

# 2D boundary integral equations and the Nyström method

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## Integral equations on 1D interval

Given: i) function  $\sigma(t)$  defined on interval  $[0, 2\pi)$ ,    periodic:  $\sigma(2\pi) = \sigma(0)$ , etc  
ii) “kernel” function  $k(t, s)$  defined on square  $[0, 2\pi)^2$ ,

Integral *operator*  $K$  acts on  $\sigma$  to give another function  $K\sigma$ :

$$(K\sigma)(t) := \int_0^{2\pi} k(t, s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

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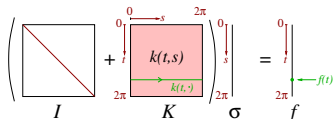
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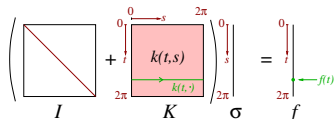
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- eigenvalues  $(K\phi_k = \lambda_k\phi_k)$  discrete, with  $\lim_{k \rightarrow \infty} \lambda_k = 0$   
unless some  $\lambda_k = -1$ , 2nd kind IE has at most one soln:  $\text{Nul}(I + K) = \{0\}$
- $\text{Nul}(I + K) = \{0\} \Rightarrow$  existence of solution for any  $f$  Fredholm Alternative  
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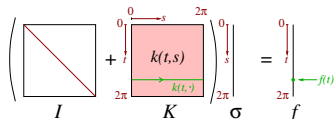
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Contrast 1st kind IE  $K\sigma = f$  is ill-posed problem (unstable)!

## Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t) dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j) \quad w_j = \text{weights}, \quad t_j = \text{nodes}$$

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solve? dense direct  $\mathcal{O}(N^3)$ ; dense iter.  $\mathcal{O}(N^2)$ ; fast iter.  $\approx \mathcal{O}(N)$ ; fast direct  $\approx \mathcal{O}(N^{(d+1)/2})$

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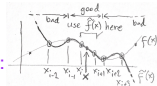
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Sometimes for BIE (eg, far-field eval), node values  $\{\sigma_j\}_{j=1}^N$  suffice.

If not, interpolate from them to any  $t \in [0, 2\pi)$ . Two approaches:

- either: rearrange (\*) to give  $\tilde{\sigma}(t) = \dots$ , called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

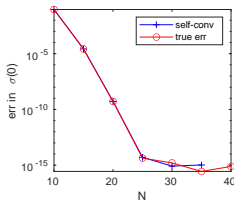
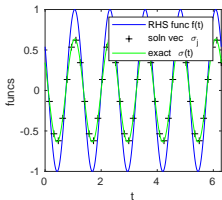
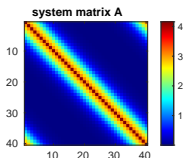


