Notations for Comp. Tools 2024 BIE workshop

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Introduction

We aim for notations common in the field of high-order Nyström BIE methods.
These follow Kress, and various Stokes BIE literature.

If you need to break with one notation convention in your slides, so be it, but try not to break many conventions at once (it will guaranteed confuse the audience).

If you want to change our actual convention in this document, discuss in GitHub Issues, or latex comments here.

Geometry and PDE

Coordinate $\boldsymbol{x} = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$. For us $d = 2$ or 3.

Laplacian $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$

Bounded domain is Ω (usually this is open, not including boundary)

Boundary curve or surface (interface) is $\partial\Omega$ or Γ.

Unit outward surface normal $\boldsymbol{n}$. Normal deriv $u_n = \frac{\partial u}{\partial n} = \partial_n u = \boldsymbol{n} \cdot \nabla u$.

We avoid ν for normal. We avoid γ_0 or γ_1 for boundary trace operators since too abstract.

Exterior domain is Ω_e or $\mathbb{R}^d \backslash \overline{\Omega}$ or E.

Multiple domains are Ω_r, $r = 1, 2, \ldots$ r for "region"? not conflict j, k later

A good symbol for the PDE operator is $\mathcal{L} = -\Delta$, $-(\Delta + \omega^2)$, etc.

so homogeneous PDE $\mathcal{L}u = 0$. This is completely optional.

Scalar-valued solution u preferred (various component potentials can be v, ϕ etc)

Vector-valued: $\boldsymbol{u}$ (flow velocity or displacement), $\boldsymbol{E}$, $\boldsymbol{H}$ for Maxwell...

Stokes: $(\boldsymbol{u}, p)$ is a solution to the Stokes system, $p =$ pressure

In short: use **bold** for vectors.

Incident field u^{inc}; but Stokes background flow usually $\boldsymbol{u}_0$

Arc length parameter s; general parameterization $\boldsymbol{x}(t)$

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Potential theory (scalar case)

Crucial: $\boldsymbol{x}$ = target, vs $\boldsymbol{y}$ = source. Surface quadrature nodes $\boldsymbol{y}_j \in \Gamma$.

will need to teach that (x_1, x_2) is a 2D location, confusion vs (x_1, y_1) common

Fundamental solutions $\Phi(\boldsymbol{x}, \boldsymbol{y})$ or $\Phi(\boldsymbol{x} - \boldsymbol{y})$ if translationally invariant.

Or: $G(\boldsymbol{x}, \boldsymbol{y})$. Use subscript for parameter, eg, G_ω at wavenumber ω.

Aim to normalized fund. sol. so $\mathcal{L}\Phi = \delta$ so, $1/4\pi r$ for Laplace 3D. $r := \|\boldsymbol{x} - \boldsymbol{y}\|$

Density functions various greek symbols: σ, τ, μ ...

Scalar single-layer potential $(\mathcal{S}\sigma)(\boldsymbol{x}) := \int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$

note: $\boldsymbol{x}$ is target. surface element in 3D could be dS_y. The subscript $\boldsymbol{y}$ is needed to indicate element wrt which coordinate.

Scalar double-layer potential $(\mathcal{D}\sigma)(\boldsymbol{x}) := \int_\Gamma \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial n_{\boldsymbol{y}}} \sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$

note: normal derivative is wrt 2nd argument, if the 2-argument form is used. Needs explaining to newcomers.

$\mathcal{D}\sigma$, or $\mathcal{D}[\sigma]$, is a potential, which can be evaluated anywhere, viz $(\mathcal{D}\sigma)(\boldsymbol{x})$.

Boundary integral operators: S is restriction of $\mathcal{S}$ to targets on Γ.

crucial: roman S for on-surface *operator*, vs script $\mathcal{S}\sigma$ is a *representation* (potential)

roman S maps functions on Γ to functions on Γ

BIO S_{Γ_2,Γ_1} has source on Γ_1, target restricted to Γ_2

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Kernels and jumps

Kernel of any BIO eg S can be notated $k(\boldsymbol{x}, \boldsymbol{y})$

Target-derivative operators S' or D^T has kernel $\frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial n_{\boldsymbol{x}}}$

Hypersingular D' or T has kernel $\frac{\partial^2 \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}$

Euro Galerkin notation for S, D, D^T, T is respectively: V, K, K' and W

K for a general BIO

If $\boldsymbol{x} \in \Gamma$, $u^+(\boldsymbol{x}) = \lim_{h \to 0^+} u(\boldsymbol{x} + h\boldsymbol{n})$ exterior limit

$u^-(\boldsymbol{x}) = \lim_{h \to 0^+} u(\boldsymbol{x} - h\boldsymbol{n})$ interior limit. Jump $[u] = u^+ - u^-$.

$u_n^{\pm}(\boldsymbol{x}) = \lim_{h \to 0^+} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla u(\boldsymbol{x} \pm h\boldsymbol{n})$ normal deriv limits

Example jump relation $[\mathcal{D}\sigma] = \sigma$

D is taken in the principal value sense.

Stokes traction is $\boldsymbol{T}(\boldsymbol{u}, p)$, the equivalent of normal derivative. Abbreviate by $\boldsymbol{T}$.

Vector PDEs

Stokes

Would be nice to stick to $\mathcal{S}$, $\mathcal{D}$ for representations, and S and D for BIOs, even though they are tensors, as in (Malhotra–Barnett JCP '24)

But whatever consistent notations you want

Vector densities $\boldsymbol{\sigma}$, etc,

although appreciate $\boldsymbol{f}$ sometimes used for SLP density (= physical force density on fluid)

Maxwell

[Mike, Felipe, etc to do] Use whatever notations you want

Miscellaneous

N for total number of (scalar or vector) unknowns

A for a system matrix. I for identity operator or matrix.

quadrature weights w_j. j for source quadrature node index, i for target.

this makes, e.g., SLP matrix A has elements $a_{ij} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) w_j$, which is logical indexing

$\boldsymbol{\sigma} = \{\sigma(\boldsymbol{y}_j)\}_{j=1}^N$ for discretized density vector.

This conflicts with vector densities eg $\boldsymbol{\sigma}$ Stokes SLP density. Workarounds welcome

Pieces of boundary (patches) can be Γ_k or γ_k. k is common "patch" index

Approximate numerical solution $\tilde{u}$ (or $\tilde{u}^{(N)}$ indicates using N unknowns) vs mathematical solution u.

k for iterates? p for quadr nodes per panel (order of scheme)

κ or ω for wavenumber. μ for viscosity.

ϵ_r permittivity, n_r refractive index, of material domain index r