MATH 2070 - Differential Equations

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Contents

1 Introduction to Differential Equations

1.1 What is a Differential Equation

Differential Equation **1.1**

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation (DE).

Differential equations are foundational to studying engineering and physics. A very basic example of a differential equation is:

$$\frac{dx}{dt} + x = 2\cos(t) \tag{1}$$

Here x is the *dependent variable* and t is the *independent variable*. To solve (1) is to find x in terms of t such that the equation still holds when everything (x, t, and $\frac{dx}{dt}$) is plugged in. Consider

$$x = x(t) = \cos(t) + \sin(t)$$

as the solution for (1). Plugging this in as appropriate will verify this solution:

$$\begin{aligned} \big(-\sin(t) + \cos(t)\big) + \big(\cos(t) + \sin(t)\big) &= 2\cos(t) \\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) &= 2\cos(t) \\ \cos(t) + \cos(t) &= 2\cos(t) \\ 2\cos(t) &= 2\cos(t) \end{aligned}$$

Clearly this equality holds and a **particular solution** has been found for (1).

Particular Solution **1.2**

Some solution for a given differential equation.

1.2 Four Fundamental Equations

There exist four equations that are each very common and have solutions that can be memorized. The **first** among them is:

$$\frac{dy}{dx} = ky \tag{2}$$

For some constant $k > 0$, the general solution to (2) is:

$$y(x) = Ce^{kx}$$

The **second** among the four fundamental equations is:

$$\frac{dy}{dx} = -ky \tag{3}$$

For some constant $k > 0$, the general solution to (3) is:

$$y(x) = Ce^{-kx}$$

The **third** is a second order derivative (see 1.3.2):

$$\frac{d^2y}{dx^2} = -k^2 y \tag{4}$$

For some constant $k > 0$, the general solution for (4) is:

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

Since (3) is of the second order, there are two constants in the solution.

Lastly, the **fourth** fundamental equation is:

$$\frac{d^2y}{dx^2} = k^2 y \tag{5}$$

For some constant $k > 0$, the general solution for (5) is:

$$y(x) = C_1 e^{kx} + C_2 e^{-kx}$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(-kx)$$

Where:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

1.3 Classification of Differential Equations

There exist several types of differential equations. Consequently, they are classified according to **type**, **order**, and **linearity**.

1.3.1 Classification by Type

If a given differential equation 1) includes only ordinary derivatives of a number of unknown functions, and 2) those derivatives are all with respect to the same independent variable then it is classified as an **Ordinary Differential Equation (ODE)**.

> **Ordinary Differential Equation** **1.3**
>
> Equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.

$$\frac{dy}{dx} + 5y = e^x \; ; \frac{dy}{dx} + \frac{dr}{dx} = 14x \; ; \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} = 0 \tag{6}$$

Each equation in (6) is an example of an ODE. Notice that each equation contains only functions derived with respect to the same variable.

A **Partial Differential Equation (PDE)** differs in that it contains derivatives of functions with respect to multiple independent variables.

$$\frac{\delta y}{\delta x} + 5\frac{\delta r}{\delta t} = \ln(x) \; ; \frac{\delta y}{\delta x} + \frac{\delta r}{\delta t} = 14x \; ; \frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 b}{\delta x^2} \tag{7}$$

(7) are examples of partial differential equations. The Greek letter delta (δ) is used to denote a partial derivative. Thus, $\frac{\delta y}{\delta x}$ is the partial derivative of the function y with respect to x.

Partial Differential Equation **1.4**

Equations that depend on partial derivatives of several variables. That is, there are several independent variables.

1.3.2 Classification by Order

The **order** of a differential equation is the highest order among derivatives it contains. For example, (8) is a third-order differential equation because $\frac{d^3u}{dx^3}$ is a third derivative and is the highest derivative.

$$\frac{dy}{dx} - \frac{d^2r}{dx^2} = \frac{d^3u}{dx^3} \tag{8}$$

1.3.3 Classification by Linearity

An equation is linear if the dependent variable (or variables) and their derivatives appear linearly, that is, only as first powers, they are not multiplied together, and no other functions of the dependent variables appear.

$$e^x \frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} + x^2y = \frac{1}{x} \tag{9}$$

(9) is a linear differential equation because its *dependent* variable (y) only appears linearly. It does not matter that the independent variable (x) appears non-linearly. Conversely, (10) is non-linear because y is squared.

$$\frac{dy}{dx} = y^2 \tag{10}$$

Similarly, (11) is also non-linear because θ appears inside a sin function.

$$\frac{d^2\theta}{dx^2} + \sin(\theta) = 0 \tag{11}$$

2 Ordinary Differential Equations

Ordinary differential equations (ODE) are differential equations with just a single input, generally thought of as time (t).

Consider the relationships between position (x), velocity (v), and acceleration (a). Velocity is the derivative of position and acceleration the derivative of velocity.

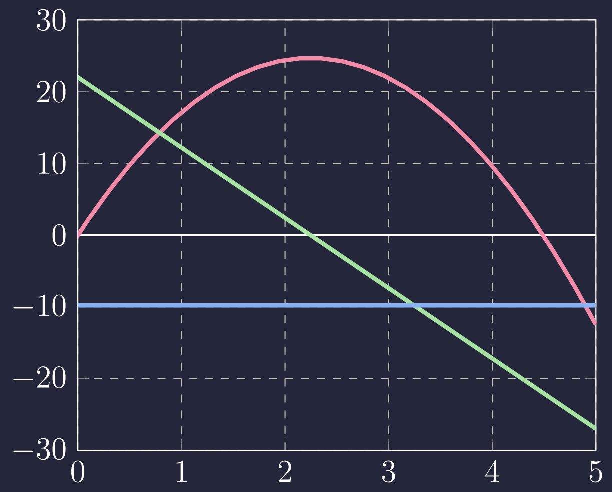


Figure 1: Position, Velocity, and Acceleration

If only the acceleration of an object across time is known ($g = 9.8\frac{m}{s^2}$), and nothing else about that object is known, then differential equations can be used to solve for the object's velocity and acceleration.

$$y''(t) = -g$$
$$\frac{d(?)}{dt}(t) = -g$$

Based on this, if a function can be found to have a derivative of $-g$ then the velocity of the object can be said to have a velocity equal to that function. In this example, it is as simple as integrating the function for acceleration:

$$\frac{d(-gt + v_0)}{dt}(t) = -g$$

Going one step further and integrating the velocity will yield the position of the object.

$$\frac{d\left(-\frac{1}{2}gt^2 + v_0t + x_0\right)}{dt}(t) = -gt + v_0$$

The last thing to consider with this example are the initial conditions of the differential equation. v_0 and x_0 are not known quantities, however, if they were to be specified then the differential would have an initial condition to satisfy.

2.1 First Order ODE

Based on the classification in Subsection 1.3, a first order ODE is a differential equation that whose highest order derivative is a first order derivative and all derivatives are with respect to the same variable.

First Order ODE **2.1**

A differential equation of the first order, with all derivatives being with respect to a single variable (usually x or t).

Previously, in Subsection 1.1, a particular solution was found for the differential equation of:

$$\frac{dx}{dt} + x = 2\cos(t)$$

However, this being a first order ODE, it has more than just a single particular solution. Consider:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - e^{-t}$$

as a solution to (2.1). To verify:

$$\begin{aligned}\big(-\sin(t) + \cos(t) - e^{-t}\big) + \big(\cos(t) + \sin(t) + e^{-t}\big) &= 2\cos(t)\\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) - e^{-t} + e^{-t} &= 2\cos(t)\\ \cos(t) + \cos(t) &= 2\cos(t)\\ 2\cos(t) &= 2\cos(t)\end{aligned}$$

As can be seen, there exists more than just the single particular solution. In fact, for (1), the entire family of solutions exists in the form:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - Ce^{-t}$$

where C is some constant. This is called a **One-Parameter Family of Solutions** for the differential equation.

One-Parameter Family of Solutions **2.2**

A one-parameter family of solutions is a solution to a differential equation containing a single constant C. This constant is arbitrary, and thus the family of solutions it represents consists of all values of C.

If a one-parameter family of solutions contains every possible solution to the differential equation, then it is referred to as the **General Solution**.

General Solution **2.3**

The entire family of solutions for a given differential equation. A general form of the solution that can be adapted to different specifications.

Each value of C gives a different solution, so really there are infinite solutions for (1).

2.1.1 First Order Linear ODE

First Order Linear ODEs follow a simple pattern for their solutions. However, before getting to that, what is a first order linear ODE?

