2D boundary integral equations and the Nyström method

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Integral equations on 1D interval

Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, periodic: $\sigma(2\pi) = \sigma(0)$, etc

ii) "kernel" function $k(t,s)$ defined on square $[0, 2\pi)^2$,

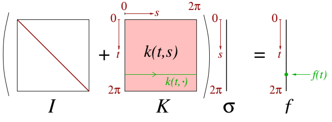
Integral *operator* K acts on σ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t,s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

continuous analog of matrix-vector prod. $A\boldsymbol{x}$

Integral *equation*: $(I + K)\sigma = f$, ie

$$\sigma(t) + \int_0^{2\pi} k(t,s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$



analog of lin. sys. $A\boldsymbol{x} = \boldsymbol{b}$

Fredholm "second kind" (due to presence of I, otherwise called "first kind")

If kernel continuous, or "weakly" singular (integrable), K is *compact*:

- eigenvalues ($K\phi_k = \lambda_k\phi_k$) discrete, with $\lim_{k\to\infty} \lambda_k = 0$

  unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: Nul $(I + K) = \{0\}$

- Nul $(I + K) = \{0\} \;\Rightarrow\;$ *existence* of solution for *any* f Fredholm Alternative

  like square matrix (finite-dim), recall: uniqueness $\Rightarrow$ consistent for any RHS

Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)!

See references for lots of beautiful theory, precise statements

Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t)dt \approx \sum_{j=1}^{N} \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^{N} w_j f(t_j)$$

w_j=weights, t_j=nodes

For f smooth, superalgebraically convergent ("spectral"): error $= \mathcal{O}(N^{-p})$ for any p

Apply quad to integral in 2nd kind IE: call the resulting approx soln $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^{N} k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \qquad (*)$$

Holds for all t; in particular, holds at each t_i, $i = 1, \ldots, N$, giving:

$$\sigma_i + \sum_{j=1}^{N} k(t_i, t_j) w_j \sigma_j = f(t_i), \quad i = 1, \ldots, N$$

where $\sigma_j := \tilde{\sigma}(t_j)$

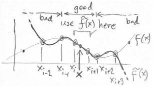
Write as: $A\boldsymbol{\sigma} = \boldsymbol{f}$ $N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$, and $f_j := f(t_j)$

solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs(A) accumulate only at $+1$, iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^{N}$ suffice.
If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

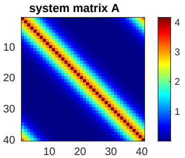
- either: rearrange (*) to give $\tilde{\sigma}(t) = \ldots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

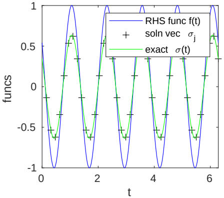


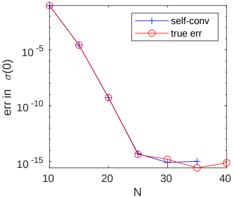
Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));                 % smooth convolutional kernel, periodic domain [0,2pi)
ffun = @(t) cos(5*t+1);                         % smooth data (RHS) func
N = 30;                                         % number of unknowns: should study convergence as N grows...
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);        % PTR nodes and weights, row vecs
A = eye(N) + bsxfun(kfun,t',t)*diag(w);        % identity plus fill k(t_i,t_j)w_j for i,j=1..N
rhs = ffun(t');                                 % col vec
sigmaj = A\rhs;                                 % dense direct square solve (pivoted LU), gives col vec
```







"self-convergence": use N=40 as "true"

f and k smooth

$\Rightarrow$ σ smooth

$\Rightarrow$ spectral conv?

Thm. (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t,\cdot)\sigma$.

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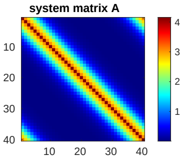

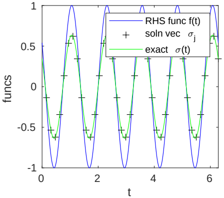

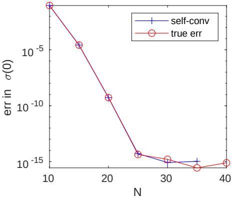

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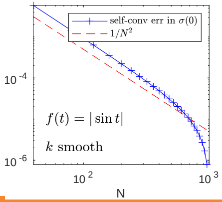
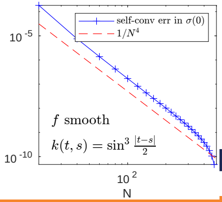
- Then, f or k nonsmooth?

worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?





Fundamental solution in $\mathbb{R}^2$

Eg PDE: Poisson eqn $\Delta u = g$ $\quad$ $\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

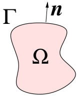
Notation: $\boldsymbol{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\boldsymbol{y} \in \mathbb{R}^2$ as another point (not y-coord!)

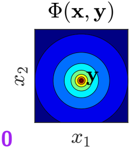
Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving $g \equiv 0$)

Eg well-posed* BVP:

$$\Delta u = 0 \text{ in } \Omega \qquad \text{PDE } (u \text{ harmonic})$$
$$u = f \text{ on } \Gamma \qquad \text{Dirichlet BC}$$

*exists, unique, continuous w.r.t. data





Laplace fundamental soln: $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ $\quad$ where $r := \|\boldsymbol{x} - \boldsymbol{y}\|$

obeys $-\Delta_{\boldsymbol{x}} \Phi = -\Delta_{\boldsymbol{y}} \Phi = \delta_{\boldsymbol{x}}$ $\quad$ Φ harmonic except unit point-mass at $\mathbf{0}$

Normal $\boldsymbol{n}$ points outwards, $\|\boldsymbol{n}\| = 1$ $\quad$ normal deriv. notation $u_n := \boldsymbol{n} \cdot \nabla u$

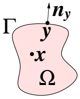
Green's 2nd identity: $\int_\Gamma v u_n - v_n u \, ds = \int_\Omega v \Delta u - (\Delta v) u \, d\boldsymbol{y}$ $\quad$ calculus

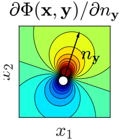
warm-up: set $u =$ BVP soln, $v \equiv 1$, G2I becomes $\int_\Gamma u_n ds - 0 = 0 - 0$: so u has zero *flux*

more fun: fix "target" $\boldsymbol{x} \in \Omega$, let $v = \Phi(\boldsymbol{x}, \cdot)$, G2I gives:

Green's representation formula:

$$\int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y}) u_n(\boldsymbol{y}) - \frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}} u(\boldsymbol{y}) \, ds_{\boldsymbol{y}} = u(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega$$



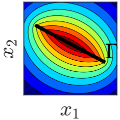


Gets soln from "Cauchy data" $(u, u_n)|_\Gamma$

also versions for Helmholtz, Stokes, Maxwell, . . .

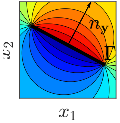
Layer potentials and their jump relations

Representations of harmonic functions off a curve Γ: "density" σ



Single-layer potential $(\mathcal{S}\sigma)(\boldsymbol{x}) := \int_\Gamma \Phi(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$ charge sheet

Double-layer potential $(\mathcal{D}\sigma)(\boldsymbol{x}) := \int_\Gamma \frac{\partial\Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$ dipole sheet



interior (-) / exterior (+) limits:

$$u^\pm(\boldsymbol{x}) := \lim_{h\to 0^+} u(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}})$$

$$u_n^\pm(\boldsymbol{x}) := \lim_{h\to 0^+} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla u(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}})$$

Jump relations:

$(\mathcal{S}\sigma)^\pm = S\sigma$ — S (Roman font) means *restriction* of $\mathcal{S}$ to Γ: a bdry int. op.

$(\mathcal{D}\sigma)^\pm = (D \pm I/2)\sigma$ — jump in potential equal to σ; D restriction to Γ in P.V. sense

$(\mathcal{S}\sigma)_n^\pm = (D^T \mp I/2)\sigma$ — jump in normal derivative

$(\mathcal{D}\sigma)_n^\pm = T\sigma$ — T hypersingular, kernel $\partial^2\Phi(\boldsymbol{x},\boldsymbol{y})/\partial\boldsymbol{n}_{\boldsymbol{x}}\partial\boldsymbol{n}_{\boldsymbol{y}} \sim 1/r^2$

- D smooth kernel on smooth Γ, while S always log (weakly) singular

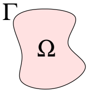
Recap GRF in LP notation: u harmonic in $\Omega \;\Rightarrow\; \mathcal{S}u_n^- - \mathcal{D}u^- = u$ in Ω

Converting BVP to BIE and solving

Say wish to solve interior Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \qquad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \qquad \text{BC}$$



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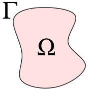

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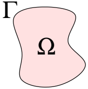

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Insert the BC to get BIE: $(I - 2D)\sigma = -2f$ scaled to 2nd kind form

This shows: let σ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns u of this form?

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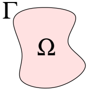

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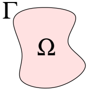

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Above BIE expressed on Γ using arc-length measure ds_y. Usually not how Γ described...

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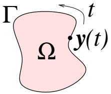

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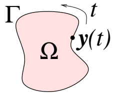

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familiar form $(I + K)\sigma = -2f$, with kernel $k(s,t) = \frac{-2}{2\pi} \frac{\mathbf{n}_{\mathbf{y}(s)} \cdot (\mathbf{y}(t) - \mathbf{y}(s))}{\|\mathbf{y}(t) - \mathbf{y}(s)\|^2} \|\mathbf{y}'(s)\|$

formula on diagonal: $k(t,t) = \lim_{s \to t} k(t,s) = \kappa(t)/2\pi$, κ curvature of Γ (check!)

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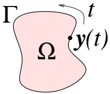

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Now Nyström discretize with PTR, solve lin. sys. for $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln: $u(\mathbf{x}) = (\mathcal{D}\sigma)(\mathbf{x}) \overset{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\mathbf{n}_{\mathbf{y}(t_j)} \cdot (\mathbf{x} - \mathbf{y}(t_j))}{2\pi \|\mathbf{x} - \mathbf{y}(t_j)\|^2} \|\mathbf{y}'(t_j)\| w_j \sigma_j$

Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

