# 2D boundary integral equations and the Nyström method

**Alex Barnett**[1] and **Fruzsina Agocs**[1]

Computational Tools 2024 BIE workshop. Day 1, 6/10/24

[1]Center for Computational Mathematics, Flatiron Institute, Simons Foundation

# Integral equations on 1D interval

Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, <span style="color:purple">periodic: $\sigma(2\pi) = \sigma(0)$, etc</span>
ii) "kernel" function $k(t,s)$ defined on square $[0, 2\pi)^2$,
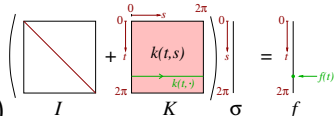
Integral *operator* $K$ acts on $\sigma$ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t,s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

<span style="color:#c55a11">continuous analog of matrix-vector prod. $Ax$</span>

Integral *equation*:  $(I + K)\sigma = f$, ie

$$\sigma(t) + \int_0^{2\pi} k(t,s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$



<span style="color:#c55a11">analog of lin. sys. $Ax = b$</span>

<span style="color:purple">Fredholm "second kind" (due to presence of $I$, otherwise called "first kind")</span>

If kernel continuous, or "weakly" singular (integrable), $K$ is *compact*:
- eigenvalues ($K\phi_k = \lambda_k \phi_k$) discrete, with $\lim_{k\to\infty} \lambda_k = 0$
  <span style="color:purple">unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: $\mathrm{Nul}\,(I + K) = \{0\}$</span>
- $\mathrm{Nul}\,(I + K) = \{0\} \Rightarrow$ *existence* of solution for *any* $f$ <span style="color:#c55a11">Fredholm Alternative</span>
  <span style="color:purple">like square matrix (finite-dim), recall: uniqueness $\Rightarrow$ consistent for any RHS</span>

Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)! 

<span style="color:#c55a11">**FLATIRON** INSTITUTE</span>

<span style="color:#c55a11">See references for lots of beautiful theory, precise statements</span>

# Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t)dt \approx \sum_{j=1}^{N} \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^{N} w_j f(t_j) \qquad w_j\text{=weights}, \quad t_j\text{=nodes}$$

For $f$ smooth, superalgebraically convergent ("spectral"): error $= \mathcal{O}(N^{-p})$ for any $p$

Apply quad to integral in 2nd kind IE: $\qquad$ call the resulting approx soln $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^{N} k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \qquad (*)$$

Holds for all $t$; in particular, holds at each $t_i$, $i = 1, \ldots, N$, giving:

$$\sigma_i + \sum_{j=1}^{N} k(t_i, t_j) w_j \sigma_j = f(t_i), \quad i = 1, \ldots, N \qquad \text{where } \sigma_j := \tilde{\sigma}(t_j)$$

Write as: $\qquad A\boldsymbol{\sigma} = \boldsymbol{f}$ $\qquad$ $N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$, and $f_j := f(t_j)$
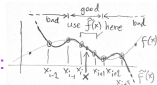
solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs($A$) accumulate only at $+1$, iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^{N}$ suffice.
If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

- either: rearrange (*) to give $\tilde{\sigma}(t) = \ldots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:
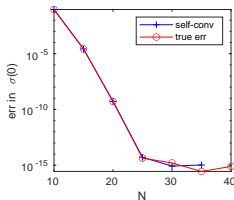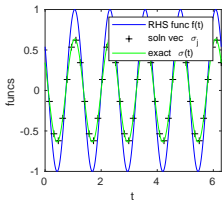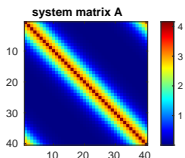
# Demo Nyström on interval (1D)

```
kfun = @(t,s) exp(3*cos(t-s));          % smooth convolutional kernel, periodic domain [0,2pi)
ffun = @(t) cos(5*t+1);                 % smooth data (RHS) func
N = 30;                                 % number of unknowns: should study convergence as N grows...
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N); % PTR nodes and weights, row vecs
A = eye(N) + bsxfun(kfun,t',t)*diag(w); % identity plus fill k(t_i,t_j)w_j for i,j=1..N
rhs = ffun(t');                         % col vec
sigmaj = A\rhs;                         % dense direct square solve (pivoted LU), gives col vec
```



"self-convergence":
use $N=40$ as "true"

$f$ and $k$ smooth
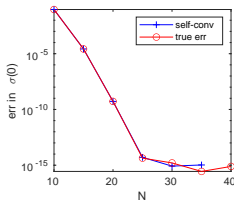$\Rightarrow$ $\sigma$ smooth
$\Rightarrow$ spectral conv?