First Order Linear ODE **2.4**

A differential equation that 1) doesn't contain a derivative beyond the first order, 2) contains only derivatives with respect to a single variable, and 3) has its dependent variable appear linearly (not part of a sin, cos, square, etc.).

If a differential equation is given in **standard form**:

$$y' + p(t)y = g(t)$$

Where $p(t)$ and $g(t)$ are arbitrary functions of t, then the general solution can be expressed as:

$$y(t) = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)}$$

Where the **integrating factor** ($\mu(t)$) is:

$$\mu(t) = e^{\int p(t)\,dt}$$

Example **2.1**

$$y' + 2y = e^{3t}, \; y(0) = 3$$

This ODE is already in standard form, so the integrating factor can be written as:

$$\begin{aligned}\mu(t) &= e^{\int 2\,dt} \\ &= e^{2t}\end{aligned}$$

And thus the general solutions is:

$$\begin{aligned}y(t) &= \frac{\int e^{2t} \cdot e^{3t}\,dt + C}{e^{2t}} \\ &= \frac{\int e^{5t}\,dt + C}{e^{2t}} \\ &= \frac{\frac{1}{5}e^{5t} + C}{e^{2t}} \\ &= \frac{1}{5}\frac{e^{5t}}{e^{2t}} + \frac{C}{e^{2t}} \\ &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}}\end{aligned}$$

Using the initial condition to find the specific solution:

$$\begin{aligned}y(t) &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}} \\ y(0) &= \frac{1}{5} \cdot e^{3\cdot 0} + \frac{C}{e^{2\cdot 0}} \\ 3 &= \frac{1}{5} + C \\ \frac{14}{5} &= C\end{aligned}$$

Thus, the solution of the IVP is:

$$y(t) = \frac{1}{5} \cdot e^{3t} + \frac{14}{15} \cdot e^{-2t}$$

2.1.2 First Order ODE

A first order ODE is an equation in the form of:

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y) \tag{1}$$

There is no strict process to find a solution to these equations. Thus, a lot of this class is spent on the various different ways to find solutions. To see one of the simpler ways, consider an equation where f is a function of only x:

$$y' = f(x)$$

Integrating both sides with respect to x, the equation becomes:

$$\int y'\,dx = \int f(x)\,dx + C$$
$$y(x) = \int f(x)\,dx + C$$

This $y(x)$ is the general solution to (1). Thus, to solve a differential equation with just a single dependent variable x, finding the antiderivative of $f(x)$ is sufficient to find the general solution.

Example **2.2**

Find the general solution for:

$$y' = 3x^2$$

Integrating each side gives:

$$\int y'\,dx = \int 3x^2\,dx + C$$
$$y(x) = x^3 + C$$

Thus, the general solution for $y' = 3x^2$ is $y(x) = x^3 + C$.

2.1.3 Initial Value Problem

Generally, there also might be a condition that the solution to a differential equation must satisfy. In general terms, this might look like:

$$y' = f(x),\ y(x_0) = y_0 \tag{2}$$

Leaving the solution to (2) as:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$

Verifying the solution, first y' is computed based on our solution.

$$\frac{d}{dx}\big(y(x)\big) = \frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt + y_0\right)$$
$$y' = f(x)$$

Second, to verify that the initial condition is satisfied:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$
$$y(x_0) = \int_{x_0}^{x_0} f(t)\,dt + y_0$$
$$y(x_0) = 0 + y_0$$
$$y(x_0) = y_0$$

This confirms the initial condition, and thus it can be seen that the solution to the differential equation with an initial condition has been found.

Example **2.3**

Solve

$$y' = e^{-x^2},\ y(0) = 1$$

First, finding the solution, both sides can be integrated:

$$y(x) = \int_0^x e^{-t^2}\,dt + 1$$

And to verify the solution:

$$\frac{d}{dx}\big(y(x)\big) = \frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt + 1\right)$$
$$y' = e^{-x^2}$$

$$y(0) = \int_0^x e^{t^2}\,dt + 1$$
$$y(0) = 0 + 1$$
$$y(0) = 1$$

The solution passes both verification tests, and so it can be safely said that the solution has been found.

Using the same method as before, equations of the form in (3) can be solved as well.

$$y' = f(y) \text{ or } \frac{dy}{dx} = f(y) \tag{3}$$

(3) can be rewritten using the inverse function theorem from calculus to switch the roles of x and y to get:

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

Finally, at this point both sides can be integrated with respect to y to get:

$$x(y) = \int \frac{1}{f(y)}\,dy + C$$

From here, it is just a matter of solving for y.

Example **2.4**

In Subsection 1.2, the claim was made that the solution for $y' = ky$ is $y = Ce^{kx}$ for $k > 0$. To show this, the method of integration can be used:

$$\frac{dy}{dx} = ky \rightarrow \frac{dx}{dy} = \frac{1}{ky}$$
$$x(y) = \int \frac{1}{ky}\, dy$$
$$x(y) = \frac{1}{k} \ln|y| + D$$

Now solving for y:

$$x(y) = \frac{1}{k} \ln|y| + D$$

Example **2.5**

$$y' + y = \cos(2t)$$

This is already in standard form, so we can go on to calculate the integrating factor:

$$\mu(t) = e^{\int 1\, dt} = e^t$$

Thus, the general solution will be:

$$y = \frac{\int e^t \cdot \cos(2t)\, dt}{e^t}$$

Integrating the numerator:

$$\int e^t \cdot \cos(2t)\, dt$$
$$\vdots$$
$$\int e^t \cdot \cos(2t)\, dt = \frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C$$

Thus:

$$y = \frac{\frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C}{e^t}$$

or:

$$y = \frac{1}{5}\left(\cos(2t) + 2\sin(2t)\right) + Ce^{-t}$$

2.2 Non-Linear ODE

Generally, non-linear ODEs are quite difficult to solve. For some of them, there are techniques that can be used to simplify the process.

2.2.1 Separable ODE

The basic form of a separable ODE is:

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

i.e., the derivative is a product of two functions, one depending on x and the other on y.

This type of ODE is solved first by separating the variables. The LHS is reserved for y and the RHS for x:

$$\frac{dy}{f(y)} = g(x)dx$$

Now that there is a single type of variable on each side (only x or only y), both sides can be integrated. This will yield a one-parameter family of *implicit solutions* (see Subsection 2.3).

Example **2.6**

$$y' = xy$$

This equation can be rewritten as:

$$\frac{dy}{dx} = x \cdot y$$

Here, it can be easily seen that this equation is separable:

$$\frac{dy}{y} = x\ dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx + C$$

$$\ln|y| = \frac{x^2}{2} + C$$

This here is the implicit solution to the ODE. It won't always be the case, but here algebra can be used to find the explicit solution:

$$e^{\ln|y|} = e^{\frac{x^2}{2}+C}$$

$$y = e^{\frac{x^2}{2}} \cdot e^C$$

$$y = De^{\frac{x^2}{2}}$$

Where $D > 0$. Since $y = 0$ is a solution as well, this can be simplified to:

$$y = De^{\frac{x^2}{2}}$$

Principle: If an ODE is given without initial values, a one-parameter family of solutions is sufficient. However, if an initial value is given (IVP), an explicit solution might be possible with an interval of existence.

Example **2.7**

$$\frac{dy}{dx} = \frac{e^x - x}{e^{-y} + y}$$

First, separating the variables to each side:

$$\frac{dy}{dx} = (e^x - x) \cdot \frac{1}{e^{-y} + y}$$
$$\left(e^{-y} + y\right) dy = (e^x - x)\, dx$$

Then integrating each side appropriately:

$$\int e^{-y} + y\, dy = \int e^x - x\, dx$$
$$-e^{-y} + \frac{1}{2}y^2 = e^x - \frac{1}{2}x^2 + C$$

And thus, a one-parameter family of implicit solutions has been found.

2.2.2 Singular Solutions

Consider:

$$y' = xy^3 \left(1 + x^2\right)^{-\frac{1}{2}} \tag{4}$$

(4) is a separable ODE that can be reorganized as:

$$y' = y^3 \cdot \frac{x}{\sqrt{1 + x^2}}$$

And subsequently separated into:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1 + x^2}}\, dx$$

And solved:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1 + x^2}}\, dx$$
$$\int \frac{1}{y^3}\, dy = \int \frac{x}{\sqrt{1 + x^2}}\, dx$$
$$-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$$

It would now be tempting to say that $-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$ is a general solution to (4). However, $y = 0$ is also a solution to (4)...

The true answer then would be to say that $-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$ is a *family* of solutions rather than the general solution and that $y = 0$ is a singular solution.

2.3 Implicit and Explicit Solutions

In general, anytime an ODE is given *without* initial values, then a single-parameter family of *implicit* solutions is sufficient.