```
a=0.7; b=1.0;                                                    % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                          % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];                      % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];                    % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                          % PTR nodes & weights
y = Y(t);                                                        % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;   % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                           % bdry curvatures
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Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

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a=0.7; b=1.0;                                                          % shape params (note a=1,b=0 unit circle)
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N = 100;
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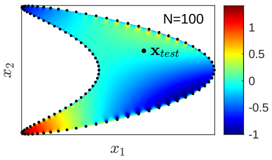

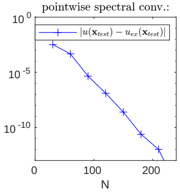

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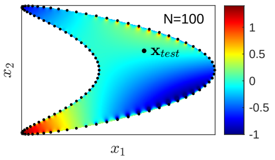

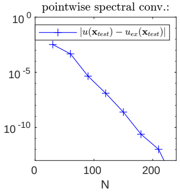

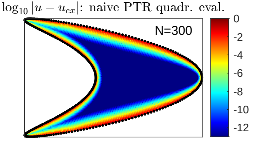

"5h" rule:
special eval.
quadratures
can fix near Γ
(Helsing, QBX...)

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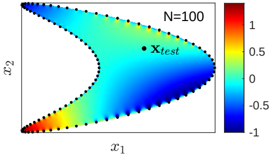

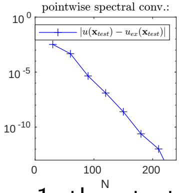

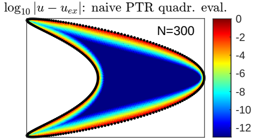

"5h" rule:
special eval. quadratures can fix near Γ
(Helsing, QBX...)

Debug: $\sigma \equiv -1 \;\Rightarrow\; u \equiv 1$, then test data from (generic!) soln u, and...

1. check/plot $\boldsymbol{n}$, κ. First test unit circle!
2. check Nyström matrix smooth at diag (*before* add I)

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Indirect vs direct formulations

using Laplace interior Dirichlet BVP

So far "indirect" BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

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GRF $u = \mathcal{S}u^- - \mathcal{D}u_n^- \quad \xrightarrow{\text{JRs}} \quad u_n^- = (D^T + I/2)u_n^- - Tu^- \quad \xrightarrow{\text{BC}} \quad (D^T - I/2)u_n^- = Tf$

Needs hypersingular apply ☹. Then solve BIE for u_n^-, eval u via GRF (needs two LP evals)

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Indirect BIE	Direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

Generalizations: Exterior, Neumann

Laplace int Dir

$\Delta u = 0$ in Ω
$u^- = f$ on Γ
uniqueness, existence $\forall f$

$u = \mathcal{D}\sigma$ representation
$(D - I/2)\sigma = f$ BIE

Laplace int Neu

$\Delta u = 0$ in Ω
$u_n^- = g$ on Γ
require $\int_\Gamma g ds = 0$ and unique only up to a const.

$u = \mathcal{S}\sigma$ others may be used
$(D^T + I/2)\sigma = g$ nullity 1, reducible

Laplace ext Dir

$\Delta u = 0$ in $\mathbb{R}^d \backslash \bar{\Omega}$
$u^+ = f$ on Γ
uniqueness, existence $\forall f$ if
$u_\infty = \mathcal{O}(1)$ in $\mathbb{R}^2$, $u_\infty := \lim_{|\boldsymbol{x}| \to \infty} u(\boldsymbol{x})$
$u_\infty = 0$ in $\mathbb{R}^{d>2}$
representation
BIE

Laplace ext Neu

$\Delta u = 0$ in $\mathbb{R}^d \backslash \bar{\Omega}$
$u_n^+ = g$ on Γ
require $\int_\Gamma g ds = 0$ and $u_\infty = 0$

$u = \mathcal{S}\sigma$
$(D^T - I/2)\sigma = g$

Helmholtz — introduction and connection to Maxwell

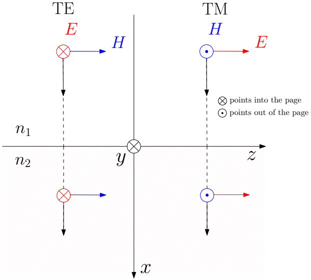
$(\Delta + \omega^2)u = 0$

Arises from scalar wave equation $u_{tt} - \Delta u = 0$ Take Fourier transform wrt t

ω is the wavenumber spatial frequency, related to wavelength via $\lambda = 2\pi/\omega$

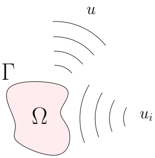
Also used for Maxwell's equations in cylindrical symm (z-invariance):

1. Assume $\mathbf{E}, \mathbf{H}(x, y, z) = \mathbf{E}, \mathbf{H}(x, y)$
2. Write Maxwell's eqs: $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, $\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}$,
3. Notice only E_z, H_z are indep $\to$ 2 polarizations, TE or TM: $E_z = 0$, $H_z = 0$ resp.
4. Choose TE and let $u := H_z$, then: $\mathbf{H} = (0, 0, u)$, $\mathbf{E} = \frac{1}{i\omega\varepsilon}(\partial_x u, -\partial_y u, 0)$, and $(\Delta + n^2\omega^2)u = 0$ with $n^2 = \varepsilon\mu$



Dirichlet BC in TE formalism = PEC perfect electric conductor; $\mathbf{E}$ || to surface

Helmholtz — scattering formalism



Split total potential into incident (known) and scattered (unknown) parts, $u^{\text{tot}} = u^{\text{inc}} + u$

BVP for u:

$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \backslash \bar{\Omega}$ PDE

$u = -u_i \quad \text{on } \Gamma$ Dirichlet BC, $u_n = -(u_i)_n$ for Neumann

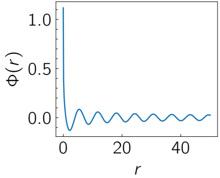
$\lim_{r\to\infty} \left(\frac{\partial u}{\partial r} - iku\right) = 0$ $r := |\boldsymbol{x} - \boldsymbol{y}|$, Sommerfeld radiation condition for uniqueness

Fundamental solution $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(\omega|\boldsymbol{x} - \boldsymbol{y}|)$

Asymptotics: $\lim_{r\to 0} \Phi(r) = \frac{1}{2\pi} \log \frac{1}{r} + \mathcal{O}(1)$

$\lim_{r\to\infty} \Phi(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \nu\pi/2 - \pi/4)} + \mathcal{O}(r^{-1})$

Same singularity as Laplace $\rightarrow$ same JRs!



Layer potentials

Helmholtz — interior resonances and how to avoid them

$u = \mathcal{D}\sigma$ has interior res prob (Leslie will have covered for the disk case)

CFIE: $u = (\mathcal{D} + i\eta\mathcal{S})\sigma$ no more unknowns, new kernel

can prove equivalence (no spurious resonances)

Helmholtz — Dirichlet demo

Dirichlet demo, plots only: Solve BVP for u via PTR + Nyström, with new diag limit for $k(t,t)$, show $1/N^3$ convergence if use naive PTR with correct diag limit (see M126 HW?)

4. debug BVP with known data from a radiative soln sources inside Ω

(don't demo CFIE since requires S w/ log-singularity).

Helmholtz transmission BVP

refractive indices in Ω vs exterior

Matching? (more effort, needs $\mathcal{S}\sigma + \mathcal{D}\tau \ \ldots$)

difference kernels at most log-singular.

Helmholtz

Getting spectral-acc Nyström for log-singular kernels: beyond today.

eg kernel-split or product quadratures (Kress, Helsing,. . .)

close-eval: kernel-split, QBX, etc.

see libraries: chunkie, BIE2D, etc

More debug ideas

TO DISCARD

Other tests:

5. Test SLP & DLP evaluators via GRF for any harmonic u in Ω

Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs — exterior need condition as $\|\boldsymbol{x}\| \to \infty$
- Choosing representation — to get 2nd kind BIE if poss., equivalent to BVP if poss.

  Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization — high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases — also for libraries

practise! write out theory yourself + try HW exer. in repo + run demos

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Useful 2D tools we did not yet cover: — in libraries, eg `chunkie`, `BIE2D`

- panel (composite) quadratures — essential for adaptivity
- high-order quadratures for log-singular kernel — SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve

  some need interpolation of $\sigma(t)$ off the nodes t_j, some not
- corners, open arcs, slits, multi-material junctions

Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- "High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane", S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

  focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- `https://users.flatironinstitute.org/~ahb/BIE/`
- `https://github.com/ahbarnett/BIEbook` in progress...