**Thm.** (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t,\cdot)\sigma$.

# Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));              % smooth convolutional kernel, periodic domain [0,2pi]
ffun = @(t) cos(5*t+1);                      % smooth data (RHS) func
N = 30;                                       % number of unknowns: should study convergence as N grows...
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);      % PTR nodes and weights, row vecs
A = eye(N) + bsxfun(kfun,t',t)*diag(w);      % identity plus fill k(t_i,t_j)w_j for i,j=1..N
rhs = ffun(t');                               % col vec
sigmaj = A\rhs;                               % dense direct square solve (pivoted LU), gives col vec
```



"self-convergence": use $N{=}40$ as "true"

$f$ and $k$ smooth
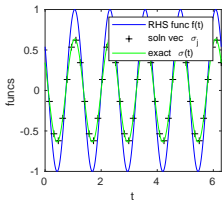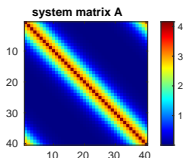$\Rightarrow$ $\sigma$ smooth
$\Rightarrow$ spectral conv?

**Thm.** (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t,\cdot)\sigma$.

• Then, $f$ or $k$ nonsmooth? worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



$f(t) = |\sin t|$
$k$ smooth

$f$ smooth
$k(t,s) = \sin^3 \frac{|t-s|}{2}$
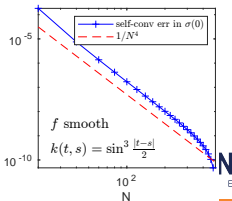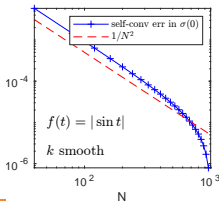
# Fundamental solution in $\mathbb{R}^2$

Eg PDE: Poisson eqn $\Delta u = g$ $\qquad$ $\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

Notation: $\boldsymbol{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\boldsymbol{y} \in \mathbb{R}^2$ as another point (not y-coord!)

Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving $g \equiv 0$)
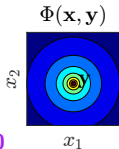
Eg well-posed* BVP:

$$\Delta u = 0 \text{ in } \Omega \qquad \text{PDE (}u\text{ harmonic)}$$
$$u = f \text{ on } \Gamma \qquad \text{Dirichlet BC}$$

*exists, unique, continuous
w.r.t. data



Laplace fundamental soln: $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ $\quad$ where $r := \|\boldsymbol{x} - \boldsymbol{y}\|$


$\Phi(\mathbf{x}, \mathbf{y})$

obeys $-\Delta_{\boldsymbol{x}}\Phi = -\Delta_{\boldsymbol{y}}\Phi = \delta_{\boldsymbol{x}}$ $\quad$ $\Phi$ harmonic except unit point-mass at $\boldsymbol{0}$

Normal $\boldsymbol{n}$ points outwards, $\|\boldsymbol{n}\| = 1$ $\quad$ normal deriv. notation $u_n := \boldsymbol{n} \cdot \nabla u$

Green's 2nd identity: $\int_\Gamma v u_n - v_n u \, ds = \int_\Omega v \Delta u - (\Delta v) u \, d\boldsymbol{y}$ $\qquad$ calculus

warm-up: set $u = $ BVP soln, $v \equiv 1$, G2I becomes $\int_\Gamma u_n ds - 0 = 0 - 0$: so $u$ has zero *flux*
more fun: fix "target" $\boldsymbol{x} \in \Omega$, let $v = \Phi(\boldsymbol{x}, \cdot)$, G2I gives:

Green's representation formula:
$\int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y}) u_n(\boldsymbol{y}) - \frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{y})}{\partial n_{\boldsymbol{y}}} u(\boldsymbol{y}) \, ds_{\boldsymbol{y}} = u(\boldsymbol{x})$ $\quad$ for $\boldsymbol{x} \in \Omega$


$\partial \Phi(\mathbf{x}, \mathbf{y})/\partial n_{\mathbf{y}}$

Gets soln from "Cauchy data" $(u, u_n)|_\Gamma$
also versions for Helmholtz, Stokes, Maxwell,...
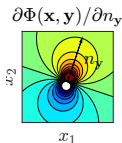
# Layer potentials and their jump relations

Representations of harmonic functions off a curve $\Gamma$: "density" $\sigma$



Single-layer potential $(\mathcal{S}\sigma)(\boldsymbol{x}) := \int_\Gamma \Phi(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$  charge sheet

Double-layer potential $(\mathcal{D}\sigma)(\boldsymbol{x}) := \int_\Gamma \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_y}}\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$  dipole sheet



interior (-) / exterior (+) limits:

$$u^{\pm}(\boldsymbol{x}) := \lim_{h\to 0^+} u(\boldsymbol{x} \pm h\boldsymbol{n_x})$$
$$u_n^{\pm}(\boldsymbol{x}) := \lim_{h\to 0^+} \boldsymbol{n_x} \cdot \nabla u(\boldsymbol{x} \pm h\boldsymbol{n_x})$$

Jump relations:

$(\mathcal{S}\sigma)^{\pm} = \quad\quad S\sigma$   $S$ (Roman font) means *restriction* of $\mathcal{S}$ to $\Gamma$: a bdry int. op.

$(\mathcal{D}\sigma)^{\pm} = \quad (D \pm I/2)\sigma$   jump in potential equal to $\sigma$;   $D$ restriction to $\Gamma$ in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = \quad (D^T \mp I/2)\sigma$   jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = \quad\quad T\sigma$   $T$ hypersingular, kernel $\partial^2\Phi(\boldsymbol{x},\boldsymbol{y})/\partial \boldsymbol{n_x}\partial \boldsymbol{n_y} \sim 1/r^2$

- $D$ smooth kernel on smooth $\Gamma$, while $S$ always log (weakly) singular

Recap GRF in LP notation:   $u$ harmonic in $\Omega$ $\Rightarrow$ $\mathcal{S}u_n^- - \mathcal{D}u^- = u$ in $\Omega$
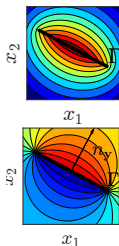
## Converting BVP to BIE and solving

Say wish to solve interior       $\Delta u = 0$ in $\Omega$     PDE

Dirichlet Laplace BVP:       $u^- = f$ on $\Gamma$     BC

# Converting BVP to BIE and solving

Say wish to solve interior

Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \quad \text{BC}$$



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

# Converting BVP to BIE and solving

Say wish to solve interior Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \qquad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \qquad \text{BC}$$



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

Insert the BC to get BIE: $\qquad (I - 2D)\sigma = -2f$ $\qquad$ scaled to 2nd kind form

This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form?

# Converting BVP to BIE and solving

Say wish to solve interior        $\Delta u = 0$ in $\Omega$    PDE

Dirichlet Laplace BVP:          $u^- = f$ on $\Gamma$    BC



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

   Insert the BC to get BIE:        $(I - 2D)\sigma = -2f$        scaled to 2nd kind form

   This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form?   Fred. Alt.: BIE *has* soln $\forall f$! BVP & BIE equivalent ☺

   (had we picked $u = \mathcal{S}\sigma$, would get 1st kind, poorly conditioned but can have its uses)

# Converting BVP to BIE and solving

Say wish to solve interior $\qquad\Delta u = 0$ in $\Omega$ $\quad$ PDE

Dirichlet Laplace BVP: $\qquad u^- = f$ on $\Gamma$ $\quad$ BC



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

Insert the BC to get BIE: $\qquad (I - 2D)\sigma = -2f$ $\qquad$ scaled to 2nd kind form

This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form? Fred. Alt.: BIE *has* soln $\forall f$! BVP & BIE equivalent ☺

(had we picked $u = \mathcal{S}\sigma$, would get 1st kind, poorly conditioned but can have its uses)

Above BIE expressed on $\Gamma$ using arc-length measure $ds$. Usually not how $\Gamma$ described...

# Converting BVP to BIE and solving

Say wish to solve interior
Dirichlet Laplace BVP:

$\Delta u = 0$ in $\Omega$   PDE

$u^- = f$ on $\Gamma$   BC



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

  Insert the BC to get BIE:    $(I - 2D)\sigma = -2f$     scaled to 2nd kind form

  This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form?   Fred. Alt.: BIE *has* soln $\forall f$! BVP & BIE equivalent ☺

  (had we picked $u = \mathcal{S}\sigma$, would get 1st kind, poorly conditioned but can have its uses)

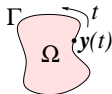Above BIE expressed on $\Gamma$ using arc-length measure $ds$. Usually not how $\Gamma$ described...

**Parameterize** the bdry $\mathbf{y}(t)$       $\mathbf{y} : \mathbb{R} \to \mathbb{R}^2$, $2\pi$-periodic, $\Gamma = \{\mathbf{y}(t) : t \in [0, 2\pi)\}$

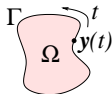  change variable $ds_y = \|\mathbf{y}'(t)\| dt$    abuse notation $\sigma(t) = \sigma(\mathbf{y}(t))$

## Converting BVP to BIE and solving

Say wish to solve interior
Dirichlet Laplace BVP:

$\Delta u = 0$ in $\Omega$    PDE

$u^- = f$ on $\Gamma$    BC



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

   Insert the BC to get BIE:    $(I - 2D)\sigma = -2f$     scaled to 2nd kind form

   This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form? Fred. Alt.: BIE *has* soln $\forall f$! BVP & BIE equivalent ☺

   (had we picked $u = \mathcal{S}\sigma$, would get 1st kind, poorly conditioned but can have its uses)

Above BIE expressed on $\Gamma$ using arc-length measure $ds$. Usually not how $\Gamma$ described...

**Parameterize** the bdry $\boldsymbol{y}(t)$      $\boldsymbol{y} : \mathbb{R} \to \mathbb{R}^2$, $2\pi$-periodic, $\Gamma = \{\boldsymbol{y}(t) : t \in [0, 2\pi)\}$

   change variable $ds_y = \|\boldsymbol{y}'(t)\| dt$    abuse notation $\sigma(t) = \sigma(\boldsymbol{y}(t))$

Get 1D IE:   $\sigma(t) - 2 \int_0^{2\pi} \frac{\partial \Phi(\boldsymbol{y}(t), \boldsymbol{y}(s))}{\partial \boldsymbol{n}_{\boldsymbol{y}(s)}} \sigma(s) \|\boldsymbol{y}'(s)\| ds = -2f(t), \ \ t \in [0, 2\pi)$

   familiar form $(I + K)\sigma = -2f$,   with kernel $k(s, t) = \frac{-2}{2\pi} \frac{\boldsymbol{n}_{\boldsymbol{y}(s)} \cdot (y(t) - y(s))}{\|\boldsymbol{y}(t) - \boldsymbol{y}(s)\|^2} \|\boldsymbol{y}'(s)\|$

     formula on diagonal: $k(t, t) = \lim_{s \to t} k(t, s) = \kappa(t)/2\pi$,   $\kappa$ curvature of $\Gamma$   (check!)

# Converting BVP to BIE and solving

Say wish to solve interior     $\Delta u = 0$ in $\Omega$    PDE

Dirichlet Laplace BVP:     $u^- = f$ on $\Gamma$    BC



Pick **representation**: $u = \mathcal{D}\sigma$, look up its **JR** for BC: $u^- = (D - I/2)\sigma$

    Insert the BC to get BIE:     $(I - 2D)\sigma = -2f$     scaled to 2nd kind form

    This shows: let $\sigma$ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns $u$ of this form? Fred. Alt.: BIE *has* soln $\forall f$! BVP & BIE equivalent ☺

    (had we picked $u = \mathcal{S}\sigma$, would get 1st kind, poorly conditioned but can have its uses)

Above BIE expressed on $\Gamma$ using arc-length measure $ds$. Usually not how $\Gamma$ described. . .

**Parameterize** the bdry $\boldsymbol{y}(t)$      $\boldsymbol{y} : \mathbb{R} \to \mathbb{R}^2$, $2\pi$-periodic, $\Gamma = \{\boldsymbol{y}(t) : t \in [0, 2\pi)\}$

    change variable $ds_y = \|\boldsymbol{y}'(t)\| dt$    abuse notation $\sigma(t) = \sigma(\boldsymbol{y}(t))$

Get 1D IE:   $\sigma(t) - 2\int_0^{2\pi} \frac{\partial \Phi(\boldsymbol{y}(t), \boldsymbol{y}(s))}{\partial \boldsymbol{n}_{\boldsymbol{y}(s)}} \sigma(s) \|\boldsymbol{y}'(s)\| ds = -2f(t)$,   $t \in [0, 2\pi)$

    familiar form $(I + K)\sigma = -2f$,   with kernel $k(s, t) = \frac{-2}{2\pi} \frac{\boldsymbol{n}_{y(s)} \cdot (y(t) - y(s))}{\|y(t) - y(s)\|^2} \|\boldsymbol{y}'(s)\|$

      formula on diagonal: $k(t, t) = \lim_{s \to t} k(t, s) = \kappa(t)/2\pi$,   $\kappa$ curvature of $\Gamma$   (check!)
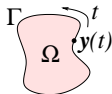
Now Nyström discretize with PTR, solve lin. sys. for $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln: $u(\boldsymbol{x}) = (\mathcal{D}\sigma)(\boldsymbol{x}) \overset{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\boldsymbol{n}_{y(t_j)} \cdot (\boldsymbol{x} - \boldsymbol{y}(t_j))}{2\pi \|\boldsymbol{x} - \boldsymbol{y}(t_j)\|^2} \|\boldsymbol{y}'(t_j)\| w_j \sigma_j$

# Interior Laplace Dirichlet BVP solve demo