If given an IVP, however, then finding an *explicit* solution and an *interval of existence* should be attempted. In some situations, this won't be possible, but it should be attempted at least.

2.3.1 Implicit Solutions

Sometimes a wall is reached even if the integration is possible. Consider:

$$y' = \frac{xy}{y^2+1} \tag{5}$$

Using the technique described in Subsection 2.2.1, this can be separated into:

$$\begin{aligned}
y' &= \frac{xy}{y^2+1}\\
\frac{dy}{dx} &= \frac{xy}{y^2+1}\\
\frac{y^2+1}{y}dy &= x\ dx\\
\left(y+\frac{1}{y}\right)dy &= x\ dx
\end{aligned}$$

Integrating both sides gives:

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C$$

The integration of this equation is quite simple. However, try to solve for y and see how difficult that will be. Though solving for y itself is too difficult, this form is still a solution and can still be verified.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{y^2}{2}+\ln|y|\right) &= \frac{d}{dx}\left(\frac{x^2}{2}+C\right)\\
y'\left(y+\frac{1}{y}\right) &= x\\
y\cdot\left(y'\left(y+\frac{1}{y}\right)\right) &= y\cdot x\\
y'\left(y^2+1\right) &= y\cdot x\\
y' &= \frac{xy}{y^2+1}
\end{aligned}$$

Producing the exact same equation in (5), thus verifying the solution.

Since these solutions are implicit, they might not be able to be graphed as a valid function. In those cases, other information, such as an initial condition, can be used to further inform the appropriate solution.

2.4 Pathological and Reasonable IVP

2.4.1 Pathological IVP

Consider:

$$\frac{dy}{dx} = x\sqrt{y},\ y(0) = 0 \tag{6}$$

Solving this IVP, it can be seen that both:

$$y = \frac{1}{16}x^2 \quad \text{and} \quad y = 0$$

are both solutions to (6). This would then be referred to as a **Pathological IVP**.

Pathological IVP **2.5**

An IVP with zero solutions, more than one solution, or infinitely many solutions.

Example **2.8**

$$ty' + (t-1)y = -e^{-t},\ y(0) = 1$$

First, putting the ODE into standard form:

$$y' + \frac{t-1}{t}y = \frac{-e^{-t}}{t}$$

Thus:

$$p(t) = \frac{t-1}{t} \qquad g(t) = \frac{-e^{-t}}{t} \qquad \mu(t) = \frac{e^t}{t}$$

Giving the general solution as:

$$y = \frac{\int \frac{e^t}{t} \cdot \frac{-e^{-t}}{t}\,dt}{\frac{e^t}{t}} = \frac{-\int \frac{1}{t^2}\,dt}{\frac{e^t}{t}} = \frac{\frac{1}{t} + C}{\frac{e^t}{t}} = e^{-t} + Cte^{-t}$$

Now to solve the IVP, plugging in the values of $y(0)$ and $t = 0$:

$$y = e^{-t} + Cte^{-t} \quad \Rightarrow \quad 1 = e^0 + C \cdot 0 \cdot e^0 = 1 + 0$$

The disappearance of C (due to being multiplied by 0) reveals that there are infinitely many solutions to this IVP, thus showing its pathological nature.

Example **2.9**

$$ty' + (t-1)y = -e^{-t},\ y(0) = 0$$

By taking the same IVP as previous, but changing the initial condition from $y(0) = 1$ to $y(0) = 0$, the behavior changes:

$$y = e^{-t} + Cte^{-t} \quad \Rightarrow \quad 0 = e^0 + C \cdot 0 \cdot e^0 = 1 + 0$$

Clearly, $0 \neq 1$, showing that this IVP has zero solutions. Again, this is a pathological IVP.

2.4.2 Reasonable IVP

It might be tempting to say that any IVP that is *not* pathological is then reasonable. While exclusivity exists between the two, the proper method to determine if an IVP is reasonably posed is by using the **Existence and Uniqueness Theorem**.

Existence and Uniqueness Theorem **2.6**

Consider:

$$y' + p(t)y = g(t) + C,\ y(t_0) = y_0$$

where the ODE is *linear* and in its *standard form*. Assuming that:

1. Both $p(t)$ and $g(t)$ are continuous over the open interval (a, b)
2. The open interval (a, b) contains t_0

Then there exists a unique function $y = y(t)$ over the interval (a, b) that solves the IVP.

Notice that, to determine if a unique solutions exists based on the existence and uniqueness theorem, the IVP need not be solved.

Example **2.10**

$$ty' + (t-1)y = -e^{-t},\ y(\ln|2|) = \frac{1}{2}$$

Putting into standard form:

$$y' + \frac{t-1}{t}y = \frac{-e^{-t}}{t}$$

From here, $p(t)$ and $g(t)$ can be seen to be:

$$p(t) = \frac{t-1}{t} \qquad g(t) = \frac{-e^{-t}}{t}$$

Both $p(t)$ and $g(t)$ are continuous everywhere except for at $t = 0$, where both of their denominators will equal 0. This says that there are two open intervals over which there could possbily be a single unique solution: $(-\infty, 0)$ or $(0, \infty)$.

Because the initial value is $y(\ln|2|) = \frac{1}{2}$, the second interval is chosen as $t_0 = \ln|2|$ lays on the interval $(0, \infty)$.

Based on this theorem, it can be asserted that there exists a unique solution over an interval (a, b) provided that this interval contains t_0 from the IVP and both functions $p(t)$ and $g(t)$ are continuous over this interval.

2.5 Intervals of Existence

2.5.1 Linear IVPs

Based on the existence and uniqueness theorem as outlined in Subsubsection 2.4.2, the general steps for finding the interval of existence is:

1. Find the standard form
$$y' + p(t)y = g(t),\ y(t_0) = y_0$$
2. Locate the singular points (where $p(t)$ or $g(t)$ are not continuous)
3. Based on the singular points, find the open intervals over which both $p(t)$ and $g(t)$ are continuous
4. Pick the interval that contains t_0

Example **2.11**

$$ty' + (t-1)y = -e^{-t},\ y\left(\ln(2)\right) = \frac{1}{2}$$

To use the theorem, first the equation must be expressed in its standard form:

$$y' + \frac{t-1}{t} \cdot y = -\frac{e^{-t}}{t}$$

Where:

$$p(t) = \frac{t-1}{t}$$
$$g(t) = -\frac{e^{-t}}{t}$$

Since, for both of these functions, the only point of discontinuity is $t = 0$, then it is known that the only two possible intervals of existence are either $(-\infty, 0)$ or $(0, \infty)$.

Furthermore, since the initial condition is given as $y\left(\ln(2)\right) = \frac{1}{2}$, $t_0 = \ln(2)$. Thus, the interval of existence for the solution of this ODE would be the interval of $(0, \infty)$.

2.5.2 Non-Linear IVPs

Consider:

$$y' = f(t, y),\, y(t_0) = y_0$$

Assume that:

1. The function $f(t, y)$ is continuous **near** (t_0, y_0)
2. The function $\frac{\delta f}{\delta y}(t, y)$ is continuous **near** (t_0, y_0)

Then a unique solution $y = y(t)$ exists **near** $t = t_0$, i.e., there exists a small number $\epsilon > 0$ such that a solution exists on $(t_0 - \epsilon, t_0 + \epsilon)$.

Example **2.12**

$$y' = y^{\frac{1}{3}},\, y(0) = 1$$

First, find the behavior of $f(t, y)$:

$$f(t, y) = y^{\frac{1}{3}} \text{ is continuous everywhere.}$$

Then, find the behavior of $\frac{\delta f}{\delta y}(t, y)$:

$$\frac{\delta f}{\delta y}(t, y) = \frac{d}{dx}\left(y^{\frac{1}{3}}\right) = \frac{1}{3}y^{-\frac{2}{3}} \text{ is continuous everywhere when } y \neq 0.$$

Thus, since the IVP is $y(0) = 1 \Rightarrow (t_0, y_0) = (0, 1)$, the IVP has a unique solution near $t = 0$. The discontinuity when $y = 0$ is irrelevant because it is not near $y_0 = 1$.

Though still very helpful, this theorem is not as powerful as the linear version as it only concludes local existence rather than an interval of existence. However, it can still be used to determine whether an IVP is reasonably or pathologically formulated.

2.6 Autonomous ODE

$$y' = f(y)$$

Such that the right hand side (RHS) does not involve t. In other words, it is a function purely in terms of y.

The equilibrium solution:

$$y = y_0$$

Such that:

$$f(y_0) = 0$$

This means that the equilibrium solutions must be a value such that $y'(y_0) = 0$.

