ENGR 2541 - Mechanics of Materials

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Contents

1 Types of Stress

Mechanics of materials provides a means to analyze the effects of *stresses* and *deformations*. Statics covered finding a balance of forces, including internal forces such as *shear*, *bending*, and *tension/compression*. Finding these internal forces is imperative to be able to determine the integrity of a structure. In addition to the internal forces, the integrity of a structure is also partially determined by the *dimensions* and *materials* of that structure.

1.1 Axial Stress

1.1.1 Overview

Axial Stress **1.1**

Internal stress experienced by a member due to forces applied along its axis. Specifically, stress is a force per area, where the area is the cross section normal to the axis.

In the analysis of a rod, for example, the ability for that rod to withstand the internal forces (its structural integrity) is determined by the cross-sectional area and the material of the rod.

(a) Resultant Force (b) Elementary Forces

Figure 1: Axial force is the resultant of distributed elementary forces

1.1.2 Analyzing Axial Stress

In Figure 1, the *stress* being experiences by the member is the **force per unit area**, denoted by the Greek letter sigma (σ).

Axial Stress

Stress can be calculated by dividing the total axial force by the cross-sectional area:

$$\sigma = \frac{P}{A}$$

1.1

This formula gives the *average* axial stress over the cross section of a member. Although in reality the distribution of the stress throughout the member varies, stress can be assumed to be *uniform* throughout the cross section.

(a) Tensile Stress (b) Compressive Stress

Figure 2: Tensile vs. Compressive Force

By convention, a positive force indicates **tensile stress** while a negative force indicates **compressive stress**.

The cross section, as seen in Figure 3, is perpendicular to the axial forces. The corresponding stress in the member in described as axial/normal stress.

Thus, the formula of $\sigma = \frac{P}{A}$ gives the normal stress of an member under axial loading.

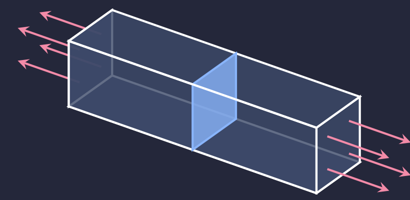


Figure 3: Normal Stress

1.1.3 Stress Points

The formula of $\sigma = \frac{P}{A}$ is only useful for *averages* or *ranges*. This can calculate the average value of the stress over the entire cross section. However, what about calculating at specific points?

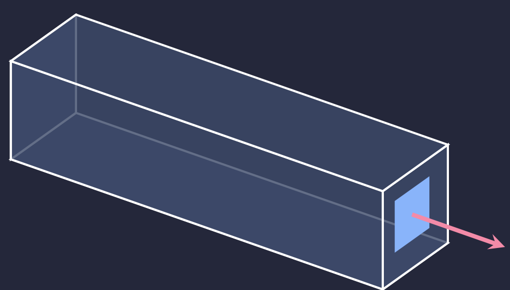


Figure 4: Stress Points

In Figure 4, to find the stress of the highlighted area, the working equation can still be applied, just now with a smaller area. To find the stress at a single point, rather than just a smaller area, stress must be calculated as the area approaches zero.

Stress at a Single Point — **1.2**

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

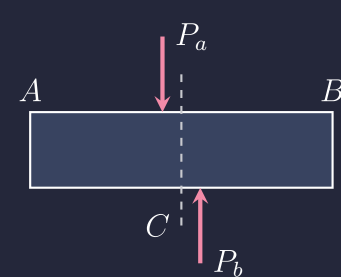
1.2 Shearing Stress

1.2.1 Overview

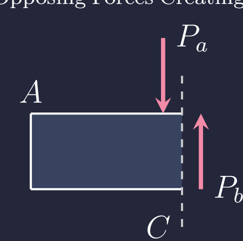
In Section 1.1, the forces creating stress acted perpendicular to the face being analyzed (see Figure 3). This created axial stress. In the case of transverse forces acting parallel to the face, shearing stress is created.

Shear Stress — **1.2**

Internal stresses caused by transverse forces acting parallel to a plane.



(a) Opposing Forces Creating Shear



(b) Opposing Forces Creating Shear

Figure 5: Shear Stress

In Figure 5, a member has two opposing forces, P_a and P_b, acting upon it. These two forces create internal shear in the section between them as seen in Figure 5b.

By analyzing the section at C in the member, it can be seen that the internal force created by the two forces in the section is equal to P_b. This resultant force is called a **shear force** (P). Dividing P by the area (A) results in the **average shearing stress** in the section.

Average Shearing Stress — **1.3**

$$\tau_{avg} = \frac{P}{A}$$

The average shearing stress cannot be assumed to be uniform throughout the section. Though not the full story, the shearing stress is generally distributed throughout the section such that it is zero at the surface and greater than the average near the center.

1.2.2 Single and Double Shear

The member in Figure 5 is said to be in **single shear** because the shear stress is entirely on a single plane. If forces were to be applied to a member as in Figure 6, there are now two sections and thus the member is said to be in **double shear**.

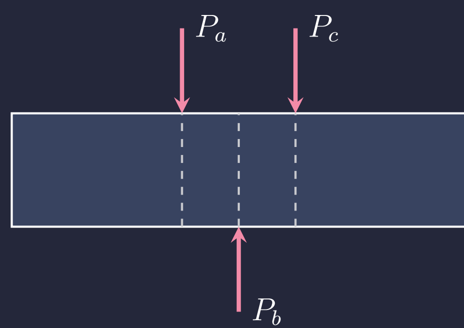


Figure 6: Double Shear

In a case of double shear, the shearing force is divided in half between the planes one between P_a and P_b, and one between P_b and P_c. Thus, to find the shear stress on either of these planes, only half the force will be used.

Double Shear **1.4**

$$\tau = \frac{P}{2A}$$

Since there are two faces of shear stress, they will add together to $\frac{P}{A}$.

1.2.3 Bearing Stress

Often, rather than singular forces creating axial stress, the surfaces of multiple objects will apply distributed forces to each other, thus creating **bearing stress**.

Bearing Stress **1.3**

Similar to axial stress, bearing stress is the result of two members interfacing with each other and applying force along an axis.

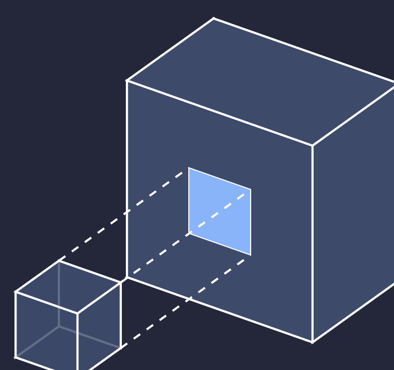


Figure 7: Bearing Shear

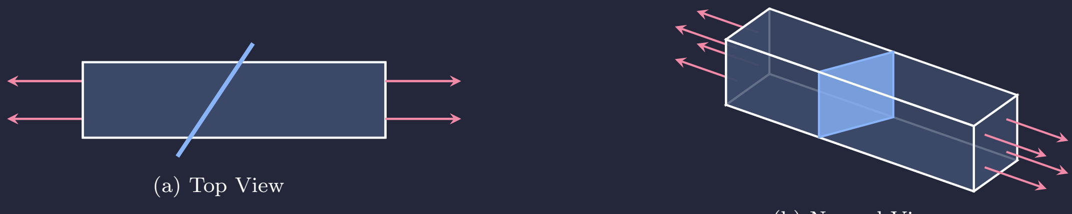
In Figure 7, two objects are applying reciprocal forces to each other. Throughout the area the two objects interface over, the distribution of the force is not even. So, in practice, the average nominal value of the stress (σ_b) called the **bearing stress** is used.

This value is obtained by dividing the load P by the area of the surface of contact.

Bearing Stress	
$$\sigma_b = \frac{P}{A} = \frac{P}{td}$$	**1.5**

1.3 Stress on an Oblique Plane

Thus far, only stresses perpendicular and parallel to planes have been considered. However, what if an object such as the one in Figure 8 were to be considered where the plane is at an angle?



(a) Top View

(b) Normal View

Figure 8: Oblique Plane

In this situation, the "axial" force applied to the member is no longer just an axial force. There is a component of the force perpendicular to the plane just as there is one parallel. This can be seen in Figure 9, where the **axial stress** to the plane is purple and the **shear stress** to the plane is yellow.

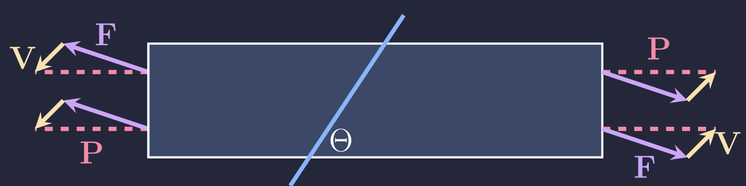


Figure 9: Components of the Applied Force

This begins to illustrate the idea that shear and axial stresses aren't very different, and largely differ based on the perspective they are analyzed from.

