2D boundary integral equations and the Nyström method

Alex Barnett[1] and **Fruzsina Agocs**[1]

Computational Tools 2024 BIE workshop. Day 1, 6/10/24

[1]Center for Computational Mathematics, Flatiron Institute, Simons Foundation

Integral equations on 1D interval

Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, periodic: $\sigma(2\pi) = \sigma(0)$, etc

ii) "kernel" function $k(t, s)$ defined on square $[0, 2\pi)^2$,

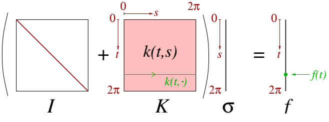
Integral *operator* K acts on σ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t,s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

continuous analog of matrix-vector prod. $A\boldsymbol{x}$

Integral *equation*: $(I + K)\sigma = f$, ie

$$\sigma(t) + \int_0^{2\pi} k(t,s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$



analog of lin. sys. $A\boldsymbol{x} = \boldsymbol{b}$

Fredholm "second kind" (due to presence of I, otherwise called "first kind")

If kernel continuous, or "weakly" singular (integrable), K is *compact*:

- eigenvalues ($K\phi_k = \lambda_k \phi_k$) discrete, with $\lim_{k\to\infty} \lambda_k = 0$

  unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: Nul $(I + K) = \{0\}$

- Nul $(I + K) = \{0\} \;\Rightarrow\;$ *existence* of solution for *any* f Fredholm Alternative

  like square matrix (finite-dim), recall: uniqueness $\Rightarrow$ consistent for any RHS

Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)!

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See references for lots of beautiful theory, precise statements

Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t)dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j)$$

w_j=weights, t_j=nodes

For f smooth, superalgebraically convergent ("spectral"): error $= \mathcal{O}(N^{-p})$ for any p

Apply quad to integral in 2nd kind IE: call the resulting approx soln $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^N k(t,t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \qquad (*)$$

Holds for all t; in particular, holds at each t_i, $i = 1, \ldots, N$, giving:

$$\sigma_i + \sum_{j=1}^N k(t_i,t_j) w_j \sigma_j = f(t_i), \quad i = 1, \ldots, N$$

where $\sigma_j := \tilde{\sigma}(t_j)$

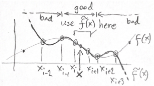
Write as: $A\boldsymbol{\sigma} = \boldsymbol{f}$ $\quad N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i,t_j)w_j$, and $f_j := f(t_j)$

solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs(A) accumulate only at $+1$, iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^N$ suffice.
If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

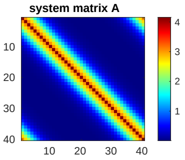
- either: rearrange (*) to give $\tilde{\sigma}(t) = \ldots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

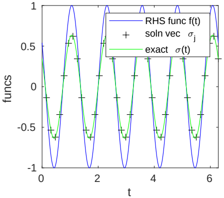


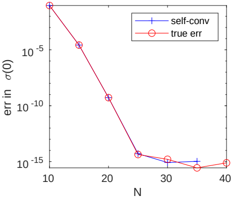
Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));                 % smooth convolutional kernel, periodic domain [0,2pi)
ffun = @(t) cos(5*t+1);                        % smooth data (RHS) func
N = 30;                                        % number of unknowns: should study convergence as N grows...
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);        % PTR nodes and weights, row vecs
A = eye(N) + bsxfun(kfun,t',t)*diag(w);        % identity plus fill k(t_i,t_j)w_j for i,j=1..N
rhs = ffun(t');                                % col vec
sigmaj = A\rhs;                                % dense direct square solve (pivoted LU), gives col vec
```







"self-convergence": use N=40 as "true"

f and k smooth

$\Rightarrow$ σ smooth

$\Rightarrow$ spectral conv?

Thm. (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t,\cdot)\sigma$.

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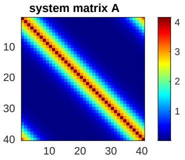

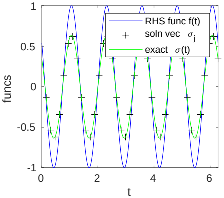

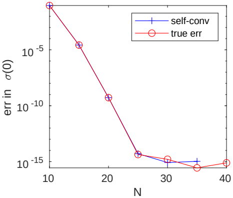

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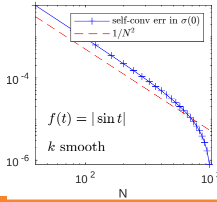
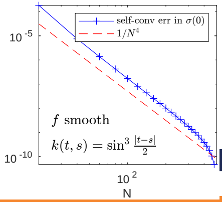
- Then, f or k nonsmooth?

worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?





Fundamental solution in $\mathbb{R}^2$

Eg PDE: Poisson eqn $\Delta u = g$ $\quad\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

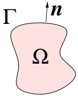
Notation: $\boldsymbol{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\boldsymbol{y} \in \mathbb{R}^2$ as another point (not y-coord!)

Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving $g \equiv 0$)

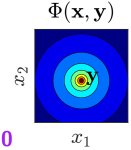
Eg well-posed* BVP:

$$\Delta u = 0 \text{ in } \Omega \qquad \text{PDE } (u \text{ harmonic})$$
$$u = f \text{ on } \Gamma \qquad \text{Dirichlet BC}$$

*exists, unique, continuous w.r.t. data



Laplace fundamental soln: $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ where $r := \|\boldsymbol{x} - \boldsymbol{y}\|$



obeys $-\Delta_{\boldsymbol{x}} \Phi = -\Delta_{\boldsymbol{y}} \Phi = \delta_{\boldsymbol{x}}$ $\quad\Phi$ harmonic except unit point-mass at $\mathbf{0}$

Normal $\boldsymbol{n}$ points outwards, $\|\boldsymbol{n}\| = 1$ $\quad$ normal deriv. notation $u_n := \boldsymbol{n} \cdot \nabla u$

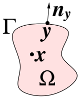
Green's 2nd identity: $\int_\Gamma v u_n - v_n u \, ds = \int_\Omega v \Delta u - (\Delta v) u \, d\boldsymbol{y}$ $\quad$ calculus

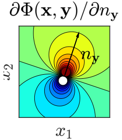
warm-up: set $u =$ BVP soln, $v \equiv 1$, G2I becomes $\int_\Gamma u_n ds - 0 = 0 - 0$: so u has zero *flux*

more fun: fix "target" $\boldsymbol{x} \in \Omega$, let $v = \Phi(\boldsymbol{x}, \cdot)$, G2I gives:

Green's representation formula:

$$\int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y}) u_n(\boldsymbol{y}) - \frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}} u(\boldsymbol{y}) \, ds_{\boldsymbol{y}} = u(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega$$





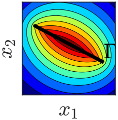
Gets soln from "Cauchy data" $(u, u_n)|_\Gamma$

also versions for Helmholtz, Stokes, Maxwell,...

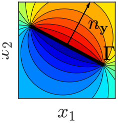
Layer potentials and their jump relations

Representations of harmonic functions off a curve Γ: "density" σ

Single-layer potential $(\mathcal{S}\sigma)(\boldsymbol{x}) := \int_\Gamma \Phi(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$ charge sheet



Double-layer potential $(\mathcal{D}\sigma)(\boldsymbol{x}) := \int_\Gamma \frac{\partial\Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$ dipole sheet



interior (-) / exterior (+) limits:

$$u^\pm(\boldsymbol{x}) := \lim_{h\to 0^+} u(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}})$$

$$u_n^\pm(\boldsymbol{x}) := \lim_{h\to 0^+} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla u(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}})$$

Jump relations:

$(\mathcal{S}\sigma)^\pm = S\sigma$ — S (Roman font) means *restriction* of $\mathcal{S}$ to Γ: a bdry int. op.

$(\mathcal{D}\sigma)^\pm = (D \pm I/2)\sigma$ — jump in potential equal to σ; D restriction to Γ in P.V. sense

$(\mathcal{S}\sigma)_n^\pm = (D^T \mp I/2)\sigma$ — jump in normal derivative

$(\mathcal{D}\sigma)_n^\pm = T\sigma$ — T hypersingular, kernel $\partial^2\Phi(\boldsymbol{x},\boldsymbol{y})/\partial\boldsymbol{n}_{\boldsymbol{x}}\partial\boldsymbol{n}_{\boldsymbol{y}} \sim 1/r^2$

- D smooth kernel on smooth Γ, while S always log (weakly) singular

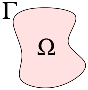
Recap GRF in LP notation: u harmonic in $\Omega \;\Rightarrow\; \mathcal{S}u_n^- - \mathcal{D}u^- = u$ in Ω

Converting BVP to BIE and solving

Say wish to solve interior Dirichlet Laplace BVP:

$\Delta u = 0$ in Ω PDE

$u^- = f$ on Γ BC



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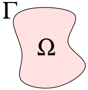

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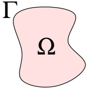

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Insert the BC to get BIE: $(I - 2D)\sigma = -2f$ scaled to 2nd kind form

This shows: let σ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns u of this form?

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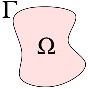

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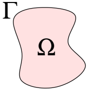

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Above BIE expressed on Γ using arc-length measure ds. Usually not how Γ described...

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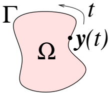

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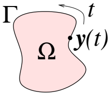

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familiar form $(I + K)\sigma = -2f$, with kernel $k(s,t) = \frac{-2}{2\pi} \frac{\boldsymbol{n}_{\boldsymbol{y}(s)} \cdot (\boldsymbol{y}(t) - \boldsymbol{y}(s))}{\|\boldsymbol{y}(t) - \boldsymbol{y}(s)\|^2} \|\boldsymbol{y}'(s)\|$

formula on diagonal: $k(t,t) = \lim_{s \to t} k(t,s) = \kappa(t)/2\pi$, κ curvature of Γ (check!)

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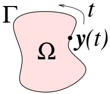

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Now Nyström discretize with PTR, solve lin. sys. for $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln: $u(\boldsymbol{x}) = (\mathcal{D}\sigma)(\boldsymbol{x}) \overset{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\boldsymbol{n}_{\boldsymbol{y}(t_j)} \cdot (\boldsymbol{x} - \boldsymbol{y}(t_j))}{2\pi \|\boldsymbol{x} - \boldsymbol{y}(t_j)\|^2} \|\boldsymbol{y}'(t_j)\| w_j \sigma_j$

Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

```
a=0.7; b=1.0;                                               % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                    % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];                % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];              % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                    % PTR nodes & weights
y = Y(t);                                                  % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed; % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                     % bdry curvatures
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sigma = A\rhs;                                             % solve. Leave u = D.sigma eval to reader
```

Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

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Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                          % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];                      % y', analytic
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N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                          % PTR nodes & weights
y = Y(t);                                                        % bdry nodes, 2-by-N
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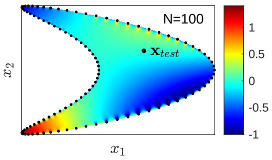

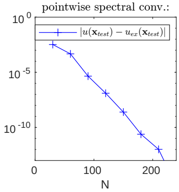

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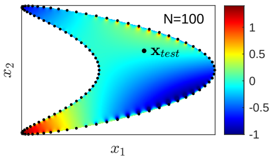

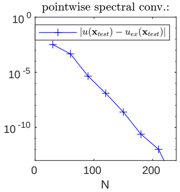

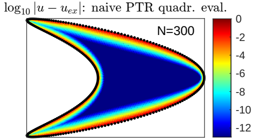

"5h" rule:
special eval.
quadratures
can fix near Γ
(Helsing, QBX...)

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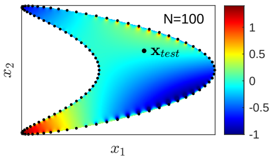

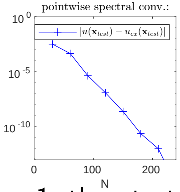

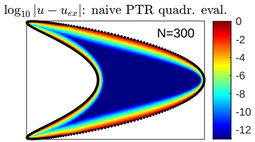

"5h" rule:
special eval.
quadratures
can fix near Γ
(Helsing, QBX...)

Debug: $\sigma \equiv -1 \;\Rightarrow\; u \equiv 1$, then test data from (generic!) soln u, and...

1. check/plot $\boldsymbol{n}$, κ. First test unit circle!
2. check Nyström matrix smooth at diag (*before* add I)

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Indirect vs direct formulations

using Laplace interior Dirichlet BVP

So far “indirect” BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

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GRF $u = \mathcal{S}u^- - \mathcal{D}u_n^- \quad \xrightarrow{\text{JRs}} \quad u_n^- = (D^T + I/2)u_n^- - Tu^- \quad \xrightarrow{\text{BC}} \quad (D^T - I/2)u_n^- = Tf$

Needs hypersingular apply ☹. Then solve BIE for u_n^-, eval u via GRF (needs two LP evals)

Notice BIO $(D^T - I/2)$ **adjoint** of that for indirect $(D - I/2)$ general fact

spectra the same, hence iterative convergence rates

Indirect vs direct formulations

using Laplace interior Dirichlet BVP

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spectra the same, hence iterative convergence rates

indirect BIE	direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

Generalizations: Exterior, Neumann

some laplace aspects, confusions w/ BVP, briefly

notation for exterior domain

subtlety of decay in 2D

For int Neu or ext Dir, need kernel $k(t,s)+1$

3. Neumann BC case: contour lines orthog to Γ

Helmholtz

$(\Delta + \kappa^2)u = 0$ arises from scalar wave equation $u_{tt} - \Delta u = 0$
κ "wavenumber"; wavelength $\lambda = 2\pi/\kappa$
Also used for 2D Maxwell (z-invar); TE vs TM
Dirichlet BC = PEC in TE (check)
Scattering formalism:
$u^{\text{tot}} = u^{\text{inc}} + u$.
BVP for u (3 lines: PDE, BC, SRC)
Solve BVP for u via PTR + Nyström

4. debug BVP with known data from a radiative soln sources inside Ω

Helmholtz transmission BVP

refractive indices in Ω vs exterior
DECIDE IF TRANSMISSION
Matching? (more effort, needs $\mathcal{S}\sigma + \mathcal{D}\tau \ldots$)

Helmholtz

fund sol,

Dirichlet demo, plots only

(don't demo CFIE since requires S w/ log-singularity). Demo DLP only, see $1/N^3$ conv if use naive PTR with correct diag limit (see M126 HW?)

Getting spectral-acc Nyström for log-singular kernels: beyond today.

eg kernel-split or product quadratures (Kress, Helsing,...)

close-eval: kernel-split, QBX, etc.

see libraries: chunkie, BIE2D, etc

More debug ideas

TO DISCARD

Other tests:

5. Test SLP & DLP evaluators via GRF for any harmonic u in Ω

Recap

*** TO EDIT WITH FRUZSINA:

BIE involves several steps: write out yourself + try HW exer. in repo

- we covered basics for one smooth curve in 2D

  code yourself, if only to understand what libraries do better!
- Interior/exterior, Laplace/Helmholtz/etc, somewhat simple changes

  spectral conv once you have good log-singular kernel quadrature
- Fancier quadratures needed for singular kernels and/or close eval
- Transmission BVPs may need twice the unknowns
- Nyström is not only way to discretize: Galerkin, collocation

  but: simplest and no less accurate

Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- "High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane", S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- `https://users.flatironinstitute.org/~ahb/BIE/`
- `https://github.com/ahbarnett/BIEbook` in progress. . .