

2D boundary integral equations and the Nyström method

Alex Barnett¹ and **Fruzsina Agocs¹**

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Integral equations on 1D interval

Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, periodic: $\sigma(2\pi) = \sigma(0)$, etc
ii) “kernel” function $k(t, s)$ defined on square $[0, 2\pi)^2$,

Integral *operator* K acts on σ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t, s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

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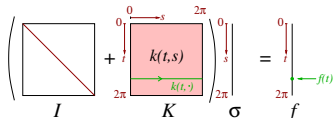
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Integral equation: $(I + K)\sigma = f$, ie

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Fredholm “second kind” (due to presence of I , otherwise called “first kind”)

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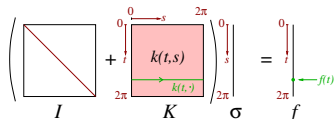
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If kernel continuous, or “weakly” singular (integrable), K is **compact**:

- eigenvalues $(K\phi_k = \lambda_k\phi_k)$ discrete, with $\lim_{k \rightarrow \infty} \lambda_k = 0$
 unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: $\text{Nul}(I + K) = \{0\}$
- $\text{Nul}(I + K) = \{0\} \Rightarrow$ existence of solution for any f Fredholm Alternative
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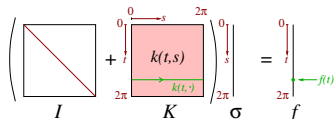
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Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)!

Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t) dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j) \quad w_j = \text{weights}, \quad t_j = \text{nodes}$$

For f smooth, superalgebraically convergent ("spectral"): error = $\mathcal{O}(N^{-p})$ for any p

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$$\tilde{\sigma}(t) + \sum_{j=1}^N k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \quad (*)$$

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Write as: $A\sigma = f$ $N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$, and $f_j := f(t_j)$

solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why want 2nd kind? eigs(A) accumulate only at $+1 \Rightarrow$ iterative converges fast

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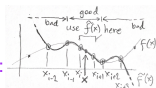
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Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^N$ suffice.

If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

- either: rearrange $(*)$ to give $\tilde{\sigma}(t) = \dots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

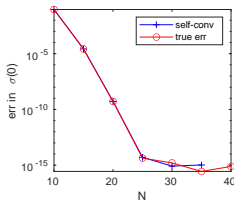
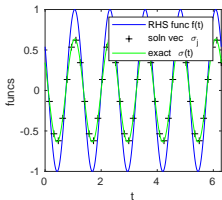
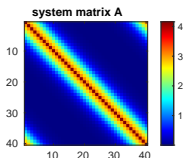


Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));  
ffun = @(t) cos(5*t+1);  
N = 30;  
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);  
A = eye(N) + bsxfun(kfun,t',t)*diag(w);  
rhs = ffun(t');  
sigmaj = A\rhs;
```

% smooth convolutional kernel, periodic domain $[0, 2\pi)$
% smooth data (RHS) func
% number of unknowns: should study convergence as N grows...
% PTR nodes and weights, row vecs
% identity plus fill $k(t_i, t_j)w_j$ for $i, j=1..N$
% col vec
% dense direct square solve (pivoted LU), gives col vec



“self-convergence”:
use $N=40$ as “true”

f and k smooth

$\Rightarrow \sigma$ smooth

\Rightarrow spectral conv?

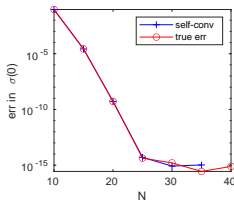
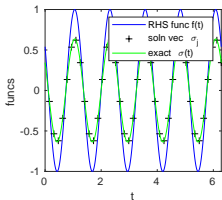
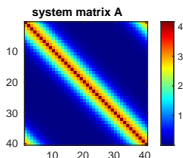
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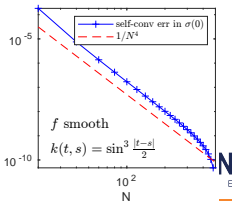
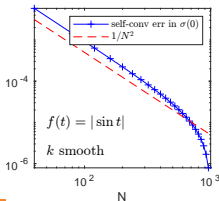
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- Then, f or k nonsmooth?
worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



Laplace fundamental solution in \mathbb{R}^2

Eg PDE: Poisson eqn $\Delta u = g$

$\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

notation: $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\mathbf{y} \in \mathbb{R}^2$ as another point (not y-coord!)

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A well-posed* BVP:

$$\Delta u = 0 \text{ in } \Omega$$

PDE (u harmonic)

$$u = f \text{ on } \Gamma$$

Dirichlet BC

*exists, unique,
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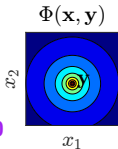
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Laplace fundamental soln: $\Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ where $r := \|\mathbf{x} - \mathbf{y}\|$

obeys $-\Delta_{\mathbf{x}} \Phi = -\Delta_{\mathbf{y}} \Phi = \delta_{\mathbf{x}}$ Φ harmonic except unit point-mass at $\mathbf{0}$

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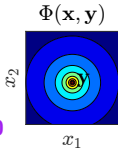
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warm-up: set $u = \text{BVP soln}$, $v \equiv 1$, G2I becomes $\int_{\Gamma} u_n \, ds - 0 = 0 - 0$: so u has zero flux

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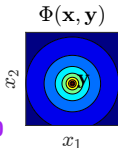
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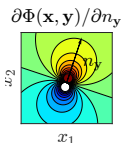
Now some fun: fix "target" $\mathbf{x} \in \Omega$, let $v = \Phi(\mathbf{x}, \cdot)$, G2I gives:

Green's representation formula:

$$\int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_{\mathbf{y}}} u(\mathbf{y}) \, ds_{\mathbf{y}} = u(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega$$

recovers soln from "Cauchy data" $(u, u_n)|_{\Gamma}$

also versions for Helmholtz, Stokes, Maxwell,...

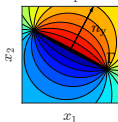
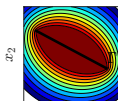


Layer potentials and their jump relations

Representations of harmonic functions off a curve Γ : “density” σ

Single-layer potential $(\mathcal{S}\sigma)(\mathbf{x}) := \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) ds_{\mathbf{y}}$ charge sheet

Double-layer potential $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y}) ds_{\mathbf{y}}$ dipole sheet



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interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

Jump relations:

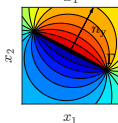
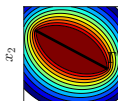
$(\mathcal{S}\sigma)^{\pm} = S\sigma$ S (Roman font) means *restriction* of S to Γ : a bdry int. op.

$(\mathcal{D}\sigma)^{\pm} = (D \pm I/2)\sigma$ jump in potential equal to σ ; D restriction to Γ in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = (D^T \mp I/2)\sigma$ jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = T\sigma$ T hypersingular, kernel $\partial^2 \Phi(\mathbf{x}, \mathbf{y}) / \partial \mathbf{n}_{\mathbf{x}} \partial \mathbf{n}_{\mathbf{y}} \sim 1/r^2$

- D smooth kernel on smooth Γ , while S always log (weakly) singular

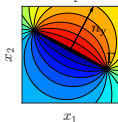
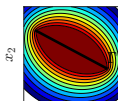


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Double-layer potential $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y}) d\mathbf{s}_{\mathbf{y}}$ dipole sheet



interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h \mathbf{n}_{\mathbf{x}})$$

Jump relations:

$(\mathcal{S}\sigma)^{\pm} = S\sigma$ S (Roman font) means *restriction* of S to Γ : a bdry int. op.

$(\mathcal{D}\sigma)^{\pm} = (D \pm I/2)\sigma$ jump in potential equal to σ ; D restriction to Γ in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = (D^T \mp I/2)\sigma$ jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = T\sigma$ T hypersingular, kernel $\partial^2 \Phi(\mathbf{x}, \mathbf{y}) / \partial \mathbf{n}_{\mathbf{x}} \partial \mathbf{n}_{\mathbf{y}} \sim 1/r^2$

- D smooth kernel on smooth Γ , while S always log (weakly) singular

Recap GRF in LP notation: u harmonic in $\Omega \Rightarrow \mathcal{S}u_n^- - \mathcal{D}u^- = u$ in Ω

Converting BVP to BIE and solving

Say wish to solve interior

Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \quad \text{BC}$$



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This shows: let σ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns u of this form?

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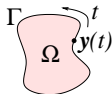
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Parameterize the bdry $\mathbf{y}(t)$ $\mathbf{y} : \mathbb{R} \rightarrow \mathbb{R}^2$, 2π -periodic, $\Gamma = \{\mathbf{y}(t) : t \in [0, 2\pi)\}$

change variable $ds_y = \|\mathbf{y}'(t)\|dt$ abuse notation $\sigma(t) = \sigma(\mathbf{y}(t))$

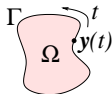
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familiar form $(I + K)\sigma = -2f$, with kernel $k(s, t) = \frac{-2}{2\pi} \frac{\mathbf{n}_{\mathbf{y}(s)} \cdot (\mathbf{y}(t) - \mathbf{y}(s))}{\|\mathbf{y}(t) - \mathbf{y}(s)\|^2} \|\mathbf{y}'(s)\|$

formula on diagonal: $k(t, t) = \lim_{s \rightarrow t} k(t, s) = \kappa(t)/2\pi$, κ curvature of Γ (check!)

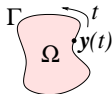
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Now Nyström discretize with PTR, solve lin. sys. for $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln: $u(\mathbf{x}) = (\mathcal{D}\sigma)(\mathbf{x}) \stackrel{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\mathbf{n}_{\mathbf{y}(t_j)} \cdot (\mathbf{x} - \mathbf{y}(t_j))}{2\pi \|\mathbf{x} - \mathbf{y}(t_j)\|^2} \|\mathbf{y}'(t_j)\| w_j \sigma_j$

Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

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kappa = -sum(Ypp(t) .* n,1)./speed.^2;
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% y', analytic
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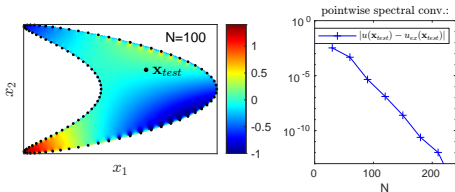
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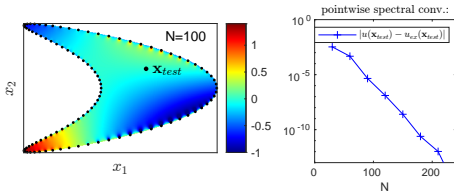
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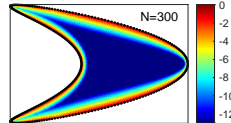
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$\log_{10} |u - u_{ex}|$: naive PTR quadr. eval.



error: "5h" rule.

Note: special quadratures can fix near Γ (Helsing, QBX...)

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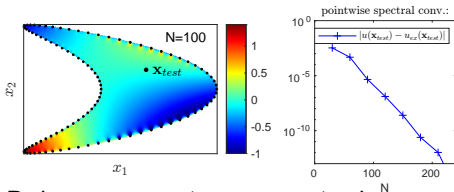
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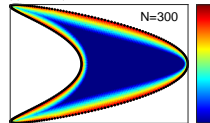
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Debug: $\sigma \equiv -1 \Rightarrow u \equiv 1$, then test data from (generic!) soln u , and...

- ① check/plot n, κ . First test unit circle!
- ② check Nyström matrix smooth at diag (before add I)

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using Laplace interior Dirichlet BVP

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Indirect BIE	Direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

Indirect 2nd-kind BIE for Neumann, exterior

recap: Laplace int. Dir.

$$\Delta u = 0 \text{ in } \Omega$$

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uniqueness, existence $\forall f$

- $u = \mathcal{D}\sigma$ rep.
 $(D - I/2)\sigma = f$ BIE: well-cond.

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$$u^+ = f \text{ on } \Gamma$$

$$u_{\infty} := \lim_{\|x\| \rightarrow \infty} u(x) \text{ exists}$$

uniqueness, existence $\forall f$

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complementary BVPs

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• $u = \mathcal{D}\sigma$ rep.

$(D - I/2)\sigma = f$ BIE: well-cond.

Laplace int. Neu.

$$\Delta u = 0 \text{ in } \Omega$$

$$u_n^- = g \text{ on } \Gamma$$

require $\int_{\Gamma} g ds = 0$

unique only up to a const.

• $u = \mathcal{S}\sigma$

$(D^T + I/2 + 11^T)\sigma = g$ well-cond.

Laplace ext. Dir.

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}$$

$$u^+ = f \text{ on } \Gamma$$

$$u_{\infty} := \lim_{\|x\| \rightarrow \infty} u(x) \text{ exists}$$

uniqueness, existence $\forall f$

• $u = \mathcal{D}\sigma + \int_{\Gamma} \sigma ds$ modified rep.

$(D + I/2 + 11^T)\sigma = f$ well-cond.

Laplace ext. Neu.

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}$$

$$u_n^+ = g \text{ on } \Gamma$$

require $\int_{\Gamma} g ds = 0$ and $u_{\infty} = 0$

unique

• $u = \mathcal{S}\sigma$

$(D^T - I/2)\sigma = g$ well-cond.

complementary BVPs

Indirect 2nd-kind BIE for Neumann, exterior

recap: Laplace int. Dir.

$$\Delta u = 0 \text{ in } \Omega$$

$$u^- = f \text{ on } \Gamma$$

uniqueness, existence $\forall f$

- $u = \mathcal{D}\sigma$ rep.

$$(D - I/2)\sigma = f \quad \text{BIE: well-cond.}$$

Laplace int. Neu.

$$\Delta u = 0 \text{ in } \Omega$$

$$u_n^- = g \text{ on } \Gamma$$

require $\int_{\Gamma} g ds = 0$

unique only up to a const.

- $u = \mathcal{S}\sigma$

$$(\mathcal{D}^T + I/2 + 11^T)\sigma = g \quad \text{well-cond.} \quad \checkmark \text{kernel} \equiv 1, \text{ kills nullspace}$$

Laplace ext. Dir.

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}$$

$$u^+ = f \text{ on } \Gamma$$

$$u_{\infty} := \lim_{\|x\| \rightarrow \infty} u(x) \text{ exists}$$

uniqueness, existence $\forall f$

- $u = \mathcal{D}\sigma + \int_{\Gamma} \sigma ds$ modified rep.

$$(D + I/2 + 11^T)\sigma = f \quad \text{well-cond.}$$

Laplace ext. Neu.

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}$$

$$u_n^+ = g \text{ on } \Gamma$$

require $\int_{\Gamma} g ds = 0$ and $u_{\infty} = 0$

unique

- $u = \mathcal{S}\sigma$

$$(D^T - I/2)\sigma = g \quad \text{well-cond.}$$

complementary BVPs

③ Exterior: don't test with $u = \log r$ why not? BVPs enforce zero net charge

Helmholtz — introduction and connection to Maxwell

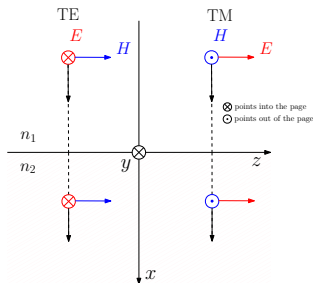
$$(\Delta + \omega^2)u = 0$$

time-harmonic scalar waves

comes from scalar wave equation $\Delta u - u_{tt} = 0$ when $u(\mathbf{x}, t) = u(\mathbf{x})e^{-i\omega t}$
 ω is the wavenumber spatial frequency, related to wavelength via $\lambda = 2\pi/\omega$

Also used for Maxwell's equations in cylindrical symm (z-invariance):

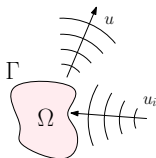
1. Assume $\mathbf{E}, \mathbf{H}(x, y, z) = \mathbf{E}, \mathbf{H}(x, y)$
2. Write Maxwell's eqs: $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, $\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}$,
3. Notice only E_z, H_z are indep \rightarrow 2 polarizations, TE or TM: $E_z = 0, H_z = 0$ resp.
4. Choose TE and let $u := H_z$, then: $\mathbf{H} = (0, 0, u)$,
 $\mathbf{E} = \frac{1}{i\omega\varepsilon}(\partial_x u, -\partial_y u, 0)$, and $(\Delta + n^2\omega^2)u = 0$ with $n^2 = \varepsilon\mu$



Dirichlet BC in TE formalism = PEC perfect electric conductor; $\mathbf{E} \perp$ to surface

Helmholtz — scattering formalism

Split total potential into incident (known) and scattered (unknown) parts, $u^{\text{tot}} = u^{\text{inc}} + u$



BVP for u :

$$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \quad \text{PDE}$$

$$u = -u_i \quad \text{on } \Gamma \quad \text{Dirichlet BC, } u_n = -(u_i)_n \text{ for Neumann}$$

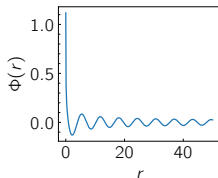
$$\lim_{r \rightarrow \infty} \left(\frac{\partial u}{\partial r} - i\omega u \right) = 0 \quad r := |\mathbf{x} - \mathbf{y}|, \text{ Sommerfeld radiation condition for uniqueness}$$

Fundamental solution $\Phi(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\omega |\mathbf{x} - \mathbf{y}|)$

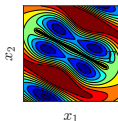
$$\text{Asymptotics: } \lim_{r \rightarrow 0} \Phi(r) = \frac{1}{2\pi} \log \frac{1}{r} + \mathcal{O}(1)$$

$$\lim_{r \rightarrow \infty} \Phi(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \nu\pi/2 - \pi/4)} + \mathcal{O}(r^{-1})$$

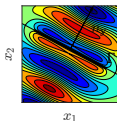
Same singularity as Laplace \rightarrow same JRs!



Layer potentials



SLP



DLP

Helmholtz — interior resonances and how to avoid them

Try the ext Dir BVP with $u = \mathcal{D}\sigma$ ($(\Delta + \omega)^2 u = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, $u = -u_i$ on Γ , SRC for u)

Observe that for some ω , condition # of BIE blows up, not always solvable

Why? Suppose $\phi \not\equiv 0$ s.t.
$$\begin{cases} (\Delta + \omega^2)\phi = 0 & \text{in } \Omega \\ \phi_n = 0 & \text{on } \Gamma \end{cases}$$
 ϕ is interior Neumann eigenfunction with eigenvalue ω^2

Then by (interior) GRF (same as for Laplace), $\mathcal{S}\phi|_{\Gamma} - \mathcal{D}\phi|_{\Gamma} = u$ in Ω .

Take $\mathbf{x} \rightarrow \Gamma^-$ and use JR: $(-D - I/2)\phi|_{\Gamma} = \phi_{\Gamma}$, i.e. $(I + 2D)\phi|_{\Gamma} = 0$.

Since $\phi|_{\Gamma}$ was nontrivial (otherwise $\phi = 0$ by GRF), nullity of $I + 2D > 0$, i.e. singular, by FA not solvable $\forall f(u_i)$.

We made use of the **complementary BVP** (int Neu), this is an “internal resonance”.

Fix: $u = (\mathcal{D} - i\eta\mathcal{S})\sigma$ combined field integral eq (CFIE), same # unknowns, new kernel
ext Dir BIE becomes $(I + 2D - 2i\eta\mathcal{S})\sigma = -2u_i$ on Γ

Proof: Let τ solve $(I/2 + D - i\eta\mathcal{S})\tau = 0$, wish to show $\tau = 0$.

From τ construct potential $v := (\mathcal{D} - i\eta\mathcal{S})\tau$, then $v^+ = 0$ by construction.

Then $v = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ by uniqueness of the complementary BVP (ext Dir)

Then v_n^+ on Γ , and by JRs and Green's 1st thm (exercise for the reader ☺), $\tau = 0$.

Helmholtz — Dirichlet demo

`demo_helmextdir.m`

Solve the Helmholtz ext Dir BVP with the $u = \mathcal{D}\sigma$ repr, u_j plane wave

Diagonal limit for Nyström matrix $k(t, t)$ same as Laplace

PTR with N nodes, test via self-convergence What's the conv. rate? Why N^{-3} ?

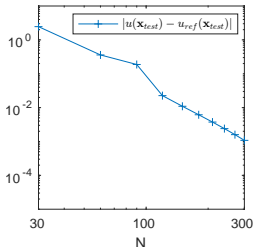
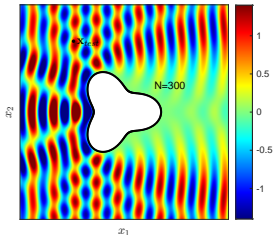
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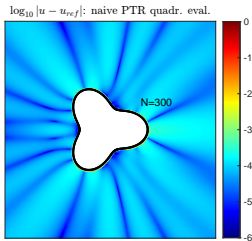
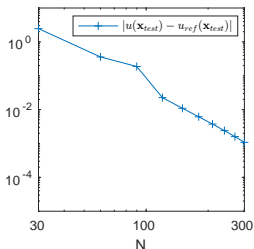
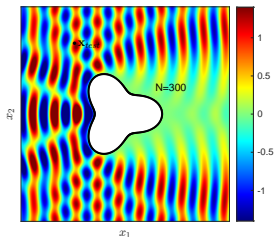
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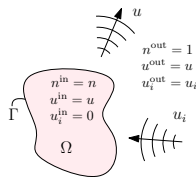
- 4 Debug BVP with known data from a radiative soln sources inside Ω
- 5 Without analytic soln, test either via self-convergence or conserved physical qty e.g. optical theorem, or no net QM flux over closed curve C containing no sources or sinks, $0 = \Im(\int_C \bar{u} u_n ds)$ (Agocs, Barnett '23)

Helmholtz – transmission BVP

If different refractive index n in Ω than outside, use usual splitting $u^{\text{tot}} = u^{\text{inc}} + u$

can always scale such that one is $n = 1$

inc wave only on outside, e.g. $u_i = \begin{cases} 0 & \text{in } \Omega \\ e^{ik \cdot x} & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \end{cases}, \mathbf{k} = \begin{bmatrix} \omega \cos \theta \\ \omega \sin \theta \end{bmatrix}$



BVP for u :

$$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \quad \text{PDE outside}$$

$$(\Delta + n^2 \omega^2)u = 0 \quad \text{in } \overline{\Omega} \quad \text{PDE inside}$$

$$[u] = -u_i \quad \text{on } \Gamma \quad [u] := u^+ - u^-, \text{ continuity of } u^{\text{tot}}$$

$$[u_n] = -(u_i)_n \quad \text{on } \Gamma \quad \text{continuity of } u_n^{\text{tot}}$$

$$\lim_{r \rightarrow \infty} \left(\frac{\partial u}{\partial r} - i\omega u \right) = 0 \quad \text{SRC outside}$$

Formulate as sys of integral eqs Rokhlin–Müller scheme, (Müller '69, Rokhlin '83)

$$u = \begin{cases} \mathcal{S}(n\omega)\sigma + \mathcal{D}(n\omega)\tau & \text{in } \Omega \\ \mathcal{S}(\omega)\sigma + \mathcal{D}(\omega)\tau & \text{in } \mathbb{R}^d \setminus \Omega \end{cases}$$

$$\begin{bmatrix} [u] \\ [u_n] \end{bmatrix} = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \mathcal{D}(\omega) - \mathcal{D}(n\omega) & \mathcal{S}(n\omega) - \mathcal{S}(\omega) \\ \mathcal{T}(\omega) - \mathcal{T}(n\omega) & \mathcal{D}(n\omega)^* - \mathcal{D}(\omega)^* \end{bmatrix} \right) \begin{bmatrix} \tau \\ -\sigma \end{bmatrix} \quad \mathcal{T} \text{ is hypersingular operator}$$

...but $\mathcal{T}(\omega) - \mathcal{T}(n\omega)$ is at most log-singular! ☺ (Show via asymptotics of $H_n^{(1)}$)

Helmholtz – high-order accuracy

Spectral accuracy Nyström for log-singular kernels: possible, but beyond today

Divide bdry into panels instead of global set of nodes, adaptive panel sizes & quadrature rules

Kernel-split: decompose kernel $G(\mathbf{x}, \mathbf{y}) = \underbrace{G^S(\mathbf{x}, \mathbf{y})}_{\text{smooth}} + \underbrace{G^L(\mathbf{x}, \mathbf{y}) \log |\mathbf{y} - \mathbf{x}|}_{\text{log singularity}} + \underbrace{G^C(\mathbf{x}, \mathbf{y}) \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}}{|\mathbf{y} - \mathbf{x}|^2}}_{\text{Cauchy singularity}}$

Product integration: target-specific quadrature rules, e.g.

$$\int_{\Gamma} f(\mathbf{x}, \mathbf{y}) \log |\mathbf{x} - \mathbf{y}| d\mathbf{s}_{\mathbf{y}} \approx \sum_{j=1}^N f(\mathbf{x}, \mathbf{y}_j) w_j^L(\mathbf{x}) \text{ (Helsing, Holst, '15), (Kress), ...}$$

Generalized Gaussian quadrature (Bremer)

Close evaluation: target close to bdry

Kernel-split approach

QBX: quadrature by expansion (Kloeckner, Barnett, Greengard, O'Neil '13), (Epstein, Greengard, Kloeckner '13)

See also libraries: chunkie, BIE2D, etc.

Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs exterior need condition as $\|\mathbf{x}\| \rightarrow \infty$
- Choosing representation to get 2nd kind BIE if poss., equivalent to BVP if poss.
Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
- Nyström discretization high-order/spectral convergence, if poss.
- Build/debug codes via well-chosen sequence of test cases also for libraries

practise! write out theory yourself + try HW exer. in repo + run demos

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practise! write out theory yourself + try HW exer. in repo + run demos

Useful 2D tools we did not yet cover: in libraries, eg chunkie, BIE2D

- panel (composite) quadratures essential for adaptivity
- high-order quadratures for log-singular kernel SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve
some need interpolation of $\sigma(t)$ off the nodes t_j , some not
- corners, open arcs, slits, multi-material junctions

Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- “High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane”, S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- <https://users.flatironinstitute.org/~ahb/BIE/>
- <https://github.com/ahbarnett/BIEbook>

in progress...