```
a=0.7; b=1.0;                                          % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];            % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];          % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                % PTR nodes & weights
y = Y(t);                                              % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;   % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                 % bdry curvatures
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);              % matrix of r=x-y (two vec cmpnts)
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);  % off-diag (-1/pi) n.r/r^2
A(diagind(A)) = kappa/(2*pi);                          % overwrite diag elements
A = eye(N) + A*diag(speed.*w);                         % note Id gets no "speed weights"
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);                 % test u(x) = x_1(x_2-0.3), not symmetric!
f = @(t) uex(Y(t));                                    % read off its Dirichlet data
rhs = -2*f(t)';
sigma = A\rhs;                                          % solve. Leave u = D.sigma eval to reader
```


FLATIRON
INSTITUTE

# Interior Laplace Dirichlet BVP solve demo

```
a=0.7; b=1.0;                                          % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];            % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];          % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                % PTR nodes & weights
y = Y(t);                                              % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;  % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                 % bdry curvatures
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);              % matrix of r=x-y (two vec cmpnts)
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);% off-diag (-1/pi) n.r/r^2
A(diagind(A)) = kappa/(2*pi);                          % overwrite diag elements
A = eye(N) + A*diag(speed.*w);                         % note Id gets no "speed weights"
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);                 % test u(x) = x_1(x_2-0.3), not symmetric!
f = @(t) uex(Y(t));                                    % read off its Dirichlet data
rhs = -2*f(t)';
sigma = A\rhs;                                          % solve. Leave u = D.sigma eval to reader
```





pointwise spectral conv.:

$|u(x_{test}) - u_{ex}(x_{test})|$

FLATIRON INSTITUTE

# Interior Laplace Dirichlet BVP solve demo

```
a=0.7; b=1.0;                                              % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                    % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];                % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];              % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                    % PTR nodes & weights
y = Y(t);                                                   % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;  % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                     % bdry curvatures
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);                  % matrix of r=x-y (two vec cmpnts)
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);    % off-diag (-1/pi) n.r/r^2
A(diagind(A)) = kappa/(2*pi);                              % overwrite diag elements
A = eye(N) + A*diag(speed.*w);                             % note Id gets no "speed weights"
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);                     % test u(x) = x_1(x_2-0.3), not symmetric!
f = @(t) uex(Y(t));                                         % read off its Dirichlet data
rhs = -2*f(t)';                                             % solve. Leave u = D.sigma eval to reader
sigma = A\rhs;
```



N=100

$x_2$

$\mathbf{x}_{test}$

$x_1$

pointwise spectral conv.:

$|u(\mathbf{x}_{test}) - u_{ex}(\mathbf{x}_{test})|$

N

$\log_{10}|u - u_{ex}|$: naive PTR quadr. eval.

N=300

"5h" rule:
special eval.
quadratures
can fix near Γ
(Helsing, QBX...)



**FLATIRON** INSTITUTE

# Interior Laplace Dirichlet BVP solve demo

```
a=0.7; b=1.0;                                              % shape params (note a=1,b=0 unit circle)
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];                    % kite parameterization y(t)
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];                % y', analytic
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];              % y'', analytic
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);                    % PTR nodes & weights
y = Y(t);                                                  % bdry nodes, 2-by-N
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;   % bdry normals
kappa = -sum(Ypp(t) .* n,1)./speed.^2;                     % bdry curvatures
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);                  % matrix of r=x-y (two vec cmpnts)
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);    % off-diag (-1/pi) n.r/r^2
A(diagind(A)) = kappa/(2*pi);                              % overwrite diag elements
A = eye(N) + A*diag(speed.*w);                             % note Id gets no "speed weights"
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);                     % test u(x) = x_1(x_2-0.3), not symmetric!
f = @(t) uex(Y(t));                                        % read off its Dirichlet data
rhs = -2*f(t)';
sigma = A\rhs;                                             % solve. Leave u = D.sigma eval to reader
```



pointwise spectral conv.:

$\log_{10}|u-u_{ex}|$: naive PTR quadr. eval.

"5h" rule: special eval. quadratures can fix near $\Gamma$ (Helsing, QBX...)

Debug: $\sigma \equiv -1 \Rightarrow u \equiv 1$, then test data from (generic!) soln $u$, and...

1. check/plot $\boldsymbol{n}$, $\kappa$. First test unit circle!
2. check Nyström matrix smooth at diag (*before* add $I$)

# Indirect vs direct formulations

So far "indirect" BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

# Indirect vs direct formulations

So far "indirect" BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

Alternative is "direct": take limit of GRF on $\Gamma$, rearrange to get BIE:

GRF $u = \mathcal{S}u^- - \mathcal{D}u_n^- \xrightarrow{\text{JRs}} u_n^- = (D^T + I/2)u_n^- - Tu^- \xrightarrow{\text{BC}} (D^T - I/2)u_n^- = Tf$

Needs hypersingular apply ☺. Then solve BIE for $u_n^-$, eval $u$ via GRF (needs two LP evals)

# Indirect vs direct formulations

So far "indirect" BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

Alternative is "direct": take limit of GRF on $\Gamma$, rearrange to get BIE:

GRF $u = \mathcal{S}u^- - \mathcal{D}u_n^- \xrightarrow{\text{JRs}} u_n^- = (D^T + I/2)u_n^- - Tu^- \xrightarrow{\text{BC}} (D^T - I/2)u_n^- = Tf$

Needs hypersingular apply ☺. Then solve BIE for $u_n^-$, eval $u$ via GRF (needs two LP evals)

Notice BIO $(D^T - I/2)$ **adjoint** of that for indirect $(D - I/2)$

generally true. So, spectra the same, thus iterative convergence rates too

# Indirect vs direct formulations

So far "indirect" BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

Alternative is "direct": take limit of GRF on $\Gamma$, rearrange to get BIE:

GRF $u = \mathcal{S}u^- - \mathcal{D}u_n^- \xrightarrow{\text{JRs}} u_n^- = (D^T + I/2)u_n^- - Tu^- \xrightarrow{\text{BC}} (D^T - I/2)u_n^- = Tf$

Needs hypersingular apply ☺. Then solve BIE for $u_n^-$, eval $u$ via GRF (needs two LP evals)

Notice BIO $(D^T - I/2)$ **adjoint** of that for indirect $(D - I/2)$

generally true. So, spectra the same, thus iterative convergence rates too

| Indirect BIE | Direct BIE |
|---|---|
| unknown density (unphysical) | unknown is physical |
| RHS is plain data | RHS needs BIO apply to data |
| eval the representation (may be simpler) | eval the GRF |

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh. . .)

FLATIRON
I N S T I T U T E

# Generalizations: Exterior, Neumann

| Laplace int Dir | Laplace int Neu |
|---|---|
| $\Delta u = 0$ in $\Omega$ | $\Delta u = 0$ in $\Omega$ |
| $u^- = f$ on $\Gamma$ | $u_n^- = g$ on $\Gamma$ |
| uniqueness, existence $\forall f$ | require $\int_\Gamma g ds = 0$ and unique only up to a const. |
| $u = \mathcal{D}\sigma$   representation | $u = \mathcal{S}\sigma$ |
| $(D - I/2)\sigma = f$   BIE | $(D^T - I/2)\sigma = g$ |

| Laplace ext Dir | Laplace ext Neu |
|---|---|
| $\Delta u = 0$ in $\mathbb{R}^d \backslash \bar{\Omega}$ | $\Delta u = 0$ in $\mathbb{R}^d \backslash \bar{\Omega}$ |
| $u^+ = f$ on $\Gamma$ | $u_n^+ = g$ on $\Gamma$ |
| uniqueness, existence $\forall f$ if | require $\int_\Gamma g ds = 0$ and $u_\infty = 0$ |
| $u_\infty = \mathcal{O}(1)$ in $\mathbb{R}^2$, $u_\infty := \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x})$ | |
| $u_\infty = 0$ in $\mathbb{R}^{m>2}$ | |
| representation | $u = \mathcal{S}\sigma$ |
| BIE | $(D^T - I/2)\sigma = g$ |

## Helmholtz

$(\Delta + \kappa^2)u = 0$ arises from scalar wave equation $u_{tt} - \Delta u = 0$

$\kappa$ "wavenumber"; wavelength $\lambda = 2\pi/\kappa$

Also used for 2D Maxwell ($z$-invar); TE vs TM

Dirichlet BC = PEC in TE (check)

Scattering formalism:

$u^{\text{tot}} = u^{\text{inc}} + u$.

BVP for $u$ (3 lines: PDE, BC, SRC, needed for uniqueness)

Helm fund sol (pic)

pics of LPs?

$u = \mathcal{D}\sigma$ has interior res prob (Leslie will have covered for the disk case)

CFIE: $u = (\mathcal{D} + i\eta\mathcal{S})\sigma$ no more unknowns, new kernel

can prove equivalence (no spurious resonances)

Dirichlet demo, plots only: Solve BVP for $u$ via PTR + Nyström, with new diag limit for $k(t,t)$, show $1/N^3$ convergence if use naive PTR with correct diag limit (see M126 HW?)

❹ debug BVP with known data from a radiative soln <span style="color:purple">sources inside $\Omega$</span>

(don't demo CFIE since requires S w/ log-singularity).

FLATIRON
I N S T I T U T E

# Helmholtz transmission BVP

refractive indices in Ω vs exterior
DECIDE IF TRANSMISSION
Matching? (more effort, needs $\mathcal{S}\sigma + \mathcal{D}\tau \ldots$)
difference kernels at most log-singular.

## Helmholtz

Getting spectral-acc Nyström for log-singular kernels: beyond today.

eg kernel-split or product quadratures (Kress, Helsing,...)

close-eval: kernel-split, QBX, etc.

see libraries: chunkie, BIE2D, etc

# More debug ideas

TO DISCARD
Other tests:

⑤ Test SLP & DLP evaluators via GRF for any harmonic $u$ in $\Omega$

# Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs      exterior need condition as $\|\mathbf{x}\| \to \infty$
- Choosing representation      to get 2nd kind BIE if poss., equivalent to BVP if poss.
    Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization                    high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases      also for libraries

practise! write out theory yourself + try HW exer. in repo + run demos

FLATIRON
I N S T I T U T E

# Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs       exterior need condition as $\|\boldsymbol{x}\| \to \infty$
- Choosing representation       to get 2nd kind BIE if poss., equivalent to BVP if poss.
    Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization       high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases       also for libraries

practise! write out theory yourself + try HW exer. in repo + run demos

Useful 2D tools we did not yet cover:       in libraries, eg `chunkie`, BIE2D

- panel (composite) quadratures       essential for adaptivity
- high-order quadratures for log-singular kernel       SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve
    some need interpolation of $\sigma(t)$ off the nodes $t_j$, some not
- corners, open arcs, slits, multi-material junctions

FLATIRON
INSTITUTE

# Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.

- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- "High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane", S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

  focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- https://users.flatironinstitute.org/~ahb/BIE/

- https://github.com/ahbarnett/BIEbook    in progress...

FLATIRON
INSTITUTE