# Demo Nyström on interval (1D)

day1/code/nyst1d\_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));  
ffun = @(t) cos(5*t+1);  
N = 30;  
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);  
A = eye(N) + bsxfun(kfun,t',t)*diag(w);  
rhs = ffun(t');  
sigmaj = A\rhs;
```

*% smooth convolutional kernel, periodic domain [0,2pi)  
% smooth data (RHS) func  
% number of unknowns: should study convergence as N grows...  
% PTR nodes and weights, row vecs  
% identity plus fill k(t\_i,t\_j)w\_j for i,j=1..N  
% col vec  
% dense direct square solve (pivoted LU), gives col vec*



“self-convergence”:  
use  $N=40$  as “true”

$f$  and  $k$  smooth

⇒  $\sigma$  smooth

⇒ spectral conv?

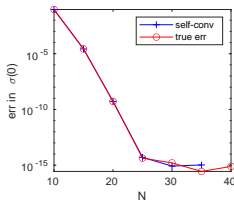
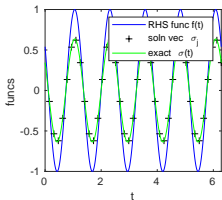
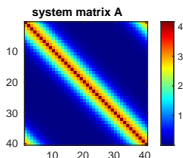
**Thm.** (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand  $k(t, \cdot)\sigma$ .

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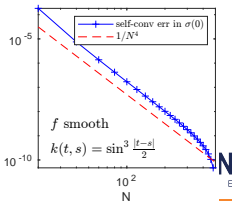
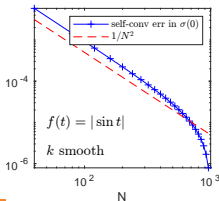
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$f$  and  $k$  smooth  
 $\Rightarrow \sigma$  smooth  
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- Then,  $f$  or  $k$  nonsmooth?  
worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



## Laplace fundamental solution in $\mathbb{R}^2$

Eg PDE: Poisson eqn  $\Delta u = g$

$\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$  Laplacian

notation:  $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$  is a point. This frees up  $\mathbf{y} \in \mathbb{R}^2$  as another point (not y-coord!)

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A well-posed\* BVP:

$$\Delta u = 0 \text{ in } \Omega$$

PDE ( $u$  harmonic)

$$u = f \text{ on } \Gamma$$

Dirichlet BC

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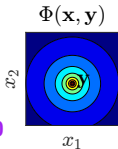
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Laplace fundamental soln:  $\Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{r}$  where  $r := \|\mathbf{x} - \mathbf{y}\|$

obeys  $-\Delta_{\mathbf{x}} \Phi = -\Delta_{\mathbf{y}} \Phi = \delta_{\mathbf{x}}$   $\Phi$  harmonic except unit point-mass at  $\mathbf{0}$

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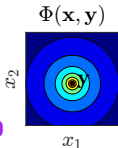
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warm-up: set  $u = \text{BVP soln}$ ,  $v \equiv 1$ , G2I becomes  $\int_{\Gamma} u_n \, ds - 0 = 0 - 0$ : so  $u$  has zero flux



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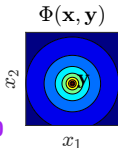
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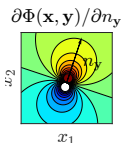
Now some fun: fix "target"  $\mathbf{x} \in \Omega$ , let  $v = \Phi(\mathbf{x}, \cdot)$ , G2I gives:

Green's representation formula:

$$\int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_{\mathbf{y}}} u(\mathbf{y}) \, ds_{\mathbf{y}} = u(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega$$

recovers soln from "Cauchy data"  $(u, u_n)|_{\Gamma}$

also versions for Helmholtz, Stokes, Maxwell,...

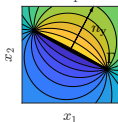
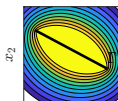


## Layer potentials and their jump relations

Representations of harmonic functions off a curve  $\Gamma$ : “density”  $\sigma$

Single-layer potential  $(\mathcal{S}\sigma)(\mathbf{x}) := \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) ds_{\mathbf{y}}$  charge sheet

Double-layer potential  $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y}) ds_{\mathbf{y}}$  dipole sheet



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interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

Jump relations:

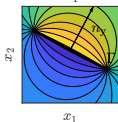
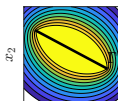
$(\mathcal{S}\sigma)^{\pm} = S\sigma$   $S$  (Roman font) means *restriction* of  $S$  to  $\Gamma$ : a bdry int. op.

$(\mathcal{D}\sigma)^{\pm} = (D \pm I/2)\sigma$  jump in potential equal to  $\sigma$ ;  $D$  restriction to  $\Gamma$  in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = (D^T \mp I/2)\sigma$  jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = T\sigma$   $T$  hypersingular, kernel  $\partial^2 \Phi(\mathbf{x}, \mathbf{y}) / \partial \mathbf{n}_{\mathbf{x}} \partial \mathbf{n}_{\mathbf{y}} \sim 1/r^2$

- $D$  smooth kernel on smooth  $\Gamma$ , while  $S$  always log (weakly) singular

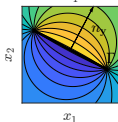
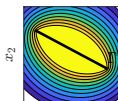


# Layer potentials and their jump relations

Representations of harmonic functions off a curve  $\Gamma$ : “density”  $\sigma$

Single-layer potential  $(\mathcal{S}\sigma)(\mathbf{x}) := \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) d\mathbf{s}_{\mathbf{y}}$  charge sheet

Double-layer potential  $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y}) d\mathbf{s}_{\mathbf{y}}$  dipole sheet



interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

Jump relations:

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Recap GRF in LP notation:  $u$  harmonic in  $\Omega \Rightarrow \mathcal{S}u_n^- - \mathcal{D}u^- = u$  in  $\Omega$

## Converting BVP to BIE and solving

Say wish to solve interior

Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE}$$

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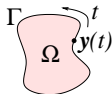
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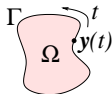
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familiar form  $(I + K)\sigma = -2f$ , with kernel  $k(s, t) = \frac{-2}{2\pi} \frac{\mathbf{n}_{\mathbf{y}(s)} \cdot (\mathbf{y}(t) - \mathbf{y}(s))}{\|\mathbf{y}(t) - \mathbf{y}(s)\|^2} \|\mathbf{y}'(s)\|$

formula on diagonal:  $k(t, t) = \lim_{s \rightarrow t} k(t, s) = \kappa(t)/2\pi$ ,  $\kappa$  curvature of  $\Gamma$  (check!)

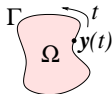
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Now Nyström discretize with PTR, solve lin. sys. for  $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln:  $u(\mathbf{x}) = (\mathcal{D}\sigma)(\mathbf{x}) \stackrel{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\mathbf{n}_{\mathbf{y}(t_j)} \cdot (\mathbf{x} - \mathbf{y}(t_j))}{2\pi \|\mathbf{x} - \mathbf{y}(t_j)\|^2} \|\mathbf{y}'(t_j)\| w_j \sigma_j$

# Interior Laplace Dirichlet BVP solve demo

demo\_lapintdir.m

```
a=0.7; b=1.0;
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];
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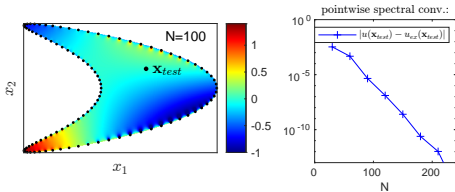
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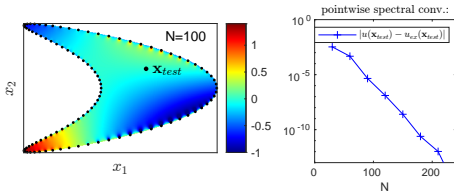
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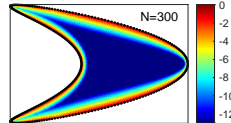
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$\log_{10} |u - u_{ex}|$ : naive PTR quadr. eval.



error: "5h" rule.

Note: special quadratures can fix near  $\Gamma$  (Helsing, QBX...)

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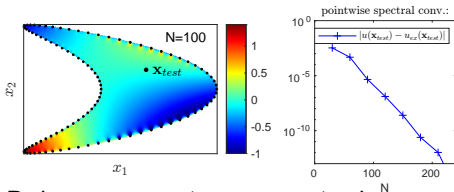
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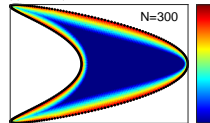
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Debug:  $\sigma \equiv -1 \Rightarrow u \equiv 1$ , then test data from (generic!) soln  $u$ , and...

- ① check/plot  $n, \kappa$ . First test unit circle!
- ② check Nyström matrix smooth at diag (before add I)



## Indirect vs direct formulations

using Laplace interior Dirichlet BVP

So far “indirect” BIE: pick representation (eg  $u = \mathcal{D}\sigma$ ), get BIE from JRs

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Indirect BIE	Direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

## Indirect 2nd-kind BIE for Neumann, exterior

recap: Laplace int. Dir.

$$\Delta u = 0 \text{ in } \Omega$$

$$u^- = f \text{ on } \Gamma$$

uniqueness, existence  $\forall f$

- $u = \mathcal{D}\sigma$       rep.  
 $(D - I/2)\sigma = f$       BIE: well-cond.

Laplace int. Neu.

$$\Delta u = 0 \text{ in } \Omega$$

$$u_n^- = g \text{ on } \Gamma$$

$$\text{require } \int_{\Gamma} g ds = 0$$

unique only up to a const.

- $u = \mathcal{S}\sigma$        $\swarrow$  kernel  $\equiv 1$ , kills nullspace  
 $(D^T + I/2 + 11^T)\sigma = g$       well-cond.

## Indirect 2nd-kind BIE for Neumann, exterior

recap: Laplace int. Dir.

$$\Delta u = 0 \text{ in } \Omega$$

$$u^- = f \text{ on } \Gamma$$

uniqueness, existence  $\forall f$

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$$\text{require } \int_{\Gamma} g ds = 0 \text{ and } u_{\infty} = 0$$

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complementary BVPs

③ Exterior: don't test with  $u = \log r$  why not? BVPs enforce zero net charge

# Helmholtz — introduction and connection to Maxwell

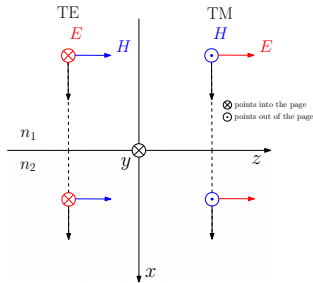
$$(\Delta + \omega^2)u = 0$$

time-harmonic scalar waves

comes from scalar wave equation  $\Delta u - u_{tt} = 0$  when  $u(\mathbf{x}, t) = u(\mathbf{x})e^{-i\omega t}$   
 $\omega$  is the wavenumber spatial frequency, related to wavelength via  $\lambda = 2\pi/\omega$

Also used for Maxwell's equations in cylindrical symm (z-invariance):

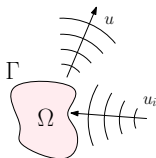
1. Assume  $\mathbf{E}, \mathbf{H}(x, y, z) = \mathbf{E}, \mathbf{H}(x, y)$
2. Write Maxwell's eqs:  $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$ ,  $\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}$ ,
3. Notice only  $E_z, H_z$  are indep  $\rightarrow$  2 polarizations, TE or TM:  $E_z = 0, H_z = 0$  resp.
4. Choose TE and let  $u := H_z$ , then:  $\mathbf{H} = (0, 0, u)$ ,  
 $\mathbf{E} = \frac{1}{i\omega\varepsilon}(\partial_x u, -\partial_y u, 0)$ , and  $(\Delta + n^2\omega^2)u = 0$  with  $n^2 = \varepsilon\mu$



Dirichlet BC in TE formalism = PEC perfect electric conductor;  $\mathbf{E} \perp$  to surface

# Helmholtz — scattering formalism

Split total potential into incident (known) and scattered (unknown) parts,  $u^{\text{tot}} = u^{\text{inc}} + u$



BVP for  $u$ :

$$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \quad \text{PDE}$$

$$u = -u_i \quad \text{on } \Gamma \quad \text{Dirichlet BC, } u_n = -(u_i)_n \text{ for Neumann}$$

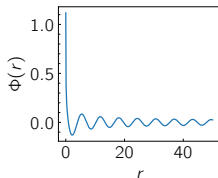
$$\lim_{r \rightarrow \infty} \left( \frac{\partial u}{\partial r} - i\omega u \right) = 0 \quad r := |\mathbf{x} - \mathbf{y}|, \text{ Sommerfeld radiation condition for uniqueness}$$

Fundamental solution  $\Phi(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\omega |\mathbf{x} - \mathbf{y}|)$

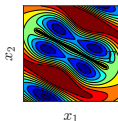
$$\text{Asymptotics: } \lim_{r \rightarrow 0} \Phi(r) = \frac{1}{2\pi} \log \frac{1}{r} + \mathcal{O}(1)$$

$$\lim_{r \rightarrow \infty} \Phi(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \nu\pi/2 - \pi/4)} + \mathcal{O}(r^{-1})$$

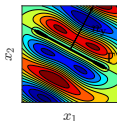
Same singularity as Laplace  $\rightarrow$  same JRs!



Layer potentials



SLP



DLP

# Helmholtz — interior resonances and how to avoid them

Try the ext Dir BVP with  $u = \mathcal{D}\sigma$  ( $(\Delta + \omega)^2 u = 0$  in  $\mathbb{R}^2 \setminus \overline{\Omega}$ ,  $u = -u_i$  on  $\Gamma$ , SRC for  $u$ )

Observe that for some  $\omega$ , condition # of BIE blows up, not always solvable

Why? Suppose  $\phi \not\equiv 0$  s.t. 
$$\begin{cases} (\Delta + \omega^2)\phi = 0 & \text{in } \Omega \\ \phi_n = 0 & \text{on } \Gamma \end{cases}$$
  $\phi$  is interior Neumann eigenfunction with eigenvalue  $\omega^2$

Then by (interior) GRF (same as for Laplace),  $\mathcal{S}\phi|_{\Gamma} - \mathcal{D}\phi|_{\Gamma} = u$  in  $\Omega$ .

Take  $\mathbf{x} \rightarrow \Gamma^-$  and use JR:  $(-D - I/2)\phi|_{\Gamma} = \phi_{\Gamma}$ , i.e.  $(I + 2D)\phi|_{\Gamma} = 0$ .

Since  $\phi|_{\Gamma}$  was nontrivial (otherwise  $\phi = 0$  by GRF), nullity of  $I + 2D > 0$ , i.e. singular, by FA not solvable  $\forall f(u_i)$ .

We made use of the **complementary BVP** (int Neu), this is an “internal resonance”.

Fix:  $u = (\mathcal{D} - i\eta\mathcal{S})\sigma$  combined field integral eq (CFIE), same # unknowns, new kernel  
ext Dir BIE becomes  $(I + 2D - 2i\eta\mathcal{S})\sigma = -2u_i$  on  $\Gamma$

Proof: Let  $\tau$  solve  $(I/2 + D - i\eta\mathcal{S})\tau = 0$ , wish to show  $\tau = 0$ .

From  $\tau$  construct potential  $v := (\mathcal{D} - i\eta\mathcal{S})\tau$ , then  $v^+ = 0$  by construction.

Then  $v = 0$  in  $\mathbb{R}^2 \setminus \overline{\Omega}$  by uniqueness of the complementary BVP (ext Dir)

Then  $v_n^+$  on  $\Gamma$ , and by JRs and Green's 1st thm (exercise for the reader ☺),  $\tau = 0$ .

# Helmholtz — Dirichlet demo

`demo_helmextdir.m`

Solve the Helmholtz ext Dir BVP with the  $u = \mathcal{D}\sigma$  repr,  $u_j$  plane wave

Diagonal limit for Nyström matrix  $k(t, t)$  same as Laplace

PTR with  $N$  nodes, test via self-convergence What's the conv. rate? Why  $N^{-3}$ ?

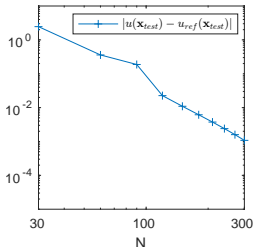
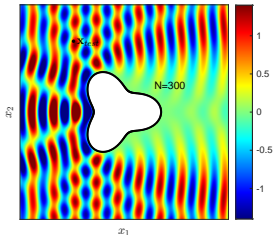
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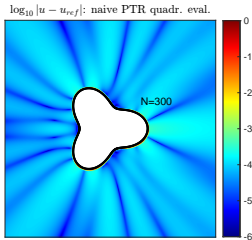
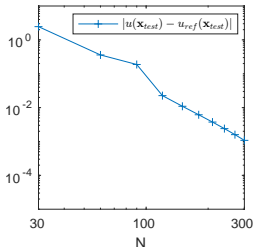
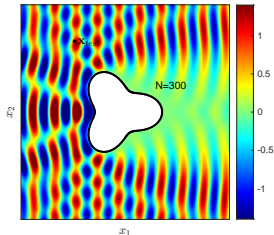
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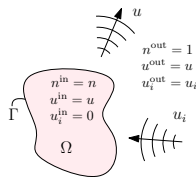
- 4 Debug BVP with known data from a radiative soln sources inside  $\Omega$
- 5 Without analytic soln, test either via self-convergence or conserved physical qty e.g. optical theorem, or no net QM flux over closed curve  $C$  containing no sources or sinks,  $\Im(\int_C \bar{u} u_n ds)$  (Agocs, Barnett '23)

# Helmholtz – transmission BVP

If different refractive index  $n$  in  $\Omega$  than outside, use usual splitting  $u^{\text{tot}} = u^{\text{inc}} + u$

can always scale such that one is  $n = 1$

inc wave only on outside, e.g.  $u_i = \begin{cases} 0 & \text{in } \Omega \\ e^{ik \cdot x} & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \end{cases}, \mathbf{k} = \begin{bmatrix} \omega \cos \theta \\ \omega \sin \theta \end{bmatrix}$



BVP for  $u$ :

$$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \quad \text{PDE outside}$$

$$(\Delta + n^2 \omega^2)u = 0 \quad \text{in } \overline{\Omega} \quad \text{PDE inside}$$

$$[u] = -u_i \quad \text{on } \Gamma \quad [u] := u^+ - u^-, \text{ continuity of } u^{\text{tot}}$$

$$[u_n] = -(u_i)_n \quad \text{on } \Gamma \quad \text{continuity of } u_n^{\text{tot}}$$

$$\lim_{r \rightarrow \infty} \left( \frac{\partial u}{\partial r} - i\omega u \right) = 0 \quad \text{SRC outside}$$

Formulate as sys of integral eqs      Rokhlin–Müller scheme, (Müller '69, Rokhlin '83)

$$u = \begin{cases} \mathcal{S}(n\omega)\sigma + \mathcal{D}(n\omega)\tau & \text{in } \Omega \\ \mathcal{S}(\omega)\sigma + \mathcal{D}(\omega)\tau & \text{in } \mathbb{R}^d \setminus \Omega \end{cases}$$

$$\begin{bmatrix} [u] \\ [u_n] \end{bmatrix} = \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \mathcal{D}(\omega) - \mathcal{D}(n\omega) & \mathcal{S}(n\omega) - \mathcal{S}(\omega) \\ \mathcal{T}(\omega) - \mathcal{T}(n\omega) & \mathcal{D}(n\omega)^* - \mathcal{D}(\omega)^* \end{bmatrix} \right) \begin{bmatrix} \tau \\ -\sigma \end{bmatrix} \quad \mathcal{T} \text{ is hypersingular operator}$$

...but  $\mathcal{T}(\omega) - \mathcal{T}(n\omega)$  is at most log-singular! ☺ (Show via asymptotics of  $H_n^{(1)}$ )



## Helmholtz – high-order accuracy

Spectral accuracy Nyström for log-singular kernels: possible, but beyond today

Divide bdry into panels instead of global set of nodes, adaptive panel sizes & quadrature rules

Kernel-split: decompose kernel  $G(\mathbf{x}, \mathbf{y}) = \underbrace{G^S(\mathbf{x}, \mathbf{y})}_{\text{smooth}} + \underbrace{G^L(\mathbf{x}, \mathbf{y}) \log |\mathbf{y} - \mathbf{x}|}_{\text{log singularity}} + \underbrace{G^C(\mathbf{x}, \mathbf{y}) \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}}{|\mathbf{y} - \mathbf{x}|^2}}_{\text{Cauchy singularity}}$

Product integration: target-specific quadrature rules, e.g.

$$\int_{\Gamma} f(\mathbf{x}, \mathbf{y}) \log |\mathbf{x} - \mathbf{y}| d\mathbf{s}_{\mathbf{y}} \approx \sum_{j=1}^N f(\mathbf{x}, \mathbf{y}_j) w_j^L(\mathbf{x}) \text{ (Helsing, Holst, '15), (Kress), ...}$$

Generalized Gaussian quadrature (Bremer)

Close evaluation: target close to bdry

Kernel-split approach

QBX: quadrature by expansion (Kloeckner, Barnett, Greengard, O'Neil '13), (Epstein, Greengard, Kloeckner '13)

See also libraries: chunkie, BIE2D, etc.

# Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs      exterior need condition as  $\|\mathbf{x}\| \rightarrow \infty$
- Choosing representation      to get 2nd kind BIE if poss., equivalent to BVP if poss.  
Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization      high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases      also for libraries

practise! write out theory yourself + try HW exer. in repo + run demos

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Useful 2D tools we did not yet cover:

in libraries, eg chunkie, BIE2D

- panel (composite) quadratures essential for adaptivity
- high-order quadratures for log-singular kernel SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve  
some need interpolation of  $\sigma(t)$  off the nodes  $t_j$ , some not
- corners, open arcs, slits, multi-material junctions

## Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- “High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane”, S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- <https://users.flatironinstitute.org/~ahb/BIE/>
- <https://github.com/ahbarnett/BIEbook>

in progress...