Trignometric interpolation with ~~(2n+1) poi~~

Given $f(x) : [-\pi, \pi) \to \mathbb{C}$

find $p(x) \in \text{span}\{e^{imt}\}_{m=-n}^{n}$

such that $f(t_j) = p(t_j)$ at $t_j = \frac{2\pi j}{2n+1}$

$j = -n, \ldots, n$

i.e. $f(t_j) = \sum_{m=-n}^{n} c_m e^{imt_j}$

Find c_m, $m = -n, \ldots, n$ s.t. for $j = -n, \ldots, n$

$$= \sum_{m=-n}^{n} c_m e^{\frac{+2imπ j}{2n+1}}$$

↑ discrete dft matrix

$$\Rightarrow \quad c_m = \frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) e^{-\frac{2\pi i m j}{(2n+1)}}$$

Question is how good is

$f(t) - \sum_{m=-n}^{n} c_m e^{imt}$ where c_m defined above

Looks like ~~the~~ a truncated fourier series of f, is there same connection to its fourier coeffs

$$f(t) = \sum_{m=-\infty}^{\infty} \hat{f}_m e^{imt}$$

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

Claim: $c_m = \hat{f}_m + \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \hat{f}_{m+l(2n+1)}$

Lemma: If $f(x) = e^{ilx}$

then $\frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) e^{\frac{-2\pi i m j}{2n+1}} = 1$ if $l \bmod (2n+1) = m$

$= 0$ otherwise

i.e. ~~periodic~~ trapezoidal

$$\frac{1}{2n+1} \sum e^{\frac{i 2\pi l j}{2n+1}} e^{-\frac{2\pi i m j}{2n+1}}$$

$$\sum_{j=-n}^{n} \left(e^{\frac{i 2\pi (l-m)}{2n+1}}\right)^{j}$$

$(l-m) \bmod (2n+1) = 0$

$$= \frac{2n+1}{2n+1} = 1$$

if not

$$\sum_{j=-n}^{n} a^{j}$$

$a = e^{\frac{i 2\pi (l-m)}{2n+1}}$

$$= \frac{a^{-n}(1 - a^{2n+1})}{1 - a}$$

$\Rightarrow a^{2n+1} = 1$

& $a \neq 1$

$$= 0$$

$$c_m = \frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) e^{-2\pi i m j/(2n+1)}$$

$f(t) = \sum_{l=-\infty}^{\infty} \hat{f}_m e^{ilt}$

& switch order of summation

$$\Rightarrow c_m = \sum_{l=-\infty}^{\infty} \hat{f}_{m + l(2n+1)}$$

$$c_m = \hat{f}_m + \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \hat{f}_{m+l(2n+1)}$$

Error in Trig. interpolant

$$|f(t) - p(t)| = \left| \sum_{m=-\infty}^{\infty} \hat{f}_m e^{imt} - \sum_{m=-n}^{n} \sum_{l=-\infty}^{\infty} \hat{f}_{m+l(2n+1)} e^{imt} \right|$$

$$= \left| \sum_{|m|>n} \hat{f}_m e^{imt} - \sum_{m=-n}^{n} \sum_{\substack{l=-\infty \\ l \neq 0}} \hat{f}_{m+l(2n+1)} e^{imt} \right|$$

$$\leq 2 \sum_{|m|>n} |\hat{f}_m|$$

Not the most optimal bound but works for now

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{imt} \, dx$$

if $\cancel{f \in C^{k+1}}$, $|f^{(k)}| \leq M$

then $\hat{f}_m = \frac{1}{2\pi} \frac{1}{-im} \int_{-\pi}^{\pi} f'(t) e^{imt}$

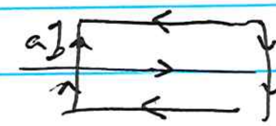
$$= \frac{1}{2\pi} \frac{1}{(-im)^k} \int_{-\pi}^{\pi} f^{(k)}(t) e^{imt} \, dt$$

$$\hat{f}_m \lesssim \frac{1}{|m|^k}$$

$$\Rightarrow |f(t) - p(t)| \lesssim 2 \frac{1}{n^{k-1}}$$

If all derivatives of f are bounded then

$$\cancel{\hat{f}_m} \quad |f(t) - p(t)| \lesssim \frac{1}{n^k} \qquad \forall k$$

Finally if f is analytic in 

then

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imt} f(t)\,dt \qquad m<0$$

$$\hat{f}_m = -\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im(t+ia)} \hat{f}(t)\,dt$$

$$\hat{f}_m \lesssim e^{-|m|a}$$

Use other contour for $m>0$

$$|f(x) - p(x)| \le |f(x) - \underset{p(x)}{p^*(x)}| + |p^*(x) - p(x)|$$

$$\lambda(x) = \max_{f,\ \max|f|=1} |\cancel{f(x) - p(x)}|$$

$p^*(x) - p(x)$ is the interpolant of $p^*(x) - f(x)$

If $p(x)$ is interpolant of $f(x)$ then for any $f(x)$ $|p(x)| \le \lambda(x)|f(x)|$

$$|f(x) - p(x)| \le |f - p^*(x)| + \lambda(x) \cdot |f - p^*(x)|$$

$$\max_{x \in [-1,1]} \lambda(x) = \Lambda$$

Lebesgue constant

New goal $f(x): [-1,1] \to \mathbb{R}$, $\mathcal{P}_{2n}$

Find $p(x) \in \text{span}\{x^n\}_{n=0}^{2n}$? such that

$f(x_j) = p(x_j)$ $\qquad \cancel{x_j = -1 + \frac{2j}{}}$

$$x_j = \frac{2(j-n)}{2n+1} \qquad j = 0, 1, \dots, 2n$$

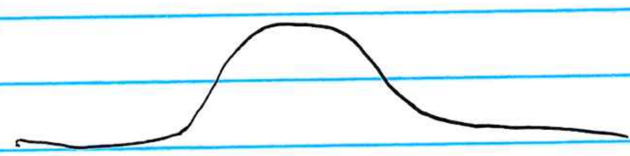
$$p(x) = \sum_{j=0}^{2n} f(x_j) \cancel{p(x)}\, l_j(x)$$

$l_j(x)$ are polynomials ~~st~~ with the property

$$l_j(x_m) = \begin{cases} 0 & \text{if } m \neq j \\ 1 & \text{if } m = j \end{cases}$$

$$l_j(x) = \frac{\prod_{\substack{m=0 \\ m \neq j}}^{2n} (x - x_m)}{\prod_{\substack{m=0 \\ m \neq j}}^{2n} (x_j - x_m)}$$

~~E~~ Behavior for $\frac{1}{1+25x^2}$



Runge phenomenon

$$\lambda(x) = \sum_{j=0}^{2n} | l_j(x) |$$

$$l_j(x) = \frac{1}{x - x_j} \frac{\prod_{m=0}^{2n} (x - x_m)}{\prod_{\substack{m=0 \\ m \neq j}}^{2n} (x_j - x_m)}$$

$$x_j = \frac{2(j-n)}{2n+1}$$

$$x_m = \frac{2(m-n)}{2n+1}$$

$$l_j \cancel{(x) =} \quad \prod_{\substack{m=0 \\ m \neq j}}^{2n} \frac{(j-m)}{(2n+1)^{2n}} \qquad j-m$$

$$= \frac{j!\,(2n-j)!}{(2n)^{2n}}$$

$$\sum |l_j(x)|$$

$$\prod_{\substack{m=0 \\ m \neq j}}^{2n} x - x_m \quad \text{for } x \sim 0$$

$$\sim \frac{n!\,n!}{(2n)^{2n}}$$

for $x \sim 1$ or -1

$$\prod_{\substack{m=0 \\ m \neq j}}^{2n} \sim \frac{2n!}{(2n)^{2n}}$$

~~for $x \sim \pm 1$~~

$$|l_j(x)| \sim \frac{2n!}{j!\,(2n-j)!}$$

$$\sim \frac{n!\,n!}{j!\,(2n-j)!} \quad \text{for } x \sim 0$$

$$n! \sim n^{n+1/2} e^{-n}$$

worst case error from $\ln(x)$

$\frac{2n!}{n!\,n!}$ for endpoints $\sim 2^n$

$\frac{n!\,n!}{n!\,n!}$ for close to origin

$$\frac{(2n)^{2n+1/2} e^{-2n}}{n^{2n+1} e^{-2n}}$$

$$\sim \frac{2^{2n}}{\sqrt{n}}$$

So if $|f - p^*| \lesssim 2^{-2n}$, then we are in some serious trouble

Unfair, can we reuse some of the fourier stuff

$g(t) = f(\cos(t))$ i.e. $f(x)$ with $x = \cos(t)$

$g \in 2\pi$ periodic & as smooth as f

use equispaced points to represent g

$t_j = \frac{-\pi j}{2n+1}$ $\Rightarrow$

$$p(t) = \sum_{m=-n}^{n} c_m e^{imt}$$

$$= c_0 + 2\sum_{m=1}^{n} c_m \cos(mt)$$

~~i.e.~~ $$p(x) = c_0 + 2\sum_{m=1}^{n} c_m \cos(m \arccos(x))$$

Chebyshev interpolation

$$f(x_j) = p(x_j) \quad \text{at} \quad x_j = \cos\left(\frac{\pi j}{2n+1}\right) \quad j = 0, 1 \ldots n$$

$$p(x) \in \operatorname{span}\{\cos(m \arccos(x))\}_{m=0}^{n}$$

$$\equiv \operatorname{span}\{x^m\}_{m=0}^{n}$$

Bonus round

$f_1(x), f_2(x) \ldots f_N(x) : [-1, 1] \to \mathbb{R}$ ← ~~Function~~ Gram-Schmidt

$q_1(x) = f_1(x)$

$q_j(x) = f_j(x) - \sum_{i=1}^{j-1} q_i(x) \int q_i(x) f_j(x) \rightarrow q_1(x), q_2(x), \ldots q_n(x)$

$\operatorname{span}\{f_1, \ldots f_N\} = \operatorname{span}\{q_1 \ldots q_m\} + O(\varepsilon)$

$\int_{-1} q_i q_j = \delta_{ij} \quad \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$

$q_1 \ldots q_N$

$q_1 \quad q_2$

x_1

x_N

Row rank = col rank

$\rightarrow$ Interpolative decomp $\begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \end{bmatrix}$

Construct ~~either~~ adaptive grid to resolve all funcs $q_j(x)$

i.e. interpolant of

$\delta(x_j) = \Sigma$