

2D boundary integral equations and the Nyström method

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Integral equations on 1D interval

- Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, periodic: $\sigma(2\pi) = \sigma(0)$, etc
ii) “kernel” function $k(t, s)$ defined on square $[0, 2\pi)^2$,

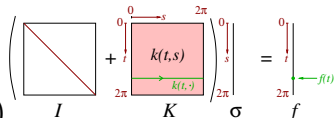
Integral operator K acts on σ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t, s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

continuous analog of matrix-vector prod. Ax

Integral equation: $(I + K)\sigma = f$, ie

$$\sigma(t) + \int_0^{2\pi} k(t, s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$




analog of lin. sys. $Ax = b$

Fredholm “second kind” (due to presence of I , otherwise called “first kind”)

If kernel continuous, or “weakly” singular (integrable), K is *compact*:

- eigenvalues $(K\phi_k = \lambda_k\phi_k)$ discrete, with $\lim_{k \rightarrow \infty} \lambda_k = 0$
unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: $\text{Nul}(I + K) = \{0\}$
- $\text{Nul}(I + K) = \{0\} \Rightarrow$ existence of solution for *any* f Fredholm Alternative
like square matrix (finite-dim), recall: uniqueness \Rightarrow consistent for any RHS

Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)!  **FLATIRON**
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See references for lots of beautiful theory, precise statements

Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t) dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j) \quad w_j = \text{weights}, \quad t_j = \text{nodes}$$

For f smooth, superalgebraically convergent ("spectral"): error = $\mathcal{O}(N^{-p})$ for any p

Apply quad to integral in 2nd kind IE:

call the resulting approx soln $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^N k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \quad (*)$$

Holds for all t ; in particular, holds at each t_i , $i = 1, \dots, N$, giving:

$$\sigma_i + \sum_{j=1}^N k(t_i, t_j) w_j \sigma_j = f(t_i), \quad i = 1, \dots, N \quad \text{where } \sigma_j := \tilde{\sigma}(t_j)$$

Write as: $A\sigma = f$ $N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$, and $f_j := f(t_j)$

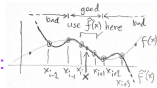
solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs(A) accumulate only at $+1$, iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^N$ suffice.

If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

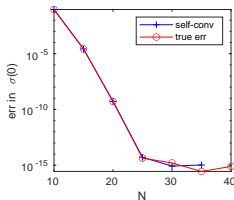
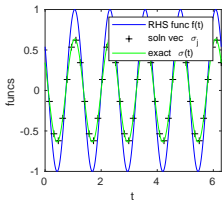
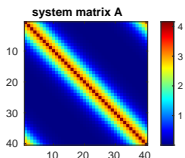
- either: rearrange $(*)$ to give $\tilde{\sigma}(t) = \dots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:



Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(s,t) exp(3*cos(t-s)); % smooth convolutional kernel, periodic domain [0,2pi)
ffun = @(t) cos(5*t+1); % smooth data (RHS) func
N = 30; % number of unknowns: should study convergence as N grows...
tj = 2*pi/N*(1:N); wj = 2*pi/N*ones(1,N); % PTR nodes and weights, row vecs
A = eye(N) + bsxfun(kfun,tj',tj)*diag(wj); % identity plus fill k(t_i,t_j)w_j for i,j=1..N
rhs = ffun(tj'); % col vec
sigmaj = A\rhs; % dense direct square solve (pivoted LU), gives col vec
```



“self-convergence”:
use $N=40$ as “true”

f and k smooth
 $\Rightarrow \sigma$ smooth
 \Rightarrow spectral conv?

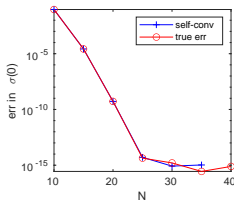
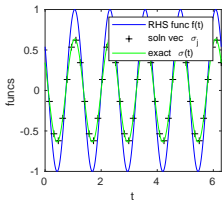
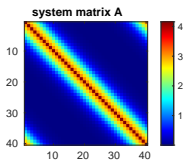
Thm. (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t, \cdot)\sigma$.

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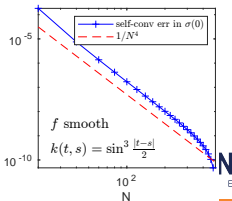
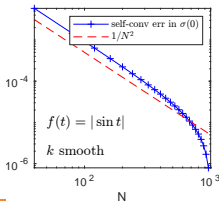
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- Then, f or k nonsmooth?
worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



Fundamental solution in \mathbb{R}^2

Eg PDE: Poisson eqn $\Delta u = g$

$\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

Notation: $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\mathbf{y} \in \mathbb{R}^2$ as another point (not y-coord!)

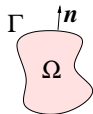
Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving $g \equiv 0$)

Eg well-posed* BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE (} u \text{ harmonic)}$$

$$* \text{exists, unique, continuous} \quad u = f \text{ on } \Gamma \quad \text{Dirichlet BC}$$

w.r.t. data



Laplace fundamental soln: $\Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ where $r := \|\mathbf{x} - \mathbf{y}\|$



obeys $-\Delta_{\mathbf{x}} \Phi = -\Delta_{\mathbf{y}} \Phi = \delta_{\mathbf{x}=\mathbf{y}}$ ie Φ harmonic except unit point-mass at origin

Normal \mathbf{n} points outwards, $\|\mathbf{n}\| = 1$ normal deriv. notation $u_n := \mathbf{n} \cdot \nabla u$

Green's 2nd identity: $\int_{\Gamma} v u_n - v_n u \, ds = \int_{\Omega} v \Delta u - (\Delta v) u \, d\mathbf{y}$ calculus

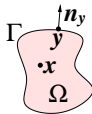
warm-up: set $u = \text{BVP soln}$, $v \equiv 1$, G2I becomes $\int_{\Gamma} u_n \, ds - 0 = 0 - 0$: so u has zero flux

more fun: fix "target" $\mathbf{x} \in \Omega$, let $v = \Phi(\mathbf{x}, \cdot)$, G2I gives:

Green's representation formula "GRF", interior version, for any $\mathbf{x} \in \Omega$:

$$\int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} u(\mathbf{y}) \, ds_{\mathbf{y}} = u(\mathbf{x})$$

Gets soln from "Cauchy data" $(u, u_n)|_{\Gamma}$. Helmholtz, Stokes, Maxwell versions



Potential theory

Recap

-
- Nyström discretization gets $\sigma(t_j)$ interpolate from them to other t

Resources

Many numerical analysis (mathematical flavor), particularly:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side:

- “High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane”, S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

goes beyond these slides for logarithmic singularities, eg SLP

- <https://users.flatironinstitute.org/~ahb/BIE/>
- <https://github.com/ahbarnett/BIEbook>

in progress...