To calculate the stresses created by the forces exerted on this member, it's just a matter of calculating the vector components of the force and using them where appropriate.

2 Design Considerations

Determining the stresses in a body, in and of itself, serves no purpose. However, these methods of analysis can be used to inform design decisions and create sturdy structures and machines. An important family of information to an engineer is how a certain material will behave under various kinds of loads.

2.1 Determining the Strength of Materials

Properties, such as the **ultimate load**, of materials can be empirically tested by pushing that material to its limits and taking measurements. To find the ultimate load, a member is put under tensile stress until it cannot go any further.

Ultimate Load **2.1**

The largest axial force that may be applied to a material before it breaks, deforms, or begins to carry less load. Ultimate load is denoted by P_U.

Since axial stress is uniformly distributed, the ultimate load divided by the cross-sectional area of the member gives the **ultimate normal stress** of the material.

Ultimate Normal Stress

$$\sigma_U = \frac{P_U}{A} \tag{2.1}$$

Also called the *Ultimate Strength in Tension*, this measures the maximum axial stress a material can undergo before failing.

Other tests can be performed to determine the **ultimate shearing stress** of a material. The same idea as prior is applied, however this time the load is applied is shear rather than axially.

Ultimate Shearing Stress

$$\tau_U = \frac{P_U}{A} \tag{2.2}$$

Also called the *Ultimate Strength in Shear*, this measures the maximum shear stress a material can undergo before failing.

2.2 Factor of Safety

Though a material can theoretically sustain stresses up to its ultimate load, in practice this is not the case. Since loads are unpredictable and vary, the maximum load a component is designed to sustain should only be a percentage of its ultimate load. This concept is reflected in the idea of a **factor of safety**.

Factor of Safety

$$\text{Factor of Safety} = \text{F.S.} = \frac{\text{ultimate load}}{\text{allowable load}} = \frac{\text{ultimate stress}}{\text{allowable stress}} \qquad \textbf{2.3}$$

The factor of safety is calculated by finding the ratio between the ultimate load/stress and the allowable load/stress.

The relationship between the F.S. as calculated by loads vs. stresses only holds up when a linear relationship exists between the load and the stress. However, most materials experience non-linear a relationship between these two values as the material approaches its ultimate load.

The F.S. to use in a design is a subjective choice, but is based on many factors such as:

1. Variations that may occur in the properties of the member
2. The number of loadings expected during the life of the structure or machine
3. The type of loadings planned for in the design or that may occur in the future
4. The type of failure (brittle and ductile materials will fail differently)
5. Uncertainty due to methods of analysis
6. Deterioration that may occur in the future because of poor maintenance or unpreventable natural causes
7. The importance of a given member in the integrity of the entire structure

3 Deformation

3.1 Normal Strain

When studying the mechanics of materials, analyzing them entirely in static equilibrium would yield little to no information about how a material changes when ample load is applied. Consider the beam in Figure 10. After having endured an axial load, the beam deforms be elongating.

In this case, the elongation undergone by the beam is considered the **strain** endured by the beam. In other situations, strain can look different, such as the bending or compression of a material.

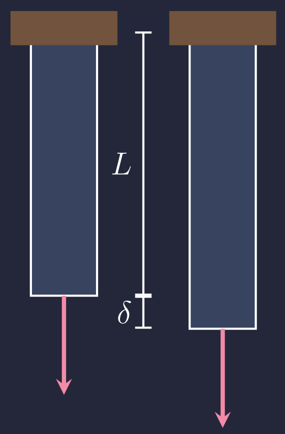


Figure 10: Deformation Due to Axial Stress

> **Strain** **3.1**
>
> Deformation experienced by an object as the result of stresses exceeding the materials ability to maintain shape.

In the case of Figure 10, the strain is specifically **normal strain**.

> **Normal Strain** **3.2**
>
> Strain specifically relating to axial/normal stress. It is specifically defined as the *deformation per unit length*. Normal strain is denoted by the Greek letter epsilon (ϵ).

Since normal strain is the deformation per unit length, to find the value of ϵ, one must divide the deformation by the *original* length of the member.

Normal Strain **3.1**

$$\epsilon = \frac{\delta}{L} \tag{3.1}$$

Though indirectly caused by an axial load, the factors in determining the amount of normal strain experienced by a material are: the materials inherent properties and the axial stress on the member.

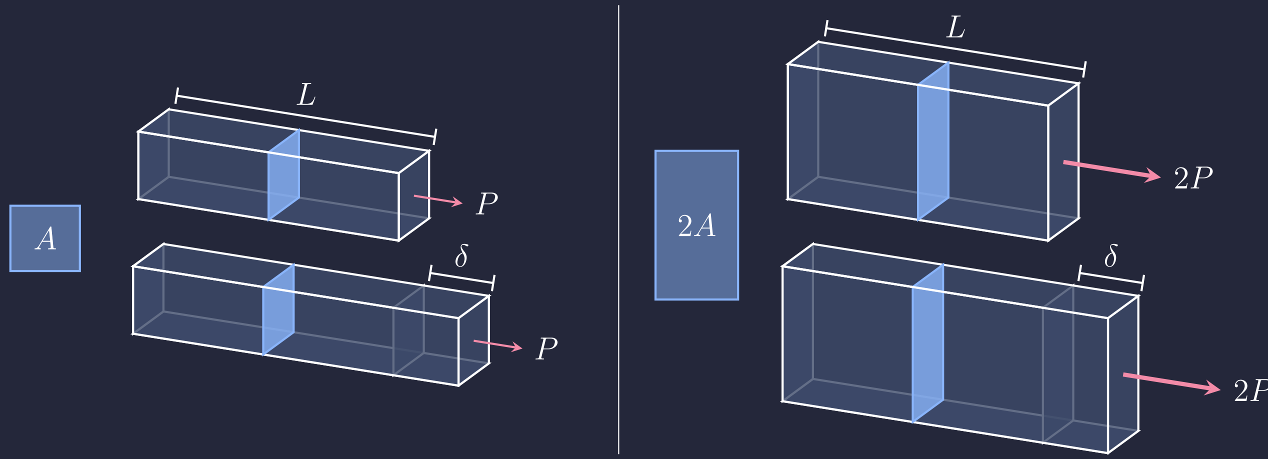


Figure 11: Strain For Two Rods of Different Cross Sections

Assuming the same material, two members of different dimensions, but undergoing the same stress, will strain the same amount. In Figure 11, this exact situation is visualized. Since stress is just a ratio of force to area, a member double the area experiencing an axial load double of another member will strain the same.

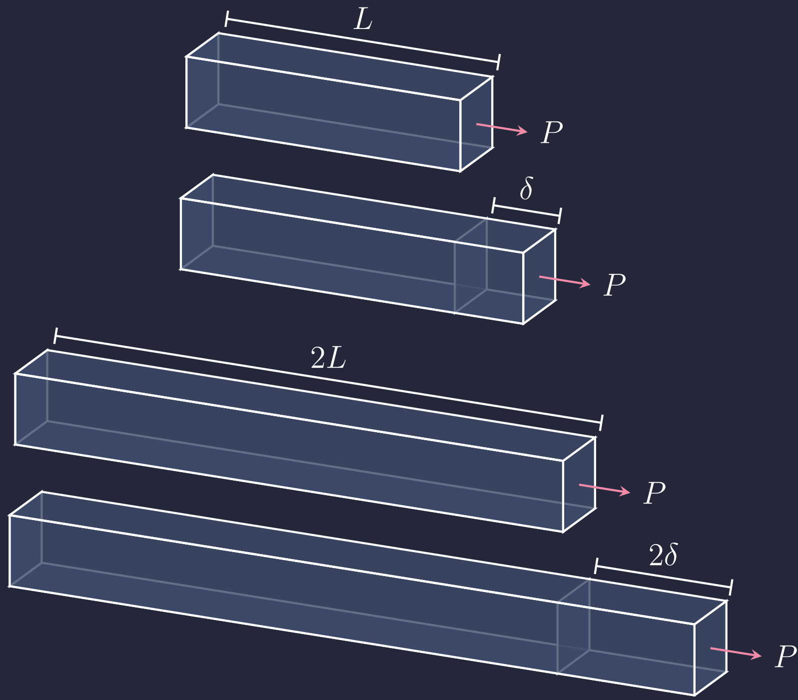


Figure 12: Strain For Two Rods of Different Lengths

Similarly, since strain is a ratio between the original length and the final length, a body double the length of another will experience the same strain assuming all else is the same. See Figure 12.

Strain so far has only been considered for uniform bodies: the same material, same cross-section, etc, and only for members loaded axially on their ends. What happens when these conditions aren't met? Conveniently, strain can be added for subsections of a member, and summed together to find the total strain:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

3.2 Stress Strain Diagrams

Having seen that strain is independent of the dimensions of a material, and rather depends on the stress experienced by the material, it follows that plotting strain as a function of stress would result in a diagram generally applicable to a specific material. This curve, called a **stress-strain diagram** characterizes the properties of a material.

Though all materials behave differently when examined through a stress-strain diagram, there are two broad categories of materials: *brittle* and *ductile*.

Ductile materials are able to yield without necessarily failing entirely. Their elongation initially increases linearly with stress until a some value σ_Y where suddenly undergoes a large deformation with a relatively small increase in stress.

Measuring Ductility

$$\text{Percent Elongation} = 100 \cdot \frac{\text{Initial Length} - \text{Length at Failure}}{\text{Initial Length}} = 100 \cdot \frac{L_B - L_0}{L_0} \qquad \textbf{3.2}$$

$$\text{Percent Reduction in Area} = 100 \cdot \frac{\text{Initial Area} - \text{Area at Failure}}{\text{Initial Area}} = 100 \cdot \frac{A_0 - A_B}{A_0}$$

Brittle materials, on the other hand, experience a very small amount of yield, after which they tend to fail suddenly. There is a distinct lack of necking in brittle materials.

3.3 True vs. Engineering Stress and Strain

When an object undergoes elongation, the cross sectional area of that object will change. Thus introduces a question: should the original or current cross-section be used as the area when calculating stress? Rather than having an answer, there are just two types of stress which each reflect one of the options: **engineering stress** and **true stress**.

True vs. Engineering Stress **3.3**

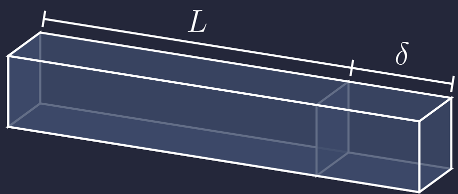
Engineering stress is stress experienced by an object calculated using the original cross-section of the body.

$$\sigma_e = \frac{P}{A_{initial}}$$

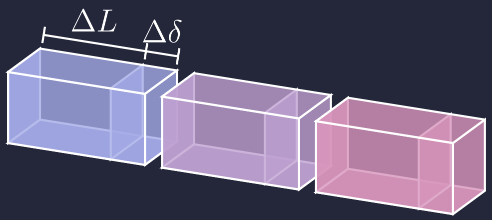
True stress is stress as calculated by using the current cross-section of the body.

$$\sigma_t = \frac{P}{A_{current}}$$

A similar situation occurs when measuring the *strain* of a body. Using the formula for strain as introduced in 3.1 produces the **engineering strain** experienced by a body. However, if rather than measuring a single value for the length and deformation of the body, one were to subdivide the body into several subsections and measure the deformation for each subsection, something closer to the **true strain** would be produced. This is seen in Figure 13b.



(a) Engineering Strain



(b) True(ish) Strain

Figure 13: Engineering vs. True Strain

Taking this subdividing to its limit, the **True Strain** will be a perfectly continuous measurement of the change in deformation over some change in length:

$$\epsilon_{true} = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}$$

3.4 Hooke's Law and the Modulus of Elasticity

Generally, structures are designed to keep any deformations within the linear portion of the stress-strain diagram. Given that some body is kept within that range, the stress (σ) is directly proportional to the strain (ϵ):

$$\sigma = E\epsilon$$

Where E is the **modulus of elasticity** of the material.

Modulus of Elasticity	**3.4**
An inherent value to a material that reflects the proportionality between the stress and strain experienced by a body.	

This linear relationship, governed by a constant coefficient (E) is known as **Hooke's Law**. This law only applies until the material reaches its **proportional limit**.

Proportional Limit	**3.5**
The upper bound of the range over which Hooke's Law applies to some material.	

3.5 Repeated Loadings and Fatigue

So far, only single instances of loading have been considered for the materials. What happens after thousands, or millions, of loads have been repeatedly applied to a material? In these cases, the material is said to have undergone **fatigue**.

Fatigue	**3.6**
The result of many repeated loadings, fatigue is a general term for the effect that these repeated loadings have on a material.	

Failures due to fatigue are generally more brittle than they are ductile. Since fatigue not only increases the chances of a material failing, but also makes it such that there is less of a yield period, it is imperative to consider fatigue when designing structures and machines that are expected to support time-varying loads.

To measure the fatigue of a material, a stress to loading curve can be made. This measures the number of repeated loads required to cause failure at different load-sizes for a given material. A generic one can be seen in Figure 14, illustrating the typical shape of such a curve.

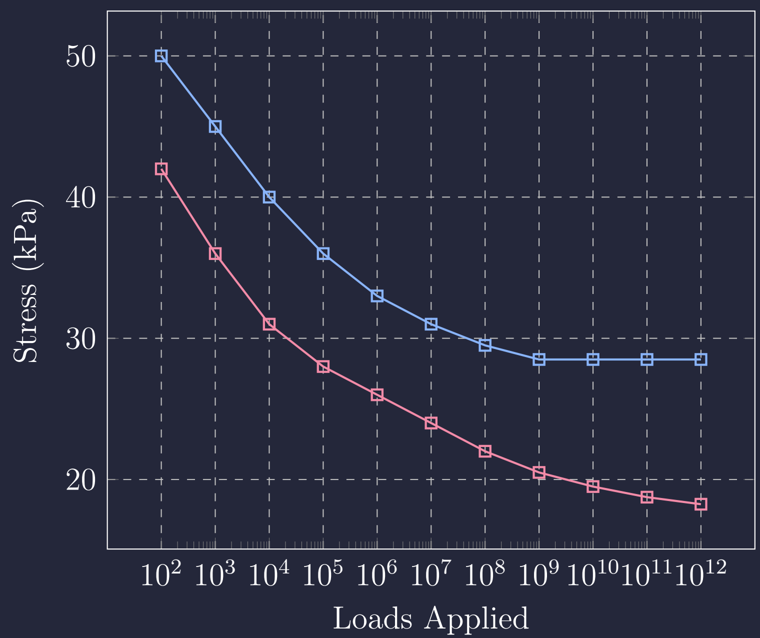


Figure 14: Stress to Loading Curve

The flat part at the end of the blue material in Figure 14 marks the **endurance limit** of the material. This is where, regardless of the number of time a load is applied, the material will never fail. Any force value below the endurance limit on the y-axis (stress) will never cause a material to fail.

However, for non-ferrous materials, there is no endurance limit. Rather, the curve just continues to go down. This is seen in the red curve in Figure 14. This behavior is caused by microscopic failure after each consecutive loading, eventually compounding into total failure.

3.6 Static Indeterminacy

Thus far, only objects in static equilibrium have been considered. However, situations in which statics alone can't determine internal forces are very common. In such cases, the geometry must be used alongside equilibrium equations. These cases are referred to as **statically indeterminate**.

> **Static Indeterminate** **3.7**
>
> Statically indeterminate structures are those which have reactions and internal forces that cannot be found through statics alone, and thus must derive results based on the material composition of the structure.

Consider the structure on Figure 15. In this body, there is a force being axially applied, but not at the ends of the structure. It is known that the structure is in static equilibrium:

$$R_A + R_B = P$$

However, there are two unknowns in this equation: R_A and R_B. In order to solve for these unknowns, it is necessary, then, to use a compatibility equation. However, before understanding that, it is important to understand the **Principal of Superposition**.

Principal of Superposition **3.8**

In materials governed by Hooke's Law, the deformation of a structure is equal to the deformation of any number of arbitrary sections of that structure:

$$\delta_{total} = \sum_{n}^{N} \delta_n$$

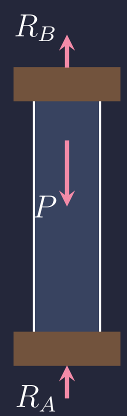


Figure 15: Basic Example

By the principal of superposition, one is able to decompose a structure into component parts, solve for the deformation in each part, then sum the deformations to find the total deformation of the structure.

Applying the Principal of Superposition, it is possible to relate the deformation of the structure based on the constraints. This process results in a compatibility equation.

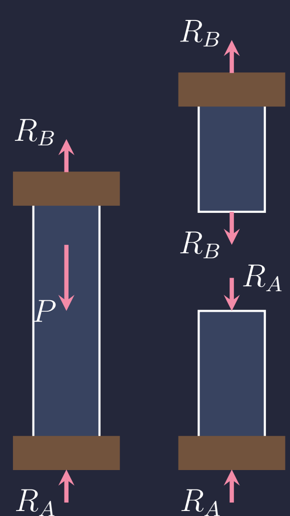


Figure 16: Sections of Figure 15

Decomposing the structure (as seen in Figure 16), the deformation of each individual component can be analyzed independently, and then be compared through a **compatibility equation** to determine the true values of each portion:

$$\delta_{top} + \delta_{bottom} = 0$$

This can then be expanded out to:

$$\frac{P_t L_t}{A_t E_t} = \frac{P_b L_b}{A_b E_b}$$

Assuming the same material throughout as well as the same cross-section, this can be taken further by canceling out the area (A) and the modulus of elasticity (E) of each side, resulting in:

$$P_t L_t = P_b L_b$$

$$\frac{L_t}{L_b} \cdot P_t = P_b$$

In situations like these, it can be clearly seen that the load on each component is directly proportional to the length of that component.

3.7 Poisson's Ratio

In all engineering materials, the relationship between the elongation of body due to axial stress is directly proportional to contraction in any transverse direction. This relationship is codified with a constant that relates the contraction with the elongation, this constant being a given material's **Poisson's Ratio**.

Poisson's Ratio

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = -\frac{\epsilon_y}{\epsilon_x} \qquad \textbf{3.3}$$

3.8 Stress Concentrations

Consider Figure 17. Due to the discontinuity in the center, the stress is distributed differently than normal. Considering the location with the smallest cross section (the exact center), the stress is distributed closer to the center of the discontinuity.

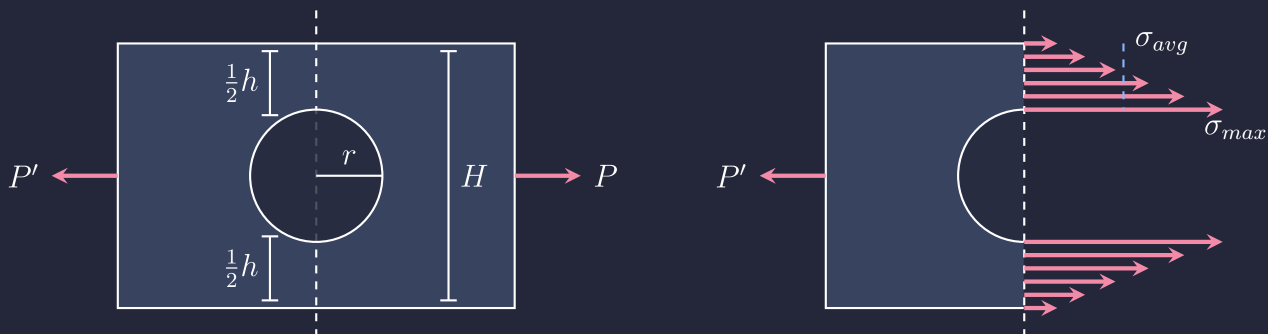


Figure 17: Stress Distribution Near Hole Discontinuity

Through experimentation, it's been determined that analyzing these structures can be done solely based on the geometry of the discontinuities. The main concern of this analysis is to find whether the allowable stress will be exceeded. Since the maximum stress (σ_{max}) will be at the narrowest cross-section, only this cross section needs to be analyzed.

To complete this analysis, an engineer must find the average stress in the narrowest cross-section:

$$\sigma_{avg} = \frac{P}{A}$$

and the ratio between the average and maximum stress at that cross-section:

$$K = \frac{\sigma_{max}}{\sigma_{avg}}$$

This ratio is referred to as the **stress concentration factor** (k). To use the (experimentally derived) stress concentration, the geometry of the discontinuity must also be known.

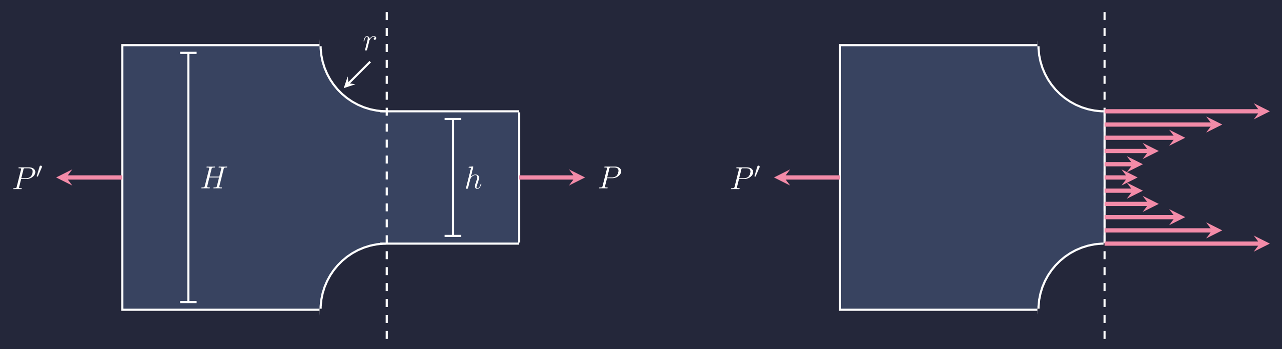


Figure 18: Stress Distribution Near Fillet Discontinuity

3.8.1 Max Stress for a Hole Discontinuity

For a hole discontinuity, the important geometry is the proportion between the smallest cross-section to the largest:

$$\frac{H-h}{H} \quad \text{or} \quad \frac{2r}{H}$$

Using this calculated value in conjunction with k, the ratio between the two values plotted on a chart results in the ratio between $\sigma_m ax$ and $\sigma_a vg$ for the specific geometry.

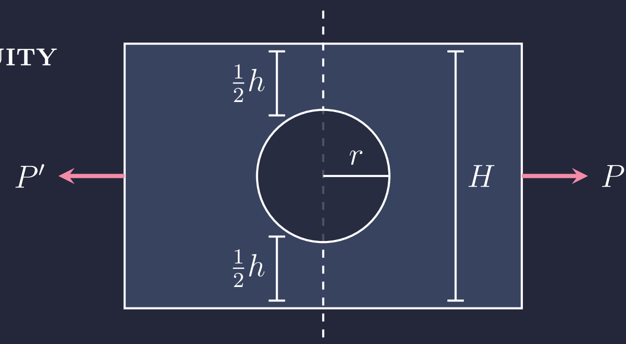


Figure 19: Hole Discontinuity

3.8.2 Max Stress for a Fillet

Similarly to Subsubsection 3.8.1, the max stress for a fillet can be found geometrically. However, the geometric values used in this case are h and r.

$$\frac{r}{h}$$

However, rather than just using a single ratio, the ratio between the two heights of the filleted member is also used.

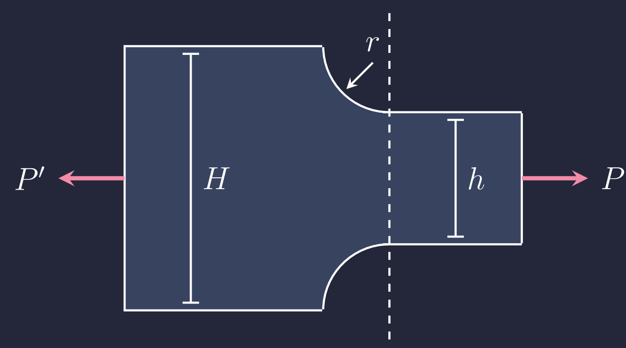


Figure 20: Fillet Discontinuity

$$\frac{H}{h}$$

3.9 Thermal Strain

When changes in shape due to changes in temperature occur, new stresses are created due to these thermal deformations. Generally, these are somewhat small. However, with large bodies or large changes in temperature these thermal strains create large enough stresses to cause a material to rupture.

(a) Initial Temperature T_i

(b) Final Temperature $T_f > T_i$

Figure 21: Expansions Due to Increased Temperature

Thermal Strain

$$\epsilon_{thermal} = \alpha \cdot \Delta T \tag{3.4}$$

α = coefficient of thermal expansion	ΔT = change in temperature

Notice that there is no stress associated with the thermal strain. The strain is directly caused by the change in temperature of the material as well as the material's inherent properties.

Similar to the axial elongation, the change in length of a body can be defined by multiplying the thermal strain (ϵ_T) by the original length of the member (L).

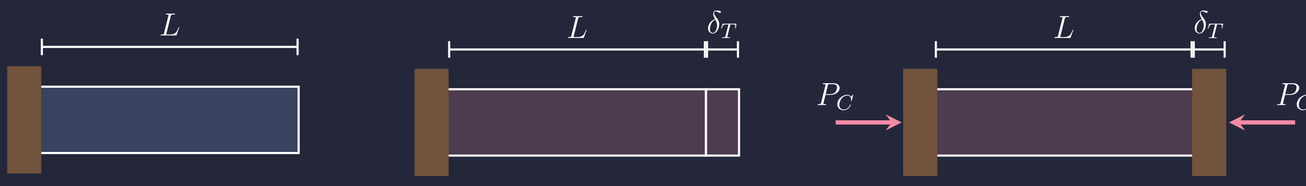
Change in Length

$$\delta_{thermal} = \epsilon_T \cdot L = \alpha \cdot \Delta T \cdot L \tag{3.5}$$

Figure 22: Constrained Body

These two formulas model the thermal strain/thermal expansion undergone by unconstrained bodies. However, how do constrained bodies (such as the one in Figure 22) react to changes in temperature?

In these situations, a **thermal stress** is induced into this bar. This thermal stress can be calculated in a similar way to statically indeterminate structures (Subsection 3.6).



(a) Initial Body

(b) Thermal Expansion

(c) Constraint

Figure 23: Constrained Thermal Expansion

The deformation due to the thermal expansion and the deformation due to the constraints can be summed to the total deformation experienced. In the example of Figure 22, since the total deformation is zero, the compatibility equation is:

$$\begin{aligned}\delta_{thermal} + \delta_{constraint} &= 0 \\ \alpha \cdot \Delta T \cdot L &= -\delta_C \\ \alpha \cdot \Delta T \cdot L &= -\frac{P_C L}{AE}\end{aligned}$$

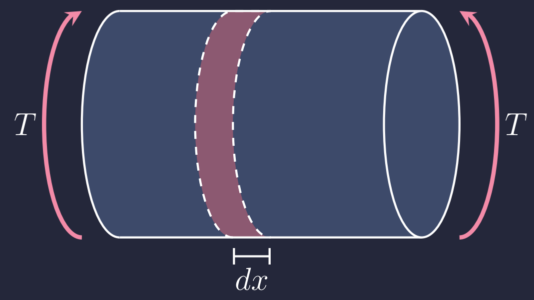
Then using the values known about the member and its composition, the force applied by the constraints can be found, leading directly to then being able to solve for the stress. This compatibility equation can be seen in steps in Figure 23.

4 Torsion

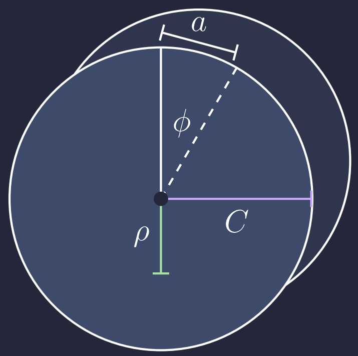
4.1 Introducing New Values

Before covering concepts, there are several new values, each with its own notation. Referring to Figure 24, all these new values are labeled, and are as follows:

Value	Symbol	Axial Counterpart
Torque	T	Force (P)
Distance From Center	ρ	Force (P)
Radius	C	Force (P)
Angle of Twist	ϕ	Force (P)
Length of Rotation	a	Force (P)



(a) Side Profile



(b) Front Profile

Figure 24: Circular Shaft in Torsion

Another important value is the angle formed by the line connecting two points some distance ρ from the center in each face of the shaft. This is the **shear strain** (γ) experienced by the body. In Figure 25, this angle can be seen between the points n, m, and m'.

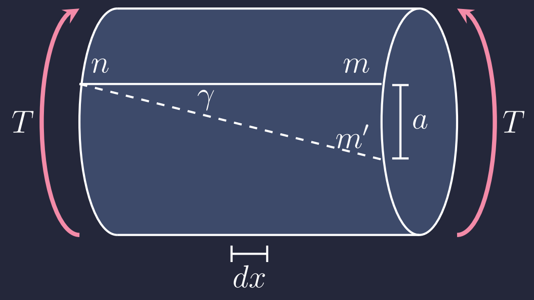


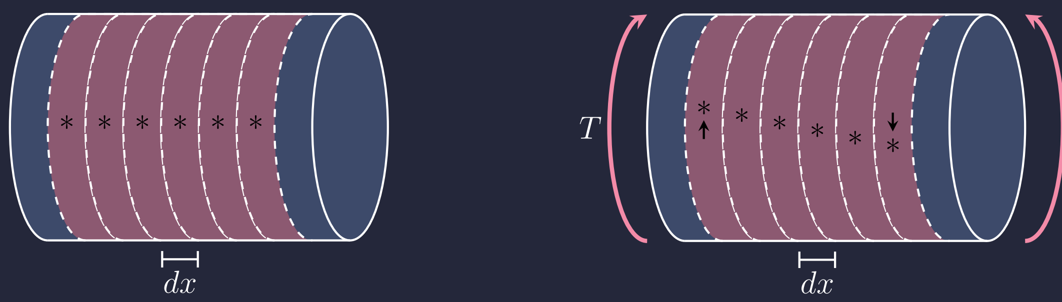
Figure 25: Shear Strain

4.2 Overview of Torsion

4.2.1 Torsion

Torsion	**4.1**
The twisting of an object caused by a moment acting about the object's longitudinal (long) axis.	

The "force" that creates the moment around the axis is referred to as torque (τ). When a shaft, such as the one if Figure 24 is put into torsion, each cross section to rotate relative to the others. However, the face of each cross section *remains plane*, which means that the cross sections remain flat and undistorted.



(a) Before Torsion (b) After Torsion

Figure 26: Shaft in Torsion

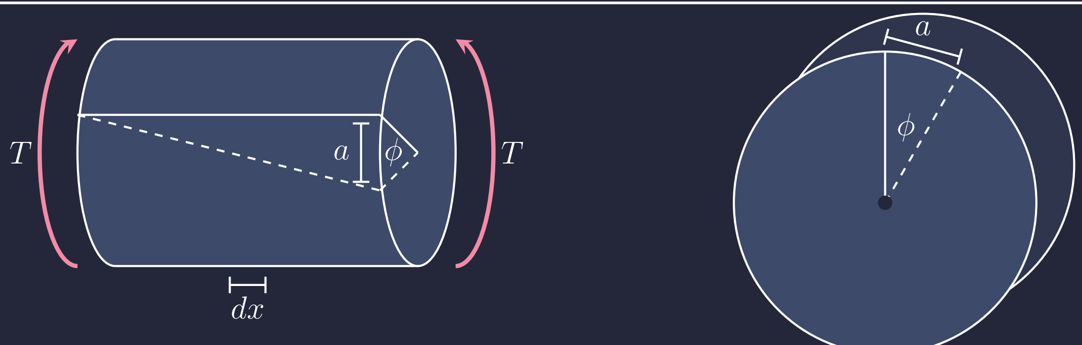
4.2.2 Angle of Twist

In Figure 27, the angle of twist (ϕ) is shown. The angle of twist refers to the angle between an initial point m, the center of the face, and the distorted point m'. The angle of twist can be calculated as:

Angle of Twist

$$\phi = \frac{TL}{GJ} \quad \text{or} \quad \int d\phi = \int \frac{T(x)}{J(x) \cdot G(x)}\, dx \tag{4.1}$$

T: Torque Applied	L: Length
G: Modulus of Rigidity	J: Polar Moment of Inertia



(a) Side Profile (b) Front Profile

Figure 27: Angle of Twist

4.2.3 Polar Moment of Inertia

The polar moment of inertia of an object is a property of an object that described its resistance to torsion. This is a value based on the geometry of the body.

Polar Moment of Inertia

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) \tag{4.2}$$

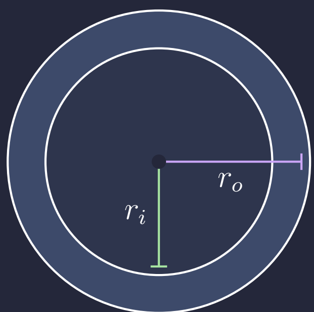


Figure 28: Geometry of a Cylinder

4.2.4 Shear Strain

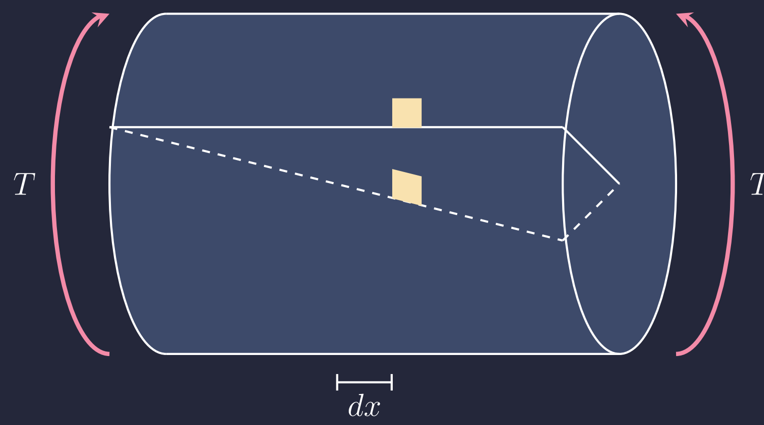
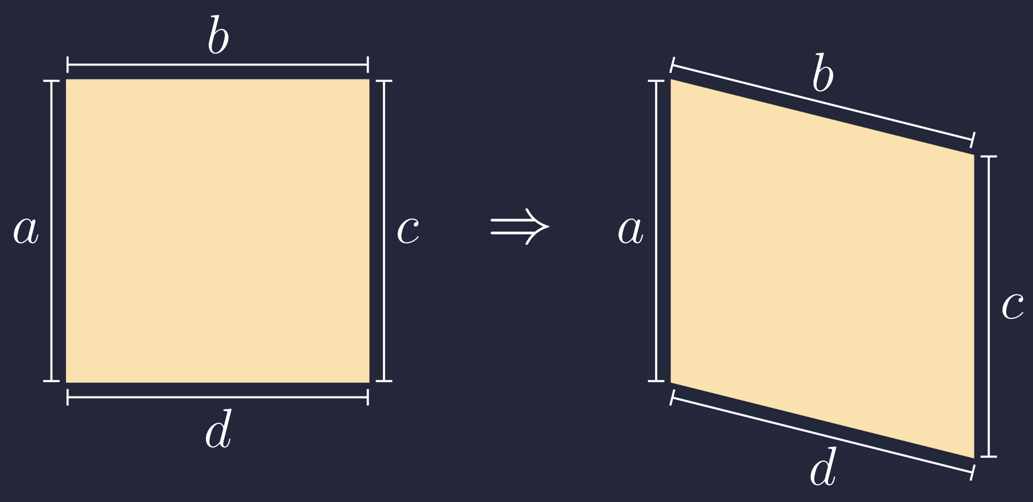


Figure 29: Front Profile

When an object undergoes shear strain one can imagine a grid across the surface of the cylinder. Each square in the grid will undergo a deformation. However, as has been said in Subsubsection 4.2.1, the circular shaft maintains it's cross-section faces' geometry intact. So, each square in the grid will have to maintain the same side lengths:



The strain caused by this can be calculated in terms of the angle of twist (4.2.2) and either the outer radius of the bar (C) or some inner radius (ρ):

Shear Strain

$$\gamma = \frac{C\phi}{L} \quad \text{or} \quad \gamma = \frac{\rho\phi}{L} \tag{4.3}$$

4.2.5 Shear Stress

To calculate the shear strain, consider Figure 30.

Shear Stress

$$
\begin{aligned}
T &= \int_0^r \tau\rho\, dA \\
&= \frac{\tau}{\rho}\int_0^r \rho^2\, dA \\
&= \frac{\tau}{\rho}\left(\frac{\pi}{2}\left(r_0^4 - r_i^4\right)\right) \\
&= \frac{\tau}{\rho} J
\end{aligned}
$$

$$\tau = \frac{T\rho}{J} \quad \text{or} \quad \tau = G\cdot\gamma \tag{4.4}$$

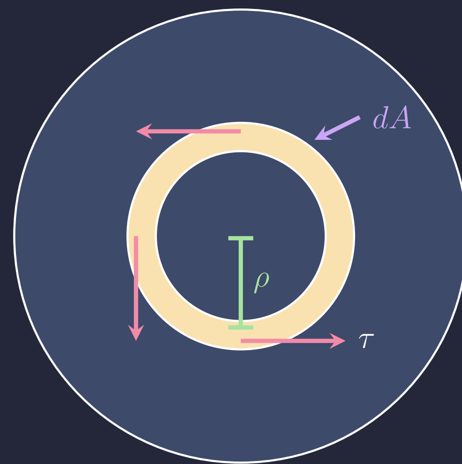
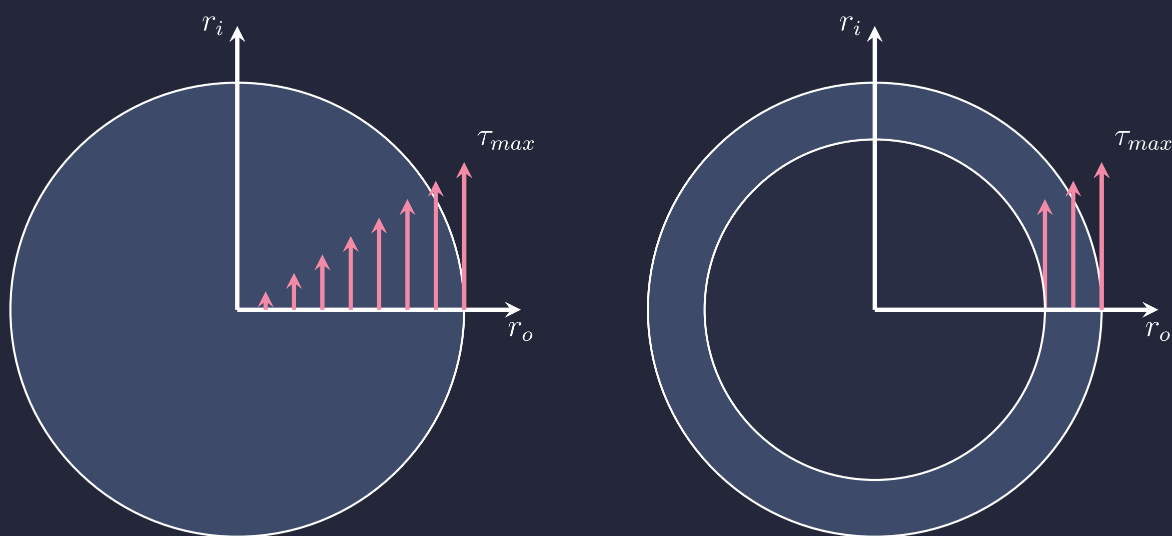


Figure 30: Calculating Shear Stress



(a) Distribution of Shear Stress

(b) On a Hollow Object

The Shear stress is down the radius of the face of the body linearly. In Figure 31b, a hollow object is experiencing shear strain. Since it is hollow, it is more efficient at holding the shear strain.

4.3 Multiple Torsions

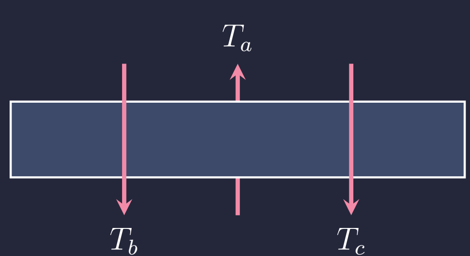


Figure 32: Multiple Torques

Consider the member in Figure 32 experiencing multiple torques simultaneously. Very similarly to previous techniques used in statics to find internal forces. The shear stress and strain can be found in any region bounded by two torques (on the ends, there would be no stress or strain). Supposing that:

$$T_b > T_a > T_c$$

Then it could be found that region with the greatest shear and stress would be between T_b and T_a.

A diagram displaying the internal torque as a function of x would look like that in Figure 33.

Figure 33: Internal Torque Diagram

4.4 Brittle vs. Ductile

(a) Brittle

(b) Ductile

Figure 34: Brittle vs. Ductile

The brittle material in Figure 34a will fail at a 45° angle from the longitudinal axis. This is due to the brittle material tend to fail in tension, thus failing along the place of maximum tension. Conversely, the material in Figure 34b is ductile and will then fail along the plane of maximum shear.

4.5 Torsional Stiffness

Similar to how an axial member will have a stiffness that helps in solving statically indeterminate problems, so too do members have torsional stiffnesses (k).

Torsional Stiffness

$$k = \frac{T}{\phi} \tag{4.5}$$

4.6 Statically Indeterminate Torsion

Just as there were situations in Subsection 3.6 where the internal forces, and in turn the stresses and shears, couldn't be solved through statics alone, there are also situations in which the internal torsion cannot be solved through statics alone. In such cases, it is necessary to analyze the geometry of the problem to solve for internal torsion.

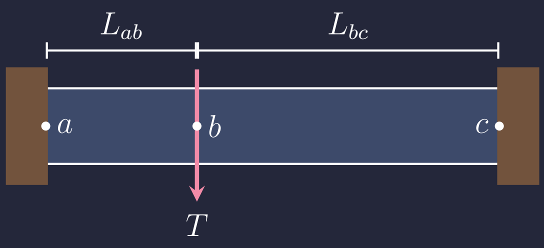


Figure 35: Statically Indeterminate Torsion

Consider Figure 35. There is a beam fixed on each end to two supports. Furthermore, there is a torque applied $\frac{1}{3}$ of the way down the beam such that $L_{ab} = \frac{1}{2}L_{bc}$.

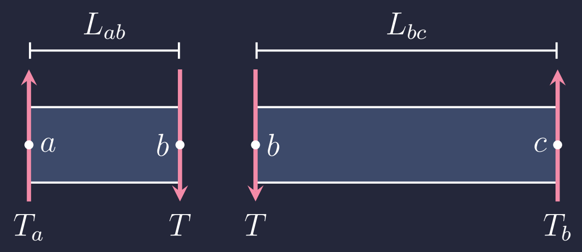


Figure 36: Free Body Diagram

The member in Figure 35 can be translated into a free body diagram as shown in Figure 36 where each component of the beam is considered individually. However, this would result in an equation with two unknowns, thus showing this situation as statically indeterminate:

$$T_a + T_b = T$$

So, how should a statically indeterminate torsion problem be approached? First, it must be noticed that, since both ends are fixed in place, then the total angle of twist (ϕ_t) should be zero.

$$\phi_t = 0$$

An analogous process to Subsection 3.6 can be applied in which the shear of each component can be calculated individually to find the total angle of twist.

$$\phi_t = \sum_{n=1}^{N} \phi_n$$

Thus:

$$\begin{aligned} \phi_t = 0 &= \phi_{ab} + \phi_{bc} \\ &= \frac{T_{ab}L_{ab}}{G_{ab}J_{ab}} + \frac{T_{bc}L_{bc}}{G_{bc}J_{bc}} \\ -\frac{T_{ab}L_{ab}}{G_{ab}J_{ab}} &= \frac{T_{bc}L_{bc}}{G_{bc}J_{bc}} \end{aligned}$$

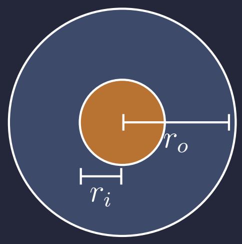
From here, different situations will give different unknowns and thus different values to solve for. In the current example, the only difference between the two sections is their lengths, and so:

$$\begin{aligned} -\frac{T_{ab}L_{ab}}{GJ} &= \frac{T_{bc}L_{bc}}{GJ} \\ -T_{ab}L_{ab} &= T_{bc}L_{bc} \\ -\frac{L_{ab}}{L_{bc}} \cdot T_{ab} &= T_{bc} \end{aligned}$$

It can then be seen that the key to solving these statically indeterminate situations is to finding the ratio between the two torques based on the summation of the angles of rotation.

Consider the shaft in Figure 37. In this situation, the shaft is experiencing two equal and opposite torques (T). Additionally, the shaft is a composite of a two materials with different rigidities (G). How could the torque in each material be found?

(a) Side Profile



(b) Front Profile

Figure 37: Statically Indeterminate Composite Shaft

In this case, the total angle of twist of the shaft *won't* be zero. However, since the shaft is a composite, it is known that the angle of twist in each material must be the same.

$$\phi_i = \phi_o \Rightarrow \frac{T_i L}{G_i J_i} = \frac{T_o L}{G_o J_o} \Rightarrow \frac{T_i}{G_i J_i} = \frac{T_o}{G_o J_o}$$

From there, there exist two unknown values: T_i and T_o. It may look impossible to solve, however, it's important to remember that the *total* torque (T) is known. Thus, there are two equations and two unknowns:

$$\frac{T_i}{G_i J_i} = \frac{T_o}{G_o J_o} \quad \text{and} \quad T_i + T_o = T$$

4.7 Power and Rotation Speed

For shafts that are designed to be rotated, there is obviously going to be a torque of some kind applied to the shaft to achieve that rotation. When designing these parts, it's important that the torque applied doesn't create shear beyond what is deemed acceptable in the specific context.

Power

$$P = T\omega = T(2\pi f) \tag{4.6}$$

To determine the amount of torque generated by the power (P) applied at a certain frequency (f) to a shaft:

Torque Generated by Power and Frequency

$$T = \frac{P}{\omega} = \frac{P}{2\pi \cdot f} \tag{4.7}$$

Once this torque has been calculated, the engineer can use other methods and formulas from Section 4 to calculate the allowable power, frequency, etc., for the specific situation.

4.8 Stress Concentrations in Circular Shafts

Discontinuities in a circular shaft will cause different stress concentrations throughout that shaft. Fillets are often used to reduce drastic concentrations, so only filleted shafts will be considered in this section.

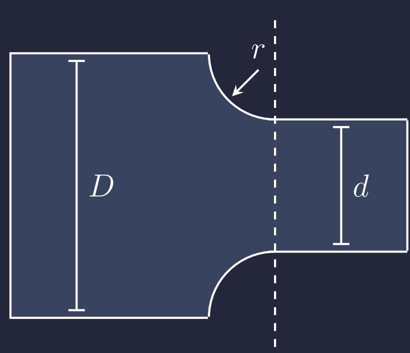


Figure 38: Filleted Shaft

The geometry of the filleted shaft will uniquely determine the maximum value of shear stress in terms of the average shear stress in the shaft. By analyzing the geometry, the **stress concentration factor** (k) can be calculated, thus giving the value of the maximum stress:

$$\tau_{max} = k\frac{T\rho}{J}$$

where $\frac{T\rho}{J}$ is calculated using the *smaller* portion of the shaft.

The same as in Subsubsection 3.8.2, k is calculated using the both the ratio of the larger to smaller diameter as well as the ratio between the radius of the fillet and the smaller diameter:

$$\frac{D}{d} \quad \text{and} \quad \frac{r}{h}$$

5 Bending

5.1 Pure Bending

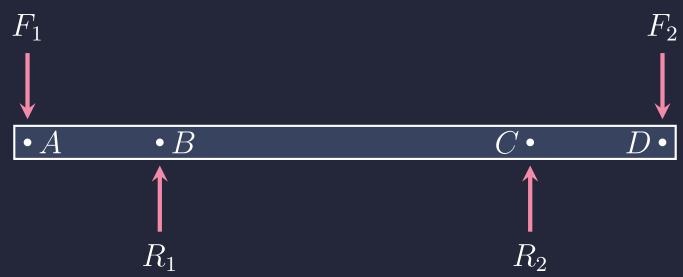


Figure 39: Member Experiencing Pure Bending

Consider the free body diagram in Figure 39. Assuming static equilibrium, the forces acting upon it would create bending moments at points B and C as is shown in Figure 40.

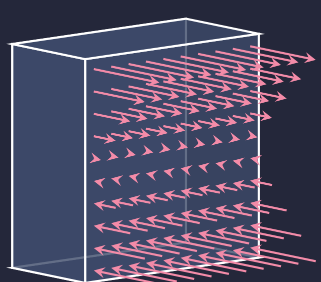
Figure 40: Bending Moments in Member

The pure symmetry of the bending of the bar ($M' = M$) results in **pure bending** in the member between B and C. This is because there is *only* bending in the member.

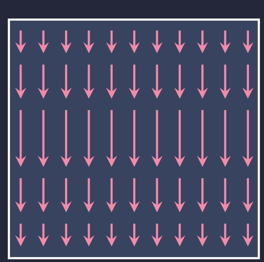
Pure Bending **5.1**

A situation in which bending exists in a member without the presence of axial, shear, or torsional forces.

If a cross section of the member were to be taken, the internal forces present would be a combination of a shear force acting in the vertical direction and a bending moment. Note these directions as correspondent to the *negative* bending seen in Figure 40.



(a) Bending Moment



(b) Shear Force

Figure 41: Internal Forces of a Bending Member

5.1.1 Bending Strain

Consider the internal tension/compression forces seen in Figure 41a. Notice that, as the forces approach the *centroid* of the beam, the magnitude of the forces decreases. To better understand this, consider the beam in Figure 42. Along its centroid in green in the neutral axis. Along this axis, the beam experiences no elongation. However, along the dotted line, the beam does elongate, implying the existence of internal axial forces.

Given the dimensions in Figure 42, the length of the arc of the neutral axis within the beam can be calculated as:

$$l_{\text{neutral arc}} = r \cdot \theta$$

Similarly, the length of the dashed arc within the beam is:

$$l_{\text{dashed}} = (r + y) \cdot \theta$$

The strain at any point can then be calculated as the ratio between these two values:

$$\epsilon = \frac{\Delta L}{L_0} = \frac{((r + y) \cdot \theta) - r \cdot \theta}{r \cdot \theta}$$

This can then be simplified into:

Bending Strain	
$\epsilon = \dfrac{y}{r}$	**5.1**

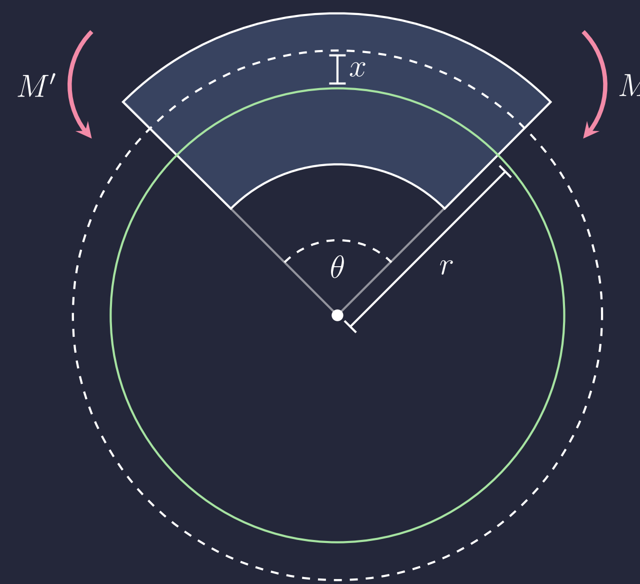


Figure 42: Neutral Axis

5.2 Bending Stress

During bending that remains within the elastic, non-plastic, range, Hooke's Law applies. This means that the bending stress can be found by multiplying the strain by some elastic constant:

Bending Stress	
$\sigma = E\epsilon = E\dfrac{y}{r}$	**5.2**

Since the internal moment is created by internal forces, the resultant forces of all of these internal forces must equal the moment. Thus, by integrating the internal forces, the moment can be found.

$$M = \int_A \sigma y \, dA = \int_A E\frac{y}{r} y \, dA = \frac{E}{r}\int_A y^2 \, dA$$

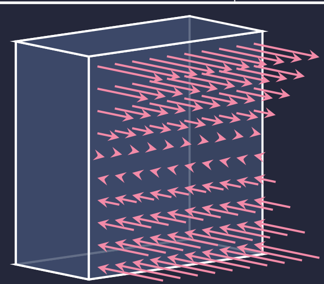


Figure 43: Bending Moment

5.3 Flexure Formula

The **bending moment of inertia** (I) of an object quantifies how much a beam resists bending. This value is calculated as:

$$\int_A y^2 \, dA$$

Thus:

$$M = \frac{E}{r}\int_A y^2 \, dA = \frac{EI}{r}$$

Combining this equation for M with the previous equation for σ results in the **Flexure Formula**.

Flexure Formula

$$\sigma = \frac{My}{I}$$

where $I_{rectangle} = \frac{bh^3}{12}$, $I_{circle} = \frac{\pi\left(r_o^4 - r_i^4\right)}{4}$ is the **area moment of inertia**

or

$$\sigma_{max} = \frac{M}{S}$$

where $S = \frac{I}{y_{max}}$ is the **Section Modulus**

5.3

5.4 Impure Bending

Just as pure bending happens when there are no shear forces present during bending, impure bending is when shear *and* bending coexist. Luckily, the flexure formula from Subsection 5.3 holds over all cases of pure bending, and most cases of impure bending.

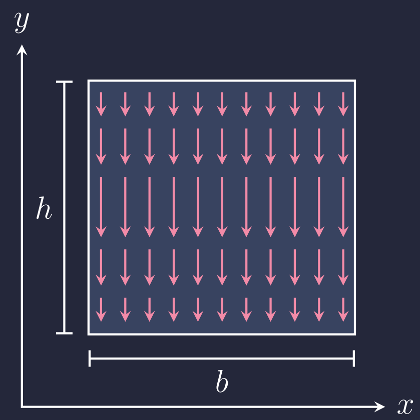


Figure 44: Bending Shear

The average shear across some cross-sectional face, such as that in Figure 44, is simply calculated as:

$$\tau = \frac{V}{A}$$

However, as can be seen, the shear forces are not distributed evenly throughout the face. So, how can the maximum shear stress be found?

Shear Stress

$$\tau(x, y) = \frac{V(x) \cdot Q(y)}{I \cdot b(y)}$$

5.4

Where V is the shear force, I is the area moment of intertia, b is the width of the face, and Q is the **first moment of area**.

First Moment of Area

$$Q = \text{area outside of axis} \cdot \text{distance to centroid from neutral axis}$$

$$Q = b\left(\frac{h}{2} - y\right) \cdot y + \frac{\frac{h}{2} - y}{2} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

5.5

In the case of Figure 44, the width is constant along the height of the cross-section just as the shear force V is constant throughout the face. With these considerations, the equation of shear stress can be expressed as:

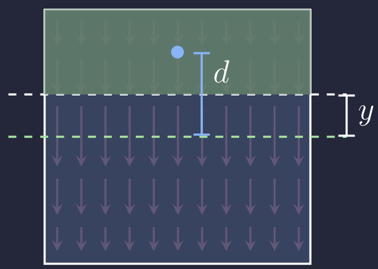


Figure 45: First Moment of Area

Shear Stress

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right) \qquad \textbf{5.6}$$

By inspecting this formula, it can be seen that the *maximum* shear stress would be found at the neutral axis of the face (where $y = 0$). By substituting in $y = 0$ to the shear stress formula, a formula for the max shear stress (τ_{max}) can be found:

Maximum Shear Stress

$$\tau_{\text{max,rectangle}} = \frac{V}{2I}\left(\frac{h^2}{4} - 0^2\right) = \frac{V}{2I}\frac{h^2}{4} = \frac{3}{2}\frac{V}{A}$$

$$\tau_{\text{max,circle}} = \frac{4}{3}\frac{V}{A} \qquad \textbf{5.7}$$

5.5 Stress Concentrations

Just as it was in Subsection 3.8, there are also stress concentrations when it comes to non-uniform members experiencing bending. Generally:

Maximum Stress

$$\sigma_{max} = k\frac{Mc}{I} = k\sigma_{avg} \qquad \textbf{5.8}$$

Where k is the experimentally derived stress concentration factor. To find k for any given member, again, only the geometry of the member must be analyzed. The relevant geometric properties for two types of non-uniform shapes are shown in Figure 46.

Figure 46: Stress Concentrations in Bending

5.6 Eccentric Bending

5.6.1 Symmetric Eccentric Bending

In all the previous axial loading analysis, it was assumed that the force creating the axial stresses was acting through the centroid of the cross-section of the member. In other words, it was assumed that the forces were perfectly centered. In those cases, only axial stresses occurred. However, what happens in a member is loaded eccentrically (not through its centroid)? This results in **eccentric loading**.

Eccentric Loading **5.2**

Eccentric loading occurs when a member is loaded such that the forces creating the load do not have a line of action through the centroid of the member's cross-section.

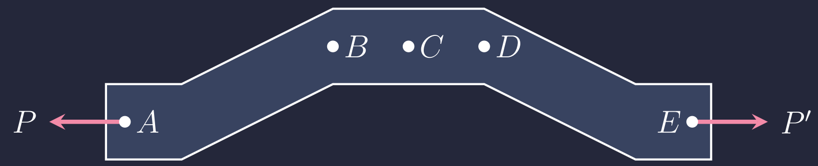


Figure 47: Eccentric Loading

When eccentric loading occurs, bending is created. Consider Figure 47. It might be intuitive to see that these forces P will create some sort of bending through the section BD. This can be seen better if an imaginary cut through the object were to be made as in Figure 48.

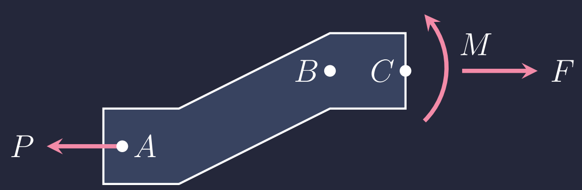


Figure 48: Eccentric Loading

Further dissecting this member, it can be seen in Figure 49 that the section BD is experiencing both centric and bending stresses.

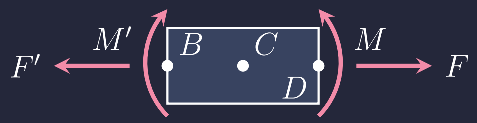
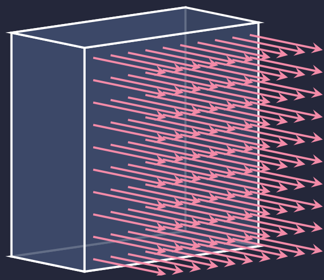
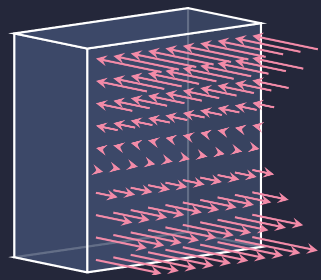


Figure 49: Eccentric Loading

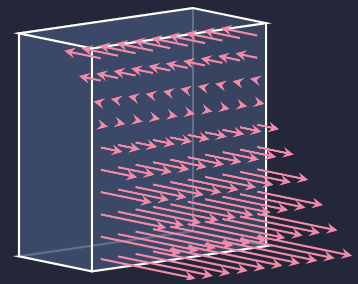
From here, the internal stresses can be expressed as the sum of the internal stresses caused be each of the two loads: the centric axial load and the bending moments.



(a) Centric Load



(b) Bending Moment



(c) Total Stress

$$\sigma = \frac{P}{A} - \frac{My}{I}$$

5.6.2 General Eccentric Bending

Applying the same principles from Subsubsection 5.6.1, a more generalized approach to eccentric bending that isn't symmetric on any plane with the member can be formulated. Consider Figure 51.

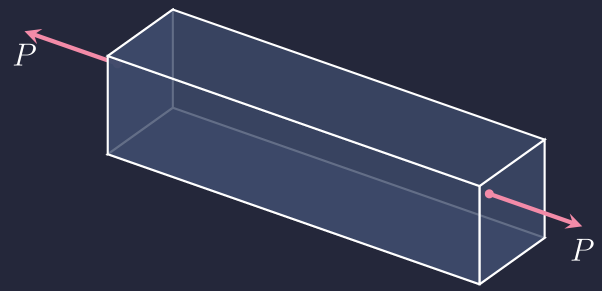


Figure 51: Asymmetric Eccentric Bending

These axial forces P and P' can be substituted for three component loads applied centrically: a moment M_z, a moment M_y, and a force P. Thus, a more generalized formula can be expressed as:

Axial Stress Under Eccentric Loading **5.9**

$$\sigma = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$