equation in Terms of Actual Factors:

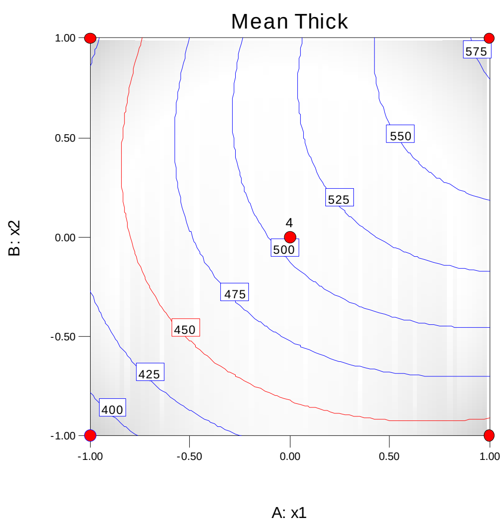
$$\begin{aligned}\text{Ln(Var Thick)} \;=\;& \\ &+2.18376 \\ &+0.23874 \;* \text{x1} \\ &-0.13165 \;* \text{x2} \\ &-0.14079 \;* \text{x1} * \text{x2}\end{aligned}$$

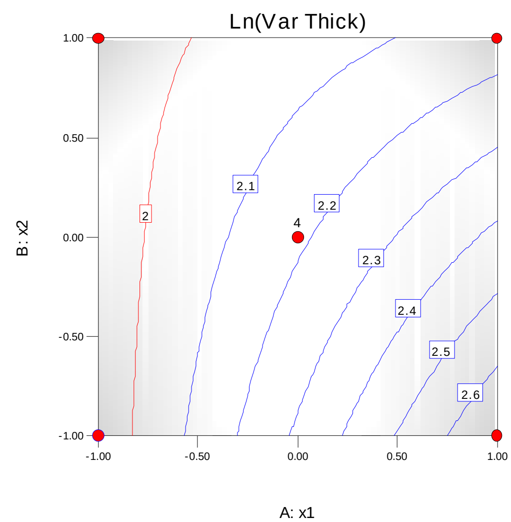


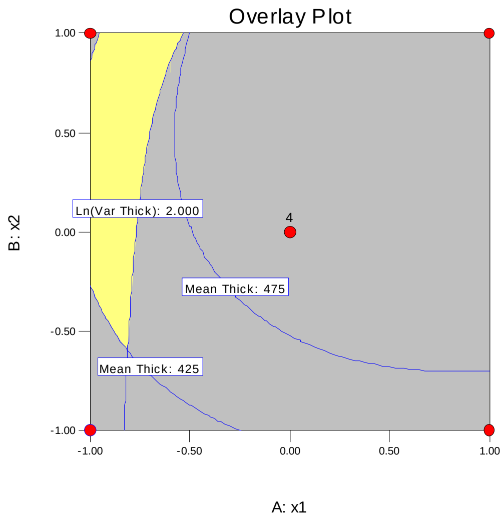


The residual plots are much improved following the natural log transformation; however, the two runs still appear to be somewhat unusual and should be investigated further. They will be retained in the analysis.

(d) Suppose you want the mean thickness to be in the interval 450±25. Find a set of operating conditions that achieve the objective and simultaneously minimize the variance.







The contour plots describe the two models while the overlay plot identifies the acceptable region for the process.

(e) Discuss the variance minimization aspects of part (d). Have you minimized total process variance?

The within run variance has been minimized; however, the run-to-run variation has not been minimized in the analysis. This may not be the most robust operating conditions for the process.

11-15 Verify that an orthogonal first-order design is also first-order rotatable.

To show that a first order orthogonal design is also first order rotatable, consider

$$V(\hat{y}) = V(\hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i) = V(\hat{\beta}_0) + \sum_{i=1}^{k} x_i^2 V(\hat{\beta}_i)$$

since all covariances between $\hat{\beta}_i$ and $\hat{\beta}_j$ are zero, due to design orthogonality. Furthermore, we have:

$$V\left(\hat{\beta}_0\right) = V(\hat{\beta}_1) = V(\hat{\beta}_2) = \ldots = V(\hat{\beta}_k) = \frac{\sigma^2}{n}\text{, so}$$

$$V(\hat{y}) = \frac{\sigma^2}{n} + \frac{\sigma^2}{n}\sum_{i=1}^{k} x_i^2$$

$$V(\hat{y}) = \frac{\sigma^2}{n}\left(1 + \frac{\sigma^2}{n}\sum_{i=1}^{k} x_i^2\right)$$

which is a function of distance from the design center (i.e. **x**=**0**), and not direction. Thus the design must be rotatable. Note that *n* is, in general, the number of points in the exterior portion of the design. If there are n_c centerpoints, then $V(\hat{\beta}_0) = \dfrac{\sigma^2}{(n + n_c)}$.

11-16 Show that augmenting a 2^k design with n_C center points does not affect the estimates of the β_i (i=1, 2, . . . , k), but that the estimate of the intercept β_0 is the average of all $2^k + n_C$ observations.

In general, the **X** matrix for the 2^k design with n_c center points and the **y** vector would be:

$$\begin{array}{cc} & \begin{array}{ccccc} \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_k \end{array} \\ \mathbf{X}= & \left[\begin{array}{ccccc} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ \hline 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

← The upper half of the matrix is the usual ± 1 notation of the 2^k design

← The lower half of the matrix represents the center points (n_c rows)

$$\mathbf{y}=\left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{2^k} \\ \hline n_{0_1} \\ n_{0_2} \\ \vdots \\ n_{0_c} \end{array}\right]$$

← 2^k+n_c observations
.

$$\mathbf{X'X}=\left[\begin{array}{cccc} 2^k+n_c & 0 & \cdots & 0 \\ & 2^k & \cdots & 0 \\ & & \ddots & \vdots \\ & & & 2^k \end{array}\right] \qquad \mathbf{X'y}=\left[\begin{array}{c} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_k \end{array}\right]$$

← Grand total of all 2^k+n_c observations

← usual contrasts from 2^k

Therefore, $\hat{\beta}_0=\dfrac{g_0}{2^k+n_c}$, which is the average of all $\left(2^k+n_c\right)$observations, while $\hat{\beta}_i=\dfrac{g_i}{2^k}$, which does not depend on the number of center points, since in computing the contrasts g_i, all observations at the center are multiplied by zero.

11-17 ***The rotatable central composite design.*** It can be shown that a second-order design is rotatable if $\sum_{u=1}^{n} x_{iu}^a x_{ju}^b = 0$ if a or b (or both) are odd and if $\sum_{u=1}^{n} x_{iu}^4 = 3\sum_{u=1}^{n} x_{iu}^2 x_{ju}^2$. Show that for the central composite design these conditions lead to $\alpha=\left(n_f\right)^{1/4}$ for rotatability, where n_f is the number of points in the factorial portion.

The balance between +1 and -1 in the factorial columns and the orthogonality among certain column in the X matrix for the central composite design will result in all odd moments being zero. To solve for α use the following relations:

$$\sum_{u=1}^{n} x_{iu}^4 = n_f + 2\alpha^4, \qquad \sum_{u=1}^{n} x_{iu}^2 x_{ju}^2 = n_f$$

then

$$\sum_{u=1}^{n} x_{iu}^4 = 3\sum_{u=1}^{n} x_{iu}^2 x_{ju}^2$$
$$n_f + 2\alpha^4 = 3(n_f)$$
$$2\alpha^4 = 2n_f$$
$$\alpha^4 = n_f$$
$$\alpha = \sqrt[4]{n_f}$$

11-18 Verify that the central composite design shown below blocks orthogonally.

Block 1			Block 2			Block 3		
x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
0	0	0	0	0	0	-1.633	0	0
0	0	0	0	0	0	1.633	0	0
1	1	1	1	1	-1	0	-1.633	0
1	-1	-1	1	-1	1	0	1.633	0
-1	-1	1	-1	1	1	0	0	-1.633
-1	1	-1	-1	-1	-1	0	0	1.633
						0	0	0
						0	0	0

Note that each block is an orthogonal first order design, since the cross products of elements in different columns add to zero for each block. To verify the second condition, choose a column, say column x_2. Now

$$\sum_{u=1}^{k} x_{2u}^2 = 13.334 \text{ , and } n=20$$

For blocks 1 and 2,

$$\sum_{m} x_{2m}^2 = 4 \text{ , } n_m=6$$

So

$$\frac{\sum_{m} x_{2m}^2}{\sum_{u=1}^{n} x_{2u}^2} = n_m = 6$$

$$\frac{4}{13.334} = \frac{6}{20}$$

$$0.3 = 0.3$$

and condition 2 is satisfied by blocks 1 and 2. For block 3, we have

$$\sum_m x_{2m}^2 = 5.334 \text{ , } n_m = 8\text{, so}$$

$$\frac{\sum_m x_{2m}^2}{\sum_{u=1}^{n} x_{2u}^2} = \frac{n_m}{n}$$

$$\frac{5.334}{13.334} = \frac{8}{20}$$

$$0.4 = 0.4$$

And condition 2 is satisfied by block 3. Similar results hold for the other columns.

11-19 ***Blocking in the central composite design***. Consider a central composite design for k = 4 variables in two blocks. Can a rotatable design always be found that blocks orthogonally?

To run a central composite design in two blocks, assign the n_f factorial points and the n_{01} center points to block 1 and the 2^k axial points plus n_{02} center points to block 2. Both blocks will be orthogonal first order designs, so the first condition for orthogonal blocking is satisfied.

The second condition implies that

$$\frac{\sum_m x_{im}^2 (block1)}{\sum_m x_{im}^2 (block2)} = \frac{n_f + n_{c1}}{2k + n_{c2}}$$

However, $\sum_m x_{im}^2 = n_f$ in block 1 and $\sum_m x_{im}^2 = 2\alpha^2$ in block 2, so

$$\frac{n_f}{2\alpha^2} = \frac{n_f + n_{c1}}{2k + n_{c2}}$$

Which gives:

$$\alpha = \left[\frac{n_f (2k + n_{c2})}{2(n_f + n_{c1})}\right]^{\frac{1}{2}}$$

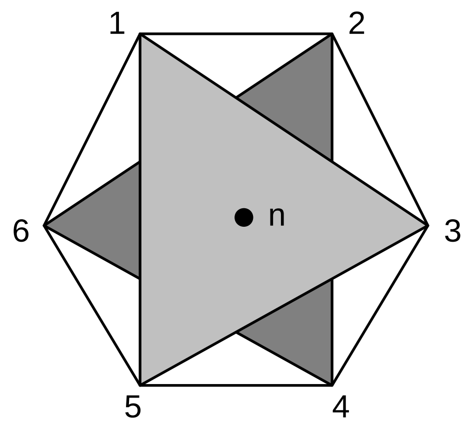
Since $\alpha = \sqrt[4]{n_f}$ if the design is to be rotatable, then the design must satisfy

$$n_f = \left[\frac{n_f (2k + n_{c2})}{2(n_f + n_{c1})}\right]^2$$

It is not possible to find rotatable central composite designs which block orthogonally for all k. For example, if k=3, the above condition cannot be satisfied. For k=2, there must be an equal number of center points in each block, i.e. $n_{c1} = n_{c2}$. For k=4, we must have $n_{c1} = 4$ and $n_{c2} = 2$.

11-20 How could a hexagon design be run in two orthogonal blocks?

The hexagonal design can be blocked as shown below. There are $n_{c1} = n_{c2} = n_c$ center points with n_c even.



Put the points 1,3,and 5 in block 1 and 2,4,and 6 in block 2. Note that each block is a simplex.

11-21 Yield during the first four cycles of a chemical process is shown in the following table. The variables are percent concentration (x_1) at levels 30, 31, and 32 and temperature (x_2) at 140, 142, and 144°F. Analyze by EVOP methods.

	Conditions				
Cycle	(1)	(2)	(3)	(4)	(5)
1	60.7	59.8	60.2	64.2	57.5
2	59.1	62.8	62.5	64.6	58.3
3	56.6	59.1	59.0	62.3	61.1
4	60.5	59.8	64.5	61.0	60.1

Cycle: n=1 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum						Previous Sum S=
(ii)	Previous Cycle Average						Previous Average =
(iii)	New Observation	60.7	59.8	60.2	64.2	57.5	New S=Range x $f_{k,n}$
(iv)	Differences						Range=
(v)	New Sums	60.7	59.8	60.2	64.2	57.5	New Sum S=
(vi)	New Averages	60.7	59.8	60.2	64.2	57.5	New average S = New Sum S/(n-1)=

Calculation of Effects		Calculation of Error Limits
$A = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 - \bar{y}_2 - \bar{y}_5\right) =$	3.55	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S =$
$B = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 - \bar{y}_2 + \bar{y}_5\right) =$	-3.55	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S =$

$AB = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 + \bar{y}_2 - \bar{y}_5\right) =$	-0.85	For CIM:	$\left(\frac{1.78}{\sqrt{n}}\right)S =$	
$CIM = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 + \bar{y}_2 + \bar{y}_5 - 4\bar{y}_1\right) =$	-0.22			

Cycle: n=2 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum	60.7	59.8	60.2	64.2	57.5	Previous Sum S=
(ii)	Previous Cycle Average	60.7	59.8	60.2	64.2	57.5	Previous Average =
(iii)	New Observation	59.1	62.8	62.5	64.6	58.3	New S=Range x $f_{k,n}$=1.38
(iv)	Differences	1.6	-3.0	-2.3	-0.4	-0.8	Range=4.6
(v)	New Sums	119.8	122.6	122.7	128.8	115.8	New Sum S=1.38
(vi)	New Averages	59.90	61.30	61.35	64.40	57.90	New average S = New Sum S/(n-1)=1.38

Calculation of Effects		Calculation of Error Limits		
$A = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 - \bar{y}_2 - \bar{y}_5\right) =$	3.28	For New Average:	$\left(\frac{2}{\sqrt{n}}\right)S =$	1.95
$B = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 - \bar{y}_2 + \bar{y}_5\right) =$	-3.23	For New Effects:	$\left(\frac{2}{\sqrt{n}}\right)S =$	1.95
$AB = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 + \bar{y}_2 - \bar{y}_5\right) =$	0.18	For CIM:	$\left(\frac{1.78}{\sqrt{n}}\right)S =$	1.74
$CIM = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 + \bar{y}_2 + \bar{y}_5 - 4\bar{y}_1\right) =$	1.07			

Cycle: n=3 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	
(i)	Previous Cycle Sum	119.8	122.6	122.7	128.8	115.8	Previous Sum S=1.38
(ii)	Previous Cycle Average	59.90	61.30	61.35	64.40	57.90	Previous Average =1.38
(iii)	New Observation	56.6	59.1	59.0	62.3	61.1	New S=Range x $f_{k,n}$=2.28
(iv)	Differences	3.30	2.20	2.35	2.10	-3.20	Range=6.5
(v)	New Sums	176.4	181.7	181.7	191.1	176.9	New Sum S=3.66
(vi)	New Averages	58.80	60.57	60.57	63.70	58.97	New average S = New Sum S/(n-1)=1.38

Calculation of Effects		Calculation of Error Limits		
$A = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 - \bar{y}_2 - \bar{y}_5\right) =$	2.37	For New Average:	$\left(\frac{2}{\sqrt{n}}\right)S =$	2.11
$B = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 - \bar{y}_2 + \bar{y}_5\right) =$	-2.37	For New Effects:	$\left(\frac{2}{\sqrt{n}}\right)S =$	2.11
$AB = \frac{1}{2}\left(\bar{y}_3 - \bar{y}_4 + \bar{y}_2 - \bar{y}_5\right) =$	-0.77	For CIM:	$\left(\frac{1.78}{\sqrt{n}}\right)S =$	1.74
$CIM = \frac{1}{2}\left(\bar{y}_3 + \bar{y}_4 + \bar{y}_2 + \bar{y}_5 - 4\bar{y}_1\right) =$	1.72			

Cycle: n=4 Phase 1

Calculation of Averages							Calculation of Standard Deviation
Operating Conditions		(1)	(2)	(3)	(4)	(5)	

(i)	Previous Cycle Sum	176.4	181.7	181.7	191.1	176.9	Previous Sum S=3.66
(ii)	Previous Cycle Average	58.80	60.57	60.57	63.70	58.97	Previous Average =1.83
(iii)	New Observation	60.5	59.8	64.5	61.0	60.1	New S=Range x $f_{k,n}$=2.45
(iv)	Differences	-1.70	0.77	-3.93	2.70	-1.13	Range=6.63
(v)	New Sums	236.9	241.5	245.2	252.1	237.0	New Sum S=6.11
(vi)	New Averages	59.23	60.38	61.55	63.03	59.25	New average S = New Sum S/(n-1)=2.04

Calculation of Effects		Calculation of Error Limits	
$A=\frac{1}{2}(\bar{y}_3+\bar{y}_4-\bar{y}_2-\bar{y}_5)=$	2.48	For New Average: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.04
$B=\frac{1}{2}(\bar{y}_3-\bar{y}_4-\bar{y}_2+\bar{y}_5)=$	-1.31	For New Effects: $\left(\frac{2}{\sqrt{n}}\right)S=$	2.04
$AB=\frac{1}{2}(\bar{y}_3-\bar{y}_4+\bar{y}_2-\bar{y}_5)=$	-0.18	For CIM: $\left(\frac{1.78}{\sqrt{n}}\right)S=$	1.82
$CIM=\frac{1}{2}(\bar{y}_3+\bar{y}_4+\bar{y}_2+\bar{y}_5-4\bar{y}_1)=$	1.46		

From studying cycles 3 and 4, it is apparent that *A* (and possibly *B*) has a significant effect. A new phase should be started following cycle 3 or 4.

11-22 Suppose that we approximate a response surface with a model of order d_1, such as $\mathbf{y}=\mathbf{X}_1\boldsymbol{\beta}_1+\boldsymbol{\varepsilon}$, when the true surface is described by a model of order $d_2>d_1$; that is $E(\mathbf{y})=\mathbf{X}_1\boldsymbol{\beta}_1+\mathbf{X}_2\boldsymbol{\beta}_2$.

(a) Show that the regression coefficients are biased, that is, that $E(\hat{\boldsymbol{\beta}}_1)=\boldsymbol{\beta}_1+\mathbf{A}\boldsymbol{\beta}_2$, where $\mathbf{A}=(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2$. **A** is usually called the alias matrix.

$$
\begin{aligned}
E\left[\hat{\boldsymbol{\beta}}_1\right] &= E\left[\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{y}\right]\\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'E[\mathbf{y}]\\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'(\mathbf{X}_1\boldsymbol{\beta}_1+\mathbf{X}_2\boldsymbol{\beta}_2)\\
&= \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_1\boldsymbol{\beta}_1+\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2\\
&= \boldsymbol{\beta}_1+\mathbf{A}\boldsymbol{\beta}_2
\end{aligned}
$$

where $\mathbf{A}=\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2$

(a) If d_1=1 and d_2=2, and a full 2^k is used to fit the model, use the result in part (a) to determine the alias structure.

In this situation, we have assumed the true surface to be first order, when it is really second order. If a full factorial is used for *k*=2, then

$$
\mathbf{X}_1=\begin{array}{c}\begin{matrix}\beta_0 & \beta_1 & \beta_2\end{matrix}\\ \begin{bmatrix}1 & -1 & -1\\ 1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\end{array}\quad
\mathbf{X}_2=\begin{array}{c}\begin{matrix}\beta_{11} & \beta_{22} & \beta_{12}\end{matrix}\\ \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\end{array}\quad\text{and}\quad
\mathbf{A}=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}
$$

$$\text{Then, } \mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right] = \mathbf{E}\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_{11} + \beta_{22} \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

The pure quadratic terms bias the intercept.

(b) If d_1=1, d_2=2 and k=3, find the alias structure assuming that a 2^{3-1} design is used to fit the model.

$$\mathbf{X}_1 = \begin{array}{c} \begin{array}{cccc} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \mathbf{X}_2 = \begin{array}{c} \begin{array}{cccccc} \beta_{11} & \beta_{22} & \beta_{33} & \beta_{12} & \beta_{13} & \beta_{23} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{Then, } \mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right] = \mathbf{E}\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{12} \\ \beta_{13} \\ \beta_{23} \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_{11} + \beta_{22} + \beta_{22} \\ \beta_1 + \beta_{23} \\ \beta_2 + \beta_{13} \\ \beta_3 + \beta_{12} \end{bmatrix}$$

(d) If d_1=1, d_2=2, k=3, and the simplex design in Problem 11-3 is used to fit the model, determine the alias structure and compare the results with part (c).

$$\mathbf{X}_1 = \begin{array}{c} \begin{array}{cccc} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \mathbf{X}_2 = \begin{array}{c} \begin{array}{cccccc} \beta_{11} & \beta_{22} & \beta_{33} & \beta_{12} & \beta_{13} & \beta_{23} \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & -\sqrt{2} \\ 2 & 0 & 1 & 0 & -\sqrt{2} & 0 \\ 0 & 2 & 1 & 0 & 0 & -\sqrt{2} \\ 2 & 0 & 1 & 0 & -\sqrt{2} & 0 \end{bmatrix} \end{array} \quad \text{and } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Then, } \mathbf{E}\left[\hat{\boldsymbol{\beta}}_1\right] = \mathbf{E}\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{12} \\ \beta_{13} \\ \beta_{23} \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_{11} + \beta_{22} + \beta_{22} \\ \beta_1 + \beta_{13} \\ \beta_2 - \beta_{23} \\ \beta_3 + \beta_{11} - \beta_{22} \end{bmatrix}$$

Notice that the alias structure is different from that found in the previous part for the 2^{3-1} design. In general, the **A** matrix will depend on which simplex design is used.

11-23 Suppose that you need to design an experiment to fit a quadratic model over the region $-1 \le x_i \le +1$, i=1,2 subject to the constraint $x_1 + x_2 \le 1$. If the constraint is violated, the process will not work properly. You can afford to make no more than n=12 runs. Set up the following designs:

(a) An "inscribed" CCD with center points at $x_1 = x_2 = 0$

x_1	x_2
-0.5	-0.5
0.5	-0.5
-0.5	0.5
0.5	0.5
-0.707	0
0.707	0
0	-0.707
0	0.707
0	0
0	0
0	0
0	0

(a)* An "inscribed" CCD with center points at $x_1 = x_2 = -0.25$ so that a larger design could be fit within the constrained region

x_1	x_2
-1	-1
0.5	-1
-1	0.5
0.5	0.5
-1.664	-0.25
1.164	-0.25
-0.25	-1.664
-0.25	1.164
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25

(a) An "inscribed" 3^2 factorial with center points at $x_1 = x_2 - 0.25$

x_1	x_2
-1	-1
-0.25	-1
0.5	-1
-1	-0.25
-0.25	-0.25
0.5	-0.25
-1	0.5
-0.25	0.5
0.5	0.5
-0.25	-0.25
-0.25	-0.25
-0.25	-0.25

(a) A D-optimal design.

x_1	x_2
-1	-1
1	-1
-1	1
1	0
0	1
0	0
-1	0
0	-1
0.5	0.5
-1	-1
1	-1
-1	1

(a) A modified D-optimal design that is identical to the one in part (c), but with all replicate runs at the design center.

x_1	x_2
1	0
0	0
0	1
-1	-1
1	-1
-1	1
-1	0
0	-1
0.5	0.5
0	0
0	0
0	0

(a) Evaluate the $\left|(\mathbf{X}'\mathbf{X})^{-1}\right|$ criteria for each design.

	(a)	(a)*	(b)	(c)	(d)
$\left|(\mathbf{X}'\mathbf{X})^{-1}\right|$	0.5	0.00005248	0.007217	0.0001016	0.0002294

(a) Evaluate the D-efficiency for each design relative to the D-optimal design in part (c).

	(a)	(a)*	(b)	(c)	(d)
D-efficiency	24.25%	111.64%	49.14%	100.00%	87.31%

(a) Which design would you prefer? Why?

The offset CCD, (a)*, is the preferred design based on the D-efficiency. Not only is it better than the D-optimal design, (c), but it maintains the desirable design features of the CCD.

11-24 Consider a 2^3 design for fitting a first-order model.

(a) Evaluate the D-criterion $\left|(\mathbf{X}'\mathbf{X})^{-1}\right|$ for this design.

$$\left|(\mathbf{X}'\mathbf{X})^{-1}\right| = 2.441\text{E-}4$$

(b) Evaluate the A-criterion $tr(\mathbf{X}'\mathbf{X})^{-1}$ for this design.

$$tr(\mathbf{X}'\mathbf{X})^{-1} = 0.5$$

(c) Find the maximum scaled prediction variance for this design. Is this design G-optimal?

$$v(\mathbf{x}) = \frac{NVar(\hat{y}(\mathbf{x}))}{\sigma^2} = N\mathbf{x}'^{(1)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}^{(1)} = 4$$. Yes, this is a G-optimal design.

11-25 Repeat Problem 11-24 using a first order model with the two-factor interaction.

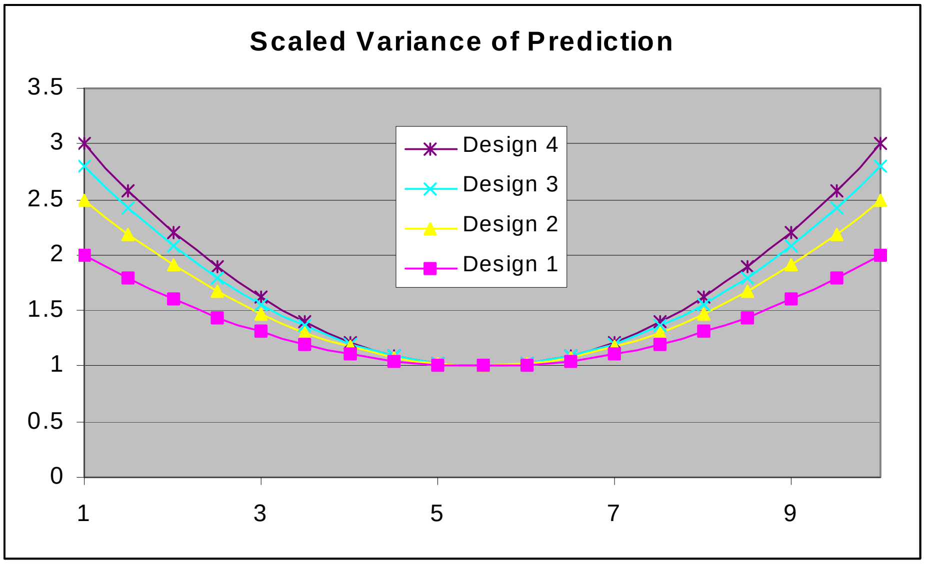
$$\left|(\mathbf{X}'\mathbf{X})^{-1}\right| = 4.768\text{E-}7$$

$$tr(\mathbf{X}'\mathbf{X})^{-1} = 0.875$$

$$v(\mathbf{x}) = \frac{NVar(\hat{y}(\mathbf{x}))}{\sigma^2} = N\mathbf{x}'^{(1)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}^{(1)} = 7$$. Yes, this is a G-optimal design.

11-26 A chemical engineer wishes to fit a calibration curve for a new procedure used to measure the concentration of a particular ingredient in a product manufactured in his facility. Twelve samples can be prepared, having known concentration. The engineer's interest is in building a model for the measured concentrations. He suspects that a linear calibration curve will be adequate to model the measured concentration as a function of the known concentrations; that is, where x is the actual concentration. Four experimental designs are under consideration. Design 1 consists of 6 runs at known concentration 1 and 6 runs at known concentration 10. Design 2 consists of 4 runs at concentrations 1, 5.5, and 10. Design 3 consists of 3 runs at concentrations 1, 4, 7, and 10. Finally, design 4 consists of 3 runs at concentrations 1 and 10 and 6 runs at concentration 5.5.

(a) Plot the scaled variance of prediction for all four designs on the same graph over the concentration range. Which design would be preferable, in your opinion?



Because it has the lowest scaled variance of prediction at all points in the design space with the exception of 5.5, Design 1 is preferred.

(b) For each design calculate the determinant of $(\mathbf{X}'\mathbf{X})^{-1}$. Which design would be preferred according to the "D" criterion?

Design	$\left\|(\mathbf{X}'\mathbf{X})^{-1}\right\|$
1	0.000343
2	0.000514
3	0.000617
4	0.000686

Design 1 would be preferred.

(c) Calculate the D-efficiency of each design relative to the "best" design that you found in part b.

Design	D-efficiency
1	100.00%
2	81.65%
3	74.55%
4	70.71%

(a) For each design, calculate the average variance of prediction over the set of points given by x = 1, 1.5, 2, 2.5, . . ., 10. Which design would you prefer according to the V-criterion?

Average Variance of Prediction		
Design	Actual	Coded
1	1.3704	0.1142
2	1.5556	0.1296
3	1.6664	0.1389
4	1.7407	0.1451

Design 1 is still preferred based on the V-criterion.

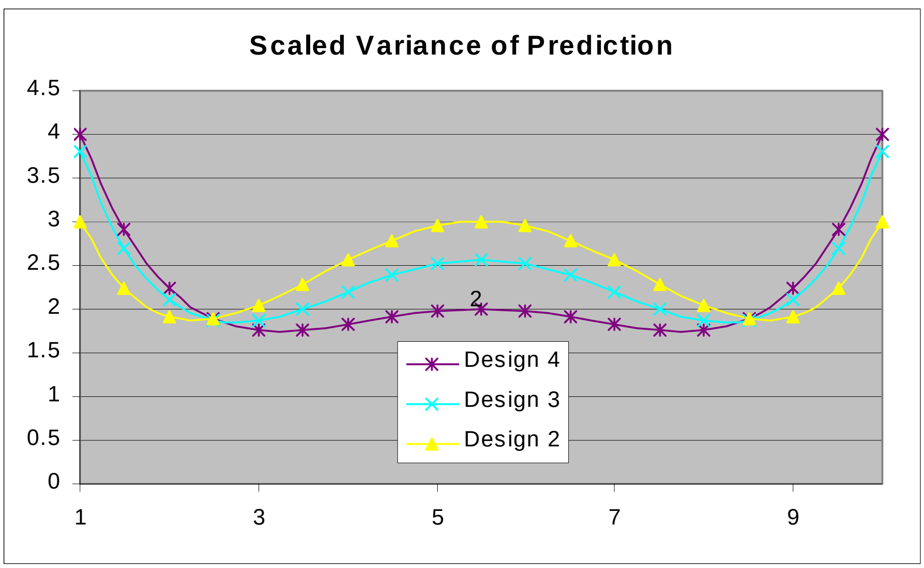
(e) Calculate the V-efficiency of each design relative to the best design you found in part (d).

Design	V-efficiency
1	100.00%
2	88.10%
3	82.24%
4	78.72%

(f) What is the G-efficiency of each design?

Design	G-efficiency
1	100.00%
2	80.00%
3	71.40%
4	66.70%

11-27 Rework Problem 11-26 assuming that the model the engineer wishes to fit is a quadratic. Obviously, only designs 2, 3, and 4 can now be considered.



Based on the plot, the preferred design would depend on the region of interest. Design 4 would be preferred if the center of the region was of interest; otherwise, Design 2 would be preferred.

Design	$\left\|(\mathbf{X}'\mathbf{X})^{-1}\right\|$
2	4.704E-07
3	6.351E-07
4	5.575E-07

Design 2 is preferred based on $\left|(\mathbf{X}'\mathbf{X})^{-1}\right|$.

Design	D-efficiency
2	100.00%

3	90.46%
4	94.49%

Average Variance of Prediction

Design	Actual	Coded
2	2.441	0.2034
3	2.393	0.1994
4	2.242	0.1869

Design 4 is preferred.

Design	V-efficiency
2	91.89%
3	93.74%
4	100.00%

Design	G-efficiency
2	100.00%
3	79.00%
4	75.00%

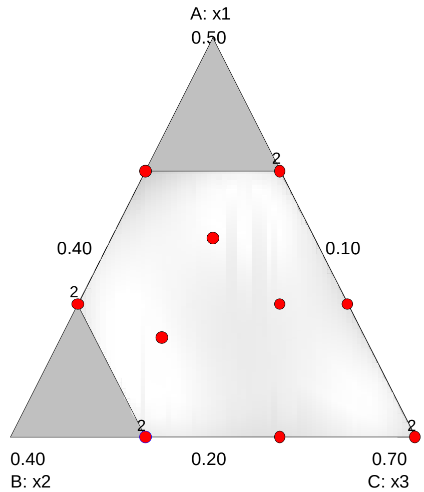
11-28 An experimenter wishes to run a three-component mixture experiment. The constraints are the components proportions are as follows:

$$0.2 \le x_1 \le 0.4$$
$$0.1 \le x_2 \le 0.3$$
$$0.4 \le x_3 \le 0.7$$

(a) Set up an experiment to fit a quadratic mixture model. Use n=14 runs, with 4 replicates. Use the D-criteria.

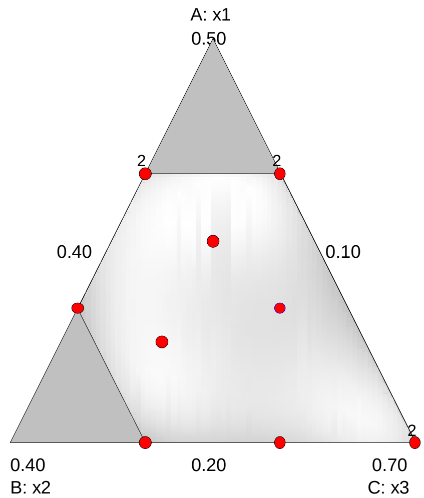
Std	x1	x2	x3
1	0.2	0.3	0.5
2	0.3	0.3	0.4
3	0.3	0.15	0.55
4	0.2	0.1	0.7
5	0.4	0.2	0.4
6	0.4	0.1	0.5
7	0.2	0.2	0.6
8	0.275	0.25	0.475
9	0.35	0.175	0.475
10	0.3	0.1	0.6
11	0.2	0.3	0.5
12	0.3	0.3	0.4
13	0.2	0.1	0.7
14	0.4	0.1	0.5

(a) Draw the experimental design region.



(c) Set up an experiment to fit a quadratic mixture model with n=12 runs, assuming that three of these runs are replicated. Use the D-criterion.

Std	x_1	x_2	x_3
1	0.3	0.15	0.55
2	0.2	0.3	0.5
3	0.3	0.3	0.4
4	0.2	0.1	0.7
5	0.4	0.2	0.4
6	0.4	0.1	0.5
7	0.2	0.2	0.6
8	0.275	0.25	0.475
9	0.35	0.175	0.475
10	0.2	0.1	0.7
11	0.4	0.1	0.5
12	0.4	0.2	0.4



(d) Comment on the two designs you have found.

The design points are the same for both designs except that the edge center on the x1-x3 edge is not included in the second design. None of the replicates for either design are in the center of the experimental region. The experimental runs are fairly uniformly spaced in the design region.

11-29 Myers and Montgomery (2002) describe a gasoline blending experiment involving three mixture components. There are no constraints on the mixture proportions, and the following 10 run design is used.

Design Point	x_1	x_2	x_3	y(mpg)
1	1	0	0	24.5, 25.1
2	0	1	0	24.8, 23.9
3	0	0	1	22.7, 23.6
4	½	½	0	25.1
5	½	0	½	24.3
6	0	½	½	23.5
7	1/3	1/3	1/3	24.8, 24.1
8	2/3	1/6	1/6	24.2
9	1/6	2/3	1/6	23.9
10	1/6	1/6	2/3	23.7

(a) What type of design did the experimenters use?

A simplex centroid design was used.

(b) Fit a quadratic mixture model to the data. Is this model adequate?

Design Expert Output

Response: y
ANOVA for Mixture Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	4.22	5	0.84	3.90	0.0435	significant
Linear Mixture	*3.92*	*2*	*1.96*	*9.06*	*0.0088*	

AB	*0.15*	*1*	*0.15*	*0.69*	*0.4289*	
AC	*0.081*	*1*	*0.081*	*0.38*	*0.5569*	
BC	*0.077*	*1*	*0.077*	*0.36*	*0.5664*	
Residual	1.73	8	0.22			
Lack of Fit	*0.50*	*4*	*0.12*	*0.40*	*0.8003*	*not significant*
Pure Error	*1.24*	*4*	*0.31*			
Cor Total	5.95	13				

The Model F-value of 3.90 implies the model is significant. There is only
a 4.35% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.47	R-Squared	0.7091
Mean	24.16	Adj R-Squared	0.5274
C.V.	1.93	Pred R-Squared	0.1144
PRESS	5.27	Adeq Precision	5.674

Component	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**
A-x1	24.74	1	0.32	24.00	25.49
B-x2	24.31	1	0.32	23.57	25.05
C-x3	23.18	1	0.32	22.43	23.92
AB	1.51	1	1.82	-2.68	5.70
AC	1.11	1	1.82	-3.08	5.30
BC	-1.09	1	1.82	-5.28	3.10

Final Equation in Terms of Pseudo Components:

```
         y  =
    +24.74  * A
    +24.31  * B
    +23.18  * C
     +1.51  * A * B
     +1.11  * A * C
     -1.09  * B * C
```

Final Equation in Terms of Real Components:

```
            y  =
    +24.74432  * x1
    +24.31098  * x2
    +23.17765  * x3
     +1.51364  * x1 * x2
     +1.11364  * x1 * x3
     -1.08636  * x2 * x3
```

The quadratic terms appear to be insignificant. The analysis below is for the linear mixture model:

Design Expert Output

Response: **y**

ANOVA for Mixture Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	3.92	2	1.96	10.64	0.0027	significant
Linear Mixture	*3.92*	*2*	*1.96*	*10.64*	*0.0027*	
Residual	2.03	11	0.18			
Lack of Fit	*0.79*	*7*	*0.11*	*0.37*	*0.8825*	*not significant*
Pure Error	*1.24*	*4*	*0.31*			
Cor Total	5.95	13				

The Model F-value of 10.64 implies the model is significant. There is only
a 0.27% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.43	R-Squared	0.6591
Mean	24.16	Adj R-Squared	0.5972
C.V.	1.78	Pred R-Squared	0.3926
PRESS	3.62	Adeq Precision	8.751

Component	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High
A-x1	24.93	1	0.25	24.38	25.48
B-x2	24.35	1	0.25	23.80	24.90
C-x3	23.19	1	0.25	22.64	23.74

Component	Effect	DF	Adjusted Std Error	Adjusted Approx t for H0 Effect=0	Prob > |t|
A-x1	1.16	1	0.33	3.49	0.0051
B-x2	0.29	1	0.33	0.87	0.4021
C-x3	-1.45	1	0.33	-4.36	0.0011

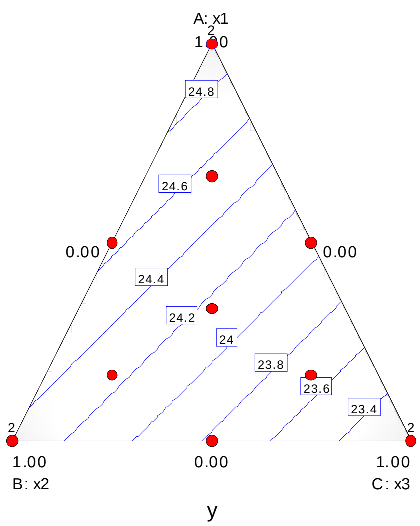
Final Equation in Terms of Pseudo Components:

$$y = +24.93 * A + 24.35 * B + 23.19 * C$$

Final Equation in Terms of Real Components:

$$y = +24.93048 * x1 + 24.35048 * x2 + 23.19048 * x3$$

(c) Plot the response surface contours. What blend would you recommend to maximize the MPG?



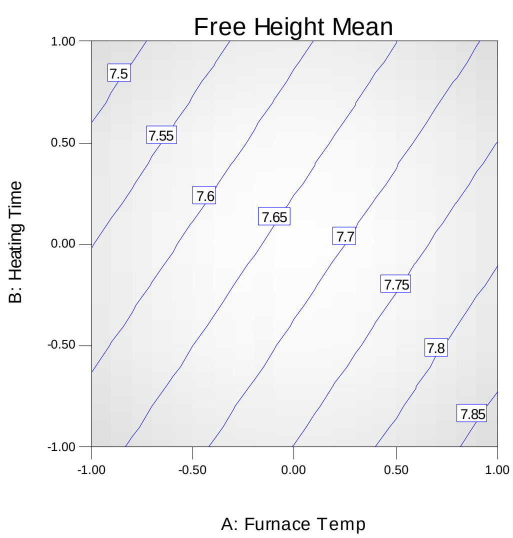
To maximize the miles per gallon, the recommended blend is $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$.

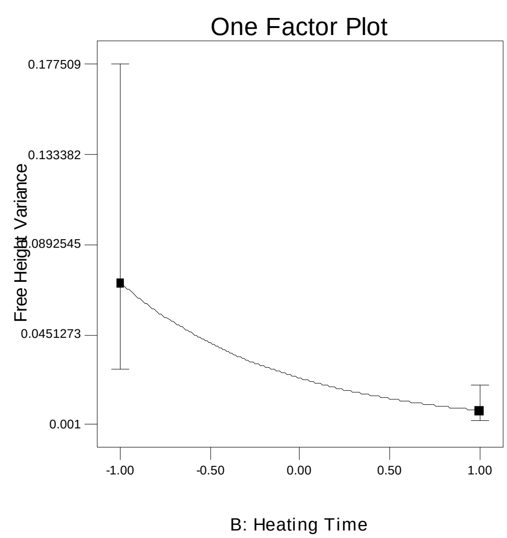
Chapter 12
Robust Parameter Design and Process Robustness Studies
Solutions

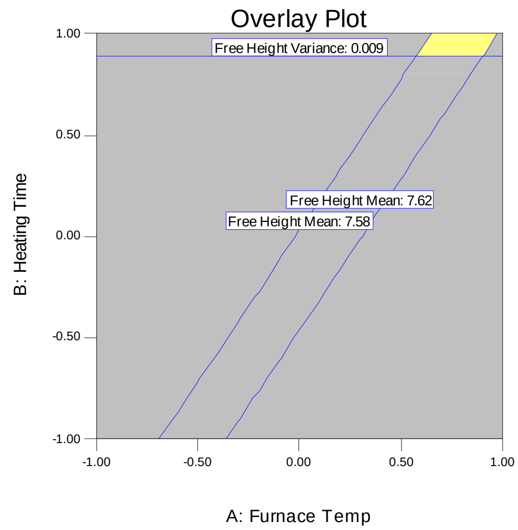
12-1 Reconsider the leaf spring experiment in Table 12-1. Suppose that the objective is to find a set of conditions where the mean free height is as close as possible to 7.6 inches with a variance of free height as small as possible. What conditions would you recommend to achieve these objectives?

A	*B*	*C*	*D*	*E*(-)	*E*(+)	$\bar{y}$	s^2
-	-	-	-	7.78,7.78, 7.81	7.50, 7,25, 7.12	7.54	0.090
+	-	-	+	8.15, 8.18, 7.88	7.88, 7.88, 7.44	7.90	0.071
-	+	-	+	7.50, 7.56, 7.50	7.50, 7.56, 7.50	7.52	0.001
+	+	-	-	7.59, 7.56, 7.75	7.63, 7.75, 7.56	7.64	0.008
-	-	+	+	7.54, 8.00, 7.88	7.32, 7.44, 7.44	7.60	0.074
+	-	+	-	7.69, 8.09, 8.06	7.56, 7.69, 7.62	7.79	0.053
-	+	+	-	7.56, 7.52, 7.44	7.18, 7.18, 7.25	7.36	0.030
+	+	+	+	7.56, 7.81, 7.69	7.81, 7.50, 7.59	7.66	0.017

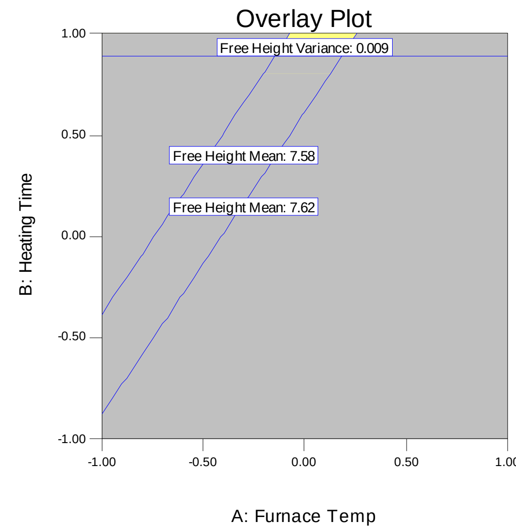
By overlaying the contour plots for Free Height Mean and the Free Height Variance, optimal solutions can be found. To minimize the variance, factor B must be at the high level while factors *A* and *D* are adjusted to assure a mean of 7.6. The two overlay plots below set factor *D* at both low and high levels. Therefore, a mean as close as possible to 7.6 with minimum variance of 0.008 can be achieved at *A* = 0.78, *B* = +1, and *D* = -1. This can also be achieved with *A* = +0.07, *B* = +1, and *D* = +1.







Factor *D* = -1



Factor D = +1

12-2 Consider the bottle filling experiment in Problem 6-18. Suppose that the percentage of carbonation (*A*) is a noise variable ($\sigma_z^2 = 1$ in coded units).

(a) Fit the response model to these data. Is there a robust design problem?

The following is the analysis of variance with all terms in the model followed by a reduced model. Because the noise factor *A* is significant, and the AB interaction is moderately significant, there is a robust design problem.

Design Expert Output

Response: Fill Deviation
ANOVA for Response Surface Reduced Cubic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Cor Total	300.05	3				
Model	73.00	7	10.43	16.69	0.0003	significant
A	*36.00*	*1*	*36.00*	*57.60*	*< 0.0001*	
B	*20.25*	*1*	*20.25*	*32.40*	*0.0005*	
C	*12.25*	*1*	*12.25*	*19.60*	*0.0022*	
AB	*2.25*	*1*	*2.25*	*3.60*	*0.0943*	
AC	*0.25*	*1*	*0.25*	*0.40*	*0.5447*	
BC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*	
ABC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*	
Pure Error	5.00	8	0.63			
Cor Total	78.00	15				

Based on the above analysis, the AB interaction is removed from the model and used as error.

Design Expert Output

Response: Fill Deviation
ANOVA for Response Surface Reduced Cubic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	70.75	4	17.69	26.84	< 0.0001	significant
A	*36.00*	*1*	*36.00*	*54.62*	*< 0.0001*	
B	*20.25*	*1*	*20.25*	*30.72*	*0.0002*	
C	*12.25*	*1*	*12.25*	*18.59*	*0.0012*	

AB	*2.25*	*1*	*2.25*	*3.41*	*0.0917*	
Residual	7.25	11	0.66			
Lack of Fit	2.25	3	0.75	1.20	0.3700	not significant
Pure Error	5.00	8	0.63			
Cor Total	78.00	15				

The Model F-value of 26.84 implies there is a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.81	R-Squared	0.9071
Mean	1.00	Adj R-Squared	0.8733
C.V.	81.18	Pred R-Squared	0.8033
PRESS	15.34	Adeq Precision	15.424

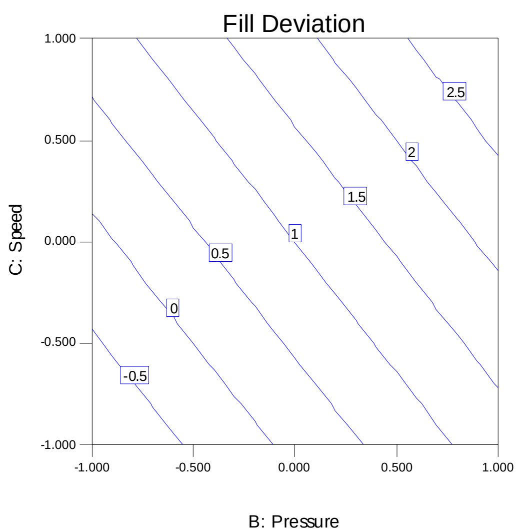
Final Equation in Terms of Coded Factors:

Fill Deviation =
+1.00
+1.50 * A
+1.13 * B
+0.88 * C
+0.38 * A * B

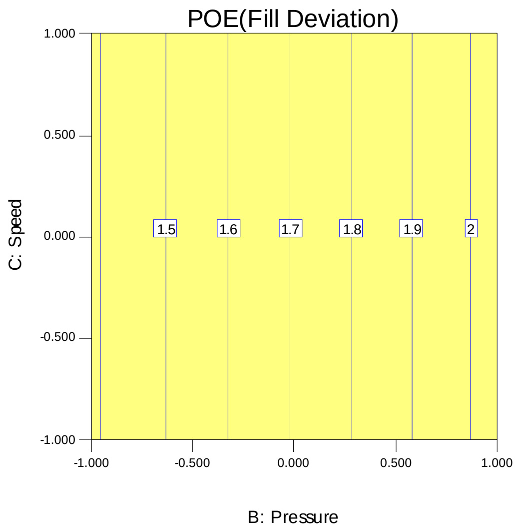
(b) Find the mean model and either the variance model or the POE.

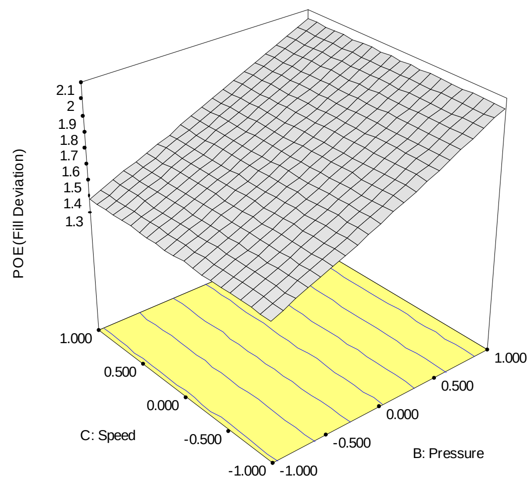
From the final equation shown in the above analysis, the mean model and corresponding contour plot is shown below.

$$E_z\left[y\left(\mathbf{x},z_1\right)\right]=1+1.13x_2+0.88x_3$$



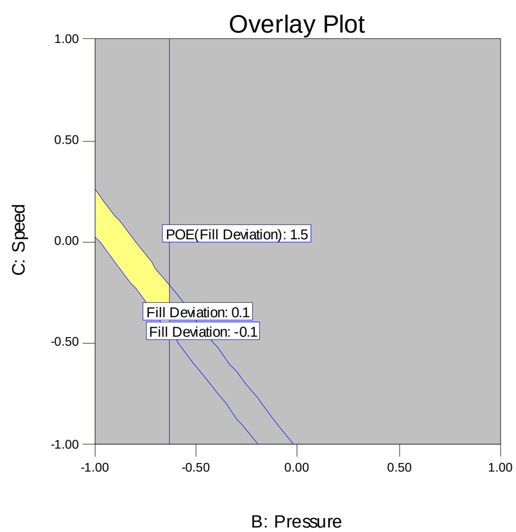
Contour and 3-D plots of the POE are shown below.





(c) Find a set of conditions that result in mean fill deviation as close to zero as possible with minimum transmitted variance.

The overlay plot below identifies a an operating region for pressure and speed that in a mean fill deviation as close to zero as possible with minimum transmitted variance.



12-3 Consider the experiment in Problem 11-12. Suppose that temperature is a noise variable ($\sigma_z^2 = 1$ in coded units). Fit response models for both responses. Is there a robust design problem with respect to both responses? Find a set of conditions that maximize conversion with activity between 55 and 60 and that minimize variability transmitted from temperature.

The analysis and models as found in problem 11-12 are shown below for both responses. There is a robust design problem with regards to the conversion response because of the significance of factor *B*, temperature, and the *BC* interaction. However, temperature is not significant in the analysis of the second response, activity.

Design Expert Output

Response: Conversion
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	2555.73	9	283.97	12.76	0.0002	significant
A	*14.44*	*1*	*14.44*	*0.65*	*0.4391*	
B	*222.96*	*1*	*222.96*	*10.02*	*0.0101*	
C	*525.64*	*1*	*525.64*	*23.63*	*0.0007*	
A^2	*48.47*	*1*	*48.47*	*2.18*	*0.1707*	
B^2	*124.48*	*1*	*124.48*	*5.60*	*0.0396*	
C^2	*388.59*	*1*	*388.59*	*17.47*	*0.0019*	
AB	*36.13*	*1*	*36.13*	*1.62*	*0.2314*	
AC	*1035.13*	*1*	*1035.13*	*46.53*	*< 0.0001*	
BC	*120.12*	*1*	*120.12*	*5.40*	*0.0425*	
Residual	222.47	10	22.25			
Lack of Fit	*56.47*	*5*	*11.29*	*0.34*	*0.8692*	*not significant*
Pure Error	*166.00*	*5*	*33.20*			
Cor Total	287.28	19				

The Model F-value of 12.76 implies the model is significant. There is only a 0.02% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	4.72	R-Squared	0.9199
Mean	78.30	Adj R-Squared	0.8479
C.V.	6.02	Pred R-Squared	0.7566
PRESS	676.22	Adeq Precision	14.239

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	81.09	1	1.92	76.81	85.38	
A-Time	1.03	1	1.28	-1.82	3.87	1.00
B-Temperature	4.04	1	1.28	1.20	6.88	1.00
C-Catalyst	6.20	1	1.28	3.36	9.05	1.00
A2	-1.83	1	1.24	-4.60	0.93	1.02
B2	2.94	1	1.24	0.17	5.71	1.02
C2	-5.19	1	1.24	-7.96	-2.42	1.02
AB	2.13	1	1.67	-1.59	5.84	1.00
AC	11.38	1	1.67	7.66	15.09	1.00
BC	-3.87	1	1.67	-7.59	-0.16	1.00

Final Equation in Terms of Coded Factors:

```
Conversion =
    +81.09
     +1.03  * A
     +4.04  * B
     +6.20  * C
     -1.83  * A2
     +2.94  * B2
     -5.19  * C2
     +2.13  * A * B
    +11.38  * A * C
     -3.87  * B * C
```

Design Expert Output

Response: Activity
ANOVA for Response Surface Quadratic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	253.20	3	84.40	39.63	< 0.0001	significant
A	*175.35*	*1*	*175.35*	*82.34*	*< 0.0001*	
C	*67.91*	*1*	*67.91*	*31.89*	*< 0.0001*	
A^2	*9.94*	*1*	*9.94*	*4.67*	*0.0463*	
Residual	34.07	16	2.13			
Lack of Fit	*30.42*	*11*	*2.77*	*3.78*	*0.0766*	*not significant*
Pure Error	*3.65*	*5*	*0.73*			

Cor Total 287.28 19

The Model F-value of 39.63 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

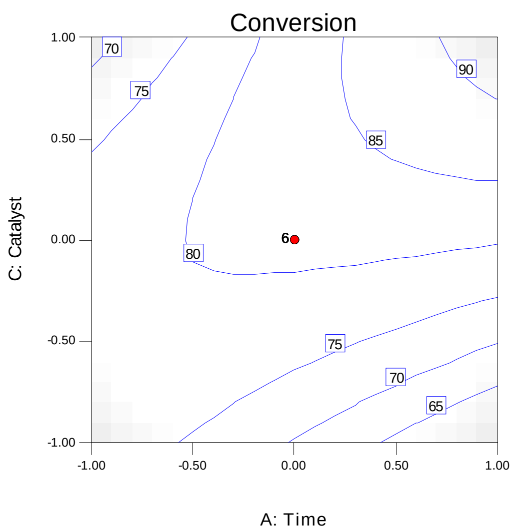
Std. Dev.	1.46	R-Squared	0.8814
Mean	60.51	Adj R-Squared	0.8591
C.V.	2.41	Pred R-Squared	0.6302
PRESS	106.24	Adeq Precision	20.447

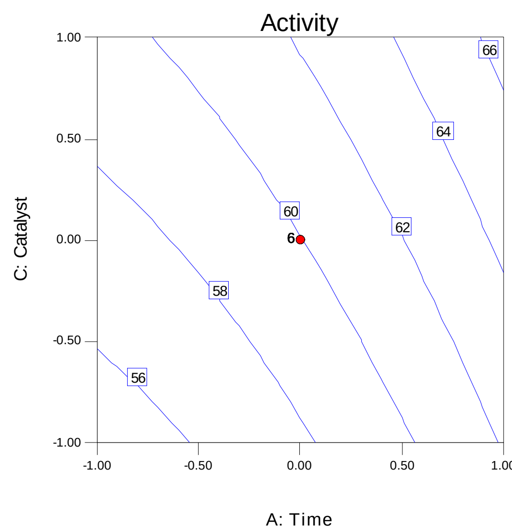
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	59.95	1	0.42	59.06	60.83	
A-Time	3.58	1	0.39	2.75	4.42	1.00
C-Catalyst	2.23	1	0.39	1.39	3.07	1.00
A^2	0.82	1	0.38	0.015	1.63	1.00

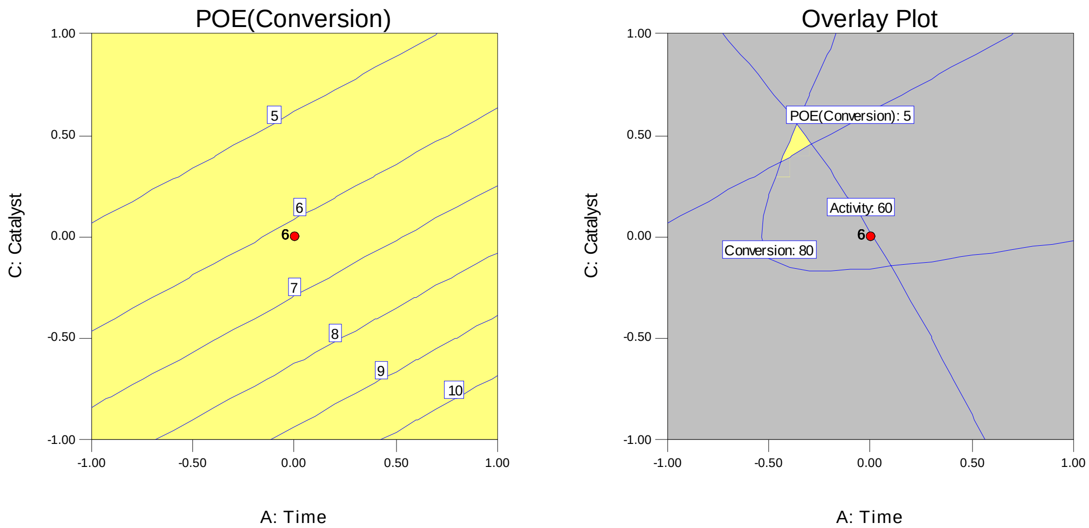
Final Equation in Terms of Coded Factors:

Activity =
+59.95
+3.58 * A
+2.23 * C
+0.82 * A2

The following contour plots of conversion, activity, and POE and the corresponding optimization plot identify a region where conversion is maximized, activity is between 55 and 60, and the transmitted variability from temperature is minimized. Factor *A* is set at 0.5 while *C* is set at 0.4.







12-4 Reconsider the leaf spring experiment from Table 12-1. Suppose that factors *A*, *B* and *C* are controllable variables, and that factors *D* and *E* are noise factors. Set up a crossed array design to investigate this problem, assuming that all of the two-factor interactions involving the controllable variables are thought to be important. What type of design have you obtained?

The following experimental design has a 2^3 inner array for the controllable variables and a 2^2 outer array for the noise factors. A total of 32 runs are required.

Inner Array			Outer Array				
			D	-1	1	-1	1
A	*B*	*C*	*E*	-1	-1	1	1
-1	-1	-1					
1	-1	-1					
-1	1	-1					
1	1	-1					
-1	-1	1					
1	-1	1					
-1	1	1					
1	1	1					

12-5 **Continuation of Problem 12-4.** Reconsider the leaf spring experiment from Table 12-1. Suppose that *A*, *B* and *C* are controllable factors and that factors *D* and *E* are noise factors. Show how a combined array design can be employed to investigate this problem that allows all two-factor interactions to be estimated and only require 16 runs. Compare this with the crossed array design from Problem 12-5. Can you see how in general combined array designs that have fewer runs than crossed array designs?

The following experiment is a 2^{5-1} fractional factorial experiment where the controllable factors are *A*, *B*, and *C* and the noise factors are *D* and *E*. Only 16 runs are required versus the 32 runs required for the crossed array design in problem 12-4.

A	*B*	*C*	*D*	*E*	Free Height
-	-	-	-	+	
+	-	-	-	-	
-	+	-	-	-	
+	+	-	-	+	
-	-	+	-	-	
+	-	+	-	+	
-	+	+	-	+	
+	+	+	-	-	
-	-	-	+	-	
+	-	-	+	+	
-	+	-	+	+	
+	+	-	+	-	
-	-	+	+	+	
+	-	+	+	-	
-	+	+	+	-	
+	+	+	+	+	

12-6 Consider the connector pull-off force experiment shown in Table 12-2. What main effects and interactions involving the controllable variables can be estimated with this design? Remember that all of the controllable variables are quantitative factors.

The design in Table 12-2 contains a 3^{4-2} inner array for the controllable variables. This is a resolution III design which aliases the main effects with two factor interactions. The alias table below identifies the alias structure for this design. Because of the partial aliasing in this design, it is difficult to interpret the interactions.

Design Expert Output

```
Alias Matrix

[Est. Terms]   Aliased Terms
 [Intercept]   = Intercept - BC - BD - CD
         [A]   = A - 0.5 * BC - 0.5 * BD - 0.5 * CD
         [B]   = B - 0.5 * AC - 0.5 * AD
         [C]   = C - 0.5 * AB - 0.5 * AD
         [D]   = D - 0.5 * AB - 0.5 * AC
        [A2]   = A2 + 0.5 * BC + 0.5 * BD + 0.5 * CD
        [B2]   = B2 + 0.5 * AC - 0.5 * AD + CD
        [C2]   = C2 - 0.5 * AB + 0.5 * AD + BD
        [D2]   = D2 + 0.5 * AB - 0.5 * AC + BC
```

12-7 Consider the connector pull-off force experiment shown in Table 12-2. Show how an experiment can be designed for this problem that will allow a full quadratic model to be fit in the controllable variables along all main effects of the noise variables and their interactions with the controllable variables. How many runs will be required in this design? How does this compare with the design in Table 12-2?

There are several designs that can be employed to achieve the requirements stated above. Below is a small central composite design with the axial points removed for the noise variables. Five center points are also included which brings the total runs to 35. As shown in the alias analysis, the full quadratic model for the controllable variables is achieved.

A	*B*	*C*	*D*	*E*	*F*	*G*
+1	+1	+1	-1	+1	+1	+1
+1	+1	-1	+1	-1	+1	-1
+1	+1	-1	+1	+1	-1	+1
+1	-1	+1	+1	-1	+1	+1
-1	+1	+1	-1	-1	+1	-1
+1	-1	-1	-1	+1	-1	-1
-1	+1	-1	+1	+1	-1	+1
+1	+1	+1	+1	-1	+1	-1
+1	-1	+1	-1	-1	-1	+1
-1	-1	-1	-1	+1	+1	-1
-1	+1	-1	+1	-1	-1	-1
+1	+1	+1	-1	+1	-1	-1
+1	-1	-1	+1	-1	-1	-1
-1	-1	+1	-1	-1	-1	+1
-1	+1	-1	-1	-1	+1	+1
+1	-1	-1	-1	-1	+1	+1
-1	+1	-1	-1	+1	+1	+1
+1	-1	-1	+1	+1	+1	+1
-1	-1	+1	+1	+1	-1	+1
-1	-1	+1	+1	+1	+1	-1
-1	+1	+1	+1	-1	-1	+1
-1	-1	-1	-1	-1	-1	-1
-2.17	0	0	0	0	0	0
2.17	0	0	0	0	0	0
0	-2.17	0	0	0	0	0
0	2.17	0	0	0	0	0
0	0	-2.17	0	0	0	0
0	0	2.17	0	0	0	0
0	0	0	-2.17	0	0	0
0	0	0	2.17	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Design Expert Output

Alias Matrix

[Est. Terms]	**Aliased Terms**
[Intercept]	= Intercept
[A]	= A
[B]	= B
[C]	= C
[D]	= D
[E]	= E + 0.211 * EG + 0.789 * FG
[F]	= F - EF - EG
[G]	= G - EF - 0.158 * EG + 0.158 * FG
[A^2]	= A^2
[B^2]	= B^2
[C^2]	= C^2
[D^2]	= D^2
[E^2]	= $E^2 + F^2 + G^2$
[AB]	= AB - 0.105 * EG - 0.895 * FG
[AC]	= AC - 0.158 * EG + 0.158 * FG
[AD]	= AD + 0.421 * EG + 0.579 * FG
[AE]	= AE - 0.474 * EG + 0.474 * FG
[AF]	= AF + EF + 1.05 * EG - 0.0526 * FG

```
[AG]    = AG + EF + 1.05 * EG - 0.0526 * FG
[BC]    = BC - 0.263 * EG + 0.263 * FG
[BD]    = BD - EF - 0.158 * EG + 0.158 * FG
[BE]    = BE - 0.368 * EG + 0.368 * FG
[BF]    = BF + 1.11 * EG - 0.105 * FG
[BG]    = BG + EF + 0.421 * EG - 0.421 * FG
[CD]    = CD - 0.421 * EG + 0.421 * FG
[CE]    = CE - EF + 0.158 * EG + 0.842 * FG
[CF]    = CF - EF - 0.211 * EG + 0.211 * FG
[CG]    = CG - 1.21 * EG + 0.211 * FG
[DE]    = DE - 0.842 * EG - 0.158 * FG
[DF]    = DF - 0.211 * EG + 0.211 * FG
[DG]    = DG - EF + 0.263 * EG - 0.263 * FG
```

12-8 Consider the experiment in Problem 11-11. Suppose that pressure is a noise variable ($\sigma_z^2 = 1$ in coded units). Fit the response model for the viscosity response. Find a set of conditions that result in viscosity as close as possible to 600 and that minimize the variability transmitted from the noise variable pressure.

Design Expert Output

Response: Viscosity

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	85467.33	6	14244.56	12.12	0.0012	significant
A	*703.12*	*1*	*703.12*	*0.60*	*0.4615*	
B	*6105.12*	*1*	*6105.12*	*5.19*	*0.0522*	
C	*5408.00*	*1*	*5408.00*	*4.60*	*0.0643*	
A2	*21736.93*	*1*	*21736.93*	*18.49*	*0.0026*	
C2	*5153.80*	*1*	*5153.80*	*4.38*	*0.0696*	
AC	*47742.25*	*1*	*47742.25*	*40.61*	*0.0002*	
Residual	9404.00	8	1175.50			
Lack of Fit	*7922.00*	*6*	*1320.33*	*1.78*	*0.4022*	*not significant*
Pure Error	*1482.00*	*2*	*741.00*			
Cor Total	94871.33	14				

The Model F-value of 12.12 implies the model is significant. There is only a 0.12% chance that a "Model F-Value" this large could occur due to noise.

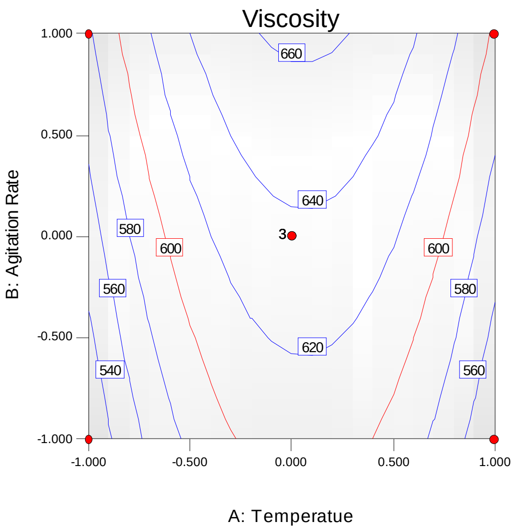
Std. Dev.	34.29	R-Squared	0.9009
Mean	575.33	Adj R-Squared	0.8265
C.V.	5.96	Pred R-Squared	0.6279
PRESS	35301.77	Adeq Precision	11.731

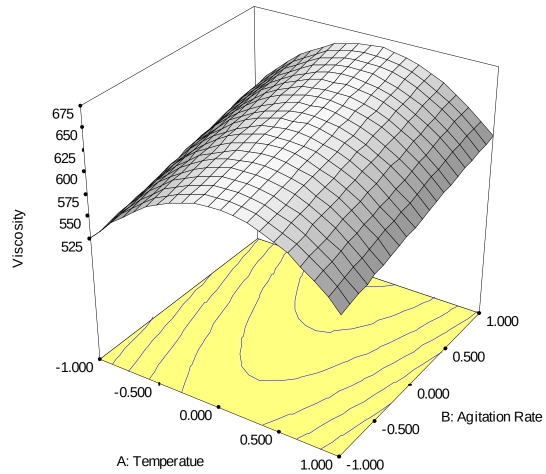
Final Equation in Terms of Coded Factors:

```
Viscosity  =
  +636.00
    +9.37  * A
   +27.62  * B
   -26.00  * C
   -76.50  * A2
   -37.25  * C2
  +109.25  * A * C
```

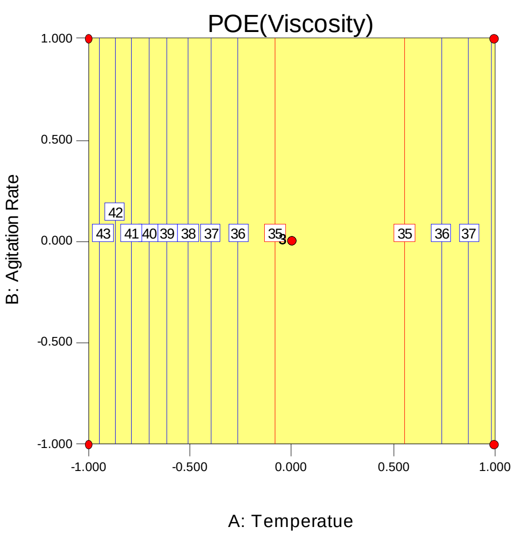
From the final equation shown in the above analysis, the mean model and corresponding contour plot is shown below.

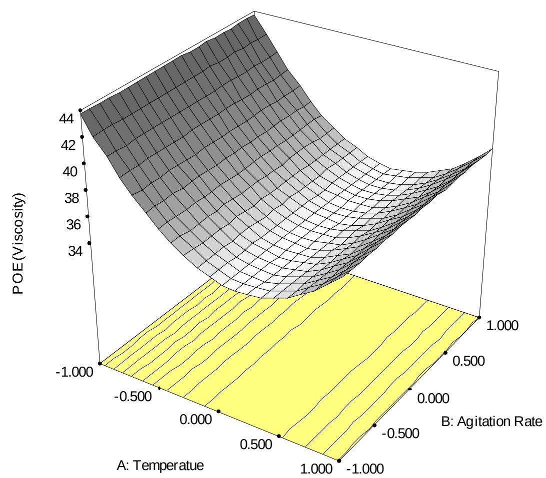
$$E_z\left[y\left(\mathbf{x}, z_1\right)\right] = 636.00 + 9.37x_1 + 27.62x_2 - 76.50x_1^2$$



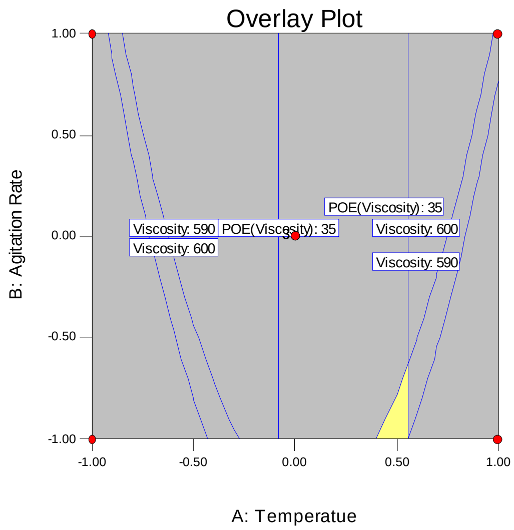


Contour and 3-D plots of the POE are shown below.





The stacked contour plots below identify a region with viscosity between 590 and 610 while minimizing the variability transmitted from the noise variable pressure. The conditions are in the region of factor A = 0.5 and factor B = -1.



12-9 A **variation of Example 12-1.** In example 12-1 (which utilized data from Example 6-2) we found that one of the process variables (B = pressure) was not important. Dropping this variable produced two replicates of a 2^3 design. The data are shown below.

C	D	$A(+)$	$A(-)$	$\bar{y}$	s^2
-	-	45, 48	71, 65	57.75	121.19
+	-	68, 80	60, 65	68.25	72.25
-	+	43, 45	100, 104	73.00	1124.67
+	+	75, 70	86, 96	81.75	134.92

Assume that C and D are controllable factors and that A is a noise factor.

(a) Fit a model to the mean response.

The following is the analysis of variance with all terms in the model:

Design Expert Output

Response: Mean
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	300.05	3	100.02		
A	*92.64*	*1*	*92.64*		
B	*206.64*	*1*	*206.64*		
AB	*0.77*	*1*	*0.77*		
Pure Error	0.000	0			
Cor Total	300.05	3			

Based on the above analysis, the AB interaction is removed from the model and used as error.

Design Expert Output

Response: Mean
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F

Model	299.28	2	149.64	195.45	0.0505	not significant
A	*92.64*	*1*	*92.64*	*121.00*	*0.0577*	
B	*206.64*	*1*	*206.64*	*269.90*	*0.0387*	
Residual	0.77	1	0.77			
Cor Total	300.05	3				

The Model F-value of 195.45 implies there is a 5.05% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	0.87	R-Squared	0.9974
Mean	70.19	Adj R-Squared	0.9923
C.V.	1.25	Pred R-Squared	0.9592
PRESS	12.25	Adeq Precision	31.672

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	70.19	1	0.44	64.63	75.75	
A-Concentration	4.81	1	0.44	-0.75	10.37	1.00
B-Stir Rate	7.19	1	0.44	1.63	12.75	1.00

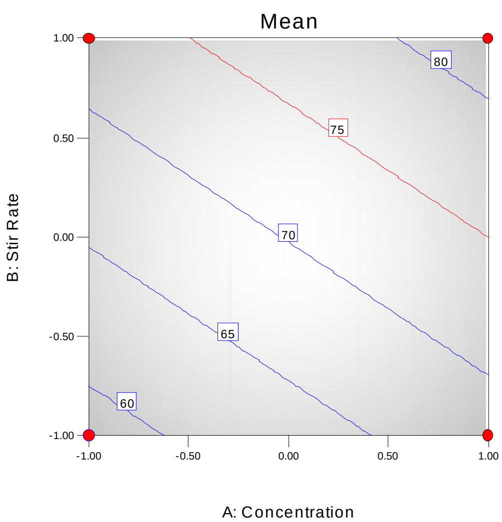
Final Equation in Terms of Coded Factors:

Mean	=
+70.19	
+4.81	* A
+7.19	* B

Final Equation in Terms of Actual Factors:

Mean	=
+70.18750	
+4.81250	* Concentration
+7.18750	* Stir Rate

The following is a contour plot of the mean model:



(b) Fit a model to the $ln(s^2)$ response.

