

# 2D boundary integral equations and the Nyström method

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## Integral equations on 1D interval

- Given: i) function  $\sigma(t)$  defined on interval  $[0, 2\pi)$ ,    periodic:  $\sigma(2\pi) = \sigma(0)$ , etc  
ii) “kernel” function  $k(t, s)$  defined on square  $[0, 2\pi)^2$ ,

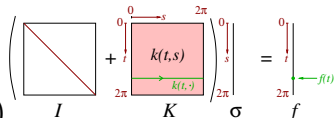
Integral operator  $K$  acts on  $\sigma$  to give another function  $K\sigma$ :

$$(K\sigma)(t) := \int_0^{2\pi} k(t, s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

continuous analog of  
matrix-vector prod.  $Ax$

Integral equation:  $(I + K)\sigma = f$ , ie

$$\sigma(t) + \int_0^{2\pi} k(t, s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$




analog of lin. sys.  $Ax = b$

Fredholm “second kind” (due to presence of  $I$ , otherwise called “first kind”)

If kernel continuous, or “weakly” singular (integrable),  $K$  is *compact*:

- eigenvalues  $(K\phi_k = \lambda_k\phi_k)$  discrete, with  $\lim_{k \rightarrow \infty} \lambda_k = 0$   
unless some  $\lambda_k = -1$ , 2nd kind IE has at most one soln:  $\text{Nul}(I + K) = \{0\}$
- $\text{Nul}(I + K) = \{0\} \Rightarrow$  existence of solution for *any*  $f$     Fredholm Alternative  
like square matrix (finite-dim), recall: uniqueness  $\Rightarrow$  consistent for any RHS

Contrast 1st kind IE  $K\sigma = f$  is ill-posed problem (unstable)!  **FLATIRON**  
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See references for lots of beautiful theory, precise statements

# Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t) dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j) \quad w_j = \text{weights}, \quad t_j = \text{nodes}$$

For  $f$  smooth, superalgebraically convergent ("spectral"): error =  $\mathcal{O}(N^{-p})$  for any  $p$

Apply quad to integral in 2nd kind IE:

call the resulting approx soln  $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^N k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \quad (*)$$

Holds for all  $t$ ; in particular, holds at each  $t_i$ ,  $i = 1, \dots, N$ , giving:

$$\sigma_i + \sum_{j=1}^N k(t_i, t_j) w_j \sigma_j = f(t_i), \quad i = 1, \dots, N \quad \text{where } \sigma_j := \tilde{\sigma}(t_j)$$

Write as:  $A\sigma = f$   $N \times N$  lin sys, entries  $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$ , and  $f_j := f(t_j)$

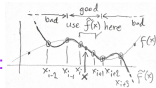
solve? dense direct  $\mathcal{O}(N^3)$ ; dense iter.  $\mathcal{O}(N^2)$ ; fast iter.  $\approx \mathcal{O}(N)$ ; fast direct  $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs( $A$ ) accumulate only at  $+1$ , iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values  $\{\sigma_j\}_{j=1}^N$  suffice.

If not, interpolate from them to any  $t \in [0, 2\pi)$ . Two approaches:

- either: rearrange  $(*)$  to give  $\tilde{\sigma}(t) = \dots$ , called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

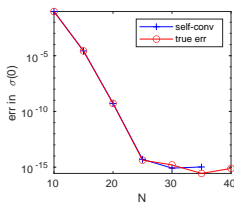
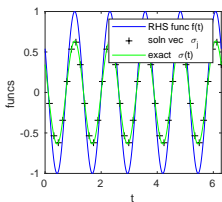
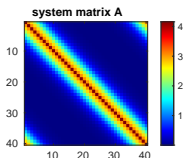


# Demo Nyström on interval (1D)

day1/code/nyst1d\_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));  
ffun = @(t) cos(5*t+1);  
N = 30;  
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);  
A = eye(N) + bsxfun(kfun,t',t)*diag(w);  
rhs = ffun(t');  
sigmaj = A\rhs;
```

*% smooth convolutional kernel, periodic domain  $[0, 2\pi)$*   
*% smooth data (RHS) func*  
*% number of unknowns: should study convergence as  $N$  grows...*  
*% PTR nodes and weights, row vecs*  
*% identity plus fill  $k(t_i, t_j)w_j$  for  $i, j=1..N$*   
*% col vec*  
*% dense direct square solve (pivoted LU), gives col vec*



“self-convergence”:  
use  $N=40$  as “true”

$f$  and  $k$  smooth

$\Rightarrow \sigma$  smooth

$\Rightarrow$  spectral conv?

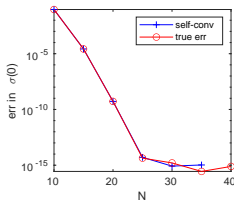
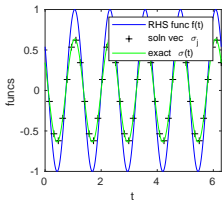
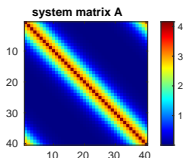
**Thm.** (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand  $k(t, \cdot)\sigma$ .

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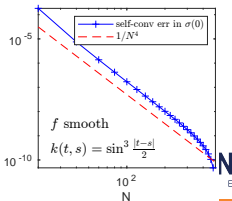
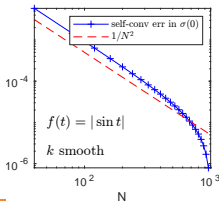
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$f$  and  $k$  smooth  
 $\Rightarrow \sigma$  smooth  
 $\Rightarrow$  spectral conv?

**Thm.** (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand  $k(t, \cdot)\sigma$ .

- Then,  $f$  or  $k$  nonsmooth?  
worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



# Fundamental solution in $\mathbb{R}^2$

Eg PDE: Poisson eqn  $\Delta u = g$

$\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$  Laplacian

Notation:  $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$  is a point. This frees up  $\mathbf{y} \in \mathbb{R}^2$  as another point (not y-coord!)

Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving  $g \equiv 0$ )

Eg well-posed\* BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE (} u \text{ harmonic)}$$

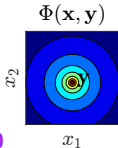
$$u = f \text{ on } \Gamma \quad \text{Dirichlet BC}$$

\*exists, unique, continuous

w.r.t. data



Laplace fundamental soln:  $\Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{r}$  where  $r := \|\mathbf{x} - \mathbf{y}\|$



obeys  $-\Delta_{\mathbf{x}} \Phi = -\Delta_{\mathbf{y}} \Phi = \delta_{\mathbf{x}}$   $\Phi$  harmonic except unit point-mass at 0

Normal  $\mathbf{n}$  points outwards,  $\|\mathbf{n}\| = 1$  normal deriv. notation  $u_n := \mathbf{n} \cdot \nabla u$

Green's 2nd identity:  $\int_{\Gamma} v u_n - v_n u \, ds = \int_{\Omega} v \Delta u - (\Delta v) u \, d\mathbf{y}$

calculus

warm-up: set  $u = \text{BVP soln}$ ,  $v \equiv 1$ , G2I becomes  $\int_{\Gamma} u_n \, ds - 0 = 0 - 0$ : so  $u$  has zero flux

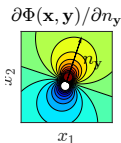
more fun: fix "target"  $\mathbf{x} \in \Omega$ , let  $v = \Phi(\mathbf{x}, \cdot)$ , G2I gives:

Green's representation formula:

$$\int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_{\mathbf{y}}} u(\mathbf{y}) \, ds_{\mathbf{y}} = u(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega$$

Gets soln from "Cauchy data"  $(u, u_n)|_{\Gamma}$

also versions for Helmholtz, Stokes, Maxwell, ...



# Layer potentials and their jump relations

Representations of harmonic functions off a curve  $\Gamma$ : “density”  $\sigma$

Single-layer potential  $(\mathcal{S}\sigma)(\mathbf{x}) := \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) d\mathbf{s}_{\mathbf{y}}$  charge sheet

Double-layer potential  $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y}) d\mathbf{s}_{\mathbf{y}}$  dipole sheet

interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

Jump relations:

$(\mathcal{S}\sigma)^{\pm} = S\sigma$   $S$  (Roman font) means *restriction* of  $S$  to  $\Gamma$ : a bdry int. op.

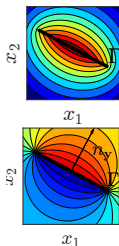
$(\mathcal{D}\sigma)^{\pm} = (D \pm I/2)\sigma$  jump in potential equal to  $\sigma$ ;  $D$  restriction to  $\Gamma$  in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = (D^T \mp I/2)\sigma$  jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = T\sigma$   $T$  hypersingular, kernel  $\partial^2 \Phi(\mathbf{x}, \mathbf{y}) / \partial \mathbf{n}_{\mathbf{x}} \partial \mathbf{n}_{\mathbf{y}} \sim 1/r^2$

- $D$  smooth kernel on smooth  $\Gamma$ , while  $S$  always log (weakly) singular

Recap GRF in LP notation:  $u$  harmonic in  $\Omega \Rightarrow \mathcal{S}u_n^- - \mathcal{D}u^- = u$  in  $\Omega$



## Converting BVP to BIE and solving

Say wish to solve interior

Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \quad \text{BC}$$





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Insert the BC to get BIE:  $(I - 2D)\sigma = -2f$  scaled to 2nd kind form

This shows: let  $\sigma$  solve BIE, then  $u = \mathcal{D}\sigma$  solves BVP (i.e., no spurious solns)

But how know *all* solns  $u$  of this form?

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Above BIE expressed on  $\Gamma$  using arc-length measure  $ds$ . Usually not how  $\Gamma$  described...

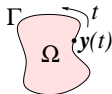
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change variable  $ds_y = \|y'(t)\|dt$  abuse notation  $\sigma(t) = \sigma(y(t))$

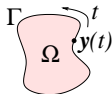
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Get 1D IE:  $\sigma(t) - 2 \int_0^{2\pi} \frac{\partial \Phi(\mathbf{y}(t), \mathbf{y}(s))}{\partial \mathbf{n}_{\mathbf{y}(s)}} \sigma(s) \|\mathbf{y}'(s)\| ds = -2f(t), \quad t \in [0, 2\pi)$

familiar form  $(I + K)\sigma = -2f$ , with kernel  $k(s, t) = \frac{-2}{2\pi} \frac{\mathbf{n}_{\mathbf{y}(s)} \cdot (\mathbf{y}(t) - \mathbf{y}(s))}{\|\mathbf{y}(t) - \mathbf{y}(s)\|^2} \|\mathbf{y}'(s)\|$

formula on diagonal:  $k(t, t) = \lim_{s \rightarrow t} k(t, s) = \kappa(t)/2\pi$ ,  $\kappa$  curvature of  $\Gamma$  (check!)

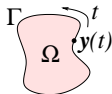
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Now Nyström discretize with PTR, solve lin. sys. for  $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln:  $u(\mathbf{x}) = (\mathcal{D}\sigma)(\mathbf{x}) \stackrel{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\mathbf{n}_{\mathbf{y}(t_j)} \cdot (\mathbf{x} - \mathbf{y}(t_j))}{2\pi \|\mathbf{x} - \mathbf{y}(t_j)\|^2} \|\mathbf{y}'(t_j)\| w_j \sigma_j$

# Interior Laplace Dirichlet BVP solve demo

demo\_lapintdir.m

```
a=0.7; b=1.0;
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);
y = Y(t);
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;
kappa = -sum(Ypp(t) .* n,1)./speed.^2;
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);
A(diagind(A)) = kappa/(2*pi);
A = eye(N) + A*diag(speed.*w);
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);
f = @(t) uex(Y(t));
rhs = -2*f(t)';
sigma = A\rhs;
```

*% shape params (note a=1,b=0 unit circle)*  
*% kite parameterization y(t)*  
*% y', analytic*  
*% y'', analytic*  
  
*% PTR nodes & weights*  
*% bdry nodes, 2-by-N*  
*% bdry normals*  
*% bdry curvatures*  
*% matrix of r=x-y (two vec cmpnts)*  
*% off-diag (-1/pi) n.r/r^2*  
*% overwrite diag elements*  
*% note Id gets no "speed weights"*  
*% test u(x) = x\_1(x\_2-0.3), not symmetric!*  
*% read off its Dirichlet data*  
  
*% solve. Leave u = D.sigma eval to reader*



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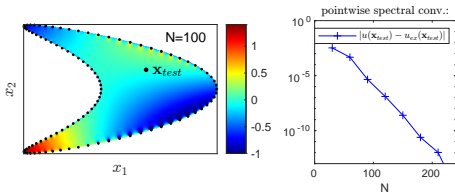
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t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);
y = Y(t);
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% shape params (note a=1,b=0 unit circle)
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% y', analytic
% y'', analytic
```

```
% PTR nodes & weights
% bdry nodes, 2-by-N
% bdry normals
% bdry curvatures
% matrix of r=x-y (two vec cmpnts)
% off-diag (-1/pi) n.r/r^2
% overwrite diag elements
% note Id gets no "speed weights"
% test u(x) = x_1(x_2-0.3), not symmetric!
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% solve. Leave u = D.sigma eval to reader
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# Interior Laplace Dirichlet BVP solve demo

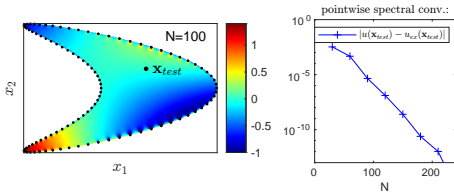
demo\_lapintdir.m

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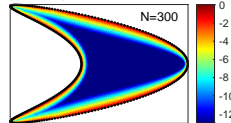
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$\log_{10} |u - u_{ex}|$ : naive PTR quadr. eval.



“5h” rule:  
special eval.  
quadratures  
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(Helsing, QBX...)

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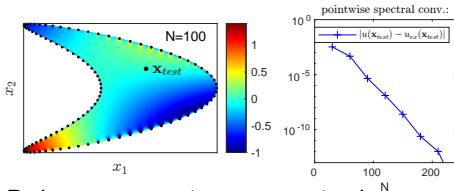
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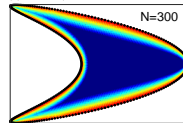
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Debug:  $\sigma \equiv -1 \Rightarrow u \equiv 1$ , then test data from (generic!) soln  $u$ , and...

- ① check/plot  $n, \kappa$ . First test unit circle!
- ② check Nyström matrix smooth at diag (before add I)

## Indirect vs direct formulations

using Laplace interior Dirichlet BVP

So far “indirect” BIE: pick representation (eg  $u = \mathcal{D}\sigma$ ), get BIE from JRs

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Indirect BIE	Direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

## Generalizations: Exterior, Neumann

some laplace aspects, confusions w/ BVP, briefly  
notation for exterior domain

subtlety of decay in 2D

For int Neu or ext Dir, need kernel  $k(t, s) + 1$

- ③ Neumann BC case: contour lines orthog to  $\Gamma$



# Helmholtz

$(\Delta + \kappa^2)u = 0$  arises from scalar wave equation  $u_{tt} - \Delta u = 0$

$\kappa$  “wavenumber”; wavelength  $\lambda = 2\pi/\kappa$

Also used for 2D Maxwell (z-invar); TE vs TM

Dirichlet BC = PEC in TE (check)

Scattering formalism:

$$u^{\text{tot}} = u^{\text{inc}} + u.$$

BVP for  $u$  (3 lines: PDE, BC, SRC, needed for uniqueness)

Helm fund sol (pic)

pics of LPs?

$u = \mathcal{D}\sigma$  has interior res prob (Leslie will have covered for the disk case)

CFIE:  $u = (\mathcal{D} + i\eta\mathcal{S})\sigma$  no more unknowns, new kernel

can prove equivalence (no spurious resonances)

Dirichlet demo, plots only: Solve BVP for  $u$  via PTR + Nyström, with new diag limit for  $k(t, t)$ , show  $1/N^3$  convergence if use naive PTR with correct diag limit (see M126 HW?)

④ debug BVP with known data from a radiative soln [sources inside  \$\Omega\$](#)   
(don't demo CFIE since requires  $S$  w/ log-singularity).

## Helmholtz transmission BVP

refractive indices in  $\Omega$  vs exterior

DECIDE IF TRANSMISSION

Matching? (more effort, needs  $\mathcal{S}\sigma + \mathcal{D}\tau \dots$ )

difference kernels at most log-singular.

# Helmholtz

Getting spectral-acc Nyström for log-singular kernels: beyond today.

eg kernel-split or product quadratures (Kress, Helsing, . . . )

close-eval: kernel-split, QBX, etc.

see libraries: chunkie, BIE2D, etc

## More debug ideas

TO DISCARD

Other tests:

- ⑤ Test SLP & DLP evaluators via GRF for any harmonic  $u$  in  $\Omega$

# Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs exterior need condition as  $\|x\| \rightarrow \infty$
- Choosing representation to get 2nd kind BIE if poss., equivalent to BVP if poss.  
Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases also for libraries

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Useful 2D tools we did not yet cover: in libraries, eg chunkie, BIE2D

- panel (composite) quadratures essential for adaptivity
- high-order quadratures for log-singular kernel SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve  
some need interpolation of  $\sigma(t)$  off the nodes  $t_j$ , some not
- corners, open arcs, slits, multi-material junctions

## Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- “High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane”, S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- <https://users.flatironinstitute.org/~ahb/BIE/>
- <https://github.com/ahbarnett/BIEbook>

in progress...