Example **2.13**

$$y' = y^3 - y$$

Clearly, the RHS is purely in terms of y, so this would be able to be solved as an autonomous ODE. So, taking the RHS and solving for 0:

$$\begin{aligned} y^3 - y &= 0 \\ y\left(y^2 - 1\right) &= 0 \\ y\left(y - 1\right)\left(y + 1\right) &= 0 \\ y = 0;\ y = 1;\ y &= -1 \end{aligned}$$

Thus, there are three equilibrium solutions.

$$@y = 0$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow 0\left(0-1\right)\left(0+1\right) = 0 \cdot -1 \cdot 1 = 0$$

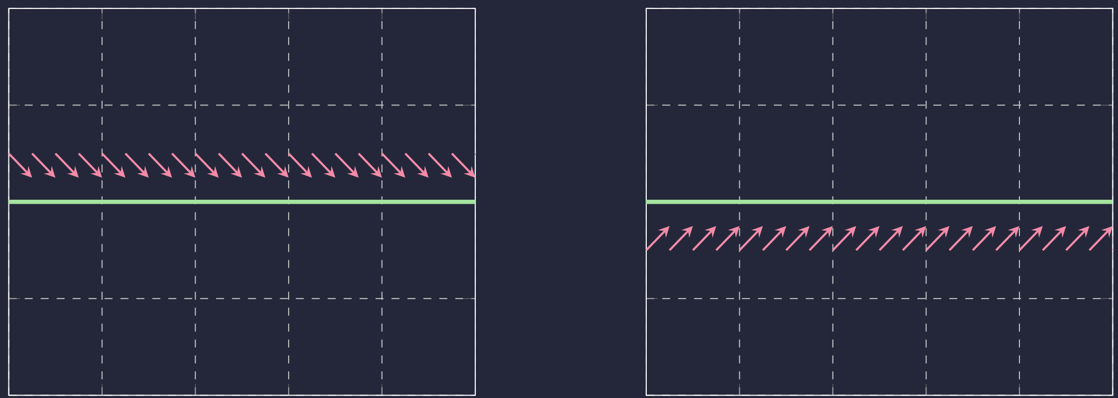
$$@y = 1$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow 1\left(1-1\right)\left(1+1\right) = 1 \cdot 0 \cdot 2 = 0$$

$$@y = -1$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow -1\left(-1-1\right)\left(-1+1\right) = -1 \cdot -2 \cdot 0 = 0$$

2.6.1 Stability of an Equilibrium Solution

Stability **2.7**

Let $y = y_0$ be an equilibrium solution of $y' = f(y)$. $y = y_0$ is *stable from above* if, for every $y > y_0$ near y_0, $f(y) < 0$.



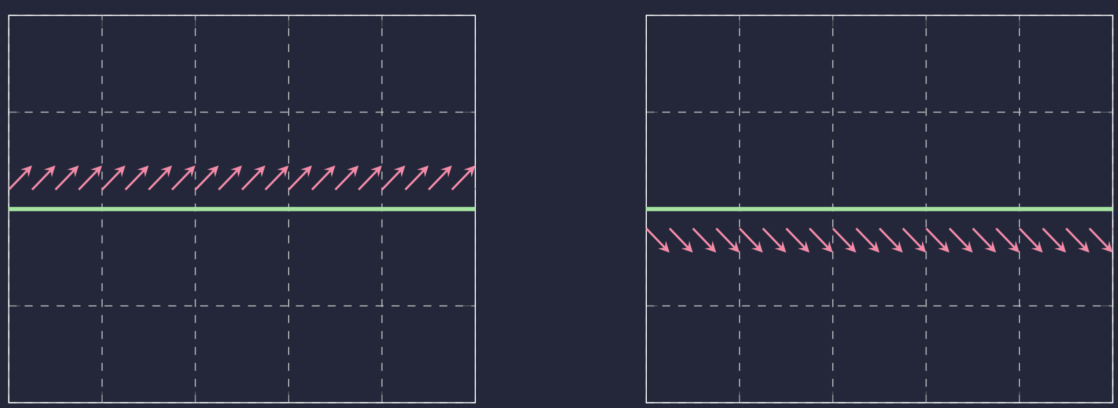
(a) Stable From Above

(b) Stable From Below

Figure 3: Stability

Using Figure 3a as a visual, the solution is stable from above because, if you were to deviate from the solution, the direction field above will guide you back down to the solution. Figure 3b is stable from below for the same reasons.

It follows then, that if the direction field around the solution points *away* from the solution, then the solution is considered to be *un*stable. See this in Figure 4.

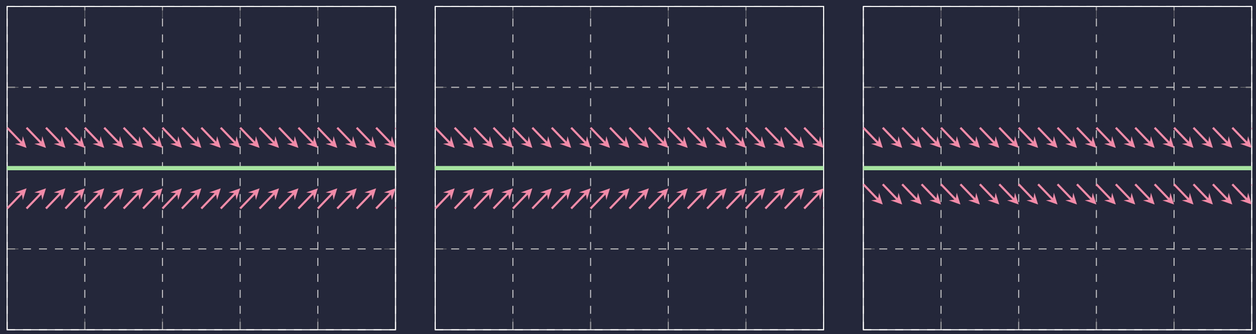


(a) Unstable From Above

(b) Unstable From Below

Figure 4: Instability

When considering a solutions behavior from both sides (top and bottom), it can be said to be either stable, unstable, or semistable.



(a) Stable

(b) Unstable

(c) Semistable

Figure 5: Types of Stability

Example **2.14**

Falling object from a great height, subject to acceleration due to gravity (mg) and air resistance (kv^2). Newton's second law:

$$m \cdot \frac{dv}{dt} = mg - kv^2;\ v(0) = 0$$

This ODE is autonomous since the RHS has no t involved, so the equilibrium solution will be obtained by setting the RHS $= 0$.

$$mg - kv^2 = 0$$
$$\vdots$$
$$v = \pm\sqrt{\frac{mg}{k}}$$

Based on this model, the terminal velocity of the falling object will be:

$$v = \sqrt{\frac{mg}{k}}$$

This ODE can also be solved by finding the explicit solution, first by rearranging the equation:

$$m \cdot \frac{dv}{dt} = mg - kv^2$$
$$\vdots$$
$$\frac{m}{mg - kv^2} dv = dt$$

Then integrate both sides. Keep in mind that g, k, and m are constants.

$$\int \frac{m}{mg - kv^2}\, dv = \int\, dt$$
$$\frac{m}{k} \int \frac{m}{\frac{mg}{k} - v^2}\, dv = \int\, dt$$
$$\frac{m}{k} \int \frac{m}{\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)}\, dv = \int\, dt$$

2.6.2 Phase Line

When there are multiple solutions to a given ODE, each solution might have a different stability. Though this information could absolutely be expressed as seen in Figure 5 with multiple solutions drawn, this information could also be visualized in a phase line.

Since these autonomous ODEs don't depend on an x, no information is gained by extending the graph into the x-axis. A phase line recognizes that by just expressing the stability of each solution along a single y-axis. This is seen in Figure 6.

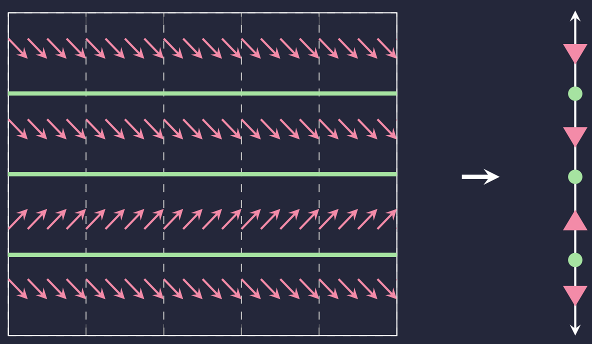


Figure 6: Phase Line

3 Numerical Methods

Oftentimes, differential equations are difficult to solve analytically. Of course, different ODEs are difficult for different reasons, but they are still difficult nonetheless. When analytical methods are difficult enough to the point where they fail, numerical methods of analysis can be used in their place.

More specifically, numerical methods are used:

- With any *reasonably formulated* IVP
- Often when analytical methods fail
- Often in engineering contexts in real world practices

3.1 Euler's Method

The core idea of Euler's Method is that a function's derivative can be used to approximate the function's value near a point. In other words, some point $f(b)$ can be approximated by knowing some other point $f(a)$ provided that 1) a and b are near each other and 2) $f'(b)$ is known.

Consider:

$$y' = f(t, y),\ y(t_0) = y_0$$

Then:

$$y(t_0 + h) \approx y(t_0) + y'(t_0) \cdot h$$

for some small step size h.

Taking this idea an extending it to any possible n for y_n, the formula for Euler's Method is:

Euler's Method	
$$y_n = y_{n-1} + h \cdot f(t_{n-1}, y_{n-1})$$ Given: $y' = f(t, y)$, for some small step size h	**3.1**

Based on this, the basic steps to computing a value based on Euler's Formula are:

1. Formulate an equation for y' in terms of t and y
2. Apply Euler's Formula to express y_n in terms of y_{n-1}, n, and h
3. Determine the number of steps to compute
4. Compute y_n incrementally (use a calculator or software)

Consider the IVP:

$$y' + y = t + 1,\ y(1) = 3 \tag{7}$$

Compute $y(3)$ with a step size of $h = 0.5$:

1. Formulate an equation for y' in terms of t and y

$$y' = t - y + 1 \Rightarrow f(t, y) = t - y + 1$$

2. Apply Euler's Formula to express y_n in terms of y_{n-1}, n, and h

$$\begin{aligned}
&y_n = y_{n-1} + h \cdot f(t_{n-1}, y_{n-1}) \Rightarrow y_n = y_{n-1} + h(t_{n-1} - y_{n-1} + 1) \\
\Rightarrow\ &y_n = y_{n-1} + h(t_{n-1} - y_{n-1} + 1) \Rightarrow y_n = y_{n-1} + h(t_0 + h(n-1) - y_{n-1} + 1) \\
\Rightarrow\ &y_n = y_{n-1} + 0.5(1 + 0.5(n-1) - y_{n-1} + 1) \Rightarrow y_n = 0.5y_{n-1} + 0.25n + 0.75
\end{aligned}$$

3. Determine the number of steps to compute

$$y(3) \Rightarrow t_0 + nh = 3 \Rightarrow 3 - t_0 = nh \Rightarrow \frac{3 - t_0}{h} = n \Rightarrow \frac{3-1}{0.5} = n \Rightarrow n = \frac{2}{0.5} \Rightarrow n = 4$$

4. Compute y_n incrementally (use a calculator or software)

$$\begin{aligned}
&y_0 = y(1) = 3 \\
&y_1 = 0.5 \cdot y_0 + 0.25 \cdot n + 0.75 \Rightarrow y_1 = 0.5 \cdot 3 + 0.25 \cdot 1 + 0.75 = 2.5 \\
&y_2 = 0.5 \cdot y_1 + 0.25 \cdot n + 0.75 \Rightarrow y_2 = 0.5 \cdot 2.5 + 0.25 \cdot 2 + 0.75 = 2.5 \\
&y_3 = 0.5 \cdot y_2 + 0.25 \cdot n + 0.75 \Rightarrow y_3 = 0.5 \cdot 2.5 + 0.25 \cdot 3 + 0.75 = 2.75 \\
&y_4 = 0.5 \cdot y_3 + 0.25 \cdot n + 0.75 \Rightarrow y_4 = 0.5 \cdot 2.75 + 0.25 \cdot 4 + 0.75 = 3.125
\end{aligned}$$

3.2 Error Analysis

In the above example of Euler's Formula (7), the approximation of $y(3)$ was calculated to be 3.125. The actual value of $y(3)$ given the same IVP, but solved directly, is ≈ 3.27067. Clearly, there is error to Euler's Formula.

A simple way to analyze the error over some interval $(t_0, t_0 + h)$, the second derivative of y can be used $(y''(t))$. If, over this interval:

1. $y''(t) > 0 \Rightarrow$ Underestimate
2. $y''(t) < 0 \Rightarrow$ Overestimate
3. $y''(t) = 0 \Rightarrow$ Proper Estimate
4. $y''(t)$ changes sign $\Rightarrow$ Can't tell

The error of (7) can be analyzed by determining if there was an under or overestimate over each interval calculated. To do so, the second derivative of the $y(t)$ must be found:

$$y' = t - y + 1 \Rightarrow y'' = t' - y' + 1 \Rightarrow y'' = 1 - (t - y + 1) \Rightarrow y'' = -t + y$$

Then using $y''(t)$, the value of the second derivative at each step can be calculated to see if there was an over or underestimate at each step. Notice that the value of y used at each step is the estimated value from Euler's Formula:

$$\begin{aligned}
y''(1.5) &= -1.5 + 2.5 = 1 \Rightarrow y''(1.5) > 0 \Rightarrow \text{underestimate} \\
y''(2) &= -2 + 2.5 = 0.5 \Rightarrow y''(2) > 0 \Rightarrow \text{underestimate} \\
y''(2.5) &= -2.5 + 2.75 = 0.25 \Rightarrow y''(2.5) > 0 \Rightarrow \text{underestimate} \\
y''(3) &= -3 + 3.125 = 0.125 \Rightarrow y''(3) > 0 \Rightarrow \text{underestimate}
\end{aligned}$$

Considering that each step is an underestimate, it is *likely* that the approximation via Euler's Formula of $y(3) = 3.125$ is also an underestimate. Unfortunately, this process is not absolute, and only provides evidence to *suggest* an underestimate.

4 Second Order ODEs

Similar to first order ODE's, second order ODE's also have a standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

Moreover, if the RHS ($g(t)$) is equal to zero, then the ODE is said to be homogeneous.

Homogeneous **4.1**

If the RHS of a second order ODE in standard form is equal to zero ($g(t) = 0$), then the ODE is said to be homogeneous. Otherwise ,the ODE is non-homogeneous.

Unfortunately, there is no standard formula to solve a second (or higher) order differential equation. However, there are different kinds of second order ODEs that benefit from different strategies when it comes to solving them. But before solving second order ODEs, it's important to know a little theory about them.

4.1 Existence and Uniqueness Theorem

Given an IVP in the standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

with:

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'$$

if $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval (a, b), and $t_0 \in (a, b)$, then the IVP has a unique solution on the interval (a, b).

4.2 Principle of Superposition

Given a second order, linear, *and* homogeneous ODE:

$$y'' + p(t)y' + q(t)y = 0$$

If some two functions y_1 and y_2 are solutions to the ODE, then for every real number c_1 and c_2, the function:

$$c_1y_1 + c_2y_2$$

is also a solution.

Furthermore, if y_1 and y_2 are linearly independent, then the general solution to the ODE is:

$$y = C_1y_1 + C_2y_2$$

4.3 Wronskian for Linear Independence

If y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$ on an interval where the existence and uniqueness theorem (**??**) holds, then y_1 and y_2 are linearly independent if and only if:

$$\left(W(y_1, y_2)\right)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = (y_1(t) \cdot y_2'(t)) - (y_2(t) \cdot y_1'(t)) \neq 0$$

on the interval in consideration. It should be noted that $\left(W(y_1, y_2)\right)(t) \neq 0$ need only be true at one point on the interval to demonstrate linear independence.

4.4 Linear Homogeneous ODE with Constant Coefficients

Consider the ODE:

$$ay'' + by' + cy = 0 \tag{8}$$

where a, b, and c are constants.

Idea: Try $y = e^{rt}$ as a solution: $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$. Using this, (8) becomes:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$\left(ar^2 + br + c\right) e^{rt} = 0$$

e^{rt} will never be 0, so to solve this, use the quadratic equation to solve for r:

$$ar^2 + br + c = 0$$

This equation is referred to as the **auxiliary equation**.

4.4.1 Distinct Real Roots Case

If the auxiliary equation has two distinct real roots $r_a \neq r_b$, then there are two solutions:

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = \left(e^{r_a t} \cdot r_b e^{r_b t}\right) - \left(e^{r_b t} \cdot r_a e^{r_a t}\right) \neq 0 \text{ (since } r_a \neq r_b)$$

Example **4.1**

$$y'' - 5y' + 6y = 0$$

Auxiliary equation:

$$r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r_a = 2, r_b = 3$$

General solution:

$$y = C_a e^{2t} + C_b e^{3t}$$

Example **4.2**

$$2y'' - 7y' + 3y = 0$$

Auxiliary equation:

$$2r^2 - 7r + 3 = 0$$
$$(2r-1)(r-3) = 0$$
$$r_a = \frac{1}{2}, r_b = 3$$

General solution:

$$y = C_a e^{\frac{1}{2}t} + C_b e^{3t}$$

Example **4.3**

$$y'' - 4y' - 6y = 0, y(0) = 1, y'(0) = 0$$

Auxiliary equation:

$$\begin{aligned} r^2 - 4r - 6 &= 0 \\ r^2 - 4r &= 6 \\ r^2 - 4r &= 6 \\ r^2 - 4r + 4 &= 10 \\ (r-2)^2 &= 10 \\ r - 2 &= \pm\sqrt{10} \\ r &= 2 \pm \sqrt{10} \end{aligned}$$

General solution:

$$y = C_a e^{2+\sqrt{10}} + C_b e^{2-\sqrt{10}}$$

To solve for the initial conditions, first find y':

$$y' = (2+\sqrt{10})C_a e^{(2+\sqrt{10})t} + (2-\sqrt{10})C_b e^{(2-\sqrt{10})t}$$

Then create a system of equations based on the initial conditions:

$$\begin{gathered} y(0) = 1 \Rightarrow C_a + C_b = 1 \\ y'(0) = 0 \Rightarrow C_a(2+\sqrt{10}) + C_b(2-\sqrt{10}) = 0 \end{gathered}$$

Solving the system of equations, C_1 and C_2 are found to be:

$$C_1 = \frac{-5-\sqrt{10}}{10} \qquad C_2 = \frac{5+\sqrt{10}}{10}$$

Giving the solution to the IVP as:

$$y = \frac{-5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} + \frac{5+\sqrt{10}}{10} e^{(2-\sqrt{10})t}$$

Analyzing the long term behavior:

$$\begin{gathered} \lim_{t\to\infty} \frac{-5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} = -\infty \\ \lim_{t\to\infty} \frac{5+\sqrt{10}}{10} e^{(2-\sqrt{10})t} = 0 \end{gathered}$$

The second term approaches 0, so the solution is dominated by the first term. The long term behavior of this solution, then, is that it approaches $-\infty$.

4.4.2 Complex Roots Case

See Section 5 for information regarding the use of complex numbers, Euler's Formula, etc., that will be used in this section.

Consider the ODE:

$$ay'' + by' + cy = 0$$

where a, b, and c are constants. By using the function $y = e^{rt}$ (see Subsection 4.4), the auxiliary equation is:

$$ar^2 + br + c = 0$$

Assuming that the quadratic equation produces two complex roots, these two roots can be written as:

$$r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta$$

Thus, the two complex solutions are:

$$y_1 = e^{(\alpha+i\beta)t},\ r_2 = e^{(\alpha-i\beta)t}$$

Euler's Formula (Subsection 5.1) allows them to be written as:

$$y_1 = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t),\ y_2 = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t)$$

At this point, each solution y_1 and y_2 exist as a complex function with a real part and an imaginary part. They can each be expressed in the form:

$$y(t) = u(t) + iv(t)$$

If

$$y(t) = u(t) + iv(t) \tag{9}$$

is a solution of

$$y'' + p(t)y' + q(t)y = 0 \tag{10}$$

then both $u(t)$ and $v(t)$ are solutions as well. This can be seen considering the first and second derivatives of (9):

$$\begin{aligned} y(t) &= u(t) + iv(t) \\ y'(t) &= u'(t) + iv'(t) \\ y''(t) &= u''(t) + iv''(t) \end{aligned}$$

and plugging them into (10):

$$\begin{aligned} y'' + p(t)y' + q(t)y &= 0 \\ u''(t) + iv''(t) + p(t)\big[u'(t) + iv'(t)\big] + q(t)\big[u(t) + iv(t)\big] &= 0 \\ u''(t) + pu'(t) + qu(t) + i\big[v''(t) + pv'(t) + qv(t)\big] &= 0 = 0 + 0i \end{aligned}$$

Since $a + bi = c + di \Leftrightarrow a = c,\ b = d$, it can be stated that:

$$u''(t) + pu'(t) + qu(t) = 0,\ i\big[v''(t) + pv'(t) + qv(t)\big] = 0$$

Therefore, both $u(t)$ and $v(t)$ are solutions.

Furthermore, if the auxiliary equation of $ay'' + by' + cy = 0$ has complex solutions $\alpha + i\beta$ and $\alpha - i\beta$, then the general solution of the ODE is:

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$

4.4.3 Repeated Roots Case

Consider an ODE such as:

$$ay'' + by' + cy = 0$$

with the auxiliary equation:

$$ar^2 + br + c = 0$$

that produces only a single root ($r_1 = r_2 = R$). In such a case, the quadratic equation only produces a single solution to the ODE:

$$y_1 = e^{Rt}$$

In these cases, another solution may be obtained as:

$$y_2 = te^{rt}$$

Thus creating the general solution of:

$$y = C_1 e^{rt} + C_2 te^{rt}$$

4.4.4 Summary

For a second order linear homogeneous ODE with constant coefficients:

$$ay'' + by' + cy = g(t)$$

The auxiliary equation is then:

$$ar^2 + br + c = 0$$

By solving the quadratic equation, three cases can occur:

1. Two distinct real roots (4.4.1): $r_1 \neq r_2$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

2. Two distinct complex roots (4.4.2): $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$

3. Two repeated real roots (4.4.3): $r_1 = r_2 = R$

$$y = C_1 e^{Rt} + C_2 te^{Rt}$$

5 Complex Numbers

Complex numbers are of the form:

$$a + bi$$

where a is the *real* part of the number, and b is the imaginary part. The multiplication of imaginary numbers is handled as:

$$(a + bi)(c + di)$$
$$\Rightarrow$$
$$ac + adi + cbi + bdi^2$$
$$\Rightarrow$$
$$(ac - bd) + i(ad + cb)$$

Furthermore, $a + bi$ may be identified with (a, b) in the complex plane where the real part (a) is represented by the x coordinate and the imaginary part (b) by the y coordinate. See Figure 7.

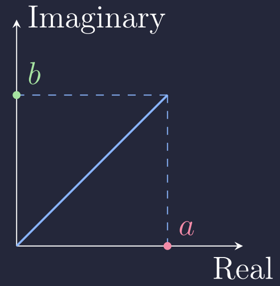


Figure 7: Geometric Interpretation

5.1 Euler's Formula

Euler's Formula provides a way to represent a complex number in terms of sin and cos:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{or}$$
$$re^{i\theta} = r\cos(\theta) + ri\sin(\theta),\ r \geq 0$$

As can be seen, Euler's Formula is simply just the polar coordinate transformation.

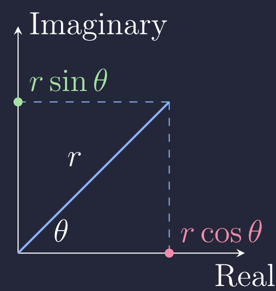
$$(r, \theta) \leftrightarrow (r\cos\theta, r\sin\theta)$$

Given a complex number:

$$a + bi$$

the amplitude (r) is the distance between $a + bi$ and the origin:

$$r = \sqrt{a^2 + b^2}$$



The angle θ, called the phase, is the angle formed between the x-axis and the line connecting $a + bi$ to the origin.

$$\begin{cases} \theta = \arctan(\frac{b}{a}),\ (a, b) \in \text{ quadrant } I \text{ or } IV \\ \theta = \arctan(\frac{b}{a}) + \pi,\ (a, b) \in \text{ quadrant } II \text{ or } III \end{cases}$$

5.2 Multiplication of Complex Numbers

Consider two complex numbers:

$$r_1 e^{i\theta_1} = r_1 \left(\cos\theta_1 + i\sin\theta_1\right),\ r_2 e^{i\theta_2} = r_2 \left(\cos\theta_2 + i\sin\theta_2\right)$$

Multiplying these two complex numbers results in:

$$\begin{aligned} r_1 e^{i\theta_1} r_2 e^{i\theta_2} &= r_1 \left(\cos\theta_1 + i\sin\theta_1\right) \cdot r_2 \left(\cos\theta_2 + i\sin\theta_2\right) \\ r_1 r_2 e^{i\theta_1 + i\theta_2} &= r_1 r_2 \Big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\Big] \\ r_1 r_2 e^{i(\theta_1+\theta_2)} &= r_1 r_2 \Big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big] \end{aligned}$$

By using the two following trig identities:

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{aligned}$$

Geometrically, this means that the new amplitude is equal to the *product* of the two original amplitudes. The new phase is the *sum* of the two original phases.

5.3 Powers of Complex Numbers

To square $re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$ using the Subsection 5.2, it results in:

$$r^2 e^{2i\theta} = r^2 \left(\cos(2\theta) + i\sin(2\theta)\right)$$

Generally, for any integer n, the n^{th} power of $re^{i\theta}$ is simply:

$$r^n e^{ni\theta} = r^n \left(\cos(n\theta) + i\sin(n\theta)\right)$$

Thus,

$$\begin{aligned} (1+i)^4 &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 \\ &= \left(\sqrt{2}\right)^4 \left(e^{i\frac{\pi}{4}}\right)^4 \\ &= 4e^{i\pi} \\ &= -4 \end{aligned}$$

5.4 Roots of Complex Numbers

The roots are somewhat more complicated. Generally, for the n^{th} root $\sqrt{z}$ has n different candidates.

Let:

$$z = Re^{i\theta},\ w = \sqrt{z}$$

means that:

$$w^n = z$$

If

$$w = re^{i\alpha}$$

then,

$$r^n e^{in\alpha} = Re^{i\theta}$$

The amplitude r is uniquely $R^{\frac{1}{n}}$, however, the phase α is *not* unique. This is because $e^{in\alpha} = e^{i\theta}$ means that:

$$n\alpha = \theta + 2k\pi, k = 0, \pm 1, \pm 2, \dots$$
$$\alpha = \frac{\theta + 2k\pi}{n}, k = 0, \pm 1, \pm 2, \dots$$

Example **5.1**

$$1^{\frac{1}{3}} = \left(e^{i \cdot 2k\pi}\right)^{\frac{1}{3}} = e^{i\frac{2k\pi}{3}}$$

Depending on the value of k, there are *three* different possibilites for e:

$$\text{If } k = 0, 3, 6, \dots \Rightarrow e^{i0} = 1$$
$$\text{If } k = 1, 4, 7, \dots \Rightarrow e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$\text{If } k = 2, 5, 8, \dots \Rightarrow e^{\frac{3\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

6 Non-Homogeneous ODE

Non-Homogeneous ODE's are of the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

These ODEs behave differently to previous ones. As such, the principle of superposition would now say:

Principle of Superposition (Non-Homogeneous) **6.1**

If Y_1, Y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_1 - Y_2$ is a solution of

$$y'' + p(t)y' + q(t)y = 0$$

The proof of this is as follows:

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= g(t) \\ y_2'' + p(t)y_2' + q(t)y_2 &= g(t) \end{aligned}$$

Then:

$$Y_1 - Y_2 = y_1'' - y_2'' + p(t)\left(y_1' - y_2'\right) + q(t)\left(y_1 - y_2\right) = g(t) - g(t) = 0$$

The general solutions of $y'' + p(t)y' + q(t)y = g(t)$ is of the form:

$$\begin{aligned} y &= y_c + Y \\ &= C_1y_1 + C_2y_2 + Y \end{aligned}$$

where $y_c = C_1y_1 + C_2y_2$ is the general solution of the *homogeneous* ODE $y'' + p(t)y' + q(t)y = 0$, and Y is a particular solution of the *non-homogeneous* $y'' + p(t)y' + q(t)y = g(t)$.

The terminology to describe the parts of this general solutions is:

- $y_c = C_1y_1 + C_2y_2$ is the **complementary solution**
- Y is the **particular solution**

Finding the complementary solution is simple. All that needs to be done is described in Subsection 4.4. To find the particular solution, two methods can be used:

1. Method of Undetermined Coefficients
2. Variation of Parameter

6.1 Method of Undetermined Coefficients

The method makes it easy to generalize to higher order ODEs, however it only works for:

$$ay'' + by' + c = g(t)$$

Where $g(t)$ is a product of exponential polynomials, sin, and cos functions. However, these limitations aren't very relevant for engineering applications.

6.1.1 Constant Functions

Consider the ODE:

$$y'' - 2y' - 3y = 3$$

Since the RHS is a constant function, set Y equal to some constant function A:

$$\begin{aligned} Y &= A \\ Y' &= 0 \\ Y'' &= 0 \end{aligned}$$

and plug in the ODE:

$$\begin{aligned} Y'' - 2Y' - 3Y = 0 - 0 - 3A &= 3 \\ \Rightarrow -3A &= 3 \\ \Rightarrow A &= -1 \end{aligned}$$

Giving $Y = -1$ as a particular solution, so the particular solution is:

$$Y = -1$$

6.1.2 Polynomial Functions

Consider the ODE:

$$y'' - y' - 2y = t^2 + 1$$

Since the RHS is a polynomial function of degree 2, Y can be set equal to a generic polynomial function of degree 2. Set:

$$\begin{aligned} Y &= At^2 + Bt + C \\ Y' &= 2At + B \\ Y'' &= 2A \end{aligned}$$

Thus:

$$\begin{aligned} Y'' - Y' - 2Y &= 2A - (2At + B) - 2\left(At^2 + Bt + C\right) \\ &= (-2A)t^2 + (-2A - 2B)t + (2A - B - 2C) \\ &= t^2 + 1 \end{aligned}$$

Setting $t^2 + 1$ and $(2A)t^2 + (-2A - 2B)t^1 + (2A - B - 2C)t^0$ equal to each other, it can be found that:

$$\begin{aligned} -2At^2 = 1t^2 &\Rightarrow -2A = 1 \\ (-2A - 2B)t^1 = (0)t^1 &\Rightarrow -2A - 2B = 0 \\ (2A - B - 2C)t^0 = (1)t^0 &\Rightarrow 2A - B - 2C = 1 \end{aligned}$$

Then solving for each value of A, B, and C:

$$A = -\frac{1}{2}$$
$$B = -A = \frac{1}{2}$$
$$C = \frac{2A - B - 1}{2} = -\frac{5}{4}$$

Giving the particular solution Y as:

$$Y = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4}$$

It should be noted that if the RHS is a polynomial of degree n, Y should be set as a generic polynomial of degree n. Even when $g(t) = t$, Y should still start from degree n, and work all the way down to the constant. For example, if:

$$g(t) = t^2 + t + 1 \Rightarrow Y = At^2 + Bt + C$$
$$g(t) = 6t^4 \Rightarrow Y = At^4 + Bt^3 + Ct^2 + Dt + E$$
$$g(t) = t \Rightarrow Y = At$$
$$g(t) = -t^3 + t - 8 \Rightarrow Y = At^3 + Bt^2 + Ct^1 + D$$

As a full example, consider the ODE:

$$y'' - y' - 2y = t^3$$

First, setting the particular solution Y as some generic polynomial of the n^{th} degree:

$$Y = At^3 - Bt^2 + Ct + D$$
$$Y' = 3At^2 - 2Bt + C$$
$$Y'' = 6At^1 - 2B$$

and plugging it in:

$$Y'' - Y' - 2Y = -2A^3 + (-3A - 2B)\,t^2 + (6A - 2B - 2C)t + (2B - C - 2D)$$

Comparing $Y = At^3 - Bt^2 + Ct + D$ with $-2A^3 + (-3A - 2B)\,t^2 + (6A - 2B - 2C)t + (2B - C - 2D)$, a system of equations can be found as:

$$-2A = 1$$
$$-3A - 2B = 0$$
$$6A - 2B - 2C = 0$$
$$2B - C - 2D = 0$$

Solving the system gives:

$$A = -\frac{1}{2}$$
$$B = -\frac{3}{2}A = \frac{3}{4}$$
$$C = \frac{1}{2}(6A - 2B) = -\frac{9}{4}$$
$$D = \frac{1}{2}(2B - C) = \frac{15}{8}$$

Thus finding the particular solution as:

$$Y = -\frac{1}{2}t^3 + \frac{3}{4}t^2 + -\frac{9}{4}t + \frac{15}{8}$$

and subsequently, the general solution as:

$$y = C_1e^{2t} + C_2e^{-t} - \frac{1}{2}t^3 + \frac{3}{4}t^2 + -\frac{9}{4}t + \frac{15}{8}$$

6.1.3 Exponential Functions

Consider the ODE:

$$y'' - y' - 2y = e^{3t}$$

Since the RHS is an exponential function, formulate Y as some generic exponential function with **the same** exponential coefficient.

$$\begin{aligned} Y &= Ae^{3t} \\ Y' &= 3Ae^{3t} \\ Y'' &= 9Ae^{3t} \end{aligned}$$

then plug into the ODE:

$$\begin{aligned} Y'' - Y' - 2Y &= 9Ae^{3t} - 3Ae^{3t} - 2\left(Ae^{3t}\right) \\ &= 9Ae^{3t} - 3Ae^{3t} - 2\left(Ae^{3t}\right) \\ &= 4Ae^{3t} = e^{3t} \end{aligned}$$

Thus showing that:

$$A = \frac{1}{4} \to Y = Ae^{3t} = \frac{1}{4}e^{3t}$$

6.1.4 Sine and Cosine Functions

Consider the ODE:

$$y'' - 2y' + y = 3\sin(3t)$$

Since the RHS is some linear combination of sin and cos functions, formulate Y as a generic summation of sin and cos with **the same** trigonometric coefficient.

$$\begin{aligned} Y &= A\sin(3t) + B\cos(3t) \\ Y' &= 3A\cos(3t) - 3B\sin(3t) \\ Y'' &= -9A\sin(3t) - 9B\cos(3t) \end{aligned}$$

then plug into the ODE:

$$\begin{aligned} Y'' - 2Y' + Y &= [-9A\sin(3t) - 9B\cos(3t)] - 2[3A\cos(3t) - 3B\sin(3t)] + [A\sin(3t) + B\cos(3t)] \\ &= (-8A + 6B)\sin(3t) + (-8B - 6A)\cos(3t) = 3\sin(3t) \end{aligned}$$

Thus showing that:

$$\begin{aligned} -8A + 6B &= 3 \\ -8B - 6A &= 0 \end{aligned}$$

and, in turn, that:

$$A = -\frac{6}{25} \; textup, \quad B = \frac{9}{50} \to Y = -\frac{6}{25}\sin(3t) + \frac{9}{50}\cos(3t)$$

Note that, for any linear combination of sin and cos, this process works. It doesn't matter that the example case of $g(t)$ was without a cos term. For example:

$$g(t) = 3\cos(2t) \Rightarrow Y = A\sin(2t) + B\cos(2t)$$
$$g(t) = \frac{1}{2}\sin(t) \Rightarrow Y = A\sin(t) + B\cos(t)$$
$$g(t) = \sin(4t) + \cos(4t) \Rightarrow Y = A\sin(4t) + B\cos(4t)$$

6.2 Ansatz

6.2.1 First Ansatz

These methods described in Subsubsections 6.1.1, 6.1.2, 6.1.3, and 6.1.4 all work because:

- Derivatives of constant functions are constant functions (0)
- Derivatives of polynomial functions are polynomials (up the same degree)
- Derivatives of exponential functions are exponential functions (with the same exponential coefficient)
- Derivatives of sine and cosine functions are sine and cosine functions (with the same trigonometric coefficient)

And so, the general approach for an ODE in the form:

$$ay'' + by' + cy = g(t)$$

If $g(t)$ is:

- A constant function, eg. 2, 16, -12:
$$Y = A$$
- A polynomial function of degree n, eg. $t^3 + 2t - 1$, $t - 12 + t^{100}$, t^1:
$$Y = At^n + Bt^{n-1} + Ct^{n-2} + \ldots + Dt + E = \sum_{i=0}^{n} C_i t^{n-i}$$
- An exponential function with an exponential coefficient of α, eg. $12e^{3t}$, e^{1t}, $32e^{-5t}$:
$$Y = Ae^{\alpha t}$$
- A linear combination of sin and cos with a trig coefficient of β, eg. $\sin(3t)$, $\sin(2t) - 12\cos(2t)$:
$$Y = A\sin(\beta t) + B\cos(\beta t)$$

This list of templates for solving for the particular solution of a non-homogeneous ODE continues with combinations of the basic types in Subsubsections 6.1.1, 6.1.2, 6.1.3, and 6.1.4.

If $g(t)$ is:

- A product of a polynomial and exponential function, eg. $4t^3e^{3t}$, $(t^3+2t)\,e^{-t}$

$$Y = \left(\sum_{i=0}^{n} C_i t^{n-i}\right) e^{\alpha t}$$

- A product of a polynomial and trigonometric function, eg. $4t^2\sin(2t)$, $t^4\sin(3t)+2t^2\cos(3t)$

$$Y = \left(\sum_{i=0}^{n} C_i t^{n-i}\right)\sin(\beta t) + \left(\sum_{i=n}^{2n} C_i t^{2n-i}\right)\cos(\beta t)$$

  It should be noted that both sin and cos are paired with a generic polynomial of the **same highest degree** from $g(t)$. Additionally, the coefficients of each term in each polynomial are entirely unique. For example, for:

$$g(t) = 2t\cos(3t) + \left(t^2+1\right)\sin(3t)$$

  The highest degree polynomial term is $n=2$, and thus:

$$Y = \left(At^2+Bt+C\right)\sin(3t) + \left(Dt^2+Et+F\right)\cos(3t)$$

- A product of a exponential and trigonometric function, eg. $3e^{3t}\cos(4t)$, $2e^{3t}\cos(4t)-3e^{3t}\sin(4t)$

$$Y = Ae^{\alpha t}\sin(\beta t) + Be^{\alpha t}\cos(\beta t)$$

  It should be noted that A and B both come from the fact that any function can be considered to have a polynomial factor of the zeroth degree, which would just be a constant function.

- A product of a polynomial, exponential, and trigonometric function, eg. $18(t^2-t)e^{3t}\cos(4t)-7e^{3t}\sin(4t)$

$$Y = \left(\sum_{i=0}^{n} C_i t^{n-i}\right) e^{\alpha t}\sin(\beta t) + \left(\sum_{i=n}^{2n} C_i t^{2n-i}\right) e^{\alpha t}\cos(\beta t)$$

The template to solve for the particular solution of a non-homogeneous ODE seen in (11) applies to all functions of $g(t)$ that any of the previous templates would apply to.

$$Y = \left(\sum_{i=0}^{n} C_i t^{n-i}\right) e^{\alpha t}\sin(\beta t) + \left(\sum_{i=n}^{2n} C_i t^{2n-i}\right) e^{\alpha t}\cos(\beta t) \tag{11}$$

Consider:

$$g(t) = 2t\cos(2t)$$

Though only a polynomial $(2t)$ and a trigonometric function $\big(\cos(2t)\big)$ are seen, there still exists an exponential function (e^0). Thus, it can be seen that $n=1$, $\alpha=0$, and $\beta=2$. Since $\alpha=0$, the exponential term would just be 1 which is why it can be essentially ignored.

This template to solve for the particular solution is referred to as the **First Ansatz**. It can be constructed generically based on the values of n, α, and β. Once the first ansatz in constructed, the particular solution can be solved through the same methods as seen in Subsection 6.1.

6.2.2 Higher Order Ansatz

Consider the ODE:

$$y'' - y' - 2y = e^{2t}$$

Based on Subsection 6.2.1, the particular solution Y should be formulated as:

$$\begin{aligned} Y &= Ae^{2t} \\ Y' &= 2Ae^{2t} \\ Y'' &= 4Ae^{2t} \end{aligned}$$

such that when it's plugged into the ODE:

$$\begin{aligned} Y'' - Y' - 2Y &= 4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} \\ &= 4Ae^{2t} - 4Ae^{2t} \\ &= 0 = g(t) = e^{2t} \end{aligned}$$

However, it can be seen that there is no such solution to the equation:

$$0 = e^{2t}$$

In such a situation where the first ansatz fails, the second ansatz should be used. Namely, by applying the variation of parameters, Y can be formulated as the first ansatz multiplied by a t factor:

$$\begin{aligned} Y = Ae^{2t} &\to Y = Ate^{2t} \\ Y' = 2Ae^{2t} &\to Y' = Ae^{2t} + 2Ate^{2t} \\ Y'' = 4Ae^{2t} &\to Y'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} \end{aligned}$$

and thus, when plugged in:

$$\begin{aligned} Y'' - Y' - 2Y &= 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} - Ae^{2t} - 2Ate^{2t} - 2Ate^{2t} \\ &= 3Ae^{2t} = g(t) = e^{2t} \end{aligned}$$

then, to solve the equation, the result is:

$$A = \frac{1}{3} \to Y = \frac{1}{3}te^{2t}$$

Now, consider:

$$y'' - 2y' + y = 2te^{t}$$

The first ansatz ($Y = (At + B)e^t$) fails, as does the second ansatz ($Y = (At^2 + Bt)e^t$). In such cases, continuing increasing the order of the ansatz applied will lead to success. Consider the third ansatz (found through multiplying by a factor of t^2):

$$\begin{aligned} Y &= (At^3 + Bt^2)e^t \\ Y' &= (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \\ Y'' &= (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \end{aligned}$$

By applying the same process as with previous ansatz:

$$\begin{aligned} Y'' - 2Y' + