

2D boundary integral equations and the Nyström method

Alex Barnett¹ and **Fruzsina Agocs¹**

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¹Center for Computational Mathematics, Flatiron Institute, Simons Foundation

Integral equations on 1D interval

- Given: i) function $\sigma(t)$ defined on interval $[0, 2\pi)$, periodic: $\sigma(2\pi) = \sigma(0)$, etc
 ii) “kernel” function $k(t, s)$ defined on square $[0, 2\pi)^2$,

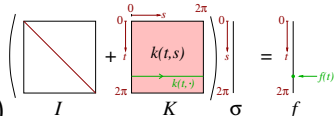
Integral operator K acts on σ to give another function $K\sigma$:

$$(K\sigma)(t) := \int_0^{2\pi} k(t, s)\sigma(s)ds, \quad t \in [0, 2\pi)$$

continuous analog of matrix-vector prod. Ax

Integral equation: $(I + K)\sigma = f$, ie

$$\sigma(t) + \int_0^{2\pi} k(t, s)\sigma(s)ds = f(t), \quad t \in [0, 2\pi)$$




analog of lin. sys. $Ax = b$

Fredholm “second kind” (due to presence of I , otherwise called “first kind”)

If kernel continuous, or “weakly” singular (integrable), K is *compact*:

- eigenvalues $(K\phi_k = \lambda_k\phi_k)$ discrete, with $\lim_{k \rightarrow \infty} \lambda_k = 0$
unless some $\lambda_k = -1$, 2nd kind IE has at most one soln: $\text{Nul}(I + K) = \{0\}$
- $\text{Nul}(I + K) = \{0\} \Rightarrow$ existence of solution for *any* f Fredholm Alternative
like square matrix (finite-dim), recall: uniqueness \Rightarrow consistent for any RHS

Contrast 1st kind IE $K\sigma = f$ is ill-posed problem (unstable)! 

See references for lots of beautiful theory, precise statements

Nyström discretization of 2nd kind IE on interval

Simplest quadrature for periodic funcs: periodic trapezoid rule (PTR)

$$\int_0^{2\pi} f(t) dt \approx \sum_{j=1}^N \frac{2\pi}{N} f\left(\frac{2\pi j}{N}\right) = \sum_{j=1}^N w_j f(t_j) \quad w_j = \text{weights}, \quad t_j = \text{nodes}$$

For f smooth, superalgebraically convergent ("spectral"): error = $\mathcal{O}(N^{-p})$ for any p

Apply quad to integral in 2nd kind IE:

call the resulting approx soln $\tilde{\sigma}$

$$\tilde{\sigma}(t) + \sum_{j=1}^N k(t, t_j) w_j \tilde{\sigma}(t_j) = f(t), \quad t \in [0, 2\pi) \quad (*)$$

Holds for all t ; in particular, holds at each t_i , $i = 1, \dots, N$, giving:

$$\sigma_i + \sum_{j=1}^N k(t_i, t_j) w_j \sigma_j = f(t_i), \quad i = 1, \dots, N \quad \text{where } \sigma_j := \tilde{\sigma}(t_j)$$

Write as: $A\sigma = f$ $N \times N$ lin sys, entries $a_{ij} = \delta_{ij} + k(t_i, t_j) w_j$, and $f_j := f(t_j)$

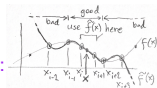
solve? dense direct $\mathcal{O}(N^3)$; dense iter. $\mathcal{O}(N^2)$; fast iter. $\approx \mathcal{O}(N)$; fast direct $\approx \mathcal{O}(N^{(d+1)/2})$

Why 2nd kind? eigs(A) accumulate only at $+1$, iterative fast conv.

Sometimes for BIE (eg, far-field eval), node values $\{\sigma_j\}_{j=1}^N$ suffice.

If not, interpolate from them to any $t \in [0, 2\pi)$. Two approaches:

- either: rearrange $(*)$ to give $\tilde{\sigma}(t) = \dots$, called "Nyström interpolant" (rare)
- or (common): use local high-order Lagrange (or global spectral) interpolation:

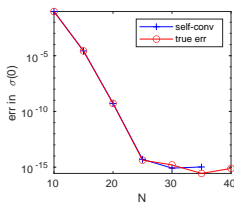
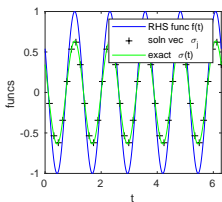
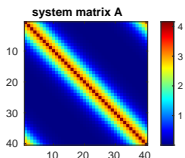


Demo Nyström on interval (1D)

day1/code/nyst1d_demos.m

```
kfun = @(t,s) exp(3*cos(t-s));  
ffun = @(t) cos(5*t+1);  
N = 30;  
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);  
A = eye(N) + bsxfun(kfun,t',t)*diag(w);  
rhs = ffun(t');  
sigmaj = A\rhs;
```

*% smooth convolutional kernel, periodic domain [0,2pi)
% smooth data (RHS) func
% number of unknowns: should study convergence as N grows...
% PTR nodes and weights, row vecs
% identity plus fill k(t_i,t_j)w_j for i,j=1..N
% col vec
% dense direct square solve (pivoted LU), gives col vec*



“self-convergence”:
use $N=40$ as “true”

f and k smooth

⇒ σ smooth

⇒ spectral conv?

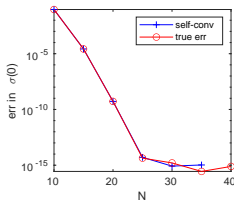
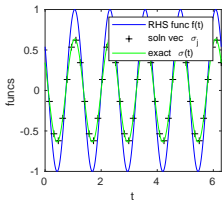
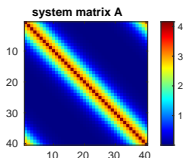
Thm. (Anselone, Kress,...): error at node values (and Nyström interpolant) same order as that of quadrature rule applied to integrand $k(t, \cdot)\sigma$.

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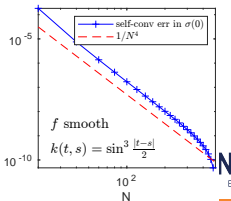
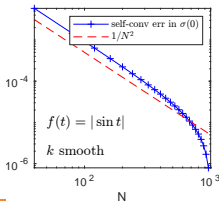
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use $N=40$ as “true”

f and k smooth
 $\Rightarrow \sigma$ smooth
 \Rightarrow spectral conv?

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- Then, f or k nonsmooth?
worse (here *algebraic*) convergence using plain PTR rule:

Qu: why does order appear to improve at end?



Fundamental solution in \mathbb{R}^2

Eg PDE: Poisson eqn $\Delta u = g$

$\Delta := (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$ Laplacian

Notation: $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$ is a point. This frees up $\mathbf{y} \in \mathbb{R}^2$ as another point (not y-coord!)

Not well-posed prob. unless add BC! BIEs are good for *homogeneous* PDEs (driving $g \equiv 0$)

Eg well-posed* BVP:

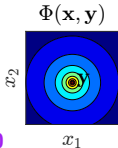
$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega && \text{PDE (} u \text{ harmonic)} \\ u &= f \text{ on } \Gamma && \text{Dirichlet BC} \end{aligned}$$

*exists, unique, continuous

w.r.t. data



Laplace fundamental soln: $\Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{r}$ where $r := \|\mathbf{x} - \mathbf{y}\|$



obeys $-\Delta_{\mathbf{x}} \Phi = -\Delta_{\mathbf{y}} \Phi = \delta_{\mathbf{x}}$ Φ harmonic except unit point-mass at 0

Normal \mathbf{n} points outwards, $\|\mathbf{n}\| = 1$ normal deriv. notation $u_n := \mathbf{n} \cdot \nabla u$

Green's 2nd identity: $\int_{\Gamma} v u_n - v_n u \, ds = \int_{\Omega} v \Delta u - (\Delta v) u \, d\mathbf{y}$

calculus

warm-up: set $u = \text{BVP soln}$, $v \equiv 1$, G2I becomes $\int_{\Gamma} u_n \, ds - 0 = 0 - 0$: so u has zero flux

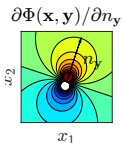
more fun: fix "target" $\mathbf{x} \in \Omega$, let $v = \Phi(\mathbf{x}, \cdot)$, G2I gives:

Green's representation formula:

$$\int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) u_n(\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_{\mathbf{y}}} u(\mathbf{y}) \, ds_{\mathbf{y}} = u(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega$$

Gets soln from "Cauchy data" $(u, u_n)|_{\Gamma}$

also versions for Helmholtz, Stokes, Maxwell, ...



Layer potentials and their jump relations

Representations of harmonic functions off a curve Γ : “density” σ

Single-layer potential $(\mathcal{S}\sigma)(\mathbf{x}) := \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})d\mathbf{s}_{\mathbf{y}}$ charge sheet

Double-layer potential $(\mathcal{D}\sigma)(\mathbf{x}) := \int_{\Gamma} \frac{\partial\Phi(\mathbf{x}, \mathbf{y})}{\partial\mathbf{n}_{\mathbf{y}}} \sigma(\mathbf{y})d\mathbf{s}_{\mathbf{y}}$ dipole sheet

interior (-) / exterior (+) limits:

$$u^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} u(\mathbf{x} \pm h\mathbf{n}_{\mathbf{x}})$$

$$u_n^{\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}_{\mathbf{x}} \cdot \nabla u(\mathbf{x} \pm h\mathbf{n}_{\mathbf{x}})$$

Jump relations:

$(\mathcal{S}\sigma)^{\pm} = S\sigma$ S (Roman font) means *restriction* of S to Γ : a bdry int. op.

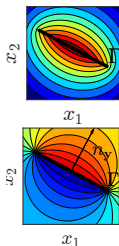
$(\mathcal{D}\sigma)^{\pm} = (D \pm I/2)\sigma$ jump in potential equal to σ ; D restriction to Γ in P.V. sense

$(\mathcal{S}\sigma)_n^{\pm} = (D^T \mp I/2)\sigma$ jump in normal derivative

$(\mathcal{D}\sigma)_n^{\pm} = T\sigma$ T hypersingular, kernel $\partial^2\Phi(\mathbf{x}, \mathbf{y})/\partial\mathbf{n}_{\mathbf{x}}\partial\mathbf{n}_{\mathbf{y}} \sim 1/r^2$

- D smooth kernel on smooth Γ , while S always log (weakly) singular

Recap GRF in LP notation: u harmonic in $\Omega \Rightarrow \mathcal{S}u_n^- - \mathcal{D}u^- = u$ in Ω



Converting BVP to BIE and solving

Say wish to solve interior

Dirichlet Laplace BVP:

$$\Delta u = 0 \text{ in } \Omega \quad \text{PDE}$$

$$u^- = f \text{ on } \Gamma \quad \text{BC}$$



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Insert the BC to get BIE: $(I - 2D)\sigma = -2f$ scaled to 2nd kind form

This shows: let σ solve BIE, then $u = \mathcal{D}\sigma$ solves BVP (i.e., no spurious solns)

But how know *all* solns u of this form?

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Above BIE expressed on Γ using arc-length measure ds_γ . Usually not how Γ described...

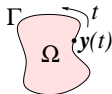
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change variable $ds_y = \|\mathbf{y}'(t)\|dt$ abuse notation $\sigma(t) = \sigma(\mathbf{y}(t))$

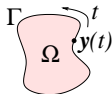
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Get 1D IE: $\sigma(t) - 2 \int_0^{2\pi} \frac{\partial \Phi(\mathbf{y}(t), \mathbf{y}(s))}{\partial \mathbf{n}_{\mathbf{y}(s)}} \sigma(s) \|\mathbf{y}'(s)\| ds = -2f(t), \quad t \in [0, 2\pi)$

familiar form $(I + K)\sigma = -2f$, with kernel $k(s, t) = \frac{-2}{2\pi} \frac{\mathbf{n}_{\mathbf{y}(s)} \cdot (\mathbf{y}(t) - \mathbf{y}(s))}{\|\mathbf{y}(t) - \mathbf{y}(s)\|^2} \|\mathbf{y}'(s)\|$

formula on diagonal: $k(t, t) = \lim_{s \rightarrow t} k(t, s) = \kappa(t)/2\pi$, κ curvature of Γ (check!)

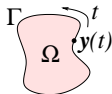
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Now Nyström discretize with PTR, solve lin. sys. for $\boldsymbol{\sigma} := \{\sigma_j\}_{j=1}^N$

Finally evaluate soln: $u(\mathbf{x}) = (\mathcal{D}\sigma)(\mathbf{x}) \stackrel{\text{PTR}}{\approx} \sum_{j=1}^N \frac{\mathbf{n}_{\mathbf{y}(t_j)} \cdot (\mathbf{x} - \mathbf{y}(t_j))}{2\pi \|\mathbf{x} - \mathbf{y}(t_j)\|^2} \|\mathbf{y}'(t_j)\| w_j \sigma_j$

Interior Laplace Dirichlet BVP solve demo

demo_lapintdir.m

```
a=0.7; b=1.0;
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);
y = Y(t);
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;
kappa = -sum(Ypp(t) .* n,1)./speed.^2;
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);
A(diagind(A)) = kappa/(2*pi);
A = eye(N) + A*diag(speed.*w);
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);
f = @(t) uex(Y(t));
rhs = -2*f(t)';
sigma = A\rhs;
```

% shape params (note a=1,b=0 unit circle)
% kite parameterization y(t)
% y', analytic
% y'', analytic

% PTR nodes & weights
% bdry nodes, 2-by-N
% bdry normals
% bdry curvatures
% matrix of r=x-y (two vec cmpnts)
% off-diag (-1/pi) n.r/r^2
% overwrite diag elements
% note Id gets no "speed weights"
% test u(x) = x_1(x_2-0.3), not symmetric!
% read off its Dirichlet data

% solve. Leave u = D.sigma eval to reader

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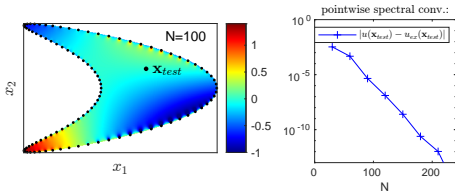
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```



Interior Laplace Dirichlet BVP solve demo

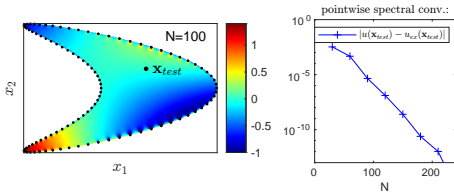
demo_lapintdir.m

```
a=0.7; b=1.0;
Y = @(t) [a*cos(t)+b*cos(2*t); sin(t)];
Yp = @(t) [-a*sin(t)-2*b*sin(2*t); cos(t)];
Ypp = @(t) [-a*cos(t)-4*b*cos(2*t); -sin(t)];
N = 100;
t = 2*pi/N*(1:N); w = 2*pi/N*ones(1,N);
y = Y(t);
n = [0 1;-1 0]*Yp(t); speed = sqrt(sum(n.^2,1)); n = n./speed;
kappa = -sum(Ypp(t) .* n,1)./speed.^2;
r1 = y(1,:)'-y(1,:); r2 = y(2,:)'-y(2,:);
A = (-1/pi)*(n(1,:).*r1 + n(2,:).*r2) ./ (r1.^2+r2.^2);
A(diagind(A)) = kappa/(2*pi);
A = eye(N) + A*diag(speed.*w);
uex = @(x) ([1 0]*x) .* ([0 1]*x-0.3);
f = @(t) uex(Y(t));
rhs = -2*f(t)';
sigma = A\rhs;
```

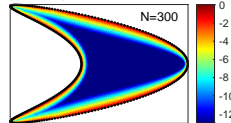
```
% shape params (note a=1,b=0 unit circle)
% kite parameterization y(t)
% y', analytic
% y'', analytic
```

```
% PTR nodes & weights
% bdry nodes, 2-by-N
% bdry normals
% bdry curvatures
% matrix of r=x-y (two vec cmpnts)
% off-diag (-1/pi) n.r/r^2
% overwrite diag elements
% note Id gets no "speed weights"
% test u(x) = x_1(x_2-0.3), not symmetric!
% read off its Dirichlet data
```

```
% solve. Leave u = D.sigma eval to reader
```



$\log_{10} |u - u_{ex}|$: naive PTR quadr. eval.



“5h” rule:
special eval.
quadratures
can fix near Γ
(Helsing, QBX...)

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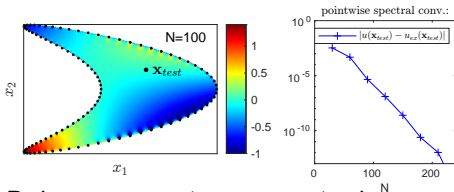
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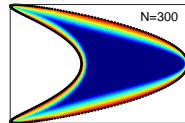
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Debug: $\sigma \equiv -1 \Rightarrow u \equiv 1$, then test data from (generic!) soln u , and...

- ① check/plot n , κ . First test unit circle!
- ② check Nyström matrix smooth at diag (before add I)

Indirect vs direct formulations

using Laplace interior Dirichlet BVP

So far “indirect” BIE: pick representation (eg $u = \mathcal{D}\sigma$), get BIE from JRs

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Indirect BIE	Direct BIE
unknown density (unphysical)	unknown is physical
RHS is plain data	RHS needs BIO apply to data
eval the representation (may be simpler)	eval the GRF

- indirect: more flexibility, but need math to prove equivalence to BVP
- accuracy differences for domains with corners (Hoskins–Rachh...)

Indirect 2nd-kind BIE for Neumann, exterior

recap: Laplace int. Dir.

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$$u^- = f \text{ on } \Gamma$$

uniqueness, existence $\forall f$

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③ Exterior: don't test with $u = \log r!$ why not? BVPs enforce zero net charge

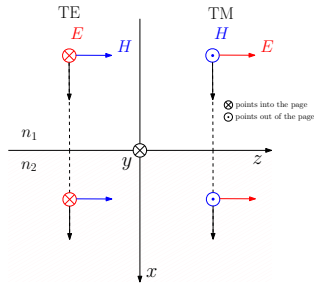
Helmholtz — introduction and connection to Maxwell

$$(\Delta + \omega^2)u = 0$$

comes from scalar wave equation $\Delta u - u_{tt} = 0$ when $u(\mathbf{x}, t) = u(\mathbf{x})e^{-i\omega t}$
 ω is the wavenumber spatial frequency, related to wavelength via $\lambda = 2\pi/\omega$

Also used for Maxwell's equations in cylindrical symm (z-invariance):

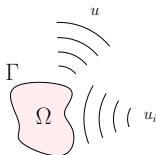
1. Assume $\mathbf{E}, \mathbf{H}(x, y, z) = \mathbf{E}, \mathbf{H}(x, y)$
2. Write Maxwell's eqs: $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, $\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}$,
3. Notice only E_z, H_z are indep \rightarrow 2 polarizations, TE or TM: $E_z = 0, H_z = 0$ resp.
4. Choose TE and let $u := H_z$, then: $\mathbf{H} = (0, 0, u)$,
 $\mathbf{E} = \frac{1}{i\omega\varepsilon}(\partial_x u, -\partial_y u, 0)$, and $(\Delta + n^2\omega^2)u = 0$ with $n^2 = \varepsilon\mu$



Dirichlet BC in TE formalism = PEC perfect electric conductor; $\mathbf{E} \parallel$ to surface

Helmholtz — scattering formalism

Split total potential into incident (known) and scattered (unknown) parts, $u^{\text{tot}} = u^{\text{inc}} + u$



BVP for u :

$$(\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^d \setminus \bar{\Omega} \quad \text{PDE}$$

$$u = -u_i \quad \text{on } \Gamma \quad \text{Dirichlet BC, } u_n = -(u_i)_n \text{ for Neumann}$$

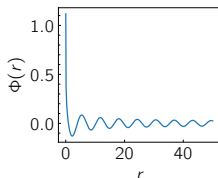
$$\lim_{r \rightarrow \infty} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \quad r := |\mathbf{x} - \mathbf{y}|, \text{ Sommerfeld radiation condition for uniqueness}$$

$$\text{Fundamental solution } \Phi(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\omega |\mathbf{x} - \mathbf{y}|)$$

$$\text{Asymptotics: } \lim_{r \rightarrow 0} \Phi(r) = \frac{1}{2\pi} \log \frac{1}{r} + \mathcal{O}(1)$$

$$\lim_{r \rightarrow \infty} \Phi(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \nu\pi/2 - \pi/4)} + \mathcal{O}(r^{-1})$$

Same singularity as Laplace \rightarrow same JRs!



Layer potentials

Helmholtz — interior resonances and how to avoid them

$u = \mathcal{D}\sigma$ has interior res prob (Leslie will have covered for the disk case)

CFIE: $u = (\mathcal{D} + i\eta\mathcal{S})\sigma$ no more unknowns, new kernel

can prove equivalence (no spurious resonances)

Helmholtz — Dirichlet demo

Dirichlet demo, plots only: Solve BVP for u via PTR + Nyström, with new diag limit for $k(t, t)$, show $1/N^3$ convergence if use naive PTR with correct diag limit (see M126 HW?)

④ debug BVP with known data from a radiative soln [sources inside \$\Omega\$](#)
(don't demo CFIE since requires S w/ log-singularity).

Helmholtz transmission BVP

refractive indices in Ω vs exterior

Matching? (more effort, needs $\mathcal{S}\sigma + \mathcal{D}\tau \dots$)

difference kernels at most log-singular.

Helmholtz

Getting spectral-acc Nyström for log-singular kernels: beyond today.

eg kernel-split or product quadratures (Kress, Helsing, . . .)

close-eval: kernel-split, QBX, etc.

see libraries: chunkie, BIE2D, etc

More debug ideas

TO DISCARD

Other tests:

- ⑤ Test SLP & DLP evaluators via GRF for any harmonic u in Ω

Summary

Covered BIE basics for smooth curves with global quadrature:

- Well-posed Laplace & Helmholtz BVPs exterior need condition as $\|x\| \rightarrow \infty$
- Choosing representation to get 2nd kind BIE if poss., equivalent to BVP if poss.
Can switch interior/exterior, Laplace/Helmholtz/etc, via simple code changes
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Useful 2D tools we did not yet cover:

in libraries, eg chunkie, BIE2D

- panel (composite) quadratures essential for adaptivity
- high-order quadratures for log-singular kernel SLP, Helmholtz, etc
- special quadratures for evaluation close to the curve
some need interpolation of $\sigma(t)$ off the nodes t_j , some not
- corners, open arcs, slits, multi-material junctions

Resources

Many numerical analysis (mathematics heavy). Somewhat accessible:

- *Linear Integral Equations*, R. Kress, (1999, Springer). Ch. 6 & 12.
- *The Numerical Solution of Integral Equations of the Second Kind*, K. E. Atkinson, (1997, CUP).

Fewer on the practical/tutorial side, few with modern devels:

- “High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane”, S Hao, AH Barnett, PG Martinsson, P Young. *Adv. Comput. Math.* **40**, 245–272 (2014).

focuses on quadrature for logarithmic singularities, eg SLP, Helmholtz

- <https://users.flatironinstitute.org/~ahb/BIE/>
- <https://github.com/ahbarnett/BIEbook>

in progress...