The following is the analysis of variance with all terms in the model:

Design Expert Output

Response:	**Variance**	**Transform:**	**Natural log**	**Constant:**	**0**
ANOVA for Selected Factorial Model					

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	4.42	3	1.47		
A	*1.74*	*1*	*1.74*		
B	*2.03*	*1*	*2.03*		
AB	*0.64*	*1*	*0.64*		
Pure Error	0.000	0			
Cor Total	4.42	3			

Based on the above analysis, the AB interaction is removed from the model and applied to the residual error.

Design Expert Output

Response: Variance Transform: Natural log Constant: 0

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	3.77	2	1.89	2.94	0.3815	not significant
A	*1.74*	*1*	*1.74*	*2.71*	*0.3477*	
B	*2.03*	*1*	*2.03*	*3.17*	*0.3260*	
Residual	0.64	1	0.64			
Cor Total	4.42	3				

The "Model F-value" of 2.94 implies the model is not significant relative to the noise. There is a 38.15 % chance that a "Model F-value" this large could occur due to noise.

Std. Dev.	0.80	R-Squared	0.8545
Mean	5.25	Adj R-Squared	0.5634
C.V.	15.26	Pred R-Squared	-1.3284
PRESS	10.28	Adeq Precision	3.954

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	5.25	1	0.40	0.16	10.34	
A-Concentration	-0.66	1	0.40	-5.75	4.43	1.00
B-Stir Rate	0.71	1	0.40	-4.38	5.81	1.00

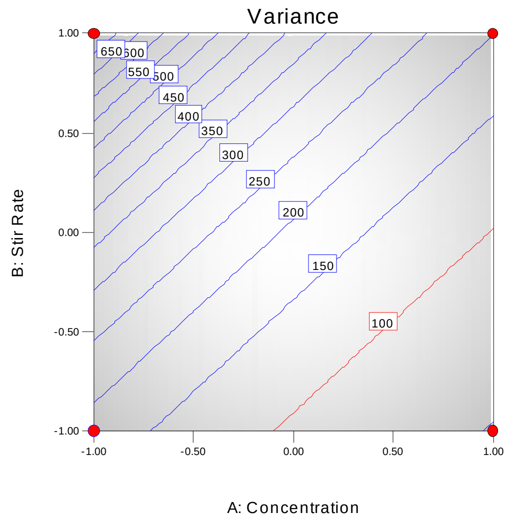
Final Equation in Terms of Coded Factors:

Ln(Variance) =
+5.25
-0.66 * A
+0.71 * B

Final Equation in Terms of Actual Factors:

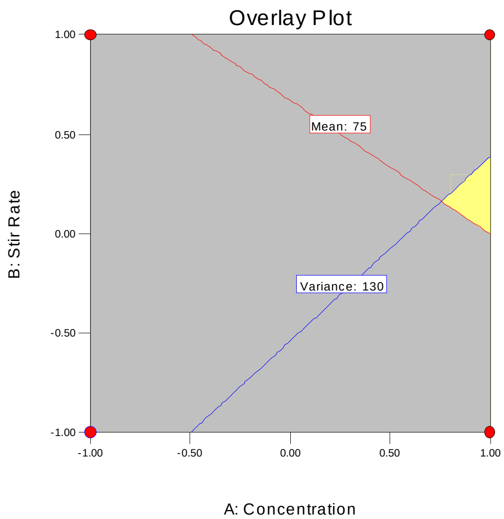
Ln(Variance) =
+5.25185
-0.65945 * Concentration
+0.71311 * Stir Rate

The following is a contour plot of the variance model in the untransformed form:



(c) Find operating conditions that result in the mean filtration rate response exceeding 75 with minimum variance.

The overlay plot shown below identifies the region required by the process:



(d) Compare your results with those from Example 12-1 which used the transmission of error approach. How similar are the two answers.

The results are very similar. Both require the Concentration to be held at the high level while the stirring rate is held near the middle.

12-10 In an article ("Let's All Beware the Latin Square," *Quality Engineering*, Vol. 1, 1989, pp. 453-465) J.S. Hunter illustrates some of the problems associated with 3^{k-p} fractional factorial designs. Factor *A* is the amount of ethanol added to a standard fuel and factor *B* represents the air/fuel ratio. The response variable is carbon monoxide (CO) emission in g/m^2. The design is shown below.

Design				Observations	
A	*B*	x_1	x_2	*y*	*y*
0	0	-1	-1	66	62
1	0	0	-1	78	81
2	0	1	-1	90	94
0	1	-1	0	72	67
1	1	0	0	80	81
2	1	1	0	75	78
0	2	-1	1	68	66
1	2	0	1	66	69
2	2	1	1	60	58

Notice that we have used the notation system of 0, 1, and 2 to represent the low, medium, and high levels for the factors. We have also used a "geometric notation" of -1, 0, and 1. Each run in the design is replicated twice.

(a) Verify that the second-order model

$$\hat{y} = 78.5 + 4.5x_1 - 7.0x_2 - 4.5x_1^2 - 4.0x_2^2 - 9.0x_1x_2$$

is a reasonable model for this experiment. Sketch the CO concentration contours in the x_1, x_2 space.

In the computer output that follows, the "coded factors" model is in the -1, 0, +1 scale.

Design Expert Output

Response: CO Emis

ANOVA for Response Surface Quadratic Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1624.00	5	324.80	50.95	< 0.0001	significant
A	*243.00*	*1*	*243.00*	*38.12*	*< 0.0001*	
B	*588.00*	*1*	*588.00*	*92.24*	*< 0.0001*	
A^2	*81.00*	*1*	*81.00*	*12.71*	*0.0039*	
B^2	*64.00*	*1*	*64.00*	*10.04*	*0.0081*	
AB	*648.00*	*1*	*648.00*	*101.65*	*< 0.0001*	
Residual	76.50	12	6.37			
Lack of Fit	*30.00*	*3*	*10.00*	*1.94*	*0.1944*	*not significant*
Pure Error	*46.50*	*9*	*5.17*			
Cor Total	1700.50	17				

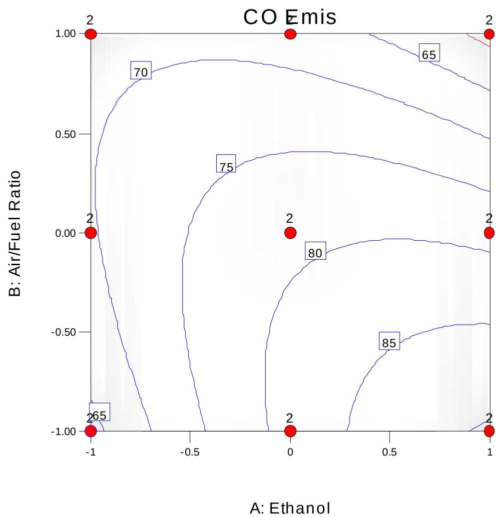
The Model F-value of 50.95 implies the model is significant. There is only a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.52	R-Squared	0.9550
Mean	72.83	Adj R-Squared	0.9363
C.V.	3.47	Pred R-Squared	0.9002
PRESS	169.71	Adeq Precision	21.952

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	78.50	1	1.33	75.60	81.40	
A-Ethanol	4.50	1	0.73	2.91	6.09	1.00
B-Air/Fuel Ratio	-7.00	1	0.73	-8.59	-5.41	1.00
A^2	-4.50	1	1.26	-7.25	-1.75	1.00
B^2	-4.00	1	1.26	-6.75	-1.25	1.00
AB	-9.00	1	0.89	-10.94	-7.06	1.00

Final Equation in Terms of Coded Factors:

CO Emis	=
+78.50	
+4.50	* A
-7.00	* B
-4.50	* A^2
-4.00	* B^2
-9.00	* A * B



(b) Now suppose that instead of only two factors, we had used *four* factors in a 3^{4-2} fractional factorial design and obtained *exactly* the same data in part (a). The design would be as follows:

Design								Observations	
A	*B*	*C*	*D*	x_1	x_2	x_3	x_4	*y*	*y*
0	0	0	0	-1	-1	-1	-1	66	62
1	0	1	1	0	-1	0	0	78	81
2	0	2	2	+1	-1	+1	+1	90	94
0	1	2	1	-1	0	+1	0	72	67
1	1	0	2	0	0	-1	+1	80	81
2	1	1	0	+1	0	0	-1	75	78
0	2	1	2	-1	+1	0	+1	68	66
1	2	2	0	0	+1	+1	-1	66	69
2	2	0	1	+1	+1	-1	0	60	58

(c) The design in part (b) allows the model

$$y = \beta_0 + \sum_{i=1}^{4} \beta_i x_i + \sum_{i=1}^{4} \beta_{ii} x_i^2 + \varepsilon$$

to be fitted. Suppose that the *true* model is

$$y = \beta_0 + \sum_{i=1}^{4} \beta_i x_i + \sum_{i=1}^{4} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \varepsilon$$

Show that if $\hat{\beta}_j$ represents the least squares estimates of the coefficients in the fitted model, then

$$
\begin{aligned}
E(\hat{\beta}_0) &= \beta_0 - \beta_{13} - \beta_{14} - \beta_{34} \\
E(\hat{\beta}_1) &= \beta_1 - (\beta_{23} + \beta_{24})/2 \\
E(\hat{\beta}_2) &= \beta_2 - (\beta_{13} + \beta_{14} + \beta_{34})/2 \\
E(\hat{\beta}_3) &= \beta_3 - (\beta_{12} + \beta_{24})/2 \\
E(\hat{\beta}_4) &= \beta_4 - (\beta_{12} + \beta_{23})/2 \\
E(\hat{\beta}_{11}) &= \beta_{11} - (\beta_{23} - \beta_{24})/2 \\
E(\hat{\beta}_{22}) &= \beta_{22} + (\beta_{13} + \beta_{14} + \beta_{34})/2 \\
E(\hat{\beta}_{33}) &= \beta_{33} - (\beta_{24} - \beta_{12})/2 + \beta_{14} \\
E(\hat{\beta}_{44}) &= \beta_{44} - (\beta_{12} - \beta_{23})/2 + \beta_{13}
\end{aligned}
$$

Does this help explain the strong effects for factors *C* and *D* observed graphically in part (b)?

Let
$$
\mathbf{X}_1 = \begin{array}{c}
\begin{matrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_{11} & \beta_{22} & \beta_{33} & \beta_{44} \end{matrix} \\
\begin{bmatrix}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
\quad \text{and} \quad
\mathbf{X}_2 = \begin{array}{c}
\begin{matrix} \beta_{12} & \beta_{13} & \beta_{14} & \beta_{23} & \beta_{24} & \beta_{34} \end{matrix} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
1 & -1 & 0 & -1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Then,
$$
\mathbf{A} = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2 \;=\; \mathbf{A} = \begin{bmatrix}
0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1/2 & -1/2 & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/2 \\
-1/2 & 0 & 0 & 0 & -1/2 & 0 \\
-1/2 & 0 & 0 & -1/2 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 1/2 & 0 \\
0 & 1/2 & 1/2 & 0 & 0 & 1/2 \\
1/2 & 0 & 1 & 0 & -1/2 & 0 \\
-1/2 & 1 & 0 & 1/2 & 0 & 0
\end{bmatrix}
$$

$$E\begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\\ \hat{\beta}_{11}\\ \hat{\beta}_{22}\\ \hat{\beta}_{33}\\ \hat{\beta}_{44}\end{bmatrix}=\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_{11}\\ \beta_{22}\\ \beta_{33}\\ \beta_{44}\end{bmatrix}+\begin{bmatrix}0 & -1 & -1 & 0 & 0 & -1\\ 0 & 0 & 0 & -1/2 & -1/2 & 0\\ 0 & -1/2 & -1/2 & 0 & 0 & -1/2\\ -1/2 & 0 & 0 & 0 & -1/2 & 0\\ -1/2 & 0 & 0 & -1/2 & 0 & 0\\ 0 & 0 & 0 & -1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2 & 0 & 0 & 1/2\\ 1/2 & 0 & 1 & 0 & -1/2 & 0\\ -1/2 & 1 & 0 & 1/2 & 0 & 0\end{bmatrix}\begin{bmatrix}\beta_{12}\\ \beta_{13}\\ \beta_{14}\\ \beta_{23}\\ \beta_{24}\\ \beta_{34}\end{bmatrix}=\begin{bmatrix}\beta_0-\beta_{13}-\beta_{14}-\beta_{34}\\ \beta_1-1/2\beta_{23}-1/2\beta_{24}\\ \beta_2-1/2\beta_{13}-1/2\beta_{14}-1/2\beta_{34}\\ \beta_3-1/2\beta_{12}-1/2\beta_{24}\\ \beta_4-1/2\beta_{12}-1/2\beta_{23}\\ \beta_{11}-1/2\beta_{23}+1/2\beta_{24}\\ \beta_{22}+1/2\beta_{13}+1/2\beta_{14}+1/2\beta_{34}\\ \beta_{33}+1/2\beta_{12}+\beta_{14}-1/2\beta_{24}\\ \beta_{44}-1/2\beta_{12}+\beta_{13}+1/2\beta_{23}\end{bmatrix}$$

12-11 An experiment has been run in a process that applies a coating material to a wafer. Each run in the experiment produced a wafer, and the coating thickness was measured several times at different locations on the wafer. Then the mean y_1, and standard deviation y_2 of the thickness measurement was obtained. The data [adapted from Box and Draper (1987)] are shown in the table below.

Run	Speed	Pressure	Distance	Mean (y_1)	Std Dev (y_2)
1	-1.000	-1.000	-1.000	24.0	12.5
2	0.000	-1.000	-1.000	120.3	8.4
3	1.000	-1.000	-1.000	213.7	42.8
4	-1.000	0.000	-1.000	86.0	3.5
5	0.000	0.000	-1.000	136.6	80.4
6	1.000	0.000	-1.000	340.7	16.2
7	-1.000	1.000	-1.000	112.3	27.6
8	0.000	1.000	-1.000	256.3	4.6
9	1.000	1.000	-1.000	271.7	23.6
10	-1.000	-1.000	0.000	81.0	0.0
11	0.000	-1.000	0.000	101.7	17.7
12	1.000	-1.000	0.000	357.0	32.9
13	-1.000	0.000	0.000	171.3	15.0
14	0.000	0.000	0.000	372.0	0.0
15	1.000	0.000	0.000	501.7	92.5
16	-1.000	1.000	0.000	264.0	63.5
17	0.000	1.000	0.000	427.0	88.6
18	1.000	1.000	0.000	730.7	21.1
19	-1.000	-1.000	1.000	220.7	133.8
20	0.000	-1.000	1.000	239.7	23.5
21	1.000	-1.000	1.000	422.0	18.5
22	-1.000	0.000	1.000	199.0	29.4
23	0.000	0.000	1.000	485.3	44.7
24	1.000	0.000	1.000	673.7	158.2
25	-1.000	1.000	1.000	176.7	55.5
26	0.000	1.000	1.000	501.0	138.9
27	1.000	1.000	1.000	1010.0	142.4

(a) What type of design did the experimenters use? Is this a good choice of design for fitting a quadratic model?

The design is a 3^3. A better choice would be a 2^3 central composite design. The CCD gives more information over the design region with fewer points.

(b) Build models of both responses.

The model for the mean is developed as follows:

Design Expert Output

Response:	**Mean**

ANOVA for Response Surface Reduced Cubic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	1.289E+006	7	1.841E+005	60.45	< 0.0001	significant
A	*5.640E+005*	*1*	*5.640E+005*	*185.16*	*< 0.0001*	
B	*2.155E+005*	*1*	*2.155E+005*	*70.75*	*< 0.0001*	
C	*3.111E+005*	*1*	*3.111E+005*	*102.14*	*< 0.0001*	
AB	*52324.81*	*1*	*52324.81*	*17.18*	*0.0006*	
AC	*68327.52*	*1*	*68327.52*	*22.43*	*0.0001*	
BC	*22794.08*	*1*	*22794.08*	*7.48*	*0.0131*	
ABC	*54830.16*	*1*	*54830.16*	*18.00*	*0.0004*	
Residual	57874.57	19	3046.03			
Cor Total	1.347E+006	26				

The Model F-value of 60.45 implies the model is significant. There is only
a 0.01% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	55.19	R-Squared	0.9570
Mean	314.67	Adj R-Squared	0.9412
C.V.	17.54	Pred R-Squared	0.9056
PRESS	1.271E+005	Adeq Precision	33.333

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	314.67	1	10.62	292.44	336.90	
A-Speed	177.01	1	13.01	149.78	204.24	1.00
B-Pressure	109.42	1	13.01	82.19	136.65	1.00
C-Distance	131.47	1	13.01	104.24	158.70	1.00
AB	66.03	1	15.93	32.69	99.38	1.00
AC	75.46	1	15.93	42.11	108.80	1.00
BC	43.58	1	15.93	10.24	76.93	1.00
ABC	82.79	1	19.51	41.95	123.63	1.00

Final Equation in Terms of Coded Factors:

Mean =
+314.67
+177.01 * A
+109.42 * B
+131.47 * C
+66.03 * A * B
+75.46 * A * C
+43.58 * B * C
+82.79 * A * B * C

Final Equation in Terms of Actual Factors:

Mean =
+314.67037
+177.01111 * Speed
+109.42222 * Pressure
+131.47222 * Distance
+66.03333 * Speed * Pressure
+75.45833 * Speed * Distance
+43.58333 * Pressure * Distance
+82.78750 * Speed * Pressure * Distance

The model for the Std. Dev. response is as follows. A square root transformation was applied to correct problems with the normality assumption.

Design Expert Output

Response: Std. Dev. Transform: Square root Constant: 0
ANOVA for Response Surface Linear Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F	
Model	116.75	3	38.92	4.34	0.0145	significant
A	*16.52*	*1*	*16.52*	*1.84*	*0.1878*	

B	*26.32*	*1*	*26.32*	*2.94*	*0.1001*
C	*73.92*	*1*	*73.92*	*8.25*	*0.0086*
Residual	206.17	23	8.96		
Cor Total	322.92	26			

The Model F-value of 4.34 implies the model is significant. There is only a 1.45% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	2.99	R-Squared	0.3616
Mean	6.00	Adj R-Squared	0.2783
C.V.	49.88	Pred R-Squared	0.1359
PRESS	279.05	Adeq Precision	7.278

Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	6.00	1	0.58	4.81	7.19	
A-Speed	0.96	1	0.71	-0.50	2.42	1.00
B-Pressure	1.21	1	0.71	-0.25	2.67	1.00
C-Distance	2.03	1	0.71	0.57	3.49	1.00

Final Equation in Terms of Coded Factors:

Sqrt(Std. Dev.) =
+6.00
+0.96 * A
+1.21 * B
+2.03 * C

Final Equation in Terms of Actual Factors:

Sqrt(Std. Dev.) =
+6.00273
+0.95796 * Speed
+1.20916 * Pressure
+2.02643 * Distance

Because Factor *A* is insignificant, it is removed from the model. The reduced linear model analysis is shown below:

Design Expert Output

Response: Std. Dev. Transform: Square root Constant: 0

ANOVA for Response Surface Reduced Linear Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	100.23	2	50.12	5.40	0.0116	significant
B	*26.32*	*1*	*26.32*	*2.84*	*0.1051*	
C	*73.92*	*1*	*73.92*	*7.97*	*0.0094*	
Residual	222.68	24	9.28			
Cor Total	322.92	26				

The Model F-value of 5.40 implies the model is significant. There is only a 1.16% chance that a "Model F-Value" this large could occur due to noise.

Std. Dev.	3.05	R-Squared	0.3104
Mean	6.00	Adj R-Squared	0.2529
C.V.	50.74	Pred R-Squared	0.1476
PRESS	275.24	Adeq Precision	6.373

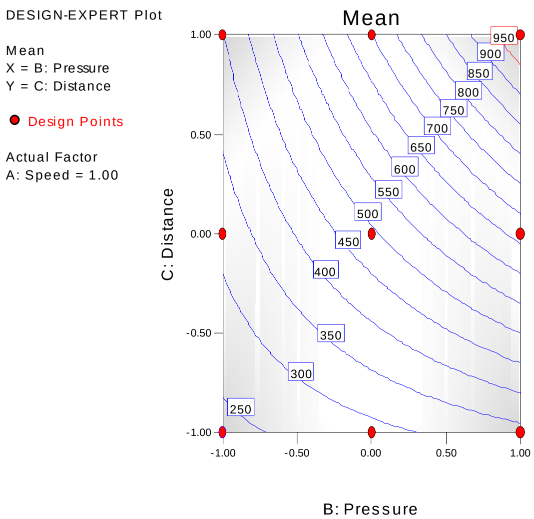
Factor	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**	**VIF**
Intercept	6.00	1	0.59	4.79	7.21	
B-Pressure	1.21	1	0.72	-0.27	2.69	1.00
C-Distance	2.03	1	0.72	0.54	3.51	1.00

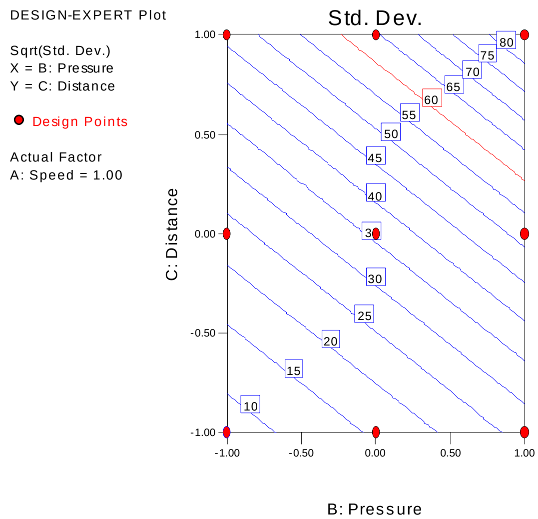
Final Equation in Terms of Coded Factors:

Sqrt(Std. Dev.) =

```
                    +6.00
                    +1.21    * B
                    +2.03    * C

Final Equation in Terms of Actual Factors:

      Sqrt(Std. Dev.)  =
                 +6.00273
                 +1.20916    * Pressure
                 +2.02643    * Distance
```

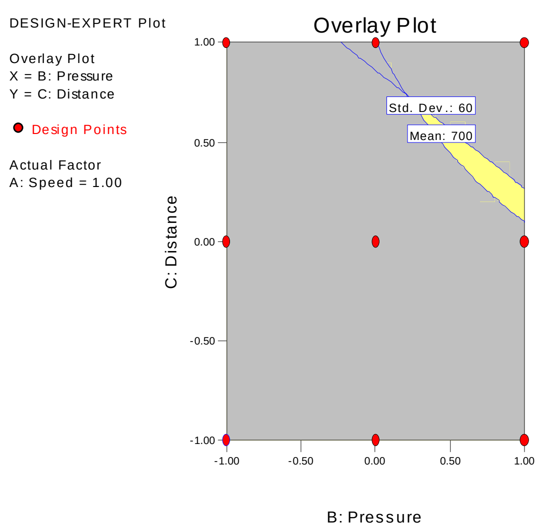
The following contour plots graphically represent the two models:





(c) Find a set of optimum conditions that result in the mean as large as possible with the standard deviation less than 60.

The overlay plot identifies a region that meets the criteria of the mean as large as possible with the standard deviation less than 60. The optimum conditions in coded terms are approximately Speed = 1.0, Pressure = 0.75 and Distance = 0.25.



12-12 Suppose that there are four controllable variables and two noise variables. It is necessary to estimate the main effects and two-factor interactions of all of the controllable variables, the main effects of the noise variables, and the two-factor interactions between all controllable and noise factors. If all factors are at two levels, what is the minimum number of runs that can be used to estimate all of the model parameters using a combined array design? Use a D-optimal algorithm to find a design.

Twenty-one runs are required for the model, with five additional runs for lack of fit, and five as replicates for a total of 31 runs as follows.

Std	*A*	*B*	*C*	*D*	*E*	*F*
1	+1	+1	-1	+1	+1	+1
2	-1	+1	-1	+1	-1	-1
3	+1	-1	-1	+1	-1	-1
4	+1	+1	-1	-1	-1	+1
5	-1	+1	-1	-1	+1	+1
6	-1	+1	+1	+1	+1	+1
7	+1	+1	-1	-1	+1	-1
8	-1	-1	+1	+1	-1	-1
9	-1	+1	+1	-1	+1	-1
10	-1	+1	+1	-1	-1	+1
11	+1	-1	+1	+1	+1	+1
12	+1	+1	+1	+1	-1	+1
13	+1	-1	-1	-1	+1	+1
14	+1	+1	+1	-1	+1	+1
15	-1	-1	-1	-1	-1	-1
16	+1	+1	+1	+1	+1	-1
17	-1	-1	-1	+1	-1	+1
18	-1	-1	-1	+1	+1	-1
19	-1	-1	+1	-1	+1	+1
20	+1	-1	+1	-1	+1	-1
21	+1	-1	+1	-1	-1	+1
22	+1	+1	+1	-1	-1	-1
23	+1	-1	-1	-1	-1	-1
24	-1	+1	-1	-1	-1	-1
25	+1	+1	-1	-1	-1	-1
26	-1	-1	+1	-1	-1	-1
27	+1	+1	+1	+1	-1	+1
28	-1	-1	-1	+1	-1	+1
29	+1	+1	+1	+1	+1	-1
30	-1	-1	-1	+1	+1	-1
31	-1	+1	-1	-1	+1	+1

12-13 Suppose that there are four controllable variables and two noise variables. It is necessary to fit a complete quadratic model in the controllable variables, the main effects of the noise variables, and the two-factor interactions between all controllable and noise factors. Set up a combined array design for this by modifying a central composite design.

The following design is a half fraction central composite design with the axial points removed from the noise factors. The total number of runs is forty-eight which includes eight center points.

Std	*A*	*B*	*C*	*D*	*E*	*F*
1	-1	-1	-1	-1	-1	-1
2	+1	-1	-1	-1	-1	+1
3	-1	+1	-1	-1	-1	+1
4	+1	+1	-1	-1	-1	-1
5	-1	-1	+1	-1	-1	+1
6	+1	-1	+1	-1	-1	-1
7	-1	+1	+1	-1	-1	-1
8	+1	+1	+1	-1	-1	+1
9	-1	-1	-1	+1	-1	+1
10	+1	-1	-1	+1	-1	-1
11	-1	+1	-1	+1	-1	-1
12	+1	+1	-1	+1	-1	+1
13	-1	-1	+1	+1	-1	-1

14	+1	-1	+1	+1	-1	+1
15	-1	+1	+1	+1	-1	+1
16	+1	+1	+1	+1	-1	-1
17	-1	-1	-1	-1	+1	+1
18	+1	-1	-1	-1	+1	-1
19	-1	+1	-1	-1	+1	-1
20	+1	+1	-1	-1	+1	+1
21	-1	-1	+1	-1	+1	-1
22	+1	-1	+1	-1	+1	+1
23	-1	+1	+1	-1	+1	+1
24	+1	+1	+1	-1	+1	-1
25	-1	-1	-1	+1	+1	-1
26	+1	-1	-1	+1	+1	+1
27	-1	+1	-1	+1	+1	+1
28	+1	+1	-1	+1	+1	-1
29	-1	-1	+1	+1	+1	+1
30	+1	-1	+1	+1	+1	-1
31	-1	+1	+1	+1	+1	-1
32	+1	+1	+1	+1	+1	+1
33	-2.378	0	0	0	0	0
34	+2.378	0	0	0	0	0
35	0	-2.378	0	0	0	0
36	0	+2.378	0	0	0	0
37	0	0	-2.378	0	0	0
38	0	0	+2.378	0	0	0
39	0	0	0	-2.378	0	0
40	0	0	0	+2.378	0	0
41	0	0	0	0	0	0
42	0	0	0	0	0	0
43	0	0	0	0	0	0
44	0	0	0	0	0	0
45	0	0	0	0	0	0
46	0	0	0	0	0	0
47	0	0	0	0	0	0
48	0	0	0	0	0	0

12-14 Reconsider the situation in Problem 12-13. Could a modified small composite design be used for this problem? Are there any disadvantages associated with the use of the small composite design?

The axial points for the noise factors were removed in following small central composite design. Five center points are included. The small central composite design does have aliasing with noise factor *E* aliased with the *AD* interaction and noise factor *F* aliased with the *BC* interaction. These aliases are corrected by leaving the axial points for the noise factors in the design.

Std	*A*	*B*	*C*	*D*	*E*	*F*
1	+1	+1	+1	+1	-1	-1
2	+1	+1	+1	-1	+1	-1
3	+1	+1	-1	+1	-1	+1
4	+1	-1	+1	-1	+1	+1
5	-1	+1	-1	+1	+1	+1
6	+1	-1	+1	+1	-1	+1
7	-1	+1	+1	-1	-1	-1
8	+1	+1	-1	-1	+1	+1
9	+1	-1	-1	+1	-1	-1
10	-1	-1	+1	-1	-1	+1
11	-1	+1	-1	-1	-1	+1
12	+1	-1	-1	-1	+1	-1
13	-1	-1	-1	+1	+1	-1
14	-1	-1	+1	+1	+1	+1
15	-1	+1	+1	+1	+1	-1
16	-1	-1	-1	-1	-1	-1
17	-2	0	0	0	0	0
18	+2	0	0	0	0	0
19	0	-2	0	0	0	0
20	0	+2	0	0	0	0
21	0	0	-2	0	0	0

22	0	0	+2	0	0	0
23	0	0	0	-2	0	0
24	0	0	0	+2	0	0
25	0	0	0	0	0	0
26	0	0	0	0	0	0
27	0	0	0	0	0	0
28	0	0	0	0	0	0
29	0	0	0	0	0	0

12-15 Reconsider the situation in Problem 12-13. What is the minimum number of runs that can be used to estimate all of the model parameters using a combined array design? Use a D-optimal algorithm to find a reasonable design for this problem.

The following design is a 36 run D-optimal design with five runs included for lack of fit and five as replicates.

Std	*A*	*B*	*C*	*D*	*E*	*F*
1	+1	+1	+1	-1	-1	-1
2	-1	+1	-1	-1	+1	+1
3	-1	+1	+1	+1	-1	+1
4	+1	+1	-1	+1	-1	-1
5	-1	-1	+1	+1	-1	-1
6	-1	+1	-1	-1	-1	-1
7	+1	-1	-1	+1	+1	-1
8	+1	-1	+1	-1	+1	-1
9	+1	+1	-1	+1	+1	+1
10	+1	-1	-1	-1	-1	-1
11	+1	-1	+1	+1	-1	+1
12	-1	+1	-1	+1	+1	-1
13	+1	+1	+1	-1	+1	+1
14	+1	+1	-1	-1	-1	+1
15	+1	+1	+1	+1	+1	-1
16	-1	-1	-1	+1	-1	+1
17	0	-1	-1	-1	+1	-1
18	0	-1	+1	-1	-1	+1
19	0	+1	0	0	0	0
20	0	0	0	-1	0	0
21	0	0	+1	0	0	0
22	-1	+1	+1	-1	+1	-1
23	-1	-1	+1	0	+1	+1
24	+1	+1	-1	-1	+1	-1
25	0	-1	+1	+1	+1	+1
26	+1	-1	-1	-1	+1	+1
27	-1	-1	-1	0	-1	-1
28	+1	-1	0	+1	-1	-1
29	-1	-1	0	+1	+1	-1
30	+1	-1	-1	0	-1	+1
31	-1	0	-1	+1	+1	+1
32	+1	+1	+1	+1	+1	-1
33	+1	-1	+1	-1	+1	-1
34	-1	+1	+1	+1	-1	+1
35	+1	+1	+1	-1	-1	-1
36	+1	+1	-1	+1	+1	+1

12-16 An experiment was run in a wave soldering process. There are five controllable variables and three noise variables. The response variable is the number of solder defects per million opportunities. The experimental design employed was the crossed array shown below.

					Outer Array				
					F	-1	1	1	-1
Inner Array					*G*	-1	1	-1	1
A	*B*	*C*	*D*	*E*	*H*	-1	-1	1	1

1	1	1	-1	-1	194	197	193	275
1	1	-1	1	1	136	136	132	136
1	-1	1	-1	1	185	261	264	264
1	-1	-1	1	-1	47	125	127	42
-1	1	1	1	-1	295	216	204	293
-1	1	-1	-1	1	234	159	231	157
-1	-1	1	1	1	328	326	247	322
-1	-1	-1	-1	-1	186	187	105	104

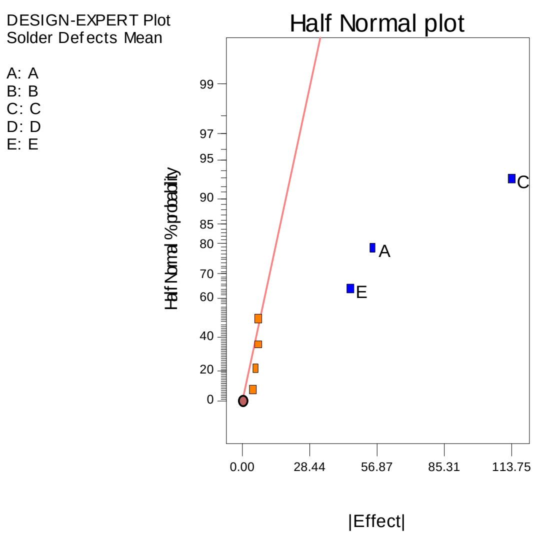
(a) What types of designs were used for the inner and outer arrays?

The inner array is a 2^{5-2} fractional factorial design with a defining relation of $I = -ACD = -BCE = ABDE$. The outer array is a 2^{3-1} fractional factorial design with a defining relation of $I = -FGH$.

(b) Develop models for the mean and variance of solder defects. What set of operating conditions would you recommend?

A	B	C	D	E	$\bar{y}$	s^2
1	1	1	-1	-1	214.75	1616.25
1	1	-1	1	1	135.00	4.00
1	-1	1	-1	1	243.50	1523.00
1	-1	-1	1	-1	85.25	2218.92
-1	1	1	1	-1	252.00	2376.67
-1	1	-1	-1	1	195.25	1852.25
-1	-1	1	1	1	305.75	1540.25
-1	-1	-1	-1	-1	145.50	2241.67

The following analysis identifies factors A, C, and E as being significant for the solder defects mean model.



Design Expert Output

Response: Solder Defects Mean
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	36068.63	3	12022.88	194.31	< 0.0001	significant
A	*6050.00*	*1*	*6050.00*	*97.78*	*0.0006*	

C	*25878.13*	*1*	*25878.13*	*418.23*	*< 0.0001*
E	*4140.50*	*1*	*4140.50*	*66.92*	*0.0012*
Residual	247.50	4	61.88		
Cor Total	36316.13	7			

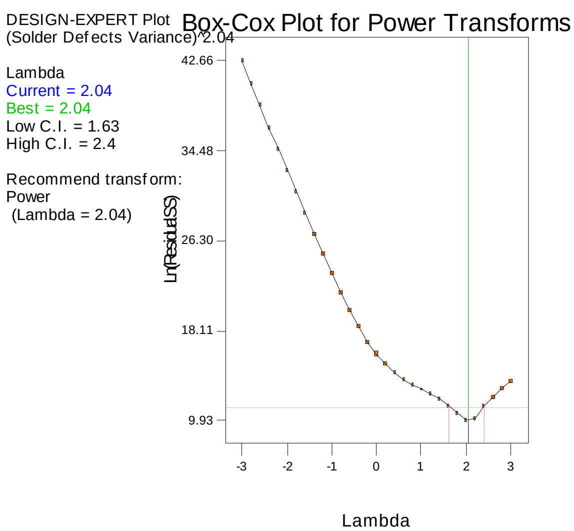
The "Model F-value" of 194.31implies the model is not significant relative to the noise. There is a 0.01 % chance that a "Model F-value" this large could occur due to noise.

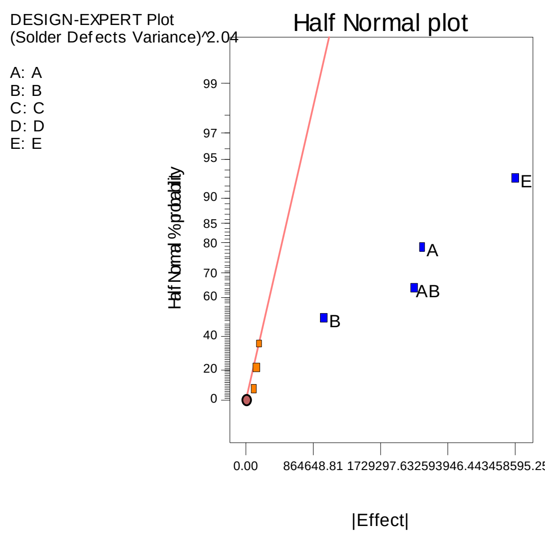
Std. Dev.	7.87	R-Squared	0.9932
Mean	197.13	Adj R-Squared	0.9881
C.V.	3.99	Pred R-Squared	0.9727
PRESS	990.00	Adeq Precision	38.519

Final Equation in Terms of Coded Factors:

Solder Defects Mean =
+197.13
-27.50 * A
+56.88 * C
+22.75 * E

Although the natural log transformation is often utilized for variance response, a power transformation actually performed better for this problem per the Box-Cox plot below. The analysis for the solder defect variance follows.





Design Expert Output

Response: Solder Defects VarianceTransform: Power Lambda: 2.04 Constant: 0

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**	
Model	4.542E+013	4	1.136E+013	325.30	0.0003	significant
A	*1.023E+013*	*1*	*1.023E+013*	*293.08*	*0.0004*	
B	*1.979E+012*	*1*	*1.979E+012*	*56.70*	*0.0049*	
E	*2.392E+013*	*1*	*2.392E+013*	*685.33*	*0.0001*	
AB	*9.289E+012*	*1*	*9.289E+012*	*266.11*	*0.0005*	
Residual	1.047E+011	3	3.491E+010			
Cor Total	4.553E+013	7				

The "Model F-value" of 325.30 implies the model is not significant relative to the noise. There is a 0.03 % chance that a "Model F-value" this large could occur due to noise.

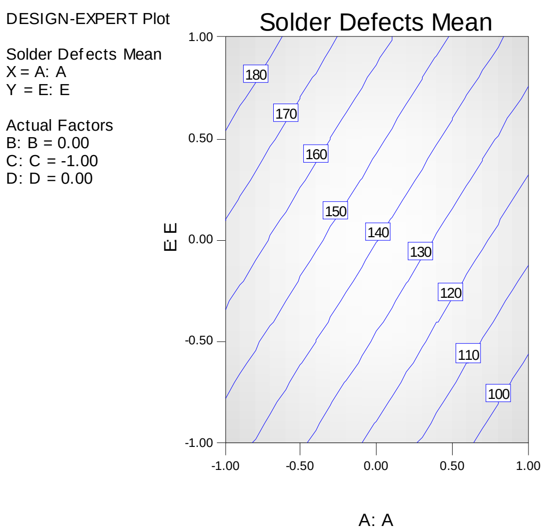
Std. Dev.	1.868E+005	R-Squared	0.9977
Mean	4.461E+006	Adj R-Squared	0.9946
C.V.	4.19	Pred R-Squared	0.9836

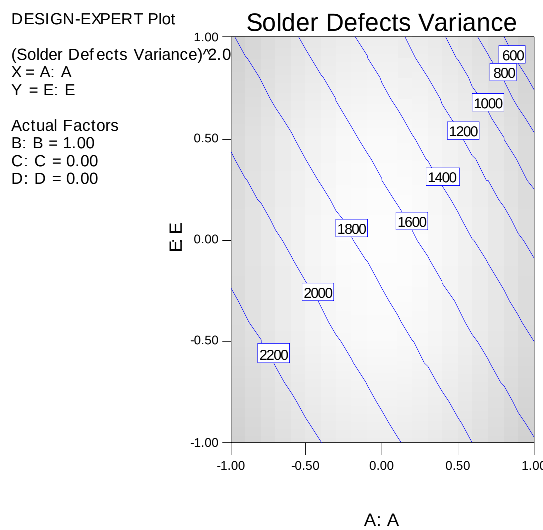
PRESS	7.447E+011	Adeq Precision	53.318

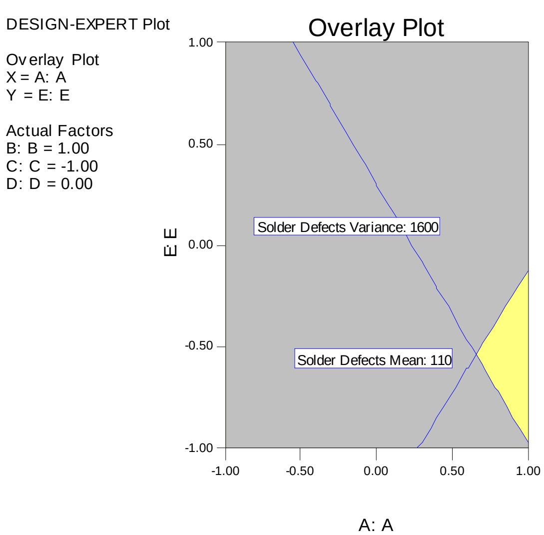
Final Equation in Terms of Coded Factors:

(Solder Defects Variance)2.04	=	
	+4.461E+006	
	-1.131E+006	* A
	-4.974E+005	* B
	-1.729E+006	* E
	-1.078E+006	* A * B

The contour plots of the mean and variance models are shown below along with the overlay plot. Assuming that we wish to minimize both solder defects mean and variance, a solution is shown in the overlay plot with factors A = +1, B = +1, C = -1, D = 0, and E near -1.







12-17 Reconsider the wave soldering experiment in Problem 12-16. Find a combined array design for this experiment that requires fewer runs.

The following experiment is a 2^{8-4}, resolution IV design with the defining relation $I = BCDE = ACDF = ABCG = ABDH$. Only 16 runs are required.

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
-1	-1	-1	-1	-1	-1	-1	-1
+1	-1	-1	-1	-1	+1	+1	+1
-1	+1	-1	-1	+1	-1	+1	+1
+1	+1	-1	-1	+1	+1	-1	-1
-1	-1	+1	-1	+1	+1	+1	-1
+1	-1	+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	-1	+1	-1	+1
+1	+1	+1	-1	-1	-1	+1	-1
-1	-1	-1	+1	+1	+1	-1	+1
+1	-1	-1	+1	+1	-1	+1	-1
-1	+1	-1	+1	-1	+1	+1	-1
+1	+1	-1	+1	-1	-1	-1	+1
-1	-1	+1	+1	-1	-1	+1	+1
+1	-1	+1	+1	-1	+1	-1	-1
-1	+1	+1	+1	+1	-1	-1	-1
+1	+1	+1	+1	+1	+1	+1	+1

12-18 Reconsider the wave soldering experiment in Problem 12-17. Suppose that it was necessary to fit a complete quadratic model in the controllable variables, all main effects of the noise variables, and all controllable variable-noise variable interactions. What design would you recommend?

The following experiment is a small central composite design with five center points; the axial points for the noise factors have been removed. A total of 45 runs are required.

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
+1	+1	+1	-1	-1	+1	+1	+1
-1	+1	+1	-1	+1	+1	+1	-1
+1	+1	-1	-1	+1	+1	-1	-1
+1	-1	-1	+1	+1	-1	+1	-1
-1	-1	+1	+1	+1	+1	-1	-1
-1	+1	+1	+1	-1	+1	-1	-1
-1	+1	+1	+1	+1	+1	-1	-1
+1	+1	+1	+1	+1	-1	+1	-1
+1	+1	-1	+1	-1	+1	-1	+1
+1	-1	+1	+1	-1	-1	+1	-1
+1	+1	+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	-1	-1	+1	+1
+1	-1	-1	-1	-1	+1	-1	-1
+1	-1	+1	-1	-1	-1	-1	+1
-1	+1	-1	-1	+1	+1	+1	-1
+1	-1	-1	+1	+1	+1	-1	+1
-1	-1	-1	-1	-1	+1	+1	-1
-1	-1	+1	+1	+1	-1	-1	+1
-1	+1	-1	-1	+1	-1	+1	+1
-1	-1	+1	+1	-1	+1	+1	+1
+1	+1	-1	+1	-1	-1	+1	-1
-1	-1	+1	-1	+1	-1	-1	-1
+1	+1	+1	-1	-1	+1	-1	-1
+1	-1	-1	-1	+1	+1	+1	+1
+1	-1	+1	-1	-1	+1	+1	+1
-1	+1	-1	+1	+1	+1	+1	+1
-1	-1	-1	-1	+1	-1	+1	+1
+1	-1	+1	+1	+1	-1	+1	-1
-1	+1	-1	+1	-1	-1	-1	+1
-1	-1	-1	-1	-1	-1	-1	-1
-2.34	0	0	0	0	0	0	0
2.34	0	0	0	0	0	0	0

0	-2.34	0	0	0	0	0	0
0	2.34	0	0	0	0	0	0
0	0	-2.34	0	0	0	0	0
0	0	2.34	0	0	0	0	0
0	0	0	-2.34	0	0	0	0
0	0	0	2.34	0	0	0	0
0	0	0	0	-2.34	0	0	0
0	0	0	0	2.34	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Chapter 13
Experiments with Random Factors
Solutions

13-1 A textile mill has a large number of looms. Each loom is supposed to provide the same output of cloth per minute. To investigate this assumption, five looms are chosen at random and their output is noted at different times. The following data are obtained:

Loom	Output (lb/min)				
1	14.0	14.1	14.2	14.0	14.1
2	13.9	13.8	13.9	14.0	14.0
3	14.1	14.2	14.1	14.0	13.9
4	13.6	13.8	14.0	13.9	13.7
5	13.8	13.6	13.9	13.8	14.0

(a) Explain why this is a random effects experiment. Are the looms equal in output? Use $\alpha = 0.05$.

The looms used in the experiment are a random sample of all the looms in the manufacturing area. The following is the analysis of variance for the data:

Minitab Output

```
ANOVA: Output versus Loom

Factor     Type Levels Values
Loom     random      5     1     2     3     4     5

Analysis of Variance for Output

Source      DF         SS         MS        F      P
Loom         4    0.34160    0.08540     5.77  0.003
Error       20    0.29600    0.01480
Total       24    0.63760

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Loom      0.01412   2   (2) + 5(1)
 2 Error     0.01480       (2)
```

(b) Estimate the variability between looms.

$$\hat{\sigma}_\tau^2 = \frac{MS_{Model} - MS_E}{n} = \frac{0.0854 - 0.0148}{5} = 0.01412$$

(c) Estimate the experimental error variance.

$$\hat{\sigma}^2 = MS_E = 0.0148$$

(d) Find a 95 percent confidence interval for $\sigma_\tau^2 / \left(\sigma_\tau^2 + \sigma^2\right)$.

$$L = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{\alpha/2,a-1,n-a}} - 1\right] = \frac{1}{5}\left[\frac{0.08540}{0.01480}\times\frac{1}{3.51} - 1\right] = 0.1288$$

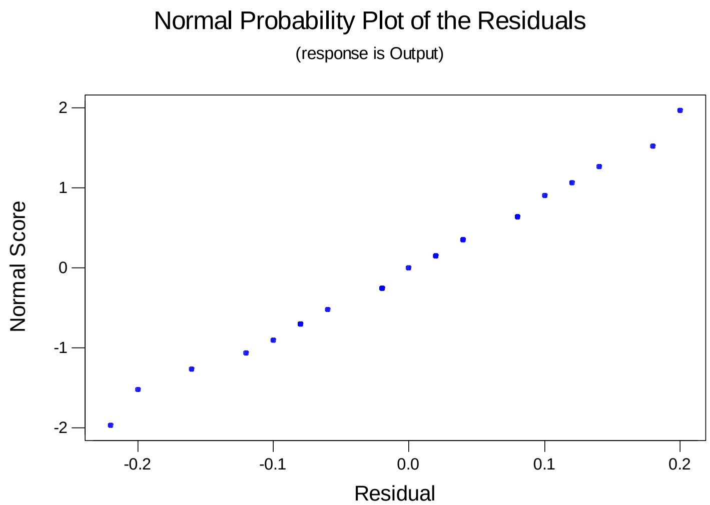
$$U = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{1-\alpha/2,a-1,n-a}}-1\right] = \frac{1}{5}\left[\frac{0.08540}{0.01480}\times 8,56-1\right] = 9.6787$$

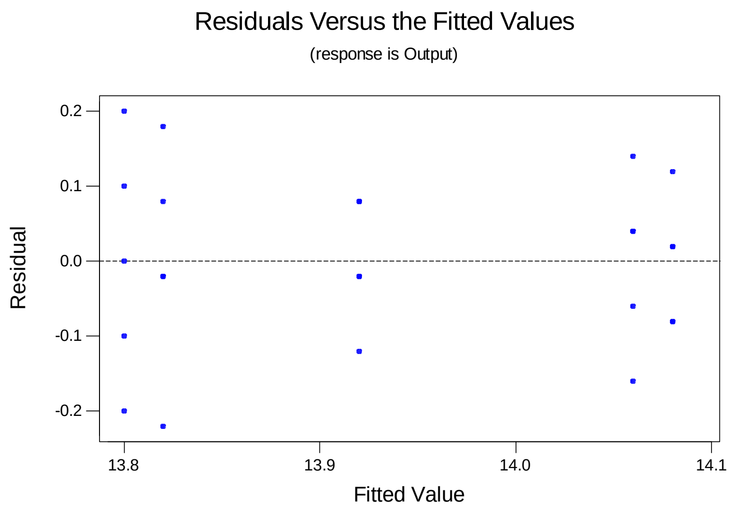
$$\frac{L}{L+1} \leq \frac{\sigma_\tau^2}{\sigma_\tau^2+\sigma^2} \leq \frac{U}{U+1}$$

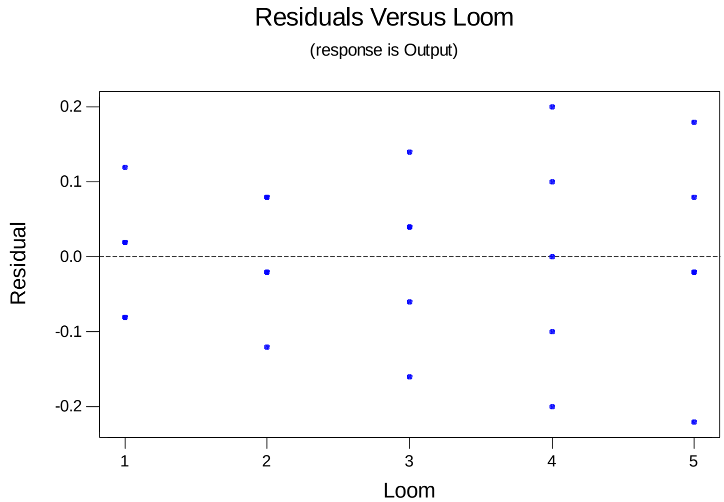
$$0.1141 \leq \frac{\sigma_\tau^2}{\sigma_\tau^2+\sigma^2} \leq 0.9064$$

(e) Analyze the residuals from this experiment. Do you think that the analysis of variance assumptions are satisfied?

There is nothing unusual about the residual plots; therefore, the analysis of variance assumptions are satisfied.







13-2 A manufacturer suspects that the batches of raw material furnished by her supplier differ significantly in calcium content. There are a large number of batches currently in the warehouse. Five of these are randomly selected for study. A chemist makes five determinations on each batch and obtains the following data:

Batch 1	Batch 2	Batch 3	Batch 4	Batch 5
23.46	23.59	23.51	23.28	23.29
23.48	23.46	23.64	23.40	23.46
23.56	23.42	23.46	23.37	23.37
23.39	23.49	23.52	23.46	23.32
23.40	23.50	23.49	23.39	23.38

(a) Is there significant variation in calcium content from batch to batch? Use $\alpha = 0.05$.

Yes, as shown in the Minitab Output below, there is a difference.

Minitab Output

```
ANOVA: Calcium versus Batch

Factor     Type Levels Values
Batch    random      5     1     2     3     4     5

Analysis of Variance for Calcium

Source      DF         SS         MS        F        P
Batch        4   0.096976   0.024244     5.54    0.004
Error       20   0.087600   0.004380
Total       24   0.184576

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Batch     0.00397   2    (2) + 5(1)
 2 Error     0.00438        (2)
```

(b) Estimate the components of variance.

$$\hat{\sigma}_{\tau}^{2} = \frac{MS_{Model} - MS_{E}}{n} = \frac{.024244 - .004380}{5} = 0.00397$$

$$\hat{\sigma}^2 = MS_E = 0.004380$$

(c) Find a 95 percent confidence interval for $\sigma_\tau^2 / \left(\sigma_\tau^2 + \sigma^2\right)$.

$$L = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{\alpha/2,a-1,n-a}} - 1\right] = 0.1154$$

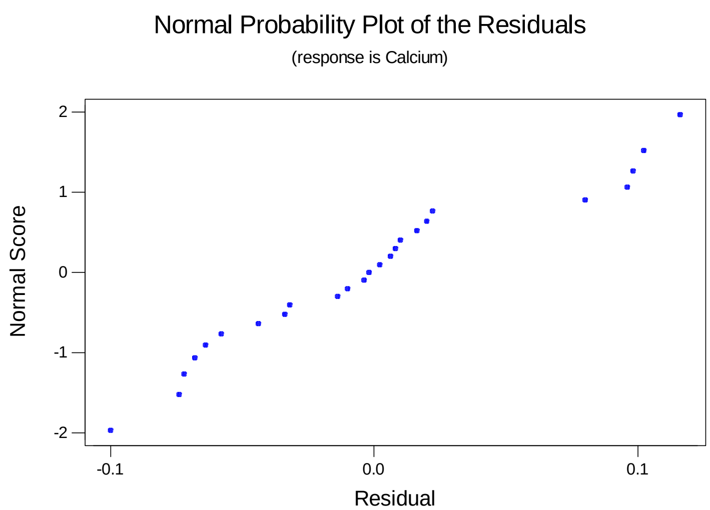
$$U = \frac{1}{n}\left[\frac{MS_{Model}}{MS_E}\frac{1}{F_{1-\alpha/2,a-1,n-a}} - 1\right] = 9.276$$

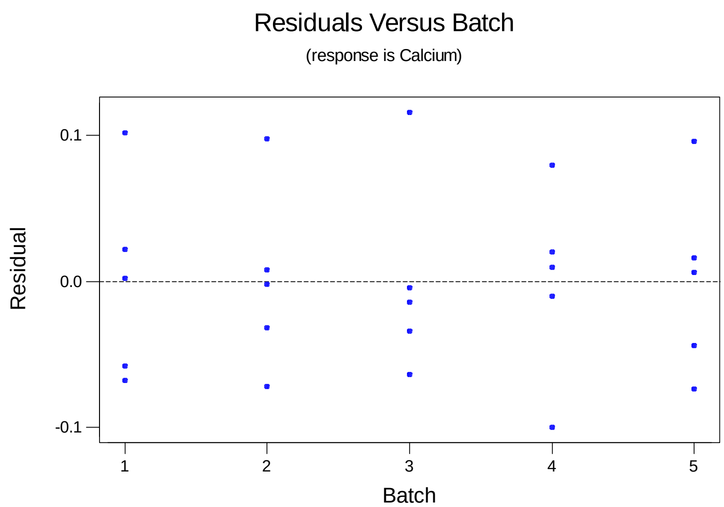
$$\frac{L}{L+1} \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le \frac{U}{U+1}$$

$$0.1035 \le \frac{\sigma_\tau^2}{\sigma_\tau^2 + \sigma^2} \le 0.9027$$

(d) Analyze the residuals from this experiment. Are the basic analysis of variance assumptions satisfied?

There are five residuals that stand out in the normal probability plot. From the Residual vs. Batch plot, we see that one point per batch appears to stand out. A natural log transformation was applied to the data but did not change the results of the residual analysis. Further investigation should probably be performed to determine if these points are outliers.





13-3 Several ovens in a metal working shop are used to heat metal specimens. All the ovens are supposed to operate at the same temperature, although it is suspected that this may not be true. Three ovens are selected at random and their temperatures on successive heats are noted. The data collected are as follows:

Oven	Temperature					
1	491.50	498.30	498.10	493.50	493.60	
2	488.50	484.65	479.90	477.35		
3	490.10	484.80	488.25	473.00	471.85	478.65

(a) Is there significant variation in temperature between ovens? Use $\alpha = 0.05$.

The analysis of variance shown below identifies significant variation in temperature between the ovens.

Minitab Output

General Linear Model: Temperature versus Oven

```
Factor     Type Levels Values
```

```
Oven      random       3 1 2 3

Analysis of Variance for Temperat, using Adjusted SS for Tests

Source      DF      Seq SS      Adj SS      Adj MS        F       P
Oven         2      594.53      594.53      297.27     8.62   0.005
Error       12      413.81      413.81       34.48
Total       14     1008.34

Expected Mean Squares, using Adjusted SS

Source        Expected Mean Square for Each Term
 1 Oven       (2) +  4.9333(1)
 2 Error      (2)

Error Terms for Tests, using Adjusted SS

Source        Error DF  Error MS  Synthesis of Error MS
 1 Oven          12.00     34.48  (2)

Variance Components, using Adjusted SS

Source     Estimated Value
Oven                 53.27
Error                34.48
```

(b) Estimate the components of variance.

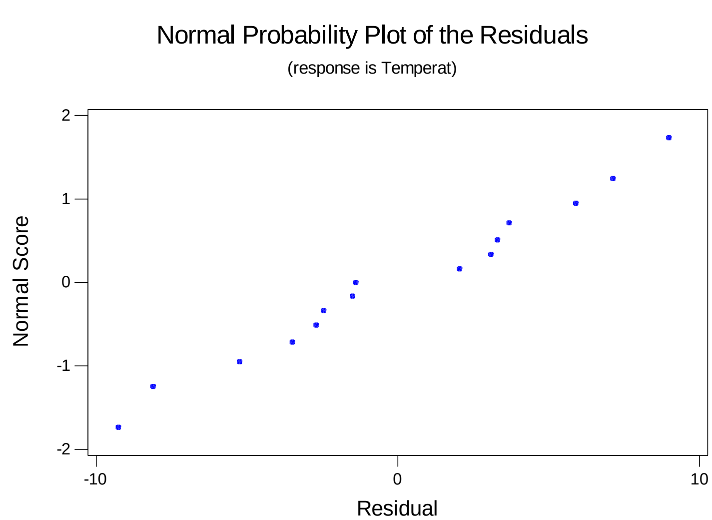
$$n_0 = \frac{1}{a-1}\left[\sum n_i - \frac{\sum n_i^2}{\sum n_i}\right] = \frac{1}{2}\left[15 - \frac{25+16+36}{15}\right] = 4.93$$

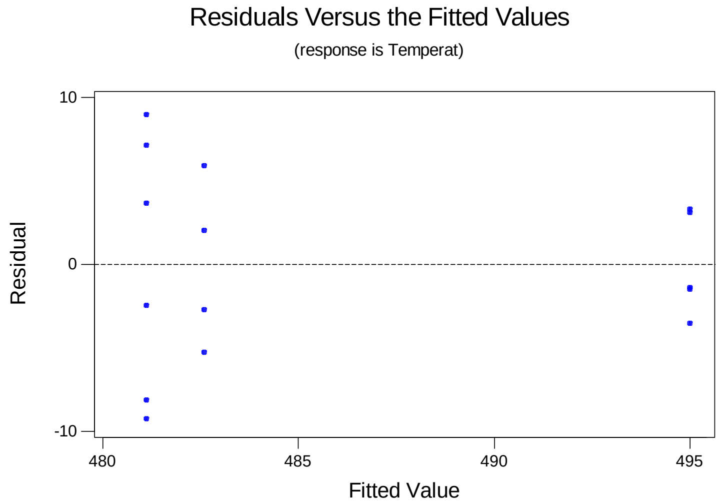
$$\hat{\sigma}_\tau^2 = \frac{MS_{Model} - MS_E}{n} = \frac{297.27 - 34.48}{4.93} = 53.30$$

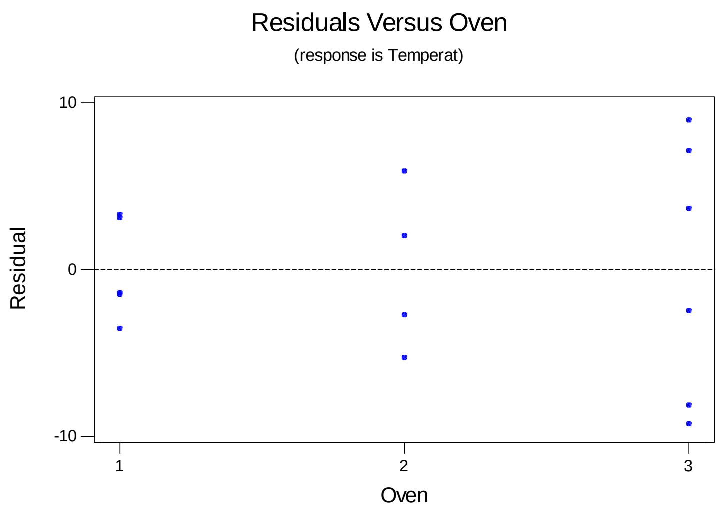
$$\hat{\sigma}^2 = MS_E = 34.48$$

(c) Analyze the residuals from this experiment. Draw conclusions about model adequacy.

There is a funnel shaped appearance in the plot of residuals versus predicted value indicating a possible non-constant variance. There is also some indication of non-constant variance in the plot of residuals versus oven. The inequality of variance problem is not severe.







13-4 An article in the *Journal of the Electrochemical Society* (Vol. 139, No. 2, 1992, pp. 524-532) describes an experiment to investigate the low-pressure vapor deposition of polysilicon. The experiment was carried out in a large-capacity reactor at Sematech in Austin, Texas. The reactor has several wafer positions, and four of these positions are selected at random. The response variable is film thickness uniformity. Three replicates of the experiments were run, and the data are as follows:

Wafer Position	Uniformity		
1	2.76	5.67	4.49
2	1.43	1.70	2.19
3	2.34	1.97	1.47
4	0.94	1.36	1.65

(a) Is there a difference in the wafer positions? Use $\alpha = 0.05$.

Yes, there is a difference.

Minitab Output

```
ANOVA: Uniformity versus Wafer Position

Factor     Type Levels Values
Wafer Po  fixed      4     1     2     3     4

Analysis of Variance for Uniformi

Source       DF         SS         MS       F      P
Wafer Po      3    16.2198     5.4066    8.29  0.008
Error         8     5.2175     0.6522
Total        11    21.4373

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Wafer Po             2   (2) + 3Q[1]
 2 Error      0.6522        (2)
```

(b) Estimate the variability due to wafer positions.

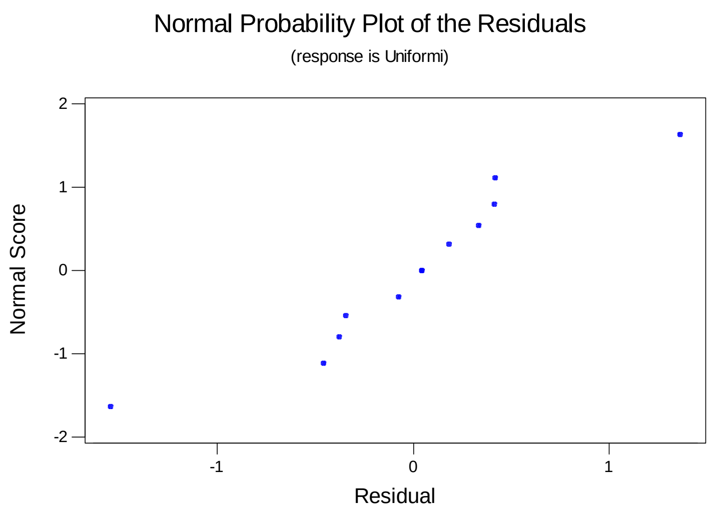
$$\hat{\sigma}_\tau^2 = \frac{MS_{Treatment} - MS_E}{n}$$
$$\hat{\sigma}_\tau^2 = \frac{5.4066 - 0.6522}{3} = 1.5844$$

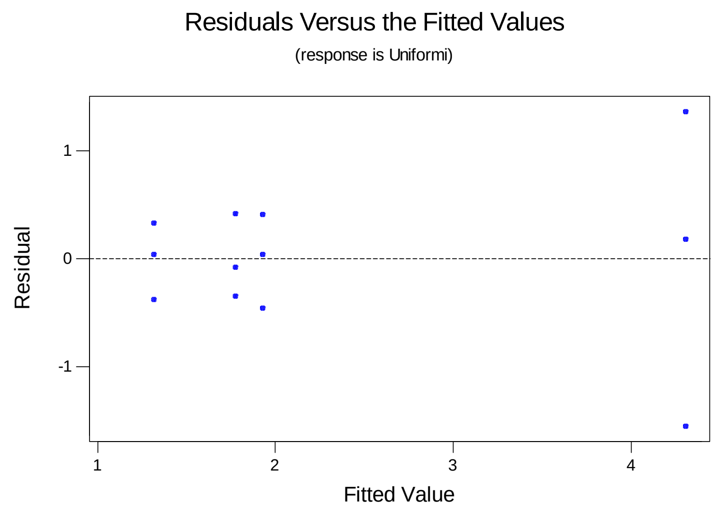
(c) Estimate the random error component.

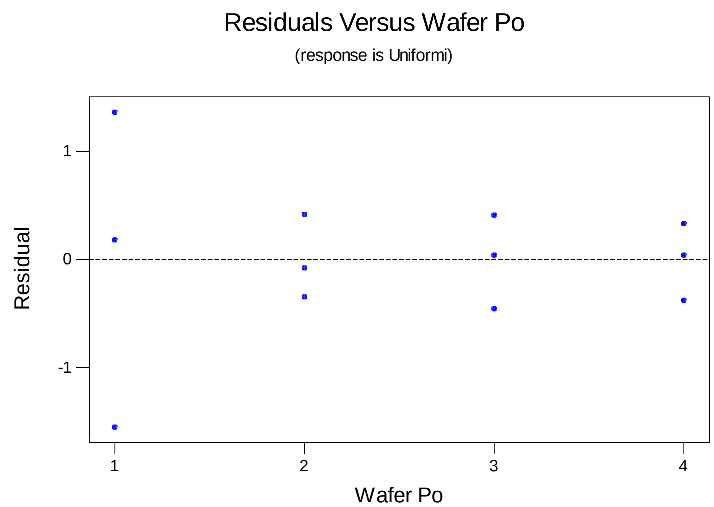
$$\hat{\sigma}^2 = 0.6522$$

(d) Analyze the residuals from this experiment and comment on model adequacy.

Variability in film thickness seems to depend on wafer position. These observations also show up as outliers on the normal probability plot. Wafer position number 1 appears to have greater variation in uniformity than the other positions.







13-5 Consider the vapor deposition experiment described in Problem 13-4.

(a) Estimate the total variability in the uniformity response.

$$\hat{\sigma}_{\tau}^{2}+\hat{\sigma}^{2}=1.5848+0.6522=2.2370$$

(b) How much of the total variability in the uniformity response is due to the difference between positions in the reactor?

$$\frac{\hat{\sigma}_{\tau}^{2}}{\hat{\sigma}^{2}+\hat{\sigma}_{\tau}^{2}}=\frac{1.5848}{2.2370}=0.70845$$

(c) To what level could the variability in the uniformity response be reduced, if the position-to-position variability in the reactor could be eliminated? Do you believe this is a significant reduction?

The variability would be reduced from 2.2370 to $\hat{\sigma}^2 = 0.6522$ which is a reduction of approximately:

$$\frac{2.2370 - 0.6522}{2.2370} = 71\%$$

13-6 An article in the *Journal of Quality Technology* (Vol. 13, No. 2, 1981, pp. 111-114) describes and experiment that investigates the effects of four bleaching chemicals on pulp brightness. These four chemicals were selected at random from a large population of potential bleaching agents. The data are as follows:

Chemical	Pulp Brightness				
1	77.199	74.466	92.746	76.208	82.876
2	80.522	79.306	81.914	80.346	73.385
3	79.417	78.017	91.596	80.802	80.626
4	78.001	78.358	77.544	77.364	77.386

(a) Is there a difference in the chemical types? Use $\alpha = 0.05$.

The computer output shows that the null hypothesis cannot be rejected. Therefore, there is no evidence that there is a difference in chemical types.

Minitab Output

```
ANOVA: Brightness versus Chemical

Factor     Type Levels Values
Chemical random      4     1     2     3     4

Analysis of Variance for Brightne

Source      DF         SS         MS       F      P
Chemical     3      53.98      17.99    0.75  0.538
Error       16     383.99      24.00
Total       19     437.97

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 Chemical   -1.201   2   (2) + 5(1)
 2 Error      23.999       (2)
```

(b) Estimate the variability due to chemical types.

$$\hat{\sigma}_\tau^2 = \frac{MS_{Treatment} - MS_E}{n}$$

$$\hat{\sigma}_\tau^2 = \frac{17.994 - 23.999}{5} = -1.201$$

which agrees with the Minitab output.

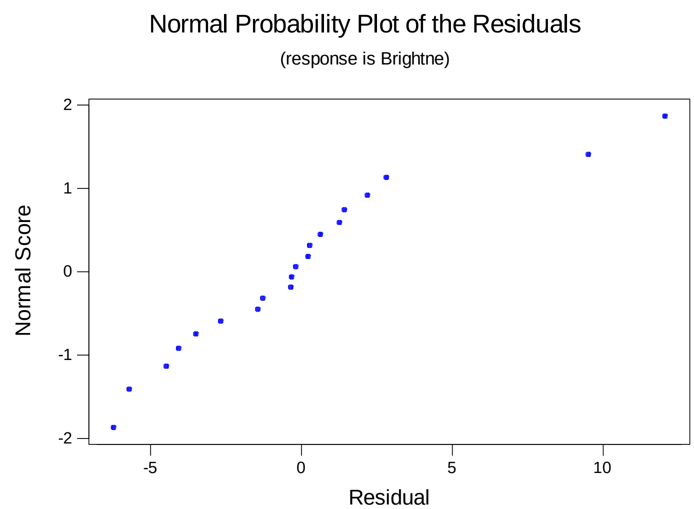
Because the variance component cannot be negative, this likely means that the variability due to chemical types is zero.

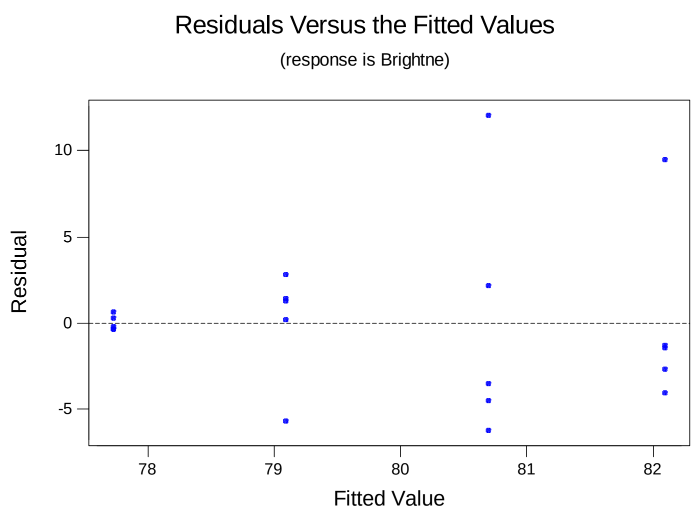
(c) Estimate the variability due to random error.

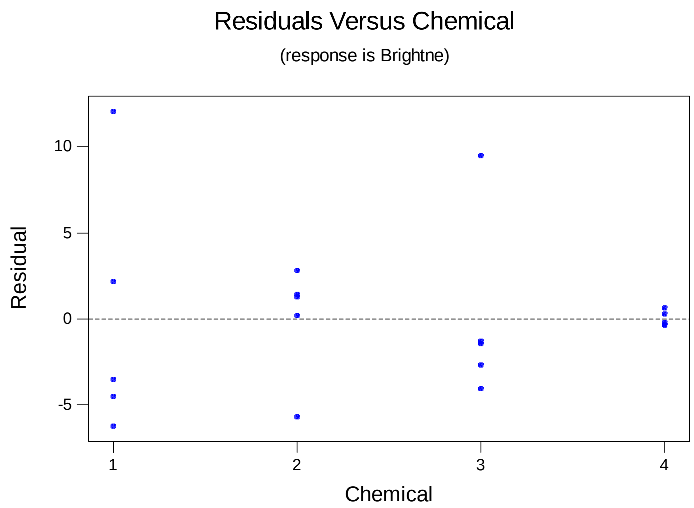
$$\hat{\sigma}^2 = 23.999$$

(d) Analyze the residuals from this experiment and comment on model adequacy.

Two data points appear to be outliers in the normal probability plot of effects. These outliers belong to chemical types 1 and 3 and should be investigated. There seems to be much less variability in brightness with chemical type 4.







13-7 Consider the one-way balanced, random effects method. Develop a procedure for finding a 100(1-α) percent confidence interval for $\sigma^2 / (\sigma_\tau^2 + \sigma^2)$.

$$\text{We know that } P\left[L \le \frac{\sigma_\tau^2}{\sigma^2} \le U \right] = 1-\alpha$$

$$P\left[L+1 \le \frac{\sigma_\tau^2}{\sigma^2} + \frac{\sigma^2}{\sigma^2} \le U+1 \right] = 1-\alpha$$

$$P\left[L+1 \le \frac{\sigma_\tau^2 + \sigma^2}{\sigma^2} \le U+1 \right] = 1-\alpha$$

$$P\left[\frac{L}{1+L} \ge \frac{\sigma^2}{\sigma_\tau^2 + \sigma^2} \ge \frac{U}{1+U} \right] = 1-\alpha$$

13-8 Refer to Problem 13-1.

(a) What is the probability of accepting H_0 if σ_τ^2 is four times the error variance σ^2?

$$\lambda = \sqrt{1+\frac{n\sigma_\tau^2}{\sigma^2}} = \sqrt{1+\frac{5(4\sigma^2)}{\sigma^2}} = \sqrt{21} = 4.6$$

$$\upsilon_1 = a-1 = 4 \qquad \upsilon_2 = N-a = 25-5 = 20 \qquad \beta \approx 0.035 \text{, from the OC curve.}$$

(b) If the difference between looms is large enough to increase the standard deviation of an observation by 20 percent, we wish to detect this with a probability of at least 0.80. What sample size should be used?

$$\upsilon_1 = a-1 = 4 \qquad \upsilon_2 = N-a = 25-5 = 20 \qquad \alpha = 0.05 \qquad P(accept) \le 0.2$$

$$\lambda = \sqrt{1+n\left[(1+0.01P)^2 - 1\right]} = \sqrt{1+n\left[(1+0.01(20))^2 - 1\right]} = \sqrt{1+0.44n}$$

Trial and Error yields:

n	υ_2	λ	P(accept)
5	20	1.79	0.6
10	45	2.32	0.3
14	65	2.67	0.2

Choose $n \geq 14$, therefore $N \geq 70$

13-9 An experiment was performed to investigate the capability of a measurement system. Ten parts were randomly selected, and two randomly selected operators measured each part three times. The tests were made in random order, and the data below resulted.

Part Number	Operator 1 Measurements			Operator 2 Measurements		
	1	2	3	1	2	3
1	50	49	50	50	48	51
2	52	52	51	51	51	51
3	53	50	50	54	52	51
4	49	51	50	48	50	51
5	48	49	48	48	49	48
6	52	50	50	52	50	50
7	51	51	51	51	50	50
8	52	50	49	53	48	50
9	50	51	50	51	48	49
10	47	46	49	46	47	48

(a) Analyze the data from this experiment.

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor     Type Levels Values
Part     random     10      1     2     3     4     5     6     7
                            8     9    10
Operator random      2      1     2

Analysis of Variance for Measurem

Source           DF         SS         MS        F      P
Part              9     99.017     11.002    18.28  0.000
Operator          1      0.417      0.417     0.69  0.427
Part*Operator     9      5.417      0.602     0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part           1.73333   3   (4) + 3(3) + 6(1)
 2 Operator      -0.00617   3   (4) + 3(3) + 30(2)
 3 Part*Operator -0.29938   4   (4) + 3(3)
 4 Error          1.50000       (4)
```

(b) Find point estimates of the variance components using the analysis of variance method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 1.5$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{0.6018519 - 1.5000000}{3} < 0 \text{, assume } \hat{\sigma}^2_{\tau\beta} = 0$$

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$

$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}^2_{\tau} = \frac{0.416667 - 0.6018519}{10(3)} < 0 \text{, assume } \hat{\sigma}^2_{\tau} = 0$$

All estimates agree with the Minitab output.

13-10 An article by Hoof and Berman ("Statistical Analysis of Power Module Thermal Test Equipment Performance", *IEEE Transactions on Components, Hybrids, and Manufacturing Technology* Vol. 11, pp. 516-520, 1988) describes an experiment conducted to investigate the capability of measurements on thermal impedance (C°/W x 100) on a power module for an induction motor starter. There are 10 parts, three operators, and three replicates. The data are shown in the following table.

Part	Inspector 1			Inspector 2			Inspector 3		
Number	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

(a) Analyze the data from this experiment, assuming both parts and operators are random effects.

Minitab Output

ANOVA: Impedance versus Inspector, Part

```
Factor     Type Levels Values
Inspecto random      3     1     2     3
Part     random     10     1     2     3     4     5     6     7
                           8     9    10


Analysis of Variance for Impedanc

Source           DF         SS         MS        F      P
Inspecto          2      39.27      19.63     7.28  0.005
Part              9    3935.96     437.33   162.27  0.000
Inspecto*Part    18      48.51       2.70     5.27  0.000
Error            60      30.67       0.51
Total            89    4054.40


Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Inspecto        0.5646   3   (4) + 3(3) + 30(1)
 2 Part           48.2926   3   (4) + 3(3) + 9(2)
 3 Inspecto*Part   0.7280   4   (4) + 3(3)
 4 Error           0.5111       (4)
```

(b) Estimate the variance components using the analysis of variance method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 0.51$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{2.70 - 0.51}{3} = 0.73$$

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{437.33 - 2.70}{3(3)} = 48.29$$

$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}^2_{\tau} = \frac{19.63 - 2.70}{10(3)} = 0.56$$

All estimates agree with the Minitab output.

13-11 Reconsider the data in Problem 5-6. Suppose that both factors, machines and operators, are chosen at random.

(a) Analyze the data from this experiment.

	Machine			
Operator	1	2	3	4
1	109	110	108	110
	110	115	109	108
2	110	110	111	114
	112	111	109	112
3	116	112	114	120
	114	115	119	117

The following Minitab output contains the analysis of variance and the variance component estimates:

Minitab Output

ANOVA: Strength versus Operator, Machine

```
Factor     Type Levels Values
Operator random      3     1     2     3
Machine  random      4     1     2     3     4

Analysis of Variance for Strength

Source              DF         SS         MS        F      P
Operator             2    160.333     80.167    10.77  0.010
Machine              3     12.458      4.153     0.56  0.662
Operator*Machine     6     44.667      7.444     1.96  0.151
Error               12     45.500      3.792
Total               23    262.958

Source              Variance Error Expected Mean Square for Each Term
                   component term (using restricted model)
 1 Operator           9.0903   3   (4) + 2(3) + 8(1)
 2 Machine           -0.5486   3   (4) + 2(3) + 6(2)
 3 Operator*Machine   1.8264   4   (4) + 2(3)
 4 Error              3.7917       (4)
```

(b) Find point estimates of the variance components using the analysis of variance method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.79167$$
$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{7.44444 - 3.79167}{2} = 1.82639$$
$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{4.15278 - 7.44444}{3(2)} < 0 \text{, assume } \hat{\sigma}^2_{\beta} = 0$$
$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}^2_{\tau} = \frac{80.16667 - 7.44444}{4(2)} = 9.09028$$

These results agree with the Minitab variance component analysis.

13-12 Reconsider the data in Problem 5-13. Suppose that both factors are random.

(a) Analyze the data from this experiment.

	Column	Factor		
Row Factor	1	2	3	4
1	36	39	36	32
2	18	20	22	20
3	30	37	33	34

Minitab Output

General Linear Model: Response versus Row, Column

```
Factor     Type Levels Values
Row      random      3 1 2 3
Column   random      4 1 2 3 4

Analysis of Variance for Response, using Adjusted SS for Tests

Source       DF     Seq SS     Adj SS     Adj MS       F       P
Row           2    580.500    580.500    290.250   60.40      **
Column        3     28.917     28.917      9.639    2.01      **
Row*Column    6     28.833     28.833      4.806      **
Error         0      0.000      0.000      0.000
Total        11    638.250

** Denominator of F-test is zero.

Expected Mean Squares, using Adjusted SS

Source          Expected Mean Square for Each Term
 1 Row          (4) +        (3) +  4.0000(1)
 2 Column       (4) +        (3) +  3.0000(2)
 3 Row*Column   (4) +        (3)
 4 Error        (4)

Error Terms for Tests, using Adjusted SS

Source          Error DF  Error MS  Synthesis of Error MS
 1 Row                 *     4.806  (3)
 2 Column              *     4.806  (3)
 3 Row*Column          *         *  (4)

Variance Components, using Adjusted SS

Source      Estimated Value
Row                71.3611
Column              1.6111
Row*Column          4.8056
Error               0.0000
```

(b) Estimate the variance components.

Because the experiment is unreplicated and the interaction term was included in the model, there is no estimate of MS_E, and therefore, no estimate of σ^2.

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{4.8056 - 0}{1} = 4.8056$$

$$\hat{\sigma}_{\beta}^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_{\beta}^2 = \frac{9.6389 - 4.8056}{3(1)} = 1.6111$$

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}_{\tau}^2 = \frac{290.2500 - 4.8056}{4(1)} = 71.3611$$

These estimates agree with the Minitab output.

13-13 Suppose that in Problem 5-11 the furnace positions were randomly selected, resulting in a mixed model experiment. Reanalyze the data from this experiment under this new assumption. Estimate the appropriate model components.

	Temperature (°C)		
Position	800	825	850
	570	1063	565
1	565	1080	510
	583	1043	590
	528	988	526
2	547	1026	538
	521	1004	532

The following analysis assumes a restricted model:

Minitab Output

```
ANOVA: Density versus Position, Temperature

Factor     Type Levels Values
Position random      2     1     2
Temperat  fixed      3   800   825   850

Analysis of Variance for Density

Source               DF         SS         MS       F      P
Position              1       7160       7160   16.00  0.002
Temperat              2     945342     472671 1155.52  0.001
Position*Temperat     2        818        409    0.91  0.427
Error                12       5371        448
Total                17     958691

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Position            745.83   4   (4) + 9(1)
 2 Temperat                     3   (4) + 3(3) + 6Q[2]
 3 Position*Temperat   -12.83   4   (4) + 3(3)
 4 Error               447.56       (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 447.56$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^2 = \frac{409 - 448}{3} < 0 \text{ assume } \hat{\sigma}_{\tau\beta}^2 = 0$$

$$\hat{\sigma}_{\tau}^{2} = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}_{\tau}^{2} = \frac{7160 - 448}{3(3)} = 745.83$$

These results agree with the Minitab output.

13-14 Reanalyze the measurement systems experiment in Problem 12-9, assuming that operators are a fixed factor. Estimate the appropriate model components.

The following analysis assumes a restricted model:

Minitab Output

ANOVA: Measurement versus Part, Operator

```
Factor     Type Levels Values
Part     random     10     1     2     3     4     5     6     7
                           8     9    10
Operator  fixed      2     1     2

Analysis of Variance for Measurem

Source           DF         SS         MS       F      P
Part              9     99.017     11.002    7.33  0.000
Operator          1      0.417      0.417    0.69  0.427
Part*Operator     9      5.417      0.602    0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part            1.5836   4   (4) + 6(1)
 2 Operator                 3   (4) + 3(3) + 30Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

$$\hat{\sigma}^{2} = MS_E \qquad \hat{\sigma}^{2} = 1.5000$$

$$\hat{\sigma}_{\tau\beta}^{2} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}_{\tau\beta}^{2} = \frac{0.60185 - 1.5000}{3} < 0 \text{ assume } \hat{\sigma}_{\tau\beta}^{2} = 0$$

$$\hat{\sigma}_{\tau}^{2} = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}_{\tau}^{2} = \frac{11.00185 - 1.50000}{2(3)} = 1.58364$$

These results agree with the Minitab output.

13-15 Reanalyze the measurement system experiment in Problem 13-10, assuming that operators are a fixed factor. Estimate the appropriate model components.

Minitab Output

ANOVA: Impedance versus Inspector, Part

```
Factor     Type Levels Values
Inspecto  fixed      3     1     2     3
Part     random     10     1     2     3     4     5     6     7
                           8     9    10

Analysis of Variance for Impedanc

Source           DF         SS         MS       F      P
Inspecto          2      39.27      19.63    7.28  0.005
Part              9    3935.96     437.33  855.64  0.000
Inspecto*Part    18      48.51       2.70    5.27  0.000
```

```
Error             60      30.67       0.51
Total             89    4054.40

Source             Variance Error Expected Mean Square for Each Term
                  component term (using restricted model)
 1 Inspecto                 3   (4) + 3(3) + 30Q[1]
 2 Part             48.5353   4   (4) + 9(2)
 3 Inspecto*Part     0.7280   4   (4) + 3(3)
 4 Error             0.5111       (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 0.51$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{2.70 - 0.51}{3} = 0.73$$

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_E}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{437.33 - 0.51}{3(3)} = 48.54$$

These results agree with the Minitab output.

13-16 In problem 5-6, suppose that there are only four machines of interest, but the operators were selected at random.

(a) What type of model is appropriate?

A mixed model is appropriate.

(b) Perform the analysis and estimate the model components.

The following analysis assumes a restricted model:

Minitab Output

```
ANOVA: Strength versus Operator, Machine

Factor     Type Levels Values
Operator random      3     1     2     3
Machine   fixed      4     1     2     3     4

Analysis of Variance for Strength

Source               DF         SS         MS        F      P
Operator              2    160.333     80.167    21.14  0.000
Machine               3     12.458      4.153     0.56  0.662
Operator*Machine      6     44.667      7.444     1.96  0.151
Error                12     45.500      3.792
Total                23    262.958

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Operator             9.547   4   (4) + 8(1)
 2 Machine                      3   (4) + 2(3) + 6Q[2]
 3 Operator*Machine     1.826   4   (4) + 2(3)
 4 Error                3.792       (4)
```

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.792$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{7.444 - 3.792}{2} = 1.826$$

$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_E}{bn} \qquad \hat{\sigma}^2_{\tau} = \frac{80.167 - 3.792}{4(2)} = 9.547$$

These results agree with the Minitab output.

13-17 By application of the expectation operator, develop the expected mean squares for the two-factor factorial, mixed model. Use the restricted model assumptions. Check your results with the expected mean squares given in Equation 13-23 to see that they agree.

The sums of squares may be written as

$$SS_A = bn\sum_{i=1}^{a}\left(\bar{y}_{i..} - \bar{y}_{...}\right)^2 , \quad SS_B = an\sum_{j=1}^{b}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2$$

$$SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2 , \quad SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{...}\right)^2$$

Using the model $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$, we may find that

$$\begin{aligned}
\bar{y}_{i..} &= \mu + \tau_i + \left(\overline{\tau}\overline{\beta}\right)_{i.} + \bar{\varepsilon}_{i..} \\
\bar{y}_{.j.} &= \mu + \beta_j + \bar{\varepsilon}_{.j.} \\
\bar{y}_{ij.} &= \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \bar{\varepsilon}_{ij.} \\
\bar{y}_{...} &= \mu + \overline{\beta}_{.} + \bar{\varepsilon}_{...}
\end{aligned}$$

Using the assumptions for the restricted form of the mixed model, $\tau_{.} = 0$, $(\tau\beta)_{.j} = 0$, which imply that $(\tau\beta)_{..} = 0$. Substituting these expressions into the sums of squares yields

$$\begin{aligned}
SS_A &= bn\sum_{i=1}^{a}\left(\tau + (\tau\beta)_{i.} + \bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...}\right)^2 \\
SS_B &= an\sum_{j=1}^{b}\left(\beta_j + \bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)^2 \\
SS_{AB} &= n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\left((\tau\beta)\right)_{ij} - (\tau\beta)_{i.} + \bar{\varepsilon}_{ij.} - \bar{\varepsilon}_{i..} - \bar{\varepsilon}_{.j.} + \bar{\varepsilon}_{...}\right)^2 \\
SS_E &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varepsilon_{ijk} - \bar{\varepsilon}_{ij.}\right)^2
\end{aligned}$$

Using the assumption that $E(\varepsilon_{ijk}) = 0$, $V(\varepsilon_{ijk}) = 0$, and $E(\varepsilon_{ijk} \cdot \varepsilon_{i'j'k'}) = 0$, we may divide each sum of squares by its degrees of freedom and take the expectation to produce

$$E(MS_A)=\sigma^2+\left[\frac{bn}{(a-1)}\right]E\sum_{i=1}^{a}\left(\tau_i+\left(\overline{\tau\beta}\right)_{i.}\right)^2$$
$$E(MS_B)=\sigma^2+\left[\frac{an}{(b-1)}\right]\sum_{j=1}^{b}\beta_j^2$$
$$E(MS_{AB})=\sigma^2+\left[\frac{n}{(a-1)(b-1)}\right]E\sum_{i=1}^{a}\sum_{j=1}^{b}\left((\tau\beta)_{ij}-\left(\overline{\tau\beta}\right)_{i.}\right)^2$$
$$E(MS_E)=\sigma^2$$

Note that $E(MS_B)$ and $E(MS_E)$ are the results given in Table 8-3. We need to simplify $E(MS_A)$ and $E(MS_{AB})$. Consider $E(MS_A)$

$$E(MS_A)=\sigma^2+\frac{bn}{a-1}\left[\sum_{i=1}^{a}E(\tau_i)^2+\sum_{i=1}^{a}E(\tau\beta)_{i.}^2+(crossproducts=0)\right]$$
$$E(MS_A)=\sigma^2+\frac{bn}{a-1}\left[\sum_{i=1}^{a}\tau_i^{\,2}+a\frac{\left[\frac{(a-1)}{a}\right]}{b}\sigma_{\tau\beta}^2\right]$$
$$E(MS_A)=\sigma^2+n\sigma_{\tau\beta}^2+\frac{bn}{a-1}\sum_{i=1}^{a}\tau_i^{\,2}$$

since $(\tau\beta)_{ij}$ is $NID\left(0,\frac{a-1}{a}\sigma_{\tau\beta}^2\right)$. Consider $E(MS_{AB})$

$$E(MS_{AB})=\sigma^2+\frac{n}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}E\left((\tau\beta)_{ij}-\left(\overline{\tau\beta}\right)_{i.}\right)^2$$
$$E(MS_{AB})=\sigma^2+\frac{n}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\frac{b-1}{b}\right)\left(\frac{a-1}{a}\right)\sigma_{\tau\beta}^2$$
$$E(MS_{AB})=\sigma^2+n\sigma_{\tau\beta}^2$$

Thus $E(MS_A)$ and $E(MS_{AB})$ agree with Equation 13-23..

13-18 Consider the three-factor factorial design in Example 13-6. Propose appropriate test statistics for all main effects and interactions. Repeat for the case where *A* and *B* are fixed and *C* is random.

If all three factors are random there are no exact tests on main effects. We could use the following:

$$A:F=\frac{MS_A+MS_{ABC}}{MS_{AB}+MS_{AC}}$$
$$B:F=\frac{MS_B+MS_{ABC}}{MS_{AB}+MS_{BC}}$$
$$C:F=\frac{MS_C+MS_{ABC}}{MS_{AC}+MS_{BC}}$$

If *A* and *B* are fixed and *C* is random, the expected mean squares are (assuming the restricted for m of the model):

Factor	F a i	F b j	R c k	R n l	E(*MS*)
τ_i	0	b	c	n	$\sigma^2 + bn\sigma_{\tau\gamma}^2 + bcn\sum\frac{\tau_i^2}{(a-1)}$
β_j	a	0	c	n	$\sigma^2 + an\sigma_{\beta\gamma}^2 + acn\sum\frac{\beta_j^2}{(b-1)}$
γ_k	a	b	1	n	$\sigma^2 + abn\sigma_{\gamma}^2$
$(\tau\beta)_{ij}$	0	0	c	n	$\sigma^2 + n\sigma_{\tau\beta\gamma}^2 + cn\sum\sum\frac{(\tau\beta)_{ji}^2}{(a-1)(b-1)}$
$(\tau\gamma)_{ik}$	0	b	1	n	$\sigma^2 + bn\sigma_{\tau\gamma}^2$
$(\beta\gamma)_{jk}$	a	0	1	n	$\sigma^2 + an\sigma_{\beta\gamma}^2$
$(\tau\beta\gamma)_{ijk}$	0	0	1	n	$\sigma^2 + n\sigma_{\tau\beta\gamma}^2$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2

These are exact tests for all effects.

13-19 Consider the experiment in Example 13-7. Analyze the data for the case where *A*, *B*, and *C* are random.

Minitab Output

ANOVA: Drop versus Temp, Operator, Gauge

```
Factor     Type Levels Values
Temp     random      3    60    75    90
Operator random      4     1     2     3     4
Gauge    random      3     1     2     3

Analysis of Variance for Drop

Source                  DF         SS         MS       F      P
Temp                     2    1023.36     511.68    2.30  0.171 x
Operator                 3     423.82     141.27    0.63  0.616 x
Gauge                    2       7.19       3.60    0.06  0.938 x
Temp*Operator            6    1211.97     202.00   14.59  0.000
Temp*Gauge               4     137.89      34.47    2.49  0.099
Operator*Gauge           6     209.47      34.91    2.52  0.081
Temp*Operator*Gauge     12     166.11      13.84    0.65  0.788
Error                   36     770.50      21.40
Total                   71    3950.32

x Not an exact F-test.

Source                  Variance Error Expected Mean Square for Each Term
                       component term (using restricted model)
 1 Temp                   12.044   *   (8) + 2(7) + 8(5) + 6(4) + 24(1)
 2 Operator               -4.544   *   (8) + 2(7) + 6(6) + 6(4) + 18(2)
 3 Gauge                  -2.164   *   (8) + 2(7) + 6(6) + 8(5) + 24(3)
 4 Temp*Operator          31.359   7   (8) + 2(7) + 6(4)
 5 Temp*Gauge              2.579   7   (8) + 2(7) + 8(5)
 6 Operator*Gauge          3.512   7   (8) + 2(7) + 6(6)
 7 Temp*Operator*Gauge    -3.780   8   (8) + 2(7)
 8 Error                  21.403       (8)
```

```
* Synthesized Test.

Error Terms for Synthesized Tests

Source                    Error DF  Error MS  Synthesis of Error MS
 1 Temp                       6.97    222.63   (4) + (5) - (7)
 2 Operator                   7.09    223.06   (4) + (6) - (7)
 3 Gauge                      5.98     55.54   (5) + (6) - (7)
```

Since all three factors are random there are no exact tests on main effects. Minitab uses an approximate *F* test for the these factors.

13-20 Derive the expected mean squares shown in Table 13-11.

Factor	F *a* *i*	R *b* *j*	R *c* *k*	R *n* *l*	E(*MS*)
τ_i	0	b	c	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma} + cn\sigma^2_{\tau\beta} + bcn\sum\frac{\tau_i^2}{(a-1)}$
β_j	a	1	c	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + acn\sigma^2_{\beta}$
γ_k	a	b	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma} + abn\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	0	1	c	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$
$(\tau\gamma)_{ik}$	0	b	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	a	1	1	n	$\sigma^2 + an\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	0	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$
ε_{ijkl}	1	1	1	1	σ^2

13-21 Consider a four-factor factorial experiment where factor *A* is at *a* levels, factor *B* is at *b* levels, factor *C* is at *c* levels, factor *D* is at *d* levels, and there are *n* replicates. Write down the sums of squares, the degrees of freedom, and the expected mean squares for the following cases. Do exact tests exist for all effects? If not, propose test statistics for those effects that cannot be directly tested. Assume the restricted model on all cases. You may use a computer package such as Minitab.

The four factor model is:

$$y_{ijklh} = \mu + \tau_i + \beta_j + \gamma_k + \delta_l + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\tau\delta)_{il} + (\beta\gamma)_{jk} + (\beta\delta)_{jl} + (\gamma\delta)_{kl} +$$
$$(\tau\beta\gamma)_{ijk} + (\tau\beta\delta)_{ijl} + (\beta\gamma\delta)_{jkl} + (\tau\gamma\delta)_{ikl} + (\tau\beta\gamma\delta)_{ijkl} + \varepsilon_{ijklh}$$

To simplify the expected mean square derivations, let capital Latin letters represent the factor effects or variance components. For example, $A = \frac{bcdn\sum\tau_i^2}{a-1}$, or $B = acdn\sigma^2_{\beta}$.

(a) *A*, *B*, *C*, and *D* are fixed factors.

Factor	F *a* *i*	F *b* *j*	F *c* *k*	F *d* *l*	R *n* *h*	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + A$
β_j	a	0	c	d	n	$\sigma^2 + B$

γ_k	a	b	0	d	n	σ^2+C
δ_l	a	b	c	0	n	σ^2+D
$(\tau\beta)_{ij}$	0	0	c	d	n	σ^2+AB
$(\tau\gamma)_{ik}$	0	b	0	d	n	σ^2+AC
$(\tau\delta)_{il}$	0	b	c	0	n	σ^2+AD
$(\beta\gamma)_{jk}$	a	0	0	d	n	σ^2+BC
$(\beta\delta)_{jl}$	a	0	c	0	n	σ^2+BD
$(\gamma\delta)_{kl}$	a	b	0	0	n	σ^2+CD
$(\tau\beta\gamma)_{ijk}$	0	0	0	d	n	σ^2+ABC
$(\tau\beta\delta)_{ijl}$	0	0	c	0	n	σ^2+ABD
$(\beta\gamma\delta)_{jkl}$	a	0	0	0	n	σ^2+BCD
$(\tau\gamma\delta)_{ikl}$	0	b	0	0	n	σ^2+ACD
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	0	0	n	σ^2+ABCD
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are exact tests for all effects. The results can also be generated in Minitab as follows:

Minitab Output

```
ANOVA: y versus A, B, C, D
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C         fixed      2     H     L
D         fixed      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    0.49  0.492
B            1       0.13       0.13    0.01  0.921
C            1       1.13       1.13    0.09  0.767
D            1       0.13       0.13    0.01  0.921
A*B          1       3.13       3.13    0.25  0.622
A*C          1       3.13       3.13    0.25  0.622
A*D          1       3.13       3.13    0.25  0.622
B*C          1       3.13       3.13    0.25  0.622
B*D          1       3.13       3.13    0.25  0.622
C*D          1       3.13       3.13    0.25  0.622
A*B*C        1       3.13       3.13    0.25  0.622
A*B*D        1      28.13      28.13    2.27  0.151
A*C*D        1       3.13       3.13    0.25  0.622
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  16   (16) + 16Q[1]
 2 B                  16   (16) + 16Q[2]
 3 C                  16   (16) + 16Q[3]
 4 D                  16   (16) + 16Q[4]
 5 A*B                16   (16) + 8Q[5]
 6 A*C                16   (16) + 8Q[6]
 7 A*D                16   (16) + 8Q[7]
 8 B*C                16   (16) + 8Q[8]
 9 B*D                16   (16) + 8Q[9]
10 C*D                16   (16) + 8Q[10]
11 A*B*C              16   (16) + 4Q[11]
12 A*B*D              16   (16) + 4Q[12]
13 A*C*D              16   (16) + 4Q[13]
14 B*C*D              16   (16) + 4Q[14]
```

```
15 A*B*C*D           16   (16) + 2Q[15]
16 Error       12.38      (16)
```

(b) *A*, *B*, *C*, and *D* are random factors.

	R	R	R	R	R	
	a	b	c	d	n	
Factor	i	j	k	l	h	E(MS)
τ_i	1	b	c	d	n	$\sigma^2 + ABCD + ACD + ABD + ABC + AD + AC + AB + A$
β_j	a	1	c	d	n	$\sigma^2 + ABCD + BCD + ABD + ABC + BD + BC + AB + B$
γ_k	a	b	1	d	n	$\sigma^2 + ABCD + ACD + BCD + ABC + AB + BC + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + ABCD + ACD + BCD + ABD + BD + AD + CD + D$
$(\tau\beta)_{ij}$	1	1	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	1	b	1	d	n	$\sigma^2 + ABCD + ABC + ACD + AC$
$(\tau\delta)_{il}$	1	b	c	1	n	$\sigma^2 + ABCD + ABD + ACD + AD$
$(\beta\gamma)_{jk}$	a	1	1	d	n	$\sigma^2 + ABCD + ABC + BCD + BC$
$(\beta\delta)_{jl}$	a	1	c	1	n	$\sigma^2 + ABCD + ABD + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + ABCD + ACD + BCD + CD$
$(\tau\beta\gamma)_{ijk}$	1	1	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	1	1	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	1	1	1	n	$\sigma^2 + ABCD + BCD$
$(\tau\gamma\delta)_{ikl}$	1	b	1	1	n	$\sigma^2 + ABCD + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	1	1	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

No exact tests exist on main effects or two-factor interactions. For main effects use statistics such as:

$$A{:}F = \frac{MS_A + MS_{ABC} + MS_{ABD} + MS_{ACD}}{MS_{AB} + MS_{AC} + MS_{AD} + MS_{ABCD}}$$

For testing two-factor interactions use statistics such as: $AB{:}F = \dfrac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A        random      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13      **
B            1       0.13       0.13      **
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.667 x
```

```
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A         1.7500   *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                           + 8(5) + 16(1)
 2 B         1.3750   *   (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
                           + 8(5) + 16(2)
 3 C        -0.1250   *   (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                           + 8(6) + 16(3)
 4 D         1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                           + 8(7) + 16(4)
 5 A*B      -3.1250   *   (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C       0.0000   *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D      -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C       0.0000   *   (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D      -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D       0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C     0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D     6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D     0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D     0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D  -4.6250  16   (16) + 2(15)
16 Error    12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.56         *   (5) + (8) + (9) - (11) - (12) - (14) + (15)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 8 B*C           0.33      3.13   (11) + (14) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

(c) *A* is fixed and *B*, *C*, and *D* are random.

Factor	F a i	R b j	R c k	R d l	R n h	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + ABCD + ACD + ABD + ABC + AD + AC + AB + A$
β_j	a	1	c	d	n	$\sigma^2 + BCD + ABD + BC + B$
γ_k	a	b	1	d	n	$\sigma^2 + BCD + BC + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + BCD + BD + CD + D$
$(\tau\beta)_{ij}$	0	1	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	1	d	n	$\sigma^2 + ABCD + ABC + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + ABCD + ABD + ACD + AD$

$(\beta\gamma)_{jk}$	a	1	1	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	1	c	1	n	$\sigma^2 + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + BCD + CD$
$(\tau\beta\gamma)_{ijk}$	0	1	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	1	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	1	1	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	1	1	n	$\sigma^2 + ABCD + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	1	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

No exact tests exist on main effects or two-factor interactions involving the fixed factor *A*. To test the fixed factor *A* use

$$A{:}\,F = \frac{MS_A + MS_{ABC} + MS_{ABD} + MS_{ACD}}{MS_{AB} + MS_{AC} + MS_{AD} + MS_{ABCD}}$$

Random main effects could be tested by, for example: $D{:}\,F = \dfrac{MS_D + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

For testing two-factor interactions involving *A* use: $AB{:}\,F = \dfrac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13      **
B            1       0.13       0.13    0.04  0.907 x
C            1       1.13       1.13    0.36  0.761 x
D            1       0.13       0.13    0.04  0.907 x
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    1.00  0.500
C*D          1       3.13       3.13    1.00  0.500
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
```

```
          component term (using restricted model)
 1 A                  *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                           + 8(5) + 16Q[1]
 2 B         -0.1875  *   (16) + 4(14) + 8(9) + 8(8) + 16(2)
 3 C         -0.1250  *   (16) + 4(14) + 8(10) + 8(8) + 16(3)
 4 D         -0.1875  *   (16) + 4(14) + 8(10) + 8(9) + 16(4)
 5 A*B       -3.1250  *   (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C        0.0000  *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250  *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000  14  (16) + 4(14) + 8(8)
 9 B*D        0.0000  14  (16) + 4(14) + 8(9)
10 C*D        0.0000  14  (16) + 4(14) + 8(10)
11 A*B*C      0.0000  15  (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15  (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15  (16) + 2(15) + 4(13)
14 B*C*D     -2.3125  16  (16) + 4(14)
15 A*B*C*D   -4.6250  16  (16) + 2(15)
16 Error     12.3750      (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.33      3.13   (8) + (9) - (14)
 3 C             0.33      3.13   (8) + (10) - (14)
 4 D             0.33      3.13   (9) + (10) - (14)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
```

(d) *A* and *B* are fixed and *C* and *D* are random.

	F	F	R	R	R	
	a	b	c	d	n	
Factor	i	j	k	l	h	E(*MS*)
τ_i	0	b	c	d	n	$\sigma^2 + ACD + AD + AC + A$
β_j	a	0	c	d	n	$\sigma^2 + BCD + BC + BD + B$
γ_k	a	b	1	d	n	$\sigma^2 + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + CD + D$
$(\tau\beta)_{ij}$	0	0	c	d	n	$\sigma^2 + ABCD + ABC + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	1	d	n	$\sigma^2 + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + ACD + AD$
$(\beta\gamma)_{jk}$	a	0	1	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	0	c	1	n	$\sigma^2 + BCD + BD$
$(\gamma\delta)_{kl}$	a	b	1	1	n	$\sigma^2 + CD$
$(\tau\beta\gamma)_{ijk}$	0	0	1	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	0	c	1	n	$\sigma^2 + ABCD + ABD$
$(\beta\gamma\delta)_{jkl}$	a	0	1	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	1	1	n	$\sigma^2 + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	1	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are no exact tests on the fixed factors *A* and *B*, or their two-factor interaction *AB*. The appropriate test statistics are:

$$A: F = \frac{MS_A + MS_{ACD}}{MS_{AC} + MS_{AD}}$$
$$B: F = \frac{MS_B + MS_{BCD}}{MS_{BC} + MS_{BD}}$$
$$AB: F = \frac{MS_{AB} + MS_{ABCD}}{MS_{ABC} + MS_{ABD}}$$

The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    1.96  0.604 x
B            1       0.13       0.13    0.04  0.907 x
C            1       1.13       1.13    0.36  0.656
D            1       0.13       0.13    0.04  0.874
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.500
A*D          1       3.13       3.13    1.00  0.500
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    1.00  0.500
C*D          1       3.13       3.13    0.25  0.622
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    0.25  0.622
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                  *   (16) + 4(13) + 8(7) + 8(6) + 16Q[1]
 2 B                  *   (16) + 4(14) + 8(9) + 8(8) + 16Q[2]
 3 C         -0.1250  10   (16) + 8(10) + 16(3)
 4 D         -0.1875  10   (16) + 8(10) + 16(4)
 5 A*B                *   (16) + 2(15) + 4(12) + 4(11) + 8Q[5]
 6 A*C        0.0000  13   (16) + 4(13) + 8(6)
 7 A*D        0.0000  13   (16) + 4(13) + 8(7)
 8 B*C        0.0000  14   (16) + 4(14) + 8(8)
 9 B*D        0.0000  14   (16) + 4(14) + 8(9)
10 C*D       -1.1563  16   (16) + 8(10)
11 A*B*C      0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D     -2.3125  16   (16) + 4(13)
14 B*C*D     -2.3125  16   (16) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A            0.33      3.13   (6) + (7) - (13)
```

```
 2 B             0.33      3.13   (8) + (9) - (14)
 5 A*B           0.98     28.13   (11) + (12) - (15)
```

(e) A, B and C are fixed and D is random.

	F	F	F	R	R	
	a	b	c	d	n	
Factor	i	j	k	l	h	E(MS)
τ_i	0	b	c	d	n	$\sigma^2 + AD + A$
β_j	a	0	c	d	n	$\sigma^2 + BD + B$
γ_k	a	b	0	d	n	$\sigma^2 + CD + C$
δ_l	a	b	c	1	n	$\sigma^2 + D$
$(\tau\beta)_{ij}$	0	0	c	d	n	$\sigma^2 + ABD + AB$
$(\tau\gamma)_{ik}$	0	b	0	d	n	$\sigma^2 + ACD + AC$
$(\tau\delta)_{il}$	0	b	c	1	n	$\sigma^2 + AD$
$(\beta\gamma)_{jk}$	a	0	0	d	n	$\sigma^2 + BCD + BC$
$(\beta\delta)_{jl}$	a	0	c	1	n	$\sigma^2 + BD$
$(\gamma\delta)_{kl}$	a	b	0	1	n	$\sigma^2 + CD$
$(\tau\beta\gamma)_{ijk}$	0	0	0	d	n	$\sigma^2 + ABCD + ABC$
$(\tau\beta\delta)_{ijl}$	0	0	c	1	n	$\sigma^2 + ABD$
$(\beta\gamma\delta)_{jkl}$	a	0	0	1	n	$\sigma^2 + BCD$
$(\tau\gamma\delta)_{ikl}$	0	b	0	1	n	$\sigma^2 + ACD$
$(\tau\beta\gamma\delta)_{ijkl}$	0	0	0	1	n	$\sigma^2 + ABCD$
$\varepsilon_{(ijkl)h}$	1	1	1	1	1	σ^2

There are exact tests for all effects. The results can also be generated in Minitab as follows:

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C         fixed      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F       P
A            1       6.13       6.13    1.96  0.395
B            1       0.13       0.13    0.04  0.874
C            1       1.13       1.13    0.36  0.656
D            1       0.13       0.13    0.01  0.921
A*B          1       3.13       3.13    0.11  0.795
A*C          1       3.13       3.13    1.00  0.500
A*D          1       3.13       3.13    0.25  0.622
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    0.25  0.622
C*D          1       3.13       3.13    0.25  0.622
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    2.27  0.151
A*C*D        1       3.13       3.13    0.25  0.622
B*C*D        1       3.13       3.13    0.25  0.622
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
```

```
 1 A                   7   (16) + 8(7) + 16Q[1]
 2 B                   9   (16) + 8(9) + 16Q[2]
 3 C                  10   (16) + 8(10) + 16Q[3]
 4 D        -0.7656  16   (16) + 16(4)
 5 A*B                12   (16) + 4(12) + 8Q[5]
 6 A*C                13   (16) + 4(13) + 8Q[6]
 7 A*D      -1.1563  16   (16) + 8(7)
 8 B*C                14   (16) + 4(14) + 8Q[8]
 9 B*D      -1.1563  16   (16) + 8(9)
10 C*D      -1.1563  16   (16) + 8(10)
11 A*B*C              15   (16) + 2(15) + 4Q[11]
12 A*B*D     3.9375  16   (16) + 4(12)
13 A*C*D    -2.3125  16   (16) + 4(13)
14 B*C*D    -2.3125  16   (16) + 4(14)
15 A*B*C*D  -4.6250  16   (16) + 2(15)
16 Error    12.3750       (16)
```

13-22 Reconsider cases (c), (d) and (e) of Problem 13-21. Obtain the expected mean squares assuming the unrestricted model. You may use a computer package such as Minitab. Compare your results with those for the restricted model.

A is fixed and *B*, *C*, and *D* are random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B        random      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13      **
B            1       0.13       0.13      **
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.667 x
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                   *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                            + 8(5) + Q[1]
 2 B          1.3750   *   (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
                            + 8(5) + 16(2)
 3 C         -0.1250   *   (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                            + 8(6) + 16(3)
 4 D          1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                            + 8(7) + 16(4)
```

```
 5 A*B       -3.1250   *   (16) + 2(15) + 4(12) + 4(11) + 8(5)
 6 A*C        0.0000   *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000   *   (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D       -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C      0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.56         *   (5) + (6) + (7) - (11) - (12) - (13) + (15)
 2 B             0.56         *   (5) + (8) + (9) - (11) - (12) - (14) + (15)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 8 B*C           0.33      3.13   (11) + (14) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

A and *B* are fixed and *C* and *D* are random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C        random      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    1.96  0.604 x
B            1       0.13       0.13    0.04  0.907 x
C            1       1.13       1.13    0.36  0.843 x
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.796 x
A*C          1       3.13       3.13    1.00  0.667 x
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.667 x
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                   *   (16) + 2(15) + 4(13) + 4(12) + 4(11) + 8(7) + 8(6)
                            + Q[1,5]
 2 B                   *   (16) + 2(15) + 4(14) + 4(12) + 4(11) + 8(9) + 8(8)
```

```
                               + Q[2,5]
 3 C         -0.1250   *   (16) + 2(15) + 4(14) + 4(13) + 4(11) + 8(10) + 8(8)
                               + 8(6) + 16(3)
 4 D          1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                               + 8(7) + 16(4)
 5 A*B                 *   (16) + 2(15) + 4(12) + 4(11) + Q[5]
 6 A*C        0.0000   *   (16) + 2(15) + 4(13) + 4(11) + 8(6)
 7 A*D       -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C        0.0000   *   (16) + 2(15) + 4(14) + 4(11) + 8(8)
 9 B*D       -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D        0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C      0.0000  15   (16) + 2(15) + 4(11)
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 1 A             0.33      3.13   (6) + (7) - (13)
 2 B             0.33      3.13   (8) + (9) - (14)
 3 C             0.14      3.13   (6) + (8) + (10) - (11) - (13) - (14) + (15)
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 5 A*B           0.98     28.13   (11) + (12) - (15)
 6 A*C           0.33      3.13   (11) + (13) - (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 8 B*C           0.33      3.13   (11) + (14) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

(e) *A*, *B* and *C* are fixed and *D* is random.

Minitab Output

ANOVA: y versus A, B, C, D

```
Factor     Type Levels Values
A         fixed      2     H     L
B         fixed      2     H     L
C         fixed      2     H     L
D        random      2     H     L

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1       6.13       6.13    1.96  0.395
B            1       0.13       0.13    0.04  0.874
C            1       1.13       1.13    0.36  0.656
D            1       0.13       0.13      **
A*B          1       3.13       3.13    0.11  0.795
A*C          1       3.13       3.13    1.00  0.500
A*D          1       3.13       3.13    0.11  0.796 x
B*C          1       3.13       3.13    1.00  0.500
B*D          1       3.13       3.13    0.11  0.796 x
C*D          1       3.13       3.13    1.00  0.667 x
A*B*C        1       3.13       3.13    1.00  0.500
A*B*D        1      28.13      28.13    9.00  0.205
A*C*D        1       3.13       3.13    1.00  0.500
B*C*D        1       3.13       3.13    1.00  0.500
A*B*C*D      1       3.13       3.13    0.25  0.622
Error       16     198.00      12.38
Total       31     264.88

x Not an exact F-test.

** Denominator of F-test is zero.
```

```
Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                  7   (16) + 2(15) + 4(13) + 4(12) + 8(7) + Q[1,5,6,11]
 2 B                  9   (16) + 2(15) + 4(14) + 4(12) + 8(9) + Q[2,5,8,11]
 3 C                 10   (16) + 2(15) + 4(14) + 4(13) + 8(10) + Q[3,6,8,11]
 4 D         1.3750   *   (16) + 2(15) + 4(14) + 4(13) + 4(12) + 8(10) + 8(9)
                           + 8(7) + 16(4)
 5 A*B               12   (16) + 2(15) + 4(12) + Q[5,11]
 6 A*C               13   (16) + 2(15) + 4(13) + Q[6,11]
 7 A*D      -3.1250   *   (16) + 2(15) + 4(13) + 4(12) + 8(7)
 8 B*C               14   (16) + 2(15) + 4(14) + Q[8,11]
 9 B*D      -3.1250   *   (16) + 2(15) + 4(14) + 4(12) + 8(9)
10 C*D       0.0000   *   (16) + 2(15) + 4(14) + 4(13) + 8(10)
11 A*B*C             15   (16) + 2(15) + Q[11]
12 A*B*D      6.2500  15   (16) + 2(15) + 4(12)
13 A*C*D      0.0000  15   (16) + 2(15) + 4(13)
14 B*C*D      0.0000  15   (16) + 2(15) + 4(14)
15 A*B*C*D   -4.6250  16   (16) + 2(15)
16 Error     12.3750       (16)

* Synthesized Test.

Error Terms for Synthesized Tests

Source       Error DF  Error MS  Synthesis of Error MS
 4 D             0.56         *   (7) + (9) + (10) - (12) - (13) - (14) + (15)
 7 A*D           0.98     28.13   (12) + (13) - (15)
 9 B*D           0.98     28.13   (12) + (14) - (15)
10 C*D           0.33      3.13   (13) + (14) - (15)
```

13-23 In Problem 5-17, assume that the three operators were selected at random. Analyze the data under these conditions and draw conclusions. Estimate the variance components.

Minitab Output

ANOVA: Score versus Cycle Time, Operator, Temperature

```
Factor     Type Levels Values
Cycle Ti  fixed      3    40    50    60
Operator random      3     1     2     3
Temperat  fixed      2   300   350

Analysis of Variance for Score

Source                        DF         SS         MS       F      P
Cycle Ti                       2    436.000    218.000    2.45  0.202
Operator                       2    261.333    130.667   39.86  0.000
Temperat                       1     50.074     50.074    8.89  0.096
Cycle Ti*Operator              4    355.667     88.917   27.13  0.000
Cycle Ti*Temperat              2     78.815     39.407    3.41  0.137
Operator*Temperat              2     11.259      5.630    1.72  0.194
Cycle Ti*Operator*Temperat     4     46.185     11.546    3.52  0.016
Error                         36    118.000      3.278
Total                         53   1357.333

Source                        Variance Error Expected Mean Square for Each Term
                             component term (using restricted model)
 1 Cycle Ti                             4   (8) + 6(4) + 18Q[1]
 2 Operator                     7.0772   8   (8) + 18(2)
 3 Temperat                             6   (8) + 9(6) + 27Q[3]
 4 Cycle Ti*Operator           14.2731   8   (8) + 6(4)
 5 Cycle Ti*Temperat                    7   (8) + 3(7) + 9Q[5]
 6 Operator*Temperat            0.2613   8   (8) + 9(6)
 7 Cycle Ti*Operator*Temperat   2.7562   8   (8) + 3(7)
 8 Error                        3.2778       (8)
```

The following calculations agree with the Minitab results:

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.27778$$

$$\hat{\sigma}^2_{\tau\beta\gamma} = \frac{MS_{ABC} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta\gamma} = \frac{11.546296 - 3.277778}{3} = 2.7562$$

$$\hat{\sigma}^2_{\beta\gamma} = \frac{MS_{BC} - MS_E}{an} \qquad \hat{\sigma}^2_{\beta\gamma} = \frac{88.91667 - 3.277778}{2(3)} = 14.27315$$

$$\hat{\sigma}^2_{\tau\gamma} = \frac{MS_{AC} - MS_E}{bn} \qquad \hat{\sigma}^2_{\tau\gamma} = \frac{5.629630 - 3.277778}{3(3)} = 0.26132$$

$$\hat{\sigma}^2_{\gamma} = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}^2_{\gamma} = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$

13-24 Consider the three-factor model

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \varepsilon_{ijk}$$

Assuming that all the factors are random, develop the analysis of variance table, including the expected mean squares. Propose appropriate test statistics for all effects.

Source	DF	E(MS)
A	a-1	$\sigma^2 + c\sigma^2_{\tau\beta} + bc\sigma^2_{\tau}$
B	b-1	$\sigma^2 + c\sigma^2_{\tau\beta} + a\sigma^2_{\beta\gamma} + ac\sigma^2_{\beta}$
C	c-1	$\sigma^2 + a\sigma^2_{\beta\gamma} + ab\sigma^2_{\gamma}$
AB	(a-1)(b-1)	$\sigma^2 + c\sigma^2_{\tau\beta}$
BC	(b-1)(c-1)	$\sigma^2 + a\sigma^2_{\beta\gamma}$
Error (AC + ABC)	b(a-1)(c-1)	σ^2
Total	abc-1	

There are exact tests for all effects except B. To test B, use the statistic $F = \frac{MS_B + MS_E}{MS_{AB} + MS_{BC}}$

13-25 The three-factor model for a single replicate is

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + (\tau\gamma)_{ik} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

If all the factors are random, can any effects be tested? If the three-factor interaction and the $(\tau\beta)_{ij}$ interaction do not exist, can all the remaining effects be tested.

The expected mean squares are found by referring to Table 12-9, deleting the line for the error term $\varepsilon_{(ijk)l}$ and setting n=1. The three-factor interaction now cannot be tested; however, exact tests exist for the two-factor interactions and approximate F tests can be conducted for the main effects. For example, to test the main effect of A, use

$$F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}$$

If $(\tau\beta\gamma)_{ijk}$ and $(\tau\beta)_{ij}$ can be eliminated, the model becomes

$$y_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + (\tau\gamma)_{ik} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

For this model, the analysis of variance is

Source	DF	E(MS)
A	a-1	$\sigma^2 + b\sigma^2_{\tau\gamma} + bc\sigma^2_{\tau}$
B	b-1	$\sigma^2 + a\sigma^2_{\beta\gamma} + ac\sigma^2_{\beta}$
C	c-1	$\sigma^2 + a\sigma^2_{\beta\gamma} + b\sigma^2_{\tau\gamma} + ab\sigma^2_{\gamma}$
AC	(a-1)(c-1)	$\sigma^2 + b\sigma^2_{\tau\gamma}$
BC	(b-1)(c-1)	$\sigma^2 + a\sigma^2_{\beta\gamma}$
Error (AB + ABC)	c(a-1)(b-1)	σ^2
Total	abc-1	

There are exact tests for all effect except C. To test the main effect of C, use the statistic:

$$F = \frac{MS_C + MS_E}{MS_{BC} + MS_{AC}}$$

13-26 In Problem 5-6, assume that both machines and operators were chosen randomly. Determine the power of the test for detecting a machine effect such that $\sigma^2_{\beta} = \sigma^2$, where σ^2_{β} is the variance component for the machine factor. Are two replicates sufficient?

$$\lambda = \sqrt{1 + \frac{an\sigma^2_{\beta}}{\sigma^2 + n\sigma^2_{\tau\beta}}}$$

If $\sigma^2_{\beta} = \sigma^2$, then an estimate of $\sigma^2 = \sigma^2_{\beta} = 3.79$, and an estimate of $\sigma^2 = n\sigma^2_{\tau\beta} = 7.45$, from the analysis of variance table. Then

$$\lambda = \sqrt{1 + \frac{(3)(2)(3.79)}{7.45}} = \sqrt{2.22} = 1.49$$

and the other OC curve parameters are $\upsilon_1 = 3$ and $\upsilon_2 = 6$. This results in $\beta \approx 0.75$ approximately, with $\alpha = 0.05$, or $\beta \approx 0.9$ with $\alpha = 0.01$. Two replicates does not seem sufficient.

13-27 In the two-factor mixed model analysis of variance, show that Cov$\left[(\tau\beta)_{ij}, (\tau\beta)_{i'j}\right] = -(1/a)^2_{\tau\beta\sigma}$ for $i \neq i'$.

Since $\sum_{i=1}^{a}(\tau\beta)_{ij} = 0$ (constant) we have $V\left[\sum_{i=1}^{a}(\tau\beta)_{ij}\right] = 0$, which implies that

$$\sum_{i=1}^{a} V(\tau\beta)_{ij} + 2\binom{a}{2} Cov\left[(\tau\beta)_{ij},(\tau\beta)_{i'j}\right] = 0$$

$$a\left[\frac{a-1}{a}\right]\sigma_{\tau\beta}^2 + \frac{a!}{2!\,(a-2)!}(2)Cov\left[(\tau\beta)_{ij},(\tau\beta)_{i'j}\right] = 0$$

$$(a-1)\sigma_{\tau\beta}^2 + a(a-1)Cov\left[(\tau\beta)_{ij},\tau(\beta)_{i'j}\right] = 0$$

$$Cov\left[\tau(\beta)_{ij},(\tau\beta)_{i'j}\right] = -\left(\frac{1}{a}\right)\sigma_{\tau\beta}^2$$

13-28 Show that the method of analysis of variance always produces unbiased point estimates of the variance component in any random or mixed model.

Let **g** be the vector of mean squares from the analysis of variance, chosen so that $E(\mathbf{g})$ does not contain any fixed effects. Let $\boldsymbol{\sigma}^2$ be the vector of variance components such that $E(\mathbf{g}) = \mathbf{A}\boldsymbol{\sigma}^2$, where **A** is a matrix of constants. Now in the analysis of variance method of variance component estimation, we equate observed and expected mean squares, i.e.

$$\mathbf{g} = \mathbf{A}\mathbf{s}^2 \Rightarrow \hat{\mathbf{s}}^2 = \mathbf{A}^{-1}\mathbf{g}$$

Since $\mathbf{A}^{-1}$ always exists then,

$$E(\mathbf{s}^2) = E(\mathbf{A}^{-1})\mathbf{g} = \mathbf{A}^{-1}E(\mathbf{g}) = \mathbf{A}^{-1}(\mathbf{A}\mathbf{s}^2) = \mathbf{s}^2$$

Thus $\hat{\boldsymbol{\sigma}}^2$ is an unbiased estimator of $\boldsymbol{\sigma}^2$. This and other properties of the analysis of variance method are discussed by Searle (1971a).

13-29 Invoking the usual normality assumptions, find an expression for the probability that a negative estimate of a variance component will be obtained by the analysis of variance method. Using this result, write a statement giving the probability that $\hat{\sigma}_\tau^2 < 0$ in a one-factor analysis of variance. Comment on the usefulness of this probability statement.

Suppose $\hat{\sigma}^2 = \dfrac{MS_1 - MS_2}{c}$, where MS_i for i=1,2 are two mean squares and c is a constant. The probability that $\hat{\sigma}_\tau^2 < 0$ (negative) is

$$P\left\{\hat{\sigma}^2 < 0\right\} = P\left\{MS_1 - MS_2 < 0\right\} = P\left\{\frac{MS_1}{MS_2} < 1\right\} = P\left\{\frac{\frac{MS_1}{E(MS_1)}}{\frac{MS_2}{E(MS_2)}} < \frac{E(MS_1)}{E(MS_2)}\right\} = P\left\{F_{u,v} < \frac{E(MS_1)}{E(MS_2)}\right\}$$

where u is the number of degrees of freedom for MS_1 and v is the number of degrees of freedom for MS_2. For the one-way model, this equation reduces to

$$P\left\{\hat{\sigma}^2 < 0\right\} = P\left\{F_{a-1,N-a} < \frac{\sigma^2}{\sigma^2 + n\sigma_\tau^2}\right\} = P\left\{F_{a-1,N-a} < \frac{1}{1+nk}\right\}$$

where $k = \dfrac{\sigma_\tau^2}{\sigma^2}$. Using arbitrary values for some of the parameters in this equation will give an experimenter some idea of the probability of obtaining a negative estimate of $\hat{\sigma}_\tau^2 < 0$.

13-30 Analyze the data in Problem 13-9, assuming that the operators are fixed, using both the unrestricted and restricted forms of the mixed models. Compare the results obtained from the two models.

The restricted model is as follows:

Minitab Output

```
ANOVA: Measurement versus Part, Operator

Factor     Type Levels Values
Part     random     10     1     2     3     4     5     6     7
                           8     9    10
Operator  fixed      2     1     2

Analysis of Variance for Measurem

Source           DF         SS         MS       F      P
Part              9     99.017     11.002    7.33  0.000
Operator          1      0.417      0.417    0.69  0.427
Part*Operator     9      5.417      0.602    0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Part            1.5836   4   (4) + 6(1)
 2 Operator                 3   (4) + 3(3) + 30Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

The second approach is the unrestricted mixed model.

Minitab Output

```
ANOVA: Measurement versus Part, Operator

Factor     Type Levels Values
Part     random     10     1     2     3     4     5     6     7
                           8     9    10
Operator  fixed      2     1     2

Analysis of Variance for Measurem

Source           DF         SS         MS       F      P
Part              9     99.017     11.002   18.28  0.000
Operator          1      0.417      0.417    0.69  0.427
Part*Operator     9      5.417      0.602    0.40  0.927
Error            40     60.000      1.500
Total            59    164.850

Source           Variance Error Expected Mean Square for Each Term
                component term (using unrestricted model)
 1 Part            1.7333   3   (4) + 3(3) + 6(1)
 2 Operator                 3   (4) + 3(3) + Q[2]
 3 Part*Operator  -0.2994   4   (4) + 3(3)
 4 Error           1.5000       (4)
```

Source	Sum of Squares	DF	Mean Square	E(MS)	*F*-test	*F*
A	0.416667	*a*-1=1	0.416667	$\sigma^2 + n\sigma_{\tau\beta}^2 + bn\frac{\sum_{i=1}^{a}\tau_i^2}{a-1}$	$F = \frac{MS_A}{MS_{AB}}$	0.692
B	99.016667	*b*-1=9	11.00185	$\sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_\beta^2$	$F = \frac{MS_B}{MS_{AB}}$	18.28
AB	5.416667	(*a*-1)(*b*-1)=9	0.60185	$\sigma^2 + n\sigma_{\tau\beta}^2$	$F = \frac{MS_{AB}}{MS_E}$	0.401
Error	60.000000	40	1.50000	σ^2		
Total	164.85000	*nabc*-1=59				

In the unrestricted model, the *F*-test for *B* is different. The *F*-test for *B* in the unrestricted model should generally be more conservative, since MS_{AB} will generally be larger than MS_E. However, this is not the case with this particular experiment.

13-31 Consider the two-factor mixed model. Show that the standard error of the fixed factor mean (e.g. *A*) is $[MS_{AB} / bn]^{1/2}$.

The standard error is often used in Duncan's Multiple Range test. Duncan's Multiple Range Test requires the variance of the difference in two means, say

$$V(\bar{y}_{i..} - \bar{y}_{m..})$$

where rows are fixed and columns are random. Now, assuming all model parameters to be independent, we have the following:

$$(\bar{y}_{i..} - \bar{y}_{m..}) = \tau_i - \tau_m + \frac{1}{b}\sum_{j=1}^{b}(\tau\beta)_{ij} - \frac{1}{b}\sum_{j=1}^{b}(\tau\beta)_{mj} + \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{mjk}$$

and

$$V(\bar{y}_{i..} - \bar{y}_{m..}) = \left(\frac{1}{b}\right)^2 b\sigma_{\tau\beta}^2 + \left(\frac{1}{b}\right)^2 b\sigma_{\tau\beta}^2 + \left(\frac{1}{bn}\right)^2 bn\sigma^2 + \left(\frac{1}{bn}\right)^2 bn\sigma^2 = \frac{2\left(\sigma^2 + n\sigma_{\tau\beta}^2\right)}{bn}$$

Since MS_{AB} estimates $\sigma^2 + n\sigma_{\tau\beta}^2$, we would use

$$\frac{2\,MS_{AB}}{bn}$$

as the standard error to test the difference. However, the table of ranges for Duncan's Multiple Range test already include the constant 2.

13-32 Consider the variance components in the random model from Problem 13-9.

(a) Find an exact 95 percent confidence interval on σ^2.

$$\frac{f_E MS_E}{\chi^2_{\alpha/2,f_E}} \le \sigma^2 \le \frac{f_E MS_E}{\chi^2_{1-\alpha/2,f_E}}$$
$$\frac{(40)(1.5)}{59.34} \le \sigma^2 \le \frac{(40)(1.5)}{24.43}$$
$$1.011 \le \sigma^2 \le 2.456$$

(b) Find approximate 95 percent confidence intervals on the other variance components using the Satterthwaite method.

$\hat{\sigma}^2_{\tau\beta}$ and $\hat{\sigma}^2_{\tau}$ are negative, and the Satterthwaithe method does not apply. The confidence interval on $\hat{\sigma}^2_{\beta}$ is

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}^2_{\beta} = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$
$$r = \frac{(MS_B - MS_{AB})^2}{\frac{MS_B^2}{(b-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)}} = \frac{(11.001852 - 0.6018519)^2}{\frac{1.001852^2}{(9)} + \frac{0.6018519^2}{(1)(9)}} = 8.01826$$
$$\frac{r\hat{\sigma}^2_{O}}{\chi^2_{\alpha/2,r}} \le \sigma^2_{\beta} \le \frac{r\hat{\sigma}^2_{\beta}}{\chi^2_{1-\alpha/2,r}}$$
$$\frac{(8.01826)(1.7333)}{17.55752} \le \sigma^2_{\beta} \le \frac{(8.01826)(1.7333)}{2.18950}$$
$$0.79157 \le \sigma^2_{\beta} \le 6.34759$$

13-33 Use the experiment described in Problem 5-6 and assume that both factor are random. Find an exact 95 percent confidence interval on σ^2. Construct approximate 95 percent confidence interval on the other variance components using the Satterthwaite method.

$$\hat{\sigma}^2 = MS_E \qquad \hat{\sigma}^2 = 3.79167$$
$$\frac{f_E MS_E}{\chi^2_{\alpha/2,f_E}} \le \sigma^2 \le \frac{f_E MS_E}{\chi^2_{1-\alpha/2,f_E}}$$
$$\frac{(12)(3.79167)}{23.34} \le \sigma^2 \le \frac{(12)(3.79167)}{4.40}$$
$$1.9494 \le \sigma^2 \le 10.3409$$

Satterthwaite Method:

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \qquad \hat{\sigma}^2_{\tau\beta} = \frac{7.44444 - 3.79167}{2} = 1.82639$$
$$r = \frac{(MS_{AB} - MS_E)^2}{\frac{MS_{AB}^2}{(a-1)(b-1)} + \frac{MS_E^2}{df_E}} = \frac{(7.44444 - 3.79167)^2}{\frac{7.44444^2}{(2)(3)} + \frac{3.79167^2}{(12)}} = 2.2940$$
$$\frac{r\hat{\sigma}^2_{\beta}}{\chi^2_{\alpha/2,r}} \le \sigma^2_{\beta} \le \frac{r\hat{\sigma}^2_{\beta}}{\chi^2_{1-\alpha/2,r}}$$

$$\frac{(2.2940)(1.82639)}{7.95918} \le \sigma_\beta^2 \le \frac{(2.2940)(1.82639)}{0.09998}$$
$$0.52640 \le \sigma_\beta^2 \le 41.90577$$

$\hat{\sigma}_\beta^2 < 0$, this variance component does not have a confidence interval using Satterthwaite's Method.

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_{AB}}{bn} \qquad \hat{\sigma}_\tau^2 = \frac{80.16667 - 7.44444}{4(2)} = 9.09028$$
$$r = \frac{(MS_A - MS_{AB})^2}{\frac{MS_A^2}{(a-1)} + \frac{MS_{AB}^2}{(a-1)(b-1)}} = \frac{(80.16667 - 7.44444)^2}{\frac{80.16667^2}{(2)} + \frac{7.44444^2}{(2)(3)}} = 1.64108$$
$$\frac{r\hat{\sigma}_\tau^2}{\chi^2_{\alpha/2,r}} \le \sigma_\tau^2 \le \frac{r\hat{\sigma}_\tau^2}{\chi^2_{1-\alpha/2,r}}$$
$$\frac{(1.64108)(9.09028)}{6.53295} \le \sigma_\tau^2 \le \frac{(1.64108)(9.09028)}{0.03205}$$
$$2.28348 \le \sigma_\tau^2 \le 465.45637$$

13-34 Consider the three-factor experiment in Problem 5-17 and assume that operators were selected at random. Find an approximate 95 percent confidence interval on the operator variance component.

$$\hat{\sigma}_\gamma^2 = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}_\gamma^2 = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$
$$r = \frac{(MS_C - MS_E)^2}{\frac{MS_C^2}{(c-1)} + \frac{MS_E^2}{df_E}} = \frac{(130.66667 - 3.27778)^2}{\frac{130.66667^2}{(2)} + \frac{3.27778^2}{(36)}} = 1.90085$$
$$\frac{r\hat{\sigma}_\gamma^2}{\chi^2_{\alpha/2,r}} \le \sigma_\gamma^2 \le \frac{r\hat{\sigma}_\gamma^2}{\chi^2_{1-\alpha/2,r}}$$
$$\frac{(1.90085)(7.07716)}{9.15467} \le \sigma_\gamma^2 \le \frac{(1.90085)(7.07716)}{0.04504}$$
$$1.46948 \le \sigma_\gamma^2 \le 4298.66532$$

13-35 Rework Problem 13-32 using the modified large-sample approach described in Section 13-7.2. Compare the two sets of confidence intervals obtained and discuss.

$$\hat{\sigma}_O^2 = \hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an} \qquad \hat{\sigma}_O^2 = \frac{11.001852 - 0.6018519}{2(3)} = 1.7333$$

$$G_1 = 1 - \frac{1}{F_{0.05,9,\infty}} = 1 - \frac{1}{1.88} = 0.46809$$

$$H_1 = \frac{1}{F_{.95,9_i,\infty}} - 1 = \frac{1}{\frac{\chi^2_{.95,9}}{9}} - 1 = .\frac{1}{0.370} - 1 = 1.7027$$

$$G_{ij} = \frac{\left(F_{\alpha,f_i,f_j} - 1\right)^2 - G_1^2 F_{\alpha,f_i,f_j} - H_1^2}{F_{\alpha,f_i,f_j}} = \frac{(3.18-1)^2 - (0.46809)^2(3.18) - 1.7027^2}{3.18} = 0.36366$$

$$V_L = G_1^2 c_1^2 MS_B^2 + H_1^2 c_2^2 MS_{AB}^2 + G_{11} c_1 c_2 MS_B MS_{AB}$$

$$V_L = (0.46809)^2 \left(\frac{1}{6}\right)^2 (11.00185)^2 + (1.7027)^2 \left(\frac{1}{6}\right)^2 (0.60185)^2 + (0.36366)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)(11.00185)(0.60185)$$

$$V_L = 0.83275$$

$$L = \hat{\sigma}_\beta^2 - \sqrt{V_L} = 1.7333 - \sqrt{0.83275} = 0.82075$$

13-36 Rework Problem 13-34 using the modified large-sample method described in Section 13-7.2. Compare this confidence interval with he one obtained previously and discuss.

$$\hat{\sigma}_\gamma^2 = \frac{MS_C - MS_E}{abn} \qquad \hat{\sigma}_\gamma^2 = \frac{130.66667 - 3.277778}{2(3)(3)} = 7.07716$$

$$G_1 = 1 - \frac{1}{F_{0.05,3,\infty}} = 1 - \frac{1}{2.60} = 0.61538$$

$$H_1 = \frac{1}{F_{.95,36,\infty}} - 1 = \frac{1}{\frac{\chi^2_{.95,36}}{36}} - 1 = .\frac{1}{0.64728} - 1 = 0.54493$$

$$G_{ij} = \frac{\left(F_{\alpha,f_i,f_j} - 1\right)^2 - G_1^2 F_{\alpha,f_i,f_j} - H_1^2}{F_{\alpha,f_i,f_j}} = \frac{(2.88-1)^2 - (0.61538)^2(2.88) - 0.54493^2}{2.88} = 0.74542$$

$$V_L = G_1^2 c_1^2 MS_B^2 + H_1^2 c_2^2 MS_{AB}^2 + G_{11} c_1 c_2 MS_B MS_{AB}$$

$$V_L = (0.61538)^2 \left(\frac{1}{18}\right)^2 (130.66667)^2 + (0.54493)^2 \left(\frac{1}{18}\right)^2 (3.27778)^2 + (0.74542)\left(\frac{1}{18}\right)\left(\frac{1}{18}\right)(130.66667)(3.27778)$$

$$V_L = 20.95112$$

$$L = \hat{\sigma}_\gamma^2 - \sqrt{V_L} = 7.07716 - \sqrt{20.95112} = 2.49992$$

Chapter 14
Nested and Split-Plot Designs
Solutions

In this chapter we have not shown residual plots and other diagnostics to conserve space. A complete analysis would, of course, include these model adequacy checking procedures.

14-1 A rocket propellant manufacturer is studying the burning rate of propellant from three production processes. Four batches of propellant are randomly selected from the output of each process and three determinations of burning rate are made on each batch. The results follow. Analyze the data and draw conclusions.

	Process 1				Process 2				Process 3			
Batch	1	2	3	4	1	2	3	4	1	2	3	4
	25	19	15	15	19	23	18	35	14	35	38	25
	30	28	17	16	17	24	21	27	15	21	54	29
	26	20	14	13	14	21	17	25	20	24	50	33

Minitab Output

ANOVA: Burn Rate versus Process, Batch

```
Factor           Type Levels Values
Process         fixed      3     1     2     3
Batch(Process) random      4     1     2     3     4

Analysis of Variance for Burn Rat

Source            DF         SS         MS       F      P
Process            2     676.06     338.03    1.46  0.281
Batch(Process)     9    2077.58     230.84   12.20  0.000
Error             24     454.00      18.92
Total             35    3207.64

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Process                  2   (3) + 3(2) + 12Q[1]
 2 Batch(Process)    70.64   3   (3) + 3(2)
 3 Error             18.92       (3)
```

There is no significant effect on mean burning rate among the different processes; however, different batches from the same process have significantly different burning rates.

14-2 The surface finish of metal parts made on four machines is being studied. An experiment is conducted in which each machine is run by three different operators and two specimens from each operator are collected and tested. Because of the location of the machines, different operators are used on each machine, and the operators are chosen at random. The data are shown in the following table. Analyze the data and draw conclusions.

	Machine 1			Machine 2			Machine 3			Machine 4		
Operator	1	2	3	1	2	3	1	2	3	1	2	3
	79	94	46	92	85	76	88	53	46	36	40	62
	62	74	57	99	79	68	75	56	57	53	56	47

Minitab Output

ANOVA: Finish versus Machine, Operator

```
Factor             Type Levels Values
Machine           fixed      4     1     2     3     4
Operator(Machine) random     3     1     2     3

Analysis of Variance for Finish

Source               DF         SS         MS       F      P
Machine               3    3617.67    1205.89    3.42  0.073
Operator(Machine)     8    2817.67     352.21    4.17  0.013
Error                12    1014.00      84.50
Total                23    7449.33

Source               Variance Error Expected Mean Square for Each Term
                    component term (using restricted model)
 1 Machine                      2   (3) + 2(2) + 6Q[1]
 2 Operator(Machine)   133.85   3   (3) + 2(2)
 3 Error                84.50       (3)
```

There is a slight effect on surface finish due to the different processes; however, the different operators running the same machine have significantly different surface finish.

14-3 A manufacturing engineer is studying the dimensional variability of a particular component that is produced on three machines. Each machine has two spindles, and four components are randomly selected from each spindle. These results follow. Analyze the data, assuming that machines and spindles are fixed factors.

	Machine 1		Machine 2		Machine 3	
Spindle	1	2	1	2	1	2
	12	8	14	12	14	16
	9	9	15	10	10	15
	11	10	13	11	12	15
	12	8	14	13	11	14

Minitab Output

ANOVA: Variability versus Machine, Spindle

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Spindle(Machine)  fixed      2     1     2

Analysis of Variance for Variabil

Source              DF         SS         MS       F      P
Machine              2     55.750     27.875   18.93  0.000
Spindle(Machine)     3     43.750     14.583    9.91  0.000
Error               18     26.500      1.472
Total               23    126.000
```

There is a significant effect on dimensional variability due to the machine and spindle factors.

14-4 To simplify production scheduling, an industrial engineer is studying the possibility of assigning one time standard to a particular class of jobs, believing that differences between jobs is negligible. To see if this simplification is possible, six jobs are randomly selected. Each job is given to a different group of three operators. Each operator completes the job twice at different times during the week, and the following results were obtained. What are your conclusions about the use of a common time standard for all jobs in this class? What value would you use for the standard?

Job	Operator 1		Operator 2		Operator 3	
1	158.3	159.4	159.2	159.6	158.9	157.8
2	154.6	154.9	157.7	156.8	154.8	156.3

3	162.5	162.6	161.0	158.9	160.5	159.5
4	160.0	158.7	157.5	158.9	161.1	158.5
5	156.3	158.1	158.3	156.9	157.7	156.9
6	163.7	161.0	162.3	160.3	162.6	161.8

Minitab Output

```
ANOVA: Time versus Job, Operator

Factor          Type Levels Values
Job           random      6     1     2     3     4     5     6
Operator(Job) random      3     1     2     3

Analysis of Variance for Time

Source           DF         SS         MS       F      P
Job               5    148.111     29.622   27.89  0.000
Operator(Job)    12     12.743      1.062    0.69  0.738
Error            18     27.575      1.532
Total            35    188.430

Source           Variance Error Expected Mean Square for Each Term
                component term (using restricted model)
 1 Job             4.7601   2   (3) + 2(2) + 6(1)
 2 Operator(Job)  -0.2350   3   (3) + 2(2)
 3 Error           1.5319       (3)
```

The jobs differ significantly; the use of a common time standard would likely not be a good idea.

14-5 Consider the three-stage nested design shown in Figure 13-5 to investigate alloy hardness. Using the data that follow, analyze the design, assuming that alloy chemistry and heats are fixed factors and ingots are random.

Alloy Chemistry	1						2					
Heats	1		2		3		1		2		3	
Ingots	1	2	1	2	1	2	1	2	1	2	1	2
	40	27	95	69	65	78	22	23	83	75	61	35
	63	30	67	47	54	45	10	39	62	64	77	42

Minitab Output

```
ANOVA: Hardness versus Alloy, Heat, Ingot

Factor              Type Levels Values
Alloy              fixed      2     1     2
Heat(Alloy)        fixed      3     1     2     3
Ingot(Alloy Heat) random      2     1     2

Analysis of Variance for Hardness

Source              DF         SS         MS       F      P
Alloy                1      315.4      315.4    0.85  0.392
Heat(Alloy)          4     6453.8     1613.5    4.35  0.055
Ingot(Alloy Heat)    6     2226.3      371.0    2.08  0.132
Error               12     2141.5      178.5
Total               23    11137.0

Source              Variance Error Expected Mean Square for Each Term
                   component term (using restricted model)
 1 Alloy                       3   (4) + 2(3) + 12Q[1]
 2 Heat(Alloy)                 3   (4) + 2(3) + 4Q[2]
 3 Ingot(Alloy Heat)    96.29   4   (4) + 2(3)
 4 Error               178.46       (4)
```

Alloy hardness differs significantly due to the different heats within each alloy.

14-6 Reanalyze the experiment in Problem 14-5 using the unrestricted form of the mixed model. Comment on any differences you observe between the restricted and unrestricted model results. You may use a computer software package.

Minitab Output

```
ANOVA: Hardness versus Alloy, Heat, Ingot

Factor              Type Levels Values
Alloy              fixed      2     1     2
Heat(Alloy)        fixed      3     1     2     3
Ingot(Alloy Heat) random      2     1     2

Analysis of Variance for Hardness

Source              DF         SS         MS       F      P
Alloy                1      315.4      315.4    0.85  0.392
Heat(Alloy)          4     6453.8     1613.5    4.35  0.055
Ingot(Alloy Heat)    6     2226.3      371.0    2.08  0.132
Error               12     2141.5      178.5
Total               23    11137.0

Source               Variance Error Expected Mean Square for Each Term
                    component term (using unrestricted model)
 1 Alloy                        3   (4) + 2(3) + Q[1,2]
 2 Heat(Alloy)                  3   (4) + 2(3) + Q[2]
 3 Ingot(Alloy Heat)    96.29   4   (4) + 2(3)
 4 Error               178.46       (4)
```

14-7 Derive the expected means squares for a balanced three-stage nested design, assuming that *A* is fixed and that *B* and *C* are random. Obtain formulas for estimating the variance components.

The expected mean squares can be generated in Minitab as follows:

Minitab Output

```
ANOVA: y versus A, B, C

Factor     Type Levels Values
A         fixed      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1      0.250      0.250    0.06  0.831
B(A)         2      8.500      4.250    0.35  0.726
C(A B)       4     49.000     12.250    2.13  0.168
Error        8     46.000      5.750
Total       15    103.750

Source      Variance Error Expected Mean Square for Each Term
           component term (using restricted model)
 1 A                    2   (4) + 2(3) + 4(2) + 8Q[1]
 2 B(A)       -2.000   3   (4) + 2(3) + 4(2)
 3 C(A B)      3.250   4   (4) + 2(3)
 4 Error       5.750       (4)
```

14-8 Repeat Problem 14-7 assuming the unrestricted form of the mixed model. You may use a computer software package. Comment on any differences you observe between the restricted and unrestricted model analysis and conclusions.

Minitab Output

```
ANOVA: y versus A, B, C

Factor     Type Levels Values
A         fixed      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1      0.250      0.250    0.06  0.831
B(A)         2      8.500      4.250    0.35  0.726
C(A B)       4     49.000     12.250    2.13  0.168
Error        8     46.000      5.750
Total       15    103.750

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A                  2   (4) + 2(3) + 4(2) + Q[1]
 2 B(A)       -2.000   3   (4) + 2(3) + 4(2)
 3 C(A B)      3.250   4   (4) + 2(3)
 4 Error       5.750       (4)
```

In this case there is no difference in results between the restricted and unrestricted models.

14-9 Derive the expected means squares for a balanced three-stage nested design if all three factors are random. Obtain formulas for estimating the variance components. Assume the restricted form of the mixed model.

The expected mean squares can be generated in Minitab as follows:

Minitab Output

```
ANOVA: y versus A, B, C

Factor     Type Levels Values
A        random      2    -1     1
B(A)     random      2    -1     1
C(A B)   random      2    -1     1

Analysis of Variance for y

Source      DF         SS         MS       F      P
A            1      0.250      0.250    0.06  0.831
B(A)         2      8.500      4.250    0.35  0.726
C(A B)       4     49.000     12.250    2.13  0.168
Error        8     46.000      5.750
Total       15    103.750

Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 A         -0.5000   2   (4) + 2(3) + 4(2) + 8(1)
 2 B(A)      -2.0000   3   (4) + 2(3) + 4(2)
 3 C(A B)     3.2500   4   (4) + 2(3)
 4 Error      5.7500       (4)
```

14-10 Verify the expected mean squares given in Table 14-1.

Factor	F a i	F b j	R n l	E(MS)
τ_i	0	b	n	$\sigma^2 + \frac{bn}{a-1}\sum \tau_i^2$

$\beta_{j(i)}$	1	0	n	$\sigma^2+\frac{n}{a(b-1)}\sum\sum\beta_{j(i)}^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

Factor	R a i	R b j	R n l	E(MS)
τ_i	1	b	n	$\sigma^2+n\sigma_\beta^2+bn\sigma_\tau^2$
$\beta_{j(i)}$	1	1	n	$\sigma^2+n\sigma_\beta^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

Factor	F a i	R b j	R n l	E(MS)
τ_i	0	b	n	$\sigma^2+n\sigma_\beta^2+\frac{bn}{a-1}\sum\tau_i^2$
$\beta_{j(i)}$	1	1	n	$\sigma^2+n\sigma_\beta^2$
$\varepsilon_{(ijk)l}$	1	1	1	σ^2

14-11 ***Unbalanced designs.*** Consider an unbalanced two-stage nested design with b_j levels of B under the ith level of A and n_{ij} replicates in the ijth cell.

(a) Write down the least squares normal equations for this situation. Solve the normal equations.

The least squares normal equations are:

$$\mu = n_{..}\hat{\mu}+\sum_{i=1}^{a}n_{i.}\hat{\tau}_i+\sum_{i=1}^{a}\sum_{j=1}^{b_i}n_{ij}\hat{\beta}_{j(i)}=y_{...}$$

$$\tau_i = n_{i.}\hat{\mu}+n_{i.}\hat{\tau}_i+\sum_{j=1}^{b_i}n_{ij}\hat{\beta}_{j(i)}=y_{i..}\text{ , for } i=1,2,\ldots,a$$

$$\beta_{j(i)}=n_{ij}\hat{\mu}+n_{ij}\hat{\tau}_i+n_{ij}\hat{\beta}_{j(i)}=y_{ij.}\text{, for } i=1,2,\ldots,a \text{ and } j=1,2,\ldots,b_i$$

There are 1+a+b equations in 1+a+b unknowns. However, there are a+1linear dependencies in these equations, and consequently, a+1 side conditions are needed to solve them. Any convenient set of a+1 linearly independent equations can be used. The easiest set is $\hat{\mu}=0$, $\hat{\tau}_i=0$, for i=1,2,…,a. Using these conditions we get

$$\hat{\mu}=0,\ \hat{\tau}_i=0,\ \hat{\beta}_{j(i)}=\bar{y}_{ij.}$$

as the solution to the normal equations. See Searle (1971) for a full discussion.

(b) Construct the analysis of variance table for the unbalanced two-stage nested design.

The analysis of variance table is

Source	SS	DF

A	$\sum_{i-1}^{a} \frac{y_{i..}^2}{n_{i.}} - \frac{y_{...}^2}{n_{..}}$	a-1
B	$\sum_{i=1}^{a}\sum_{j=1}^{b_i} \frac{y_{ij.}^2}{n_{ij}} - \sum_{i-1}^{a} \frac{y_{i..}^2}{n_{i.}}$	$b.-a$
Error	$\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n_{ij}} y_{ijk}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b_i} \frac{y_{ij.}^2}{n_{ij}}$	$n..-b$
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n_{ij}} y_{ijk}^2 - \frac{y_{...}^2}{n_{..}}$	$n..$-1

(c) Analyze the following data, using the results in part (b).

Factor A	1		2		
Factor B	1	2	1	2	3
	6	-3	5	2	1
	4	1	7	4	0
	8		9	3	-3
			6		

Note that a=2, b_1=2, b_2=3, $b.=b_1+b_2$=5, n_{11}=3, n_{12}=2, n_{21}=4, n_{22}=3 and n_{23}=3

Source	SS	DF	MS
A	0.13	1	0.13
B	153.78	3	51.26
Error	35.42	10	3.54
Total	189.33	14	

The analysis can also be performed in Minitab as follows. The adjusted sum of squares is utilized by Minitab's general linear model routine.

Minitab Output

General Linear Model: y versus A, B

```
Factor     Type Levels Values
A         fixed      2 1 2
B(A)      fixed      5 1 2 1 2 3

Analysis of Variance for y, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
A           1      0.133      0.898      0.898    0.25  0.625
B(A)        3    153.783    153.783     51.261   14.47  0.001
Error      10     35.417     35.417      3.542
Total      14    189.333
```

14-12 ***Variance components in the unbalanced two-stage nested design.*** Consider the model

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{k(ij)} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n_{ij} \end{cases}$$

where A and B are random factors. Show that

$$E(MS_A) = \sigma^2 + c_1\sigma_\beta^2 + c_2\sigma_\tau^2$$
$$E(MS_{B(A)}) = \sigma^2 + c_0\sigma_\beta^2$$
$$E(MS_E) = \sigma^2$$

where

$$c_0 = \frac{N - \sum_{i=1}^{a}\left(\sum_{j=1}^{b_i}\frac{n_{ij}^2}{n_{i.}}\right)}{b-a}$$

$$c_1 = \frac{\sum_{i=1}^{a}\left(\sum_{j=1}^{b_i}\frac{n_{ij}^2}{n_{i.}}\right) - \sum_{i=1}^{a}\sum_{j=1}^{b_i}\frac{n_{ij}^2}{N}}{a-1}$$

$$c_2 = \frac{N - \frac{\sum_{i=1}^{a} n_{i.}^2}{N}}{a-1}$$

See "Variance Component Estimation in the 2-way Nested Classification," by S.R. Searle, *Annals of Mathematical Statistics*, Vol. 32, pp. 1161-1166, 1961. A good discussion of variance component estimation from unbalanced data is in Searle (1971a).

14-13 A process engineer is testing the yield of a product manufactured on three machines. Each machine can be operated at two power settings. Furthermore, a machine has three stations on which the product is formed. An experiment is conducted in which each machine is tested at both power settings, and three observations on yield are taken from each station. The runs are made in random order, and the results follow. Analyze this experiment, assuming all three factors are fixed.

	Machine 1			Machine 2			Machine 3		
Station	1	2	3	1	2	3	1	2	3
Power Setting 1	34.1	33.7	36.2	32.1	33.1	32.8	32.9	33.8	33.6
	30.3	34.9	36.8	33.5	34.7	35.1	33.0	33.4	32.8
	31.6	35.0	37.1	34.0	33.9	34.3	33.1	32.8	31.7
Power Setting 2	24.3	28.1	25.7	24.1	24.1	26.0	24.2	23.2	24.7
	26.3	29.3	26.1	25.0	25.1	27.1	26.1	27.4	22.0
	27.1	28.6	24.9	26.3	27.9	23.9	25.3	28.0	24.8

The linear model is $y_{ijkl} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_{k(j)} + (\tau\gamma)_{ik(j)} + \varepsilon_{(ijk)l}$

Minitab Output

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power             fixed      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                  DF         SS         MS       F       P
Machine                  2     21.143     10.572    6.46  0.004
```

```
Power                     1    853.631    853.631  521.80  0.000
Station(Machine)          6     32.583      5.431    3.32  0.011
Machine*Power             2      0.616      0.308    0.19  0.829
Power*Station(Machine)    6     28.941      4.824    2.95  0.019
Error                    36     58.893      1.636
Total                    53    995.808

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using restricted model)
 1 Machine                          6   (6) + 18Q[1]
 2 Power                            6   (6) + 27Q[2]
 3 Station(Machine)                 6   (6) + 6Q[3]
 4 Machine*Power                    6   (6) + 9Q[4]
 5 Power*Station(Machine)           6   (6) + 3Q[5]
 6 Error                     1.636      (6)
```

14-14 Suppose that in Problem 14-13 a large number of power settings could have been used and that the two selected for the experiment were chosen randomly. Obtain the expected mean squares for this situation and modify the previous analysis appropriately.

The analysis of variance and the expected mean squares can be obtained from Minitab as follows:

Minitab Output

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power            random      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                   DF         SS         MS       F      P
Machine                   2     21.143     10.572   34.33  0.028
Power                     1    853.631    853.631  521.80  0.000
Station(Machine)          6     32.583      5.431    1.13  0.445
Machine*Power             2      0.616      0.308    0.19  0.829
Power*Station(Machine)    6     28.941      4.824    2.95  0.019
Error                    36     58.893      1.636
Total                    53    995.808

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using restricted model)
 1 Machine                          4   (6) + 9(4) + 18Q[1]
 2 Power                   31.5554  6   (6) + 27(2)
 3 Station(Machine)                 5   (6) + 3(5) + 6Q[3]
 4 Machine*Power           -0.1476  6   (6) + 9(4)
 5 Power*Station(Machine)   1.0625  6   (6) + 3(5)
 6 Error                    1.6359      (6)
```

14-15 Reanalyze the experiment in Problem 14-14 assuming the unrestricted form of the mixed model. You may use a computer software program to do this. Comment on any differences between the restricted and unrestricted model analysis and conclusions.

Minitab Output

ANOVA: Yield versus Machine, Power, Station

```
Factor             Type Levels Values
Machine           fixed      3     1     2     3
Power            random      2     1     2
Station(Machine)  fixed      3     1     2     3

Analysis of Variance for Yield

Source                   DF         SS         MS       F      P
Machine                   2     21.143     10.572   34.33  0.028
Power                     1    853.631    853.631 2771.86  0.000
```

```
Station(Machine)           6     32.583      5.431    1.13  0.445
Machine*Power              2      0.616      0.308    0.06  0.939
Power*Station(Machine)     6     28.941      4.824    2.95  0.019
Error                     36     58.893      1.636
Total                     53    995.808

Source                     Variance Error Expected Mean Square for Each Term
                          component term (using unrestricted model)
 1 Machine                          4   (6) + 3(5) + 9(4) + Q[1,3]
 2 Power                    31.6046  4   (6) + 3(5) + 9(4) + 27(2)
 3 Station(Machine)                 5   (6) + 3(5) + Q[3]
 4 Machine*Power           -0.5017  5   (6) + 3(5) + 9(4)
 5 Power*Station(Machine)   1.0625  6   (6) + 3(5)
 6 Error                    1.6359      (6)
```

There are differences between several of the expected mean squares. However, the conclusions that could be drawn do not differ in any meaningful way from the restricted model analysis.

14-16 A structural engineer is studying the strength of aluminum alloy purchased from three vendors. Each vendor submits the alloy in standard-sized bars of 1.0, 1.5, or 2.0 inches. The processing of different sizes of bar stock from a common ingot involves different forging techniques, and so this factor may be important. Furthermore, the bar stock if forged from ingots made in different heats. Each vendor submits two tests specimens of each size bar stock from the three heats. The resulting strength data follow. Analyze the data, assuming that vendors and bar size are fixed and heats are random.

	Vendor 1			Vendor 2			Vendor 3		
Heat	1	2	3	1	2	3	1	2	3
Bar Size: 1 inch	1.230	1.346	1.235	1.301	1.346	1.315	1.247	1.275	1.324
	1.259	1.400	1.206	1.263	1.392	1.320	1.296	1.268	1.315
1 1/2 inch	1.316	1.329	1.250	1.274	1.384	1.346	1.273	1.260	1.392
	1.300	1.362	1.239	1.268	1.375	1.357	1.264	1.265	1.364
2 inch	1.287	1.346	1.273	1.247	1.362	1.336	1.301	1.280	1.319
	1.292	1.382	1.215	1.215	1.328	1.342	1.262	1.271	1.323

$$y_{ijkl} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_{k(j)} + (\tau\gamma)_{ik(j)} + \varepsilon_{(ijk)l}$$

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
Bar Size      fixed      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                    DF         SS         MS        F      P
Vendor                     2  0.0088486  0.0044243     0.26  0.776
Heat(Vendor)               6  0.1002093  0.0167016    41.32  0.000
Bar Size                   2  0.0025263  0.0012631     1.37  0.290
Vendor*Bar Size            4  0.0023754  0.0005939     0.65  0.640
Bar Size*Heat(Vendor)     12  0.0110303  0.0009192     2.27  0.037
Error                     27  0.0109135  0.0004042
Total                     53  0.1359034

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using restricted model)
 1 Vendor                            2   (6) + 6(2) + 18Q[1]
 2 Heat(Vendor)             0.00272  6   (6) + 6(2)
 3 Bar Size                          5   (6) + 2(5) + 18Q[3]
 4 Vendor*Bar Size                   5   (6) + 2(5) + 6Q[4]
 5 Bar Size*Heat(Vendor)  0.00026   6   (6) + 2(5)
 6 Error                    0.00040      (6)
```

14-17 Reanalyze the experiment in Problem 14-16 assuming the unrestricted form of the mixed model. You may use a computer software program to do this. Comment on any differences between the restricted and unrestricted model analysis and conclusions.

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
Bar Size      fixed      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                    DF         SS         MS       F      P
Vendor                     2  0.0088486  0.0044243    0.26  0.776
Heat(Vendor)               6  0.1002093  0.0167016   18.17  0.000
Bar Size                   2  0.0025263  0.0012631    1.37  0.290
Vendor*Bar Size            4  0.0023754  0.0005939    0.65  0.640
Bar Size*Heat(Vendor)     12  0.0110303  0.0009192    2.27  0.037
Error                     27  0.0109135  0.0004042
Total                     53  0.1359034

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using unrestricted model)
 1 Vendor                            2   (6) + 2(5) + 6(2) + Q[1,4]
 2 Heat(Vendor)            0.00263   5   (6) + 2(5) + 6(2)
 3 Bar Size                          5   (6) + 2(5) + Q[3,4]
 4 Vendor*Bar Size                   5   (6) + 2(5) + Q[4]
 5 Bar Size*Heat(Vendor)  0.00026   6   (6) + 2(5)
 6 Error                   0.00040       (6)
```

There are some differences in the expected mean squares. However, the conclusions do not differ from those of the restricted model analysis.

14-18 Suppose that in Problem 14-16 the bar stock may be purchased in many sizes and that the three sizes are actually used in experiment were selected randomly. Obtain the expected mean squares for this situation and modify the previous analysis appropriately. Use the restricted form of the mixed model.

Minitab Output

ANOVA: Strength versus Vendor, Bar Size, Heat

```
Factor         Type Levels Values
Vendor        fixed      3     1     2     3
Heat(Vendor) random      3     1     2     3
Bar Size     random      3   1.0   1.5   2.0

Analysis of Variance for Strength

Source                    DF         SS         MS       F      P
Vendor                     2  0.0088486  0.0044243    0.27  0.772 x
Heat(Vendor)               6  0.1002093  0.0167016   18.17  0.000
Bar Size                   2  0.0025263  0.0012631    1.37  0.290
Vendor*Bar Size            4  0.0023754  0.0005939    0.65  0.640
Bar Size*Heat(Vendor)     12  0.0110303  0.0009192    2.27  0.037
Error                     27  0.0109135  0.0004042
Total                     53  0.1359034

x Not an exact F-test.

Source                    Variance Error Expected Mean Square for Each Term
                         component term (using restricted model)
 1 Vendor                            *   (6) + 2(5) + 6(4) + 6(2) + 18Q[1]
 2 Heat(Vendor)            0.00263   5   (6) + 2(5) + 6(2)
 3 Bar Size                0.00002   5   (6) + 2(5) + 18(3)
 4 Vendor*Bar Size        -0.00005   5   (6) + 2(5) + 6(4)
 5 Bar Size*Heat(Vendor)  0.00026   6   (6) + 2(5)
```

```
 6 Error                 0.00040       (6)

* Synthesized Test.

Error Terms for Synthesized Tests

Source                    Error DF  Error MS  Synthesis of Error MS
 1 Vendor                     5.75 0.0163762   (2) + (4) - (5)
```

Notice that a Satterthwaite type test is used for vendor.

14-19 Steel in normalized by heating above the critical temperature, soaking, and then air cooling. This process increases the strength of the steel, refines the grain, and homogenizes the structure. An experiment is performed to determine the effect of temperature and heat treatment time on the strength of normalized steel. Two temperatures and three times are selected. The experiment is performed by heating the oven to a randomly selected temperature and inserting three specimens. After 10 minutes one specimen is removed, after 20 minutes the second specimen is removed, and after 30 minutes the final specimen is removed. Then the temperature is changed to the other level and the process is repeated. Four shifts are required to collect the data, which are shown below. Analyze the data and draw conclusions, assume both factors are fixed.

		Temperature (F)	
Shift	Time(minutes)	1500	1600
1	10	63	89
	20	54	91
	30	61	62
2	10	50	80
	20	52	72
	30	59	69
3	10	48	73
	20	74	81
	30	71	69
4	10	54	88
	20	48	92
	30	59	64

This is a split-plot design. Shifts correspond to blocks, temperature is the whole plot treatment, and time is the subtreatments (in the subplot or split-plot part of the design). The expected mean squares and analysis of variance are shown below. The following Minitab Output has been modified to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Strength versus Shift, Temperature, Time

```
Factor     Type Levels Values
Shift    random      4     1     2     3     4
Temperat  fixed      2  1500  1600
Time      fixed      3    10    20    30

Analysis of Variance for Strength
                                                Standard     Split Plot
Source            DF         SS         MS       F       P        F      P
Shift              3     145.46      48.49    1.19  0.390
Temperat           1    2340.38    2340.38   29.20  0.012   29.21  0.012
Shift*Temperat     3     240.46      80.15    1.97  0.220
Time               2     159.25      79.63    1.00  0.422    1.00  0.422
Shift*Time         6     478.42      79.74    1.96  0.217
Temperat*Time      2     795.25     397.63    9.76  0.013    9.76  0.013
Error              6     244.42      40.74
Total             23    4403.63

Source            Variance Error Expected Mean Square for Each Term
```

```
                  component term (using restricted model)
 1 Shift              1.292   7   (7) + 6(1)
 2 Temperat                   3   (7) + 3(3) + 12Q[2]
 3 Shift*Temperat    13.139   7   (7) + 3(3)
 4 Time                       5   (7) + 2(5) + 8Q[4]
 5 Shift*Time        19.500   7   (7) + 2(5)
 6 Temperat*Time              7   (7) + 4Q[6]
 7 Error             40.736       (7)
```

14-20 An experiment is designed to study pigment dispersion in paint. Four different mixes of a particular pigment are studied. The procedure consists of preparing a particular mix and then applying that mix to a panel by three application methods (brushing, spraying, and rolling). The response measured is the percentage reflectance of the pigment. Three days are required to run the experiment, and the data obtained follow. Analyze the data and draw conclusions, assuming that mixes and application methods are fixed.

		Mix			
Day	App Method	1	2	3	4
1	1	64.5	66.3	74.1	66.5
	2	68.3	69.5	73.8	70.0
	3	70.3	73.1	78.0	72.3
2	1	65.2	65.0	73.8	64.8
	2	69.2	70.3	74.5	68.3
	3	71.2	72.8	79.1	71.5
3	1	66.2	66.5	72.3	67.7
	2	69.0	69.0	75.4	68.6
	3	70.8	74.2	80.1	72.4

This is a split plot design. Days correspond to blocks, mix is the whole plot treatment, and method is the subtreatment (in the subplot or split plot part of the design). The following Minitab Output has been modified to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Reflectance versus Day, Mix, Method

```
Factor     Type Levels Values
Day      random      3     1     2     3
Mix       fixed      4     1     2     3     4
Method    fixed      3     1     2     3

Analysis of Variance for Reflecta
                                                 Standard     Split Plot
Source        DF         SS          MS        F       P        F       P
Day            2      2.042      1.021    1.39  0.285
Mix            3    307.479    102.493  135.77  0. 000 135.75  0.000
Day*Mix        6      4.529      0.755    1.03  0.451
Method         2    222.095    111.047  226.24  0.000  226.16  0.000
Day*Method     4      1.963      0.491    0.67  0.625
Mix*Method     6     10.036      1.673    2.28  0.105    2.28  0.105
Error         12      8.786      0.732
Total         35    556.930

Source        Variance Error Expected Mean Square for Each Term
             component term (using restricted model)
 1 Day         0.02406   7   (7) + 12(1)
 2 Mix                   3   (7) + 3(3) + 9Q[2]
 3 Day*Mix     0.00759   7   (7) + 3(3)
 4 Method                5   (7) + 4(5) + 12Q[4]
 5 Day*Method -0.06032   7   (7) + 4(5)
 6 Mix*Method            7   (7) + 3Q[6]
 7 Error       0.73213       (7)
```

14-21 Repeat Problem 14-20, assuming that the mixes are random and the application methods are fixed.

The F-tests are the same as those in Problem 13-20. The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Again, the Error term in the analysis of variance is actually the three factor interaction.

Minitab Output

ANOVA: Reflectance versus Day, Mix, Method

```
Factor     Type Levels Values
Day      random      3     1     2     3
Mix      random      4     1     2     3     4
Method    fixed      3     1     2     3

Analysis of Variance for Reflecta
                                              Standard        Split Plot
Source        DF        SS         MS       F      P           F      P
Day            2     2.042      1.021    1.35  0.328
Mix            3   307.479    102.493  135.77  0.000     135.75  0.000
Day*Mix        6     4.529      0.755    1.03  0.451
Method         2   222.095    111.047   77.58  0.001 x   226.16  0.000
Day*Method     4     1.963      0.491    0.67  0.625
Mix*Method     6    10.036      1.673    2.28  0.105       2.28  0.105
Error         12     8.786      0.732
Total         35   556.930

x Not an exact F-test.

Source        Variance Error Expected Mean Square for Each Term
             component term (using restricted model)
 1 Day          0.0222   3   (7) + 3(3) + 12(1)
 2 Mix         11.3042   3   (7) + 3(3) + 9(2)
 3 Day*Mix      0.0076   7   (7) + 3(3)
 4 Method                *   (7) + 3(6) + 4(5) + 12Q[4]
 5 Day*Method  -0.0603   7   (7) + 4(5)
 6 Mix*Method   0.3135   7   (7) + 3(6)
 7 Error        0.7321       (7)

* Synthesized Test.

Error Terms for Synthesized Tests

Source         Error DF  Error MS  Synthesis of Error MS
 4 Method          3.59     1.431   (5) + (6) - (7)
```

14-22 Consider the split-split-plot design described in example 14-3. Suppose that this experiment is conducted as described and that the data shown below are obtained. Analyze and draw conclusions.

		Technician								
		1			2			3		
Blocks	Dose Strengths	1	2	3	1	2	3	1	2	3
	Wall Thickness									
1	1	95	71	108	96	70	108	95	70	100
	2	104	82	115	99	84	100	102	81	106
	3	101	85	117	95	83	105	105	84	113
	4	108	85	116	97	85	109	107	87	115
2	1	95	78	110	100	72	104	92	69	101
	2	106	84	109	101	79	102	100	76	104
	3	103	86	116	99	80	108	101	80	109
	4	109	84	110	112	86	109	108	86	113
3	1	96	70	107	94	66	100	90	73	98
	2	105	81	106	100	84	101	97	75	100
	3	106	88	112	104	87	109	100	82	104
	4	113	90	117	121	90	117	110	91	112
4	1	90	68	109	98	68	106	98	72	101
	2	100	84	112	102	81	103	102	78	105

3	102	85	115	100	85	110	105	80	110
4	114	88	118	118	85	116	110	95	120

Using the computer output, the F-ratios were calculated by hand using the expected mean squares found in Table 14-18. The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Notice that the Error term in the analysis of variance is actually the four factor interaction.

Minitab Output

ANOVA: Time versus Day, Tech, Dose, Thick

```
Factor     Type Levels Values
Day      random      4     1     2     3     4
Tech      fixed      3     1     2     3
Dose      fixed      3     1     2     3
Thick     fixed      4     1     2     3     4

Analysis of Variance for Time
                                                Standard     Split Plot
Source             DF          SS          MS        F      P        F       P
Day                 3       48.41       16.14     3.38  0.029
Tech                2      248.35      124.17     4.62  0.061     4.62  0.061
Day*Tech            6      161.15       26.86     5.62  0.000
Dose                2    20570.06    10285.03   550.44  0.000   550.30  0.000
Day*Dose            6      112.11       18.69     3.91  0.004
Tech*Dose           4      125.94       31.49     3.32  0.048     3.32  0.048
Day*Tech*Dose      12      113.89        9.49     1.99  0.056
Thick               3     3806.91     1268.97    36.47  0.000    36.48  0.000
Day*Thick           9      313.12       34.79     7.28  0.000
Tech*Thick          6      126.49       21.08     2.26  0.084     2.26  0.084
Day*Tech*Thick     18      167.57        9.31     1.95  0.044
Dose*Thick          6      402.28       67.05    17.13  0.000    17.15  0.000
Day*Dose*Thick     18       70.44        3.91     0.82  0.668
Tech*Dose*Thick    12      205.89       17.16     3.59  0.001     3.59  0.001
Error              36      172.06        4.78
Total             143    26644.66

Source              Variance Error Expected Mean Square for Each Term
                   component term (using restricted model)
 1 Day                0.3155  15   (15) + 36(1)
 2 Tech                        3   (15) + 12(3) + 48Q[2]
 3 Day*Tech           1.8400  15   (15) + 12(3)
 4 Dose                        5   (15) + 12(5) + 48Q[4]
 5 Day*Dose           1.1588  15   (15) + 12(5)
 6 Tech*Dose                   7   (15) + 4(7) + 16Q[6]
 7 Day*Tech*Dose      1.1779  15   (15) + 4(7)
 8 Thick                       9   (15) + 9(9) + 36Q[8]
 9 Day*Thick          3.3346  15   (15) + 9(9)
10 Tech*Thick                 11   (15) + 3(11) + 12Q[10]
11 Day*Tech*Thick     1.5100  15   (15) + 3(11)
12 Dose*Thick                 13   (15) + 3(13) + 12Q[12]
13 Day*Dose*Thick    -0.2886  15   (15) + 3(13)
14 Tech*Dose*Thick            15   (15) + 4Q[14]
15 Error              4.7793       (15)
```

14-23 Rework Problem 14-22, assuming that the dosage strengths are chosen at random. Use the restricted form of the mixed model.

The following Minitab Output has been edited to display the results of the split-plot analysis. Minitab will calculate the sums of squares correctly, but the expected mean squares and the statistical tests are not, in general, correct. Again, the Error term in the analysis of variance is actually the four factor interaction.

Minitab Output

ANOVA: Time versus Day, Tech, Dose, Thick

```
Factor     Type Levels Values
```

```
Day      random      4     1     2     3     4
Tech      fixed      3     1     2     3
Dose     random      3     1     2     3
Thick     fixed      4     1     2     3     4

Analysis of Variance for Time
                                                Standard        Split Plot
Source              DF          SS          MS       F      P          F      P
Day                  3       48.41       16.14    0.86  0.509
Tech                 2      248.35      124.17    2.54  0.155       4.62  0.061
Day*Tech             6      161.15       26.86    2.83  0.059
Dose                 2    20570.06    10285.03  550.44  0.000     550.30  0.000
Day*Dose             6      112.11       18.69    3.91  0.004
Tech*Dose            4      125.94       31.49    3.32  0.048       3.32  0.048
Day*Tech*Dose       12      113.89        9.49    1.99  0.056
Thick                3     3806.91     1268.97   12.96  0.001 x    36.48  0.000
Day*Thick            9      313.12       34.79    8.89  0.000
Tech*Thick           6      126.49       21.08    0.97  0.475 x     2.26  0.084
Day*Tech*Thick      18      167.57        9.31    1.95  0.044
Dose*Thick           6      402.28       67.05   17.13  0.000      17.15  0.000
Day*Dose*Thick      18       70.44        3.91    0.82  0.668
Tech*Dose*Thick     12      205.89       17.16    3.59  0.001       3.59  0.001
Error               36      172.06        4.78
Total              143    26644.66

x Not an exact F-test.

Source              Variance Error Expected Mean Square for Each Term
                   component term (using restricted model)
 1 Day                -0.071   5   (15) + 12(5) + 36(1)
 2 Tech                        *   (15) + 4(7) + 16(6) + 12(3) + 48Q[2]
 3 Day*Tech            1.447   7   (15) + 4(7) + 12(3)
 4 Dose              213.882   5   (15) + 12(5) + 48(4)
 5 Day*Dose            1.159  15   (15) + 12(5)
 6 Tech*Dose           1.375   7   (15) + 4(7) + 16(6)
 7 Day*Tech*Dose       1.178  15   (15) + 4(7)
 8 Thick                       *   (15) + 3(13) + 12(12) + 9(9) + 36Q[8]
 9 Day*Thick           3.431  13   (15) + 3(13) + 9(9)
10 Tech*Thick                  *   (15) + 4(14) + 3(11) + 12Q[10]
11 Day*Tech*Thick      1.510  15   (15) + 3(11)
12 Dose*Thick          5.261  13   (15) + 3(13) + 12(12)
13 Day*Dose*Thick     -0.289  15   (15) + 3(13)
14 Tech*Dose*Thick     3.095  15   (15) + 4(14)
15 Error               4.779       (15)

* Synthesized Test.

Error Terms for Synthesized Tests

Source               Error DF  Error MS  Synthesis of Error MS
 2 Tech                  6.35     48.85   (3) + (6) - (7)
 8 Thick                10.84     97.92   (9) + (12) - (13)
10 Tech*Thick           15.69     21.69   (11) + (14) - (15)
```

There are no exact tests on technicians β_j, dosage strengths γ_k, wall thickness δ_h, or the technician x wall thickness interaction $(\beta\delta)_{jh}$. The approximate *F*-tests are as follows:

H_0: $\beta_j = 0$

$$F = \frac{MS_B + MS_{ABC}}{MS_{AB} + MS_{BC}} = \frac{124.174 + 9.491}{26.859 + 31.486} = 2.291$$

$$p = \frac{(MS_B + MS_{ABC})^2}{\frac{MS_B^2}{2} + \frac{MS_{ABC}^2}{12}} = \frac{(124.174 + 9.491)^2}{\frac{124.174^2}{2} + \frac{9.491^2}{12}} = 2.315$$

$$q=\frac{\left(MS_{AB}+MS_{BC}\right)^2}{\frac{MS_{AB}^2}{6}+\frac{MS_{BC}^2}{4}}=\frac{\left(26.859+31.486\right)^2}{\frac{26.859^2}{6}+\frac{31.486^2}{4}}=9.248$$

Do not reject H_0: $\beta_j = 0$

H_0: $\gamma_k = 0$

$$F=\frac{MS_C+MS_{ACD}}{MS_{CD}+MS_{AD}}=\frac{10285.028+3.914}{67.046+34.791}=101.039$$

$$p=\frac{\left(MS_C+MS_{ACD}\right)^2}{\frac{MS_C^2}{2}+\frac{MS_{ACD}^2}{18}}=\frac{\left(10285.028+3.914\right)^2}{\frac{10285.028^2}{2}+\frac{3.914^2}{18}}=2.002$$

$$q=\frac{\left(MS_{CD}+MS_{AD}\right)^2}{\frac{MS_{CD}^2}{6}+\frac{MS_{AD}^2}{9}}=\frac{\left(67.046+34.791\right)^2}{\frac{67.046^2}{6}+\frac{34.791^2}{9}}=11.736$$

Reject H_0: $\gamma_k = 0$

H_0: $\delta_h = 0$

$$F=\frac{MS_D+MS_{ACD}}{MS_{CD}+MS_{AD}}=\frac{1268.970+3.914}{67.046+34.791}=12.499$$

$$p=\frac{\left(MS_D+MS_{ACD}\right)^2}{\frac{MS_D^2}{3}+\frac{MS_{ACD}^2}{18}}=\frac{\left(1268.970+3.914\right)^2}{\frac{1268.970^2}{3}+\frac{3.914^2}{18}}=3.019$$

$$q=\frac{\left(MS_{CD}+MS_{AD}\right)^2}{\frac{MS_{CD}^2}{6}+\frac{MS_{AD}^2}{9}}=\frac{\left(67.046+34.791\right)^2}{\frac{67.046^2}{6}+\frac{34.791^2}{9}}=11.736$$

Reject H_0: $\delta_h = 0$

H_0: $(\beta\delta)_{jh} = 0$

$$F=\frac{MS_{BD}+MS_{ABCD}}{MS_{BCD}+MS_{ABD}}=\frac{21.081+4.779}{17.157+9.309}=0.977$$

$F<1$, Do not reject H_0: $(\beta\delta)_{jh} = 0$

14-24 Suppose that in Problem 14-22 four technicians had been used. Assuming that all the factors are fixed, how many blocks should be run to obtain an adequate number of degrees of freedom on the test for differences among technicians?

The number of degrees of freedom for the test is (a-1)(4-1)=3(a-1), where a is the number of blocks used.

Number of Blocks (*a*)	DF for test
2	3
3	6
4	9
5	12

At least three blocks should be run, but four would give a better test.

14-25 Consider the experiment described in Example 14-4. Demonstrate how the order in which the treatments combinations are run would be determined if this experiment were run as (a) a split-split-plot, (b) a split-plot, (c) a factorial design in a randomized block, and (d) a completely randomized factorial design.

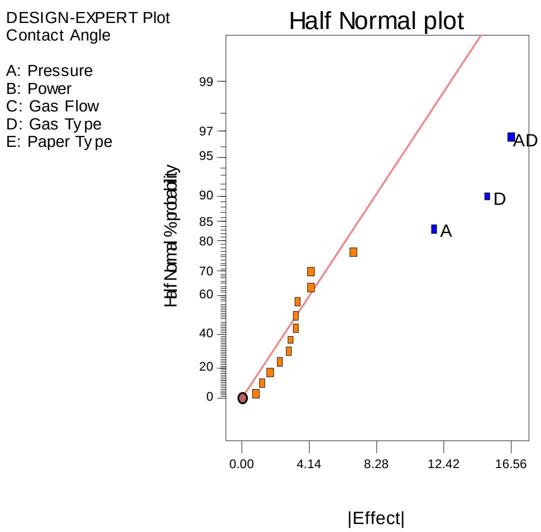
(a) Randomization for the split-split plot design is described in Example 14-4.
(b) In the split-plot, within a block, the technicians would be the main treatment and within a block-technician plot, the 12 combinations of dosage strength and wall thickness would be run in random order. The design would be a two-factor factorial in a split-plot.
(c) To run the design in a randomized block, the 36 combinations of technician, dosage strength, and wall thickness would be run in random order within each block. The design would be a three factor factorial in a randomized block.
(d) The blocks would be considered as replicates, and all 144 observations would be 4 replicates of a three factor factorial.

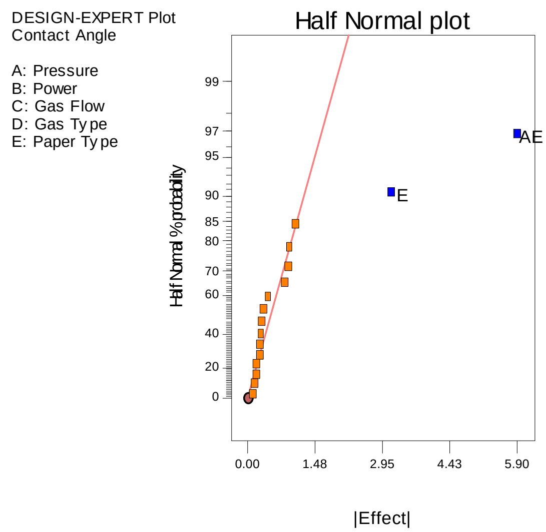
14-26 An article in *Quality Engineering* ("Quality Quandaries: Two-Level Factorials Run as Split-Plot Experiments", Bisgaard, et al, Vol. 8, No. 4, pp. 705-708, 1996) describes a 2^5 factorial experiment on a plasma process focused on making paper more susceptible to ink. Four of the factors (*A*-*D*) are difficult to change from run-to-run, so the experimenters set up the reactor at the eight sets of conditions specific by the low and high levels of these factors, and then processed the two paper types (factor *E*) together. The placement of the paper specimens in the reactors (right versus left) was randomized. This produces a split-plot design with *A*-*D* as the whole-plot factors and factor *E* as the subplot factor. The data from this experiment are shown below. Analyze the data from this experiment and draw conclusions.

Standard Order	Run Number	*A* = Pressure	*B* = Power	*C* = Gas Flow	*D* = Gas Type	*E* = Paper Type	*y* Contact Angle
1	23	-1	-1	-1	Oxygen	E1	48.6
2	3	+1	-1	-1	Oxygen	E1	41.2
3	11	-1	+1	-1	Oxygen	E1	55.8
4	29	+1	+1	-1	Oxygen	E1	53.5
5	1	-1	-1	+1	Oxygen	E1	37.6
6	15	+1	-1	+1	Oxygen	E1	47.2

7	27	-1	+1	+1	Oxygen	E1	47.2
8	25	+1	+1	+1	Oxygen	E1	48.7
9	19	-1	-1	-1	SiCl4	E1	5
10	5	+1	-1	-1	SiCl4	E1	56.8
11	9	-1	+1	-1	SiCl4	E1	25.6
12	31	+1	+1	-1	SiCl4	E1	41.8
13	13	-1	-1	+1	SiCl4	E1	13.3
14	7	+1	-1	+1	SiCl4	E1	47.5
15	21	-1	+1	+1	SiCl4	E1	11.3
16	17	+1	+1	+1	SiCl4	E1	49.5
17	24	-1	-1	-1	Oxygen	E2	57
18	4	+1	-1	-1	Oxygen	E2	38.2
19	12	-1	+1	-1	Oxygen	E2	62.9
20	30	+1	+1	-1	Oxygen	E2	51.3
21	2	-1	-1	+1	Oxygen	E2	43.5
22	16	+1	-1	+1	Oxygen	E2	44.8
23	28	-1	+1	+1	Oxygen	E2	54.6
24	26	+1	+1	+1	Oxygen	E2	44.4
25	20	-1	-1	-1	SiCl4	E2	18.1
26	6	+1	-1	-1	SiCl4	E2	56.2
27	10	-1	+1	-1	SiCl4	E2	33
28	32	+1	+1	-1	SiCl4	E2	37.8
29	14	-1	-1	+1	SiCl4	E2	23.7
30	8	+1	-1	+1	SiCl4	E2	43.2
31	22	-1	+1	+1	SiCl4	E2	23.9
32	18	+1	+1	+1	SiCl4	E2	48.2

Half normal probability plots of the effects for both the whole plot with factors *A*, *B*, *C*, *D*, and their corresponding interactions, as well as the sub-plot with factor *E* and all interactions involving *E*, are shown below. The analysis of variance is not shown because of the known errors in the calculations; however, the models are also shown below.





Design Expert Output

Response: Contact Angle

Final Equation in Terms of Coded Factors:

Contact Angle	=
+40.98	
+5.91	* A
-7.55	* D
+1.57	* E
+8.28	* A * D
-2.95	* A * E

Final Equation in Terms of Actual Factors:

Gas Type Oxygen
Paper Type E1
Contact Angle =
+46.96250
+0.58125 * Pressure

Gas Type SiCl4
Paper Type E1
Contact Angle =
+31.86250
+17.14375 * Pressure

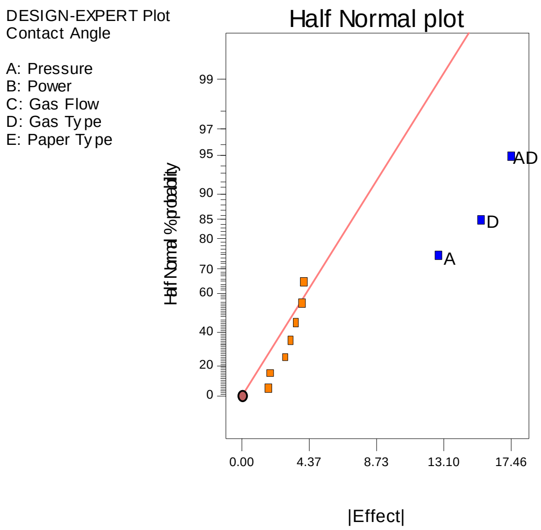
Gas Type Oxygen
Paper Type E2
Contact Angle =
+50.10000
-5.31875 * Pressure

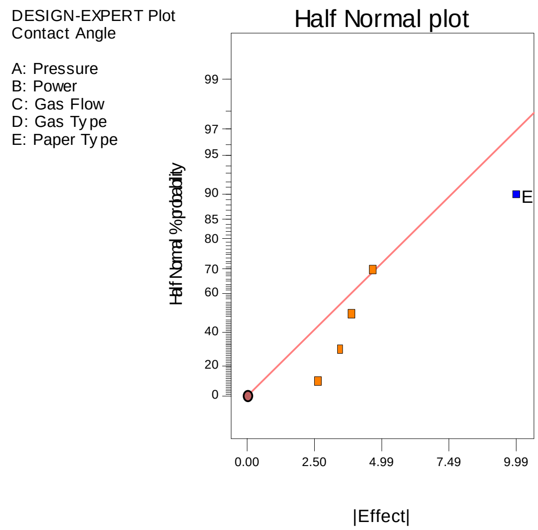
Gas Type SiCl4
Paper Type E2
Contact Angle =
+35.00000
+11.24375 * Pressure

14-27 Reconsider the experiment in problem 14-26. This is a rather large experiment, so suppose that the experimenter had used a 2^{5-1} design instead. Set up the 2^{5-1} design in a split-plot, using the principle fraction. Then select the response data using the information from the full factorial. Analyze the data and draw conclusions. Do they agree with the results of Problem 14-26?

Standard Order	Run Number	*A* = Pressure	*B* = Power	*C* = Gas Flow	*D* = Gas Type	*E* = Paper Type	*y* Contact Angle
1	12	-1	-1	-1	Oxygen	E2	57
2	2	+1	-1	-1	Oxygen	E1	41.2
3	6	-1	+1	-1	Oxygen	E1	55.8
4	15	+1	+1	-1	Oxygen	E2	51.3
5	1	-1	-1	+1	Oxygen	E1	37.6
6	8	+1	-1	+1	Oxygen	E2	44.8
7	14	-1	+1	+1	Oxygen	E2	54.6
8	13	+1	+1	+1	Oxygen	E1	48.7
9	10	-1	-1	-1	SiCl4	E1	5
10	3	+1	-1	-1	SiCl4	E2	56.2
11	5	-1	+1	-1	SiCl4	E2	33
12	16	+1	+1	-1	SiCl4	E1	41.8
13	7	-1	-1	+1	SiCl4	E2	23.7
14	4	+1	-1	+1	SiCl4	E1	47.5
15	11	-1	+1	+1	SiCl4	E1	11.3
16	9	+1	+1	+1	SiCl4	E2	48.2

Similar results are found with the half fraction other than the AE interaction is no longer significant and the effect for factor E is larger. The half normal probability plot of effects for the whole and sub-plots are shown below. The resulting model is also shown.





Design Expert Output

Response: Contact Angle

Final Equation in Terms of Coded Factors:

Contact Angle =
+41.11
+6.36 * A
-7.77 * D
+4.99 * E
+8.73 * A * D

Final Equation in Terms of Actual Factors:

Gas Type Oxygen
Paper Type E1
Contact Angle =
+43.88125
-2.37500 * Pressure

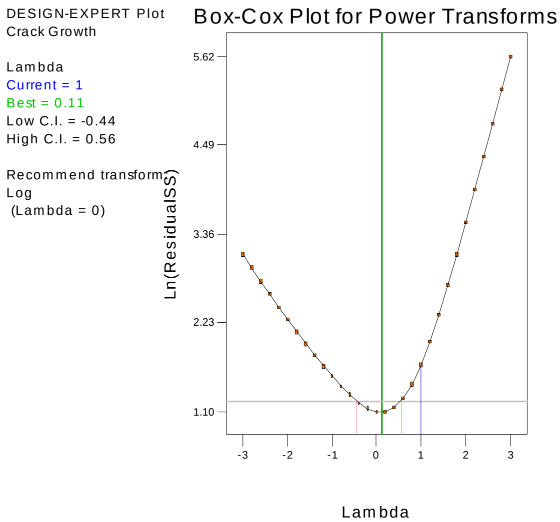
Gas Type SiCl4
Paper Type E1
Contact Angle =
+28.34375
+15.08750 * Pressure

Gas Type Oxygen
Paper Type E2
Contact Angle =
+53.86875
-2.37500 * Pressure

Gas Type SiCl4
Paper Type E2
Contact Angle =
+38.33125
+15.08750 * Pressure

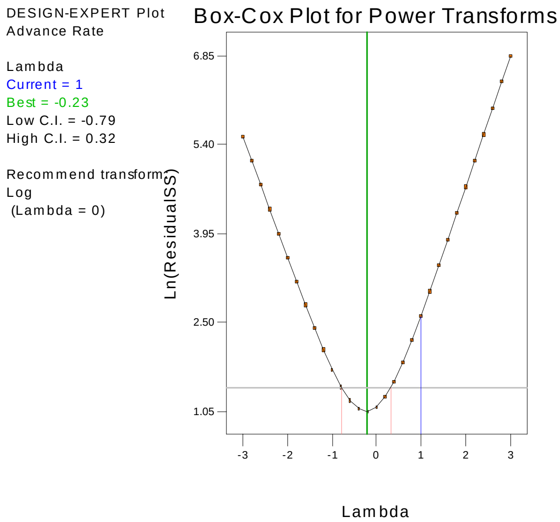
Chapter 15
Other Design and Analysis Topics
Solutions

15-1 Reconsider the experiment in Problem 5-22. Use the Box-Cox procedure to determine if a transformation on the response is appropriate (or useful) in the analysis of the data from this experiment.



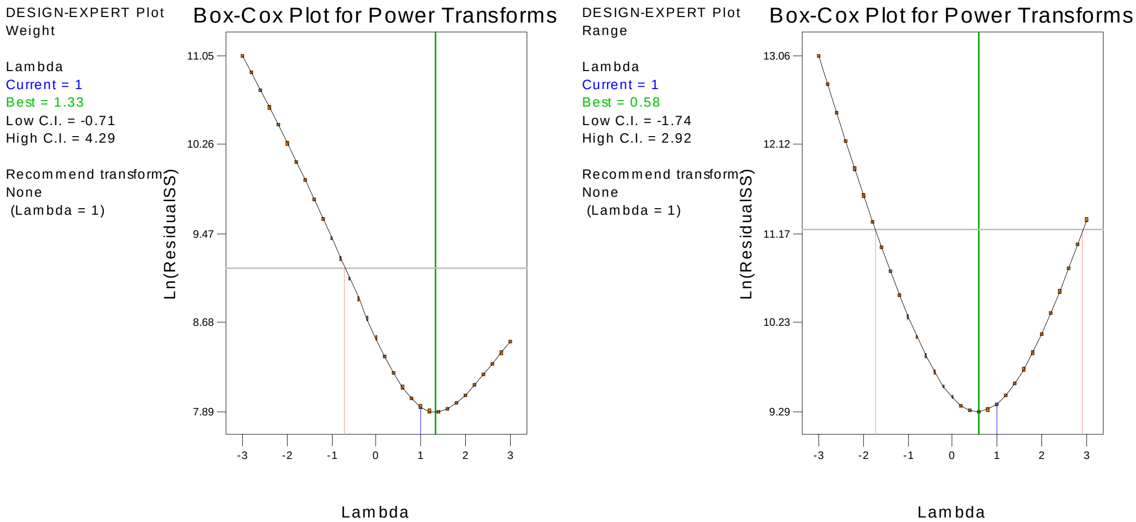
With the value of lambda near zero, and since the confidence interval does not include one, a natural log transformation would be appropriate.

15-2 In example 6-3 we selected a log transformation for the drill advance rate response. Use the Box-Cox procedure to demonstrate that this is an appropriate data transformation.



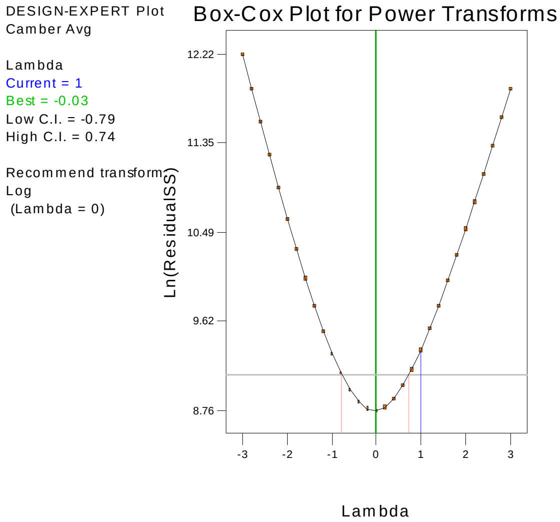
Because the value of lambda is very close to zero, and the confidence interval does not include one, the natural log was the correct transformation chosen for this analysis.

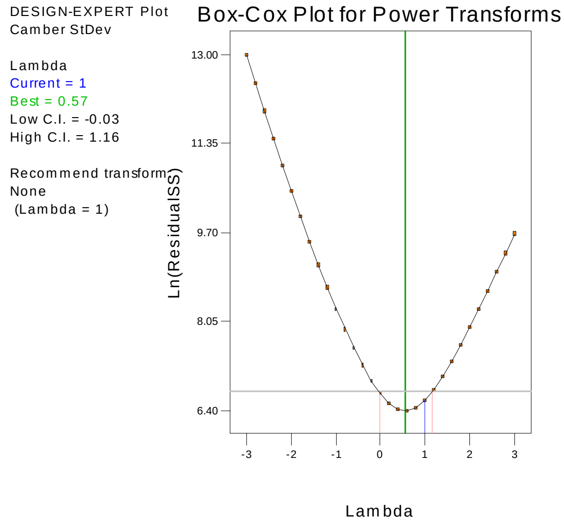
15-3 Reconsider the smelting process experiment in Problem 8-23, where a 2^{6-3} fractional factorial design was used to study the weight of packing material stuck to carbon anodes after baking. Each of the eight runs in the design was replicated three times and both the average weight and the range of the weights at each test combination were treated as response variables. Is there any indication that that a transformation is required for either response?



There is no indication that a transformation is required for either response.

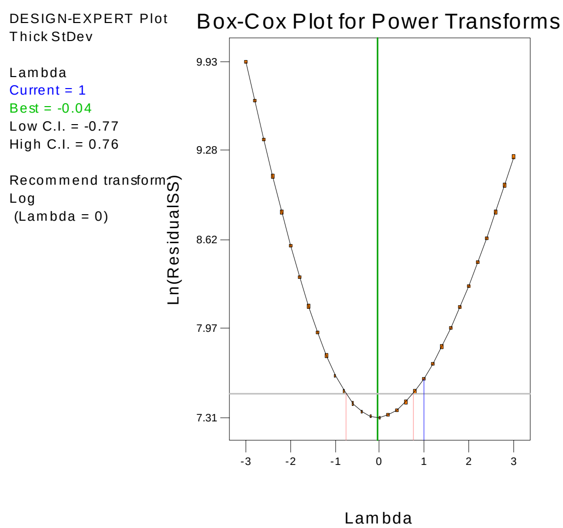
15-4 In Problem 8-25 a replicated fractional factorial design was used to study substrate camber in semiconductor manufacturing. Both the mean and standard deviation of the camber measurements were used as response variables. Is there any indication that a transformation is required for either response?





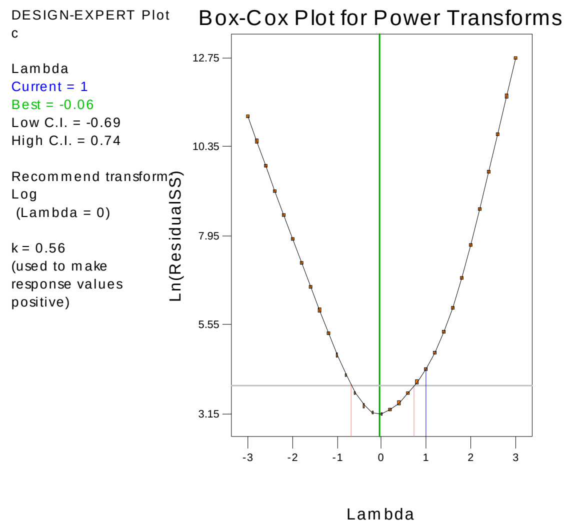
The Box-Cox plot for the Camber Average suggests a natural log transformation should be applied. This decision is based on the confidence interval for lambda not including one and the point estimate of lambda being very close to zero. With a lambda of approximately 0.5, a square root transformation could be considered for the Camber Standard Deviation; however, the confidence interval indicates that no transformation is needed.

15-5 Reconsider the photoresist experiment in Problem 8-26. Use the variance of the resist thickness at each test combination as the response variable. Is there any indication that a transformation is required?



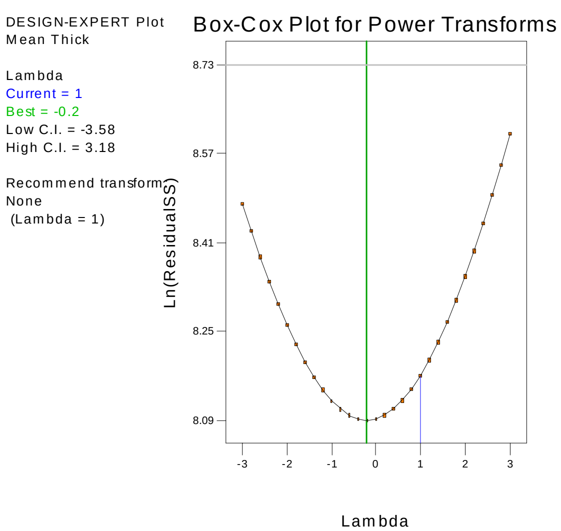
With the point estimate of lambda near zero, and the confidence interval for lambda not inclusive of one, a natural log transformation would be appropriate.

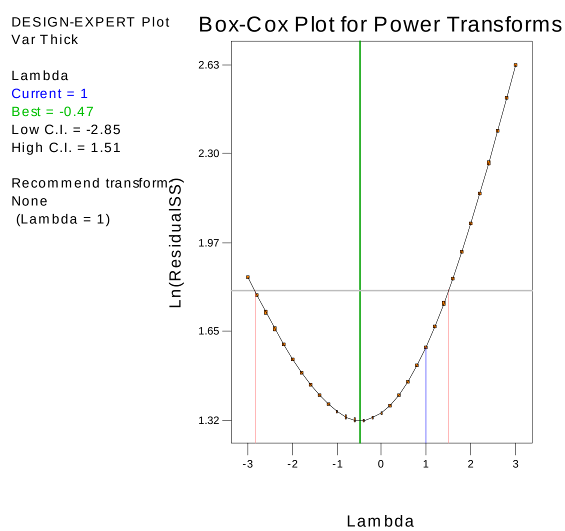
15-6 In the grill defects experiment described in Problem 8-30 a variation of the square root transformation was employed in the analysis of the data. Use the Box-Cox method to determine if this is the appropriate transformation.



Because the confidence interval for the minimum lambda does not include one, the decision to use a transformation is correct. Because the lambda point estimate is close to zero, the natural log transformation would be appropriate. This is a stronger transformation than the square root.

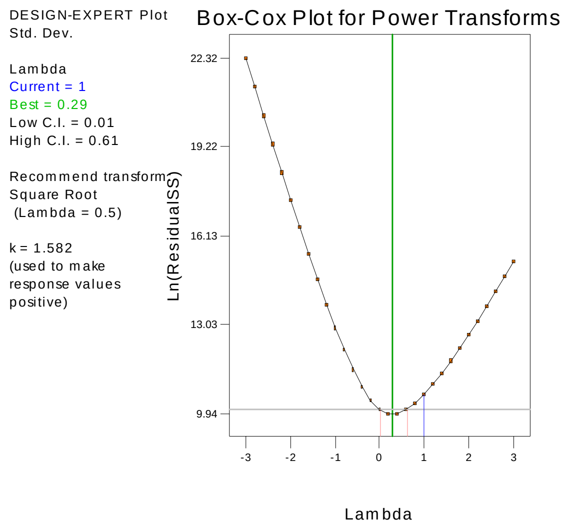
15-7 In the central composite design of Problem 11-14, two responses were obtained, the mean and variance of an oxide thickness. Use the Box-Cox method to investigate the potential usefulness of transformation for both of these responses. Is the log transformation suggested in part (c) of that problem appropriate?





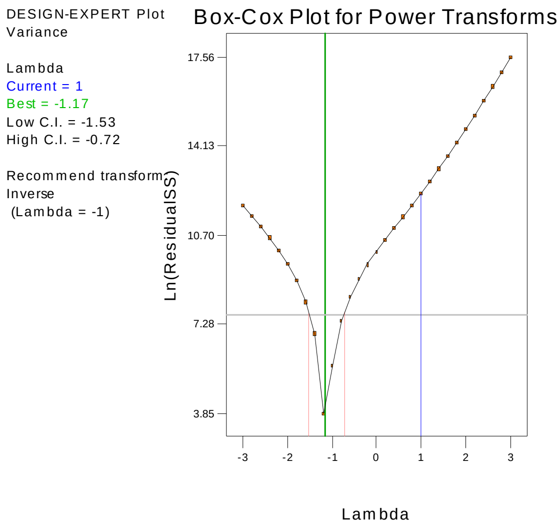
The Box-Cox plot for the Mean Thickness model suggests that a natural log transformation could be applied; however, the confidence interval for lambda includes one. Therefore, a transformation would have a minimal effect. The natural log transformation applied to the Variance of Thickness model appears to be acceptable; however, again the confidence interval for lambda includes one.

15-8 In the 3^3 factorial design of Problem 12-12 one of the responses is a standard deviation. Use the Box-Cox method to investigate the usefulness of transformations for this response. Would your answer change if we used the variance of the response?



Because the confidence interval for lambda does not include one, a transformation should be applied. The natural log transformation should not be considered due to zero not being included in the confidence interval. The square root transformation appears to be acceptable. However, notice that the value of zero is very close to the lower confidence limit, and the minimizing value of lambda is between 0 and 0.5. It is likely that either the natural log or the square root transformation would work reasonably well.

15-9 Problem 12-10 suggests using the $ln(s^2)$ as the response (refer to part b). Does the Box-Cox method indicate that a transformation is appropriate?



Because the confidence interval for lambda does not include one, a transformation should be applied. The confidence interval does not include zero; therefore, the natural log transformation is inappropriate. With the point estimate of lambda at –1.17, the reciprocal transformation is appropriate.

15-10 Myers, Montgomery and Vining (2002) describe an experiment to study spermatozoa survival. The design factors are the amount of sodium citrate, the amount of glycerol, and equilibrium time, each at two levels. The response variable is the number of spermatozoa that survive out of fifty that were tested at each set of conditions. The data are in the following table. Analyze the data from this experiment with logistical regression.

Sodium Citrate	Glycerol	Equilibrium Time	Number Survived
-	-	-	34
+	-	-	20
-	+	-	8
+	+	-	21
-	-	+	30
+	-	+	20
-	+	+	10
+	+	+	25

Minitab Output

Binary Logistic Regression: Number Survi, Freq versus Sodium Citra, Glycerol, .

```
Link Function: Logit

Response Information

Variable         Value    Count
Number Survived  Success    168
                 Failure    232
Freq             Total      400

Logistic Regression Table
                                                        Odds     95% CI
Predictor             Coef    SE Coef       Z       P  Ratio   Lower  Upper
Constant         -0.376962   0.110113   -3.42   0.001
Sodium Citrate   0.0932642   0.110103    0.85   0.397   1.10    0.88   1.36
```

```
Glycerol         -0.463247  0.110078  -4.21  0.000   0.63   0.51   0.78
Equilbrium Time  0.0259045  0.109167   0.24  0.812   1.03   0.83   1.27
AB                0.585116  0.110066   5.32  0.000   1.80   1.45   2.23
AC               0.0543714  0.109317   0.50  0.619   1.06   0.85   1.31
BC                0.112190  0.108845   1.03  0.303   1.12   0.90   1.38

Log-Likelihood = -248.028
Test that all slopes are zero: G = 48.178, DF = 6, P-Value = 0.000

Goodness-of-Fit Tests

Method           Chi-Square  DF      P
Pearson            0.113790   1  0.736
Deviance           0.113865   1  0.736
Hosmer-Lemeshow    0.113790   6  1.000
```

This analysis shows that Glycerol (*B*) and the Sodium Citrate x Glycerol (*AB*) interaction have an effect on the survival rate of spermatozoa.

15-11 A soft drink distributor is studying the effectiveness of delivery methods. Three different types of hand trucks have been developed, and an experiment is performed in the company's methods engineering laboratory. The variable of interest is the delivery time in minutes (y); however, delivery time is also strongly related to the case volume delivered (x). Each hand truck is used four times and the data that follow are obtained. Analyze the data and draw the appropriate conclusions. Use α=0.05.

		Hand	Truck	Type	
1	1	2	2	3	3
y	*x*	*y*	*x*	*y*	*x*
27	24	25	26	40	38
44	40	35	32	22	26
33	35	46	42	53	50
41	40	26	25	18	20

From the analysis performed in Minitab, hand truck does not have a statistically significant effect on delivery time. Volume, as expected, does have a significant effect.

Minitab Output

General Linear Model: Time versus Truck

```
Factor     Type Levels Values
Truck     fixed      3 1 2 3

Analysis of Variance for Time, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Volume      1    1232.07    1217.55    1217.55  232.20  0.000
Truck       2      11.65      11.65       5.82    1.11  0.375
Error       8      41.95      41.95       5.24
Total      11    1285.67

Term          Coef   SE Coef        T      P
Constant    -4.747     2.638    -1.80  0.110
Volume     1.17326   0.07699    15.24  0.000
```

15-12 Compute the adjusted treatment means and the standard errors of the adjusted treatment means for the data in Problem 15-11.

$$\text{adj } \bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}\left(\bar{x}_{i.} - \bar{x}_{..}\right)$$

$$\text{adj } \bar{y}_{1.} = \frac{145}{4} - (1.173)\left(\frac{139}{4} - \frac{398}{12}\right) = 34.39$$

$$\text{adj } \bar{y}_{2.} = \frac{132}{4} - (1.173)\left(\frac{125}{4} - \frac{398}{12}\right) = 35.25$$

$$\text{adj } \bar{y}_{3.} = \frac{133}{4} - (1.173)\left(\frac{134}{4} - \frac{398}{12}\right) = 32.86$$

$$S_{adj.\bar{y}_{i.}} = \left[MS_E\left\{\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right\}\right]^{\frac{1}{2}}$$

$$S_{adj.\bar{y}_{1.}} = \left[5.24\left\{\frac{1}{4} + \frac{(34.75 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.151$$

$$S_{adj.\bar{y}_{2.}} = \left[5.24\left\{\frac{1}{4} + \frac{(31.25 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.154$$

$$S_{adj.\bar{y}_{3.}} = \left[5.24\left\{\frac{1}{4} + \frac{(33.50 - 33.17)^2}{884.50}\right\}\right]^{\frac{1}{2}} = 1.145$$

The solutions can also be obtained with Minitab as follows:

Minitab Output
```
Least Squares Means for Time

Truck      Mean   SE Mean
1         34.39     1.151
2         35.25     1.154
3         32.86     1.145
```

15-13 The sums of squares and products for a single-factor analysis of covariance follow. Complete the analysis and draw appropriate conclusions. Use $\alpha = 0.05$.

Source of Variation	Degrees of Freedom	Sums of Squares and Products x	xy	x
Treatment	3	1500	1000	650
Error	12	6000	1200	550
Total	15	7500	2200	1200

Source	df	Sums of Squares & Products x	xy	y	Adjusted y	df	MS	F_0
Treatment	3	1500	1000	650	-	-		
Error	12	6000	1200	550	310	11	28.18	
Total	15	7500	2200	1200	559.67	14		
Adjusted	Treat.				244.67	3	81.56	2.89

Treatments differ only at 10%.

15-14 Find the standard errors of the adjusted treatment means in Example 15-5.

From Example 14-4 $\bar{y}_{1.} = 40.38$, adj $\bar{y}_{2.} = 41.42$, adj $\bar{y}_{3.} = 37.78$

$$S_{adj.\bar{y}_{1.}} = \left[2.54\left\{\frac{1}{5} + \frac{(25.20 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7231$$

$$S_{adj.\bar{y}_{2.}} = \left[2.54\left\{\frac{1}{5} + \frac{(26.00 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7439$$

$$S_{adj.\bar{y}_{3.}} = \left[2.54\left\{\frac{1}{5} + \frac{(21.20 - 24.13)^2}{195.60}\right\}\right]^{\frac{1}{2}} = 0.7871$$

15-15 Four different formulations of an industrial glue are being tested. The tensile strength of the glue when it is applied to join parts is also related to the application thickness. Five observations on strength (*y*) in pounds and thickness (*x*) in 0.01 inches are obtained for each formulation. The data are shown in the following table. Analyze these data and draw appropriate conclusions.

			Glue	Formulation			
1	1	2	2	3	3	4	4
y	*x*	*y*	*x*	*y*	*x*	*y*	*x*
46.5	13	48.7	12	46.3	15	44.7	16
45.9	14	49.0	10	47.1	14	43.0	15
49.8	12	50.1	11	48.9	11	51.0	10
46.1	12	48.5	12	48.2	11	48.1	12
44.3	14	45.2	14	50.3	10	48.6	11

From the analysis performed in Minitab, glue formulation does not have a statistically significant effect on strength. As expected, glue thickness does affect strength.

Minitab Output

General Linear Model: Strength versus Glue

```
Factor     Type Levels Values
Glue      fixed      4 1 2 3 4

Analysis of Variance for Strength, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Thick       1     68.852     59.566     59.566   42.62  0.000
Glue        3      1.771      1.771      0.590    0.42  0.740
Error      15     20.962     20.962      1.397
Total      19     91.585

Term          Coef   SE Coef        T      P
Constant    60.089     1.944    30.91  0.000
Thick      -1.0099    0.1547    -6.53  0.000

Unusual Observations for Strength

Obs  Strength       Fit      SE Fit  Residual   St Resid
  3   49.8000   47.5299      0.5508    2.2701      2.17R

R denotes an observation with a large standardized residual.

Expected Mean Squares, using Adjusted SS

Source       Expected Mean Square for Each Term
 1 Thick     (3) + Q[1]
 2 Glue      (3) + Q[2]
```

```
 3 Error     (3)

Error Terms for Tests, using Adjusted SS

Source       Error DF  Error MS  Synthesis of Error MS
 1 Thick        15.00     1.397  (3)
 2 Glue         15.00     1.397  (3)

Variance Components, using Adjusted SS

Source    Estimated Value
Error               1.397
```

15-16 Compute the adjusted treatment means and their standard errors using the data in Problem 15-15.

$$\text{adj } \bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$$
$$\text{adj } \bar{y}_{1.} = 46.52 - (-1.0099)(13.00 - 12.45) = 47.08$$
$$\text{adj } \bar{y}_{2.} = 48.30 - (-1.0099)(11.80 - 12.45) = 47.64$$
$$\text{adj } \bar{y}_{3.} = 48.16 - (-1.0099)(12.20 - 12.45) = 47.91$$
$$\text{adj } \bar{y}_{4.} = 47.08 - (-1.0099)(12.80 - 12.45) = 47.43$$

$$S_{adj.\bar{y}_{i.}} = \left[MS_E\left\{\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right\}\right]^{\frac{1}{2}}$$

$$S_{adj.\bar{y}_{1.}} = \left[1.40\left\{\frac{1}{5} + \frac{(13.00 - 12.45)^2}{58.40}\right\}\right]^{\frac{1}{2}} = 0.5360$$

$$S_{adj.\bar{y}_{2.}} = \left[1.40\left\{\frac{1}{5} + \frac{(11.80 - 12.45)^2}{58.40}\right\}\right]^{\frac{1}{2}} = 0.5386$$

$$S_{adj.\bar{y}_{3.}} = \left[1.40\left\{\frac{1}{5} + \frac{(12.20 - 12.45)^2}{58.40}\right\}\right]^{\frac{1}{2}} = 0.5306$$

$$S_{adj.\bar{y}_{4.}} = \left[1.40\left\{\frac{1}{5} + \frac{(12.80 - 12.45)^2}{58.40}\right\}\right]^{\frac{1}{2}} = 0.5319$$

The adjusted treatment means can also be generated in Minitab as follows:

Minitab Output

```
Least Squares Means for Strength

Glue      Mean   SE Mean
1        47.08    0.5355
2        47.64    0.5382
3        47.91    0.5301
4        47.43    0.5314
```

15-17 An engineer is studying the effect of cutting speed on the rate of metal removal in a machining operation. However, the rate of metal removal is also related to the hardness of the test specimen. Five observations are taken at each cutting speed. The amount of metal removed (y) and the hardness of the specimen (x) are shown in the following table. Analyze the data using and analysis of covariance. Use α=0.05.

		Cutting	Speed	(rpm)	
1000	1000	1200	1200	1400	1400
y	x	y	x	y	x
68	120	112	165	118	175
90	140	94	140	82	132
98	150	65	120	73	124
77	125	74	125	92	141
88	136	85	133	80	130

As shown in the analysis performed in Minitab, there is no difference in the rate of removal between the three cutting speeds. As expected, the hardness does have an impact on rate of removal.

Minitab Output

General Linear Model: Removal versus Speed

```
Factor     Type Levels Values
Speed     fixed      3 1000 1200 1400

Analysis of Variance for Removal, using Adjusted SS for Tests

Source     DF     Seq SS     Adj SS     Adj MS       F      P
Hardness    1     3075.7     3019.3     3019.3  347.96  0.000
Speed       2        2.4        2.4        1.2    0.14  0.872
Error      11       95.5       95.5        8.7
Total      14     3173.6

Term          Coef   SE Coef        T      P
Constant   -41.656     6.907    -6.03  0.000
Hardness   0.93426   0.05008    18.65  0.000
Speed
1000         0.478     1.085     0.44  0.668
1200         0.036     1.076     0.03  0.974

Unusual Observations for Removal

Obs   Removal       Fit      SE Fit  Residual   St Resid
  8    65.000    70.491       1.558    -5.491     -2.20R

R denotes an observation with a large standardized residual.

Expected Mean Squares, using Adjusted SS

Source       Expected Mean Square for Each Term
 1 Hardness  (3) + Q[1]
 2 Speed     (3) + Q[2]
 3 Error     (3)

Error Terms for Tests, using Adjusted SS

Source       Error DF  Error MS  Synthesis of Error MS
 1 Hardness     11.00       8.7  (3)
 2 Speed        11.00       8.7  (3)

Variance Components, using Adjusted SS

Source    Estimated Value
Error               8.677

Means for Covariates

Covariate      Mean     StDev
Hardness      137.1     15.94

Least Squares Means for Removal

Speed      Mean   SE Mean
1000      86.88     1.325
1200      86.44     1.318
```

```
1400      85.89     1.328
```

15-18 Show that in a single factor analysis of covariance with a single covariate a 100(1-α) percent confidence interval on the i_{th} adjusted treatment mean is

$$\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1}\left[MS_E\left(\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right)\right]^{\frac{1}{2}}$$

Using this formula, calculate a 95 percent confidence interval on the adjusted mean of machine 1 in Example 14-4.

The 100(1-α) percent interval on the i_{th} adjusted treatment mean would be

$$\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1}S_{adj\bar{y}_{i.}}$$

since $\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$ is an estimator of the i_{th} adjusted treatment mean. The standard error of the adjusted treatment mean is found as follows:

$$V(adj.\bar{y}_{i.}) = V\left[\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right] = V(\bar{y}_{i.}) + (\bar{x}_{i.} - \bar{x}_{..})^2 V(\hat{\beta})$$

Since the $\{\bar{y}_{i.}\}$ and $\hat{\beta}$ are independent. From regression analysis, we have $V(\hat{\beta}) = \frac{\sigma^2}{E_{xx}}$. Therefore,

$$V(adj.\bar{y}_{i.}) = \frac{\sigma^2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2\sigma^2}{E_{xx}} = \sigma^2\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right]$$

Replacing σ^2 by its estimator MS_E, yields

$$\hat{V}(adj.\bar{y}_{i.}) = MS_E\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right] \text{ or}$$

$$S(adj.\bar{y}_{i.}) = \left\{MS_E\left[\frac{1}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}}\right]\right\}^{\frac{1}{2}}$$

Substitution of this result into $\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \pm t_{\alpha/2,a(n-1)-1}S_{adj\bar{y}_{i.}}$ will produce the desired confidence interval. A 95% confidence interval on the mean of machine 1 would be found as follows:

$$\begin{gathered}
adj.\bar{y}_{i.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) = 40.38 \\
S(adj.\bar{y}_{i.}) = 0.7231 \\
\left[40.38 \pm t_{0.025,11}(0.7231)\right] \\
\left[40.38 \pm (2.20)(0.7231)\right] \\
\left[40.38 \pm 1.59\right]
\end{gathered}$$

Therefore, $38.79 \le \mu_1 \le 41.96$, where μ_1 denotes the true adjusted mean of treatment one.

15-19 Show that in a single-factor analysis of covariance with a single covariate, the standard error of the difference between any two adjusted treatment means is

$$S_{Adj\bar{y}_{i.}-Adj\bar{y}_{j.}} = \left[MS_E \left(\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}} \right) \right]^{\frac{1}{2}}$$

$$adj.\bar{y}_{i.} - adj.\bar{y}_{j.} = \bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) - \left[\bar{y}_{j.} - \hat{\beta}(\bar{x}_{j.} - \bar{x}_{..}) \right]$$

$$adj.\bar{y}_{i.} - adj.\bar{y}_{j.} = \bar{y}_{i.} - \bar{y}_{j.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{j.})$$

The variance of this statistic is

$$V\left[\bar{y}_{i.} - \bar{y}_{j.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{j.}) \right] = V(\bar{y}_{i.}) + V(\bar{y}_{j.}) + (\bar{x}_{i.} - \bar{x}_{j.})^2 V(\hat{\beta})$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{j.})^2 \sigma^2}{E_{xx}} = \sigma^2 \left[\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{j.})^2}{E_{xx}} \right]$$

Replacing σ^2 by its estimator MS_E, , and taking the square root yields the standard error

$$S_{Adj\bar{y}_{i.}-Adj\bar{y}_{j.}} = \left[MS_E \left(\frac{2}{n} + \frac{(\bar{x}_{i.} - \bar{x}_{..})^2}{E_{xx}} \right) \right]^{\frac{1}{2}}$$

15-20 Discuss how the operating characteristic curves for the analysis of variance can be used in the analysis of covariance.

To use the operating characteristic curves, fixed effects case, we would use as the parameter Φ^2,

$$\Phi^2 = \frac{a \sum \tau_i^2}{n\sigma^2}$$

The test has *a*-1 degrees of freedom in the numerator and *a*(*n*-1)-1 degrees of freedom in the denominator.