Integral equations cheat sheet

June 2024

1 Green's functions

PDE	Equation	Dimension	G
Poisson/Laplace	$\Delta G(r) = -\delta(r)$	2	$\frac{1}{2\pi}\log\frac{1}{r}$
		3	$\frac{1}{4\pi r}$
Helmholtz	$(\Delta + k^2)G(r) = -\delta(r)$	2	$\frac{i}{4}H_0^{(1)}(kr)$
		3	$\frac{e^{ikr}}{4\pi r}$
Yukawa/Klein-Gordon/ Screened Poisson	$(\Delta - k^2)G(r) = -\delta(r)$	2	$\frac{1}{4\pi}K_0(kr)$
		3	$\frac{e^{-kr}}{4\pi r}$
Stokes	$\Delta G - \nabla p = -\delta I$ $\Delta \cdot G = 0$	2	$\frac{1}{4\pi}\left(-I\log(r) + \frac{\vec{r}\otimes\vec{r}}{r^2}\right)$
		3	$\frac{1}{8\pi}\left(I\frac{1}{r} + \frac{\vec{r}\otimes\vec{r}}{r^3}\right)$
Vector wave	$\nabla\times\nabla\times G(r) - k^2G(r) = I\delta(r)$	3	$\left(I + \frac{1}{k^2}\nabla\nabla\right)\frac{e^{ikr}}{4\pi r}$

2 Layer potentials in 2D

In the following, we adopt the notation that a subscript s denotes that the variable is associated with the source location (i.e. n_s is the normal at the source location) and t denotes that the variable is associated with the target location. We use $\vec{r}_{s,t}$ to denote the locations of the source and target, and r to denote their distance. We always assume that the normal vectors $n_.$ are normalized, and denote unit tangent vectors by $\tau_.$. Singularity here refers to the nature of the singularity of the on surface kernel on a smooth surface. The +/- limits refer to the limits from above and below (with orientation determined locally by the normal - + is in the direction of the normal).

2.1 Laplace

Potential	On surface	Singularity type	Limits (+/-)
Single/ S	$\frac{1}{2\pi}\log\frac{1}{r}$	Weak	N.A.
Double / D	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s}{2\pi r^2}$	Smooth	$\frac{1}{2}, -\frac{1}{2}$
S' / $n\cdot\nabla S$	$-\frac{(\vec{r}_t-\vec{r}_s)\cdot n_t}{2\pi r^2}$	Smooth	$-\frac{1}{2}, \frac{1}{2}$
D' / $n\cdot\nabla D$	$\frac{n_t\cdot n_s}{2\pi r^2} - 2\frac{((\vec{r}_t-\vec{r}_s)\cdot n_s)((\vec{r}_t-\vec{r}_s)\cdot n_t)}{2\pi r^4}$	Hypersingular	-

2.2 Helmholtz

Potential	On surface	Singularity type	Limits (+/-)
Single / S_k	$\frac{i}{4}H_0^{(1)}(kr)$	Weak	N.A.
Double / D_k	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s}{r}\frac{ik}{4}H_1^{(1)}(kr)$	Weak	$\frac{1}{2}, -\frac{1}{2}$
S_k' / $n\cdot\nabla S_k$	$\frac{(\vec{r}_s-\vec{r}_t)\cdot n_t}{r}\frac{ik}{4}H_1^{(1)}(kr)$	Weak	$-\frac{1}{2}, \frac{1}{2}$
D_k' / $n\cdot\nabla D_k$	$\frac{ik}{4}\frac{n_t\cdot n_s}{r}H_1^{(0)}(kr) - \frac{ik^2}{4}\frac{((\vec{r}_t-\vec{r}_s)\cdot n_s)((\vec{r}_t-\vec{r}_s)\cdot n_t)}{r^2}H_2^{(0)}(kr)$	Hypersingular	-

2.3 Yukawa

Potential	On surface	Singularity type	Limits (+/-)
Single / S_{ik}	$\frac{1}{2\pi}K_0(kr)$	Weak	N.A.
Double / D_{ik}	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s}{r}\frac{k}{2\pi}K_1(kr)$	Weak	$\frac{1}{2}, -\frac{1}{2}$
S'_{ik} / $n \cdot \nabla S_{ik}$	$\frac{(\vec{r}_s-\vec{r}_t)\cdot n_t}{r}\frac{k}{2\pi}K_1(kr)$	Weak	$-\frac{1}{2}, \frac{1}{2}$
D'_{ik} / $n \cdot \nabla D_{ik}$	$\frac{k}{2\pi}\frac{n_t\cdot n_s}{r}K_1(kr) - \frac{k^2}{2\pi}\frac{((\vec{r}_t-\vec{r}_s)\cdot n_s)((\vec{r}_t-\vec{r}_s)\cdot n_t)}{r^2}K_2^{(0)}(kr)$	Hypersingular	-

2.4 Stokes

Potential	On surface	Singularity type	Limits (+/-)
Single (Stokeslet) / S_{stok}	$\frac{1}{4\pi}\left(-I\log(r) + \frac{(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^2}\right)$	Weak	N.A.
Double (Stresslet) / D_{stok}	$\frac{(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^2}\frac{n_s\cdot(\vec{r}_t-\vec{r}_s)}{\pi r^2}$	Smooth	$\frac{1}{2}, -\frac{1}{2}$
Stokeslet pressure Π_S	$\frac{(\vec{r}_t-\vec{r}_s)}{2\pi r^2}$	Singular	$(\frac{1}{2}n\otimes n), (-\frac{1}{2}n\otimes n)$
Stokeslet traction S'_{stok} $n\cdot(\Pi_S + 0.5(\nabla S + (\nabla S)^T))$	-$\frac{(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^2}\frac{n_t\cdot(\vec{r}_t-\vec{r}_s)}{\pi r^2}$	Smooth	$-\frac{1}{2}, \frac{1}{2}$
Stresslet Pressure Π_D	$-\frac{n_s}{2\pi r^2} + \frac{(\vec{r}_t-\vec{r}_s)(\vec{r}_t-\vec{r}_s)\cdot n_s}{r^4}$	Hypersingular	-

3 Layer potentials in 3D

In the following, we adopt the notation that a subscript s denotes that the variable is associated with the source location (i.e. n_s is the normal at the source location) and t denotes that the variable is associated with the target location. We use $\vec{r}_{s,t}$ to denote the locations of the source and target, and r to denote their distance. We always assume that the normal vectors $n_{\cdot}$ are normalized, and denote orthonormal tangent vectors by $\tau_{u/v,\cdot}$. Singularity here refers to the nature of the singularity of the on surface kernel on a smooth surface. The +/- limits refer to the limits from above and below (with orientation determined locally by the normal - + is in the direction of the normal).

3.1 Laplace

Potential	On surface	Singularity type	Limits (+/-)
Single / S	$\frac{1}{4\pi r}$	Weak	N.A.
Double / D	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s}{4\pi r^3}$	Weak	$\frac{1}{2}, -\frac{1}{2}$
S' / $n\cdot\nabla S$	$-\frac{(\vec{r}_t-\vec{r}_s)\cdot n_t}{4\pi r^3}$	Weak	$-\frac{1}{2}, \frac{1}{2}$
D' / $n\cdot\nabla D$	$\frac{n_t\cdot n_s}{4\pi r^3} - 3\frac{((\vec{r}_t-\vec{r}_s)\cdot n_s)((\vec{r}_t-\vec{r}_s)\cdot n_t)}{4\pi r^5}$	Hypersingular	-

3.2 Helmholtz

Potential	On surface	Singularity type	Limits (+/-)
Single / S_k	$\frac{e^{ikr}}{4\pi r}$	Weak	N.A.
Double / D_k	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s(1-ikr)}{4\pi r^3}e^{ikr}$	Weak	$\frac{1}{2}, -\frac{1}{2}$
S'_k / $n\cdot\nabla S_k$	$-\frac{(\vec{r}_t-\vec{r}_s)\cdot n_t(1-ikr)}{4\pi r^3}e^{ikr}$	Weak	$-\frac{1}{2}, \frac{1}{2}$
D'_k / $n\cdot\nabla D_k$	$\frac{e^{ikr}}{4\pi r^5}\left[r^2(n_s\cdot n_t)(1-ikr) - (r\cdot n_s)(r\cdot n_t)(r^2k^2 + 3kir - 3)\right]$	Hypersingular	-

3.3 Yukawa

Potential	On surface	Singularity type	Limits (+/-)
Single / S_k	$\frac{e^{-kr}}{4\pi r}$	Weak	N.A.
Double / D_k	$\frac{(\vec{r}_t-\vec{r}_s)\cdot n_s(1+kr)}{4\pi r^3}e^{-kr}$	Weak	$\frac{1}{2},-\frac{1}{2}$
S'_k / $n\cdot\nabla S_k$	$-\frac{(\vec{r}_t-\vec{r}_s)\cdot n_t(1+kr)}{4\pi r^3}e^{-kr}$	Weak	$-\frac{1}{2},\frac{1}{2}$
D'_k / $n\cdot\nabla D_k$	$\frac{e^{-kr}}{4\pi r^5}\left[r^2(n_s\cdot n_t)(1+kr)+(r\cdot n_s)(r\cdot n_t)(r^2k^2+3kr+3)\right]$	Hypersingular	-

3.4 Stokes

Potential	On surface	Singularity type	Limits (+/-)
Single (Stokeslet) / S_{stok}	$\frac{1}{8\pi}\left(I\frac{1}{r}+\frac{(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^3}\right)$	Weak	N.A.
Double (Stresslet) / D_{stok}	$\frac{3(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^2}\frac{n_s\cdot(\vec{r}_t-\vec{r}_s)}{4\pi r^3}$	Weak	$\frac{1}{2},-\frac{1}{2}$
Stokeslet pressure Π_S	$\frac{(\vec{r}_t-\vec{r}_s)}{4\pi r^3}$	Singular	$(\frac{1}{2}n\otimes n),(-\frac{1}{2}n\otimes n)$
Stokeslet traction S'_{stok} $n\cdot(\Pi_S+0.5(\nabla S+(\nabla S)^T))$	-$\frac{3(\vec{r}_t-\vec{r}_s)\otimes(\vec{r}_t-\vec{r}_s)}{r^2}\frac{n_t\cdot(\vec{r}_t-\vec{r}_s)}{4\pi r^3}$	Weak	$-\frac{1}{2},\frac{1}{2}$
Stresslet Pressure Π_D	$-\frac{n_s}{2\pi r^3}+\frac{6(\vec{r}_t-\vec{r}_s)(\vec{r}_t-\vec{r}_s)\cdot n_s}{r^5}$	Hypersingular	-

3.5 Maxwell

In this section, ρ is a scalar density and J is a vector tangential density

Potential	On surface	Singularity type (Tangential/normal)	Limits (+/-)
$\nabla S_k[\rho]$	$-\frac{(\vec{r}_t-\vec{r}_s)(1-ikr)}{4\pi r^3}e^{ikr}\rho$	Singular, Weak	$-\frac{1}{2}n_t\rho,\frac{1}{2}n_t\rho$
$\nabla\times S_k[J]$	$-\frac{(1-ikr)}{4\pi r^3}e^{ikr}\begin{bmatrix}(y_t-y_s)J_z-(z_t-z_s)J_y\\(z_t-z_s)J_x-(x_t-x_s)J_z\\(x_t-x_s)J_y-(y_t-y_s)J_x\end{bmatrix}$	Weak, Singular	$\frac{n\times J}{2},-\frac{n\times J}{2}$
$\nabla\cdot S_k[J]$	$-\frac{((\vec{r}_t-\vec{r}_s)\cdot J)(1-ikr)}{4\pi r^3}e^{ikr}$	Singular, Singular	-

4 Boundary value problems in 2D

In the following we adopt the convention that normals point outwards.

4.1 Laplace

Boundary condition	Interior/Exterior/Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D	$-\frac{1}{2}I+D$	-
Dirichlet	Exterior	D	$\frac{1}{2}I+D$	1 per connected component
Neumann	Interior	S	$\frac{1}{2}I+S'$	1 per connected component
Neumann	Exterior	S	$-\frac{1}{2}I+S'$	-

4.2 Helmholtz

Boundary condition	Interior / Exterior / Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D_k	$-\frac{1}{2}I + D_k$	some k (Lap. eigs.)
Dirichlet	Exterior	D_k	$\frac{1}{2}I + D_k$	some k (spur. resonances)
Dirichlet	Exterior	$D_k - i\alpha S_k$	$\frac{1}{2}I + D_k - i\alpha S_k$	-
Neumann	Interior	S_k	$\frac{1}{2}I + S'_k$	some k (Lap. eigs.)
Neumann	Exterior	S_k	$-\frac{1}{2}I + S'_k$	some k (spur. resonances)
Neumann	Exterior	Combined $\beta[S_k + i\alpha D_k S_{ik}]$, $\beta = -\frac{1}{\frac{1}{2}+\frac{i}{4}\alpha}$	$I + \beta S'_k + i\beta\alpha(D'_k - D'_{ik})(S_{ik}) + i\beta\alpha(S'_{ik})^2$	-
$\alpha u_{\mathrm{e}} - \beta u_{\mathrm{i}} = f$ $\mu\partial_n u_{\mathrm{e}} - \nu\partial_n u_{\mathrm{i}} = g$	Transmission	Ext: $\frac{1}{\nu}D_{k_{\mathrm{e}}}[\rho] + \frac{1}{\alpha}S_{k_{\mathrm{e}}}[\sigma]$, Int: $\frac{1}{\mu}D_{k_{\mathrm{i}}}[\rho] + \frac{1}{\beta}S_{k_{\mathrm{i}}}[\sigma]$	$\begin{bmatrix} I + \frac{1}{\nu}D_{k_e} - \frac{1}{\mu}D_{k_i} & \frac{1}{\alpha}S_{k_e} - \frac{1}{\beta}S_{k_i} \\ \frac{1}{\nu}D'_{k_e} - \frac{1}{\mu}D'_{k_i} & -I + \frac{1}{\alpha}S'_{k_e} - \frac{1}{\beta}S'_{k_i} \end{bmatrix}\begin{bmatrix}\rho \\ \sigma\end{bmatrix}$	-

4.3 Yukawa

Boundary condition	Interior / Exterior / Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D_{ik}	$-\frac{1}{2}I + D_{ik}$	-
Dirichlet	Exterior	D_{ik}	$\frac{1}{2}I + D_{ik}$	-
Neumann	Interior	S_{ik}	$\frac{1}{2}I + S'_{ik}$	-
Neumann	Exterior	S_{ik}	$-\frac{1}{2}I + S'_{ik}$	-

4.4 Stokes

Boundary condition	Interior/Exterior	Representation	Integral Equation	Known null space
Velocity/Resistance	Interior	D_{stok}	$-\frac{1}{2}I + D_{\mathrm{stok}}$	1 (Fixable by $n_t \int_\Gamma n_s \cdot$)
Velocity/Resistance	Interior	$\alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	$-\frac{1}{2}\alpha + \alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	1 (Fixable by $n_t \int_\Gamma n_s \cdot$)
Velocity/Resistance	Exterior	D_{stok}	$\frac{1}{2}I + D_{\mathrm{stok}}$	2 per connected component
Velocity/Resistance	Exterior	$\alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	$\frac{1}{2}\alpha + \alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	Density must be zero mean, 2 (fixable by constants)

5 Boundary value problems in 3D

In the following we adopt the convention that normals point outwards. The tangent vectors τ_u and τ_v are defined so that $\tau_u \times \tau_v = n$.

5.1 Laplace

Boundary condition	Interior/Exterior/Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D	$-\frac{1}{2}I + D$	-
Dirichlet	Exterior	D	$\frac{1}{2}I + D$	1 per connected component
Neumann	Interior	S	$\frac{1}{2}I + S'$	1 per connected component
Neumann	Exterior	S	$-\frac{1}{2}I + S'$	-

5.2 Helmholtz

Boundary condition	Interior / Exterior / Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D_k	$-\frac{1}{2}I + D_k$	some k (Lap. eigs.)
Dirichlet	Exterior	D_k	$\frac{1}{2}I + D_k$	some k (spur. resonances)
Dirichlet	Exterior	$D_k - i\alpha S_k$	$\frac{1}{2}I + D_k - i\alpha S_k$	-
Neumann	Interior	S_k	$\frac{1}{2}I + S'_k$	some k (Lap. eigs.)
Neumann	Exterior	S_k	$-\frac{1}{2}I + S'_k$	some k (spur. resonances)
Neumann	Exterior	Combined $\beta[S_k + i\alpha D_k S_{ik}]$, $\beta = -\frac{1}{\frac{1}{2}+\frac{i}{4}\alpha}$	$I + \beta S'_k + i\beta\alpha(D'_k - D'_{ik})(S_{ik})$ $+ i\beta\alpha(S'_{ik})^2$	-
$\alpha u_{\mathrm{e}} - \beta u_{\mathrm{i}} = f$ $\mu\partial_n u_{\mathrm{e}} - \nu\partial_n u_{\mathrm{i}} = g$	Transmission	Ext: $\frac{1}{\nu}D_{k_{\mathrm{e}}}[\rho] + \frac{1}{\alpha}S_{k_{\mathrm{e}}}[\sigma]$, Int: $\frac{1}{\mu}D_{k_{\mathrm{i}}}[\rho] + \frac{1}{\beta}S_{k_{\mathrm{i}}}[\sigma]$	$\begin{bmatrix} I + \frac{1}{\nu}D_{k_e} - \frac{1}{\mu}D_{k_i} & \frac{1}{\alpha}S_{k_e} - \frac{1}{\beta}S_{k_i} \\ \frac{1}{\nu}D'_{k_e} - \frac{1}{\mu}D'_{k_i} & -I + \frac{1}{\alpha}S'_{k_e} - \frac{1}{\beta}S'_{k_i} \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \end{bmatrix}$	-

5.3 Yukawa

Boundary condition	Interior / Exterior / Transmission	Representation	Integral Equation	Known null space
Dirichlet	Interior	D_{ik}	$-\frac{1}{2}I + D_{ik}$	-
Dirichlet	Exterior	D_{ik}	$\frac{1}{2}I + D_{ik}$	-
Neumann	Interior	S_{ik}	$\frac{1}{2}I + S'_{ik}$	-
Neumann	Exterior	S_{ik}	$-\frac{1}{2}I + S'_{ik}$	-

5.4 Stokes

Boundary condition	Interior/Exterior	Representation	Integral Equation	Known null space
Velocity/Resistance	Interior	D_{stok}	$-\frac{1}{2}I + D_{\mathrm{stok}}$	1 (Fixable by $n_t \int_\Gamma n_s \cdot$)
Velocity/Resistance	Interior	$\alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	$-\frac{1}{2}\alpha + \alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	1 (Fixable by $n_t \int_\Gamma n_s \cdot$)
Velocity/Resistance	Exterior	D_{stok}	$\frac{1}{2}I + D_{\mathrm{stok}}$	3 per connected component
Velocity/Resistance	Exterior	$\alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	$\frac{1}{2}\alpha + \alpha D_{\mathrm{stok}} + \beta S_{\mathrm{stok}}$	-

5.5 Maxwell

In this section, all perfect electric boundary (pec) value problems are for the exterior only. As before, J, K, M are surface vector fields, ρ, q, r are scalar functions defined on the surface.

BC	Representation	Conditions Imposed	Integral Equation	Known null space/ Failure mode
pec	$H = \nabla \times S_k[J]$ $E = ikS_k[J] - \nabla S_k[\rho]$	$n \times H - \alpha n \times n \times E = J$ $n \cdot E - \frac{\alpha}{ik}\nabla \cdot E = \rho$	$\frac{J}{2} - n \times \nabla \times S_k[J] + \alpha n \times n \times (ikS_k[J] - \nabla S_k \rho)$ $\frac{\rho}{2} + S'_k[\rho] - ikn \cdot S_k[J] + \alpha(\nabla \cdot S_k[J] - ikS_k[\rho])$	Topological low frequency breakdown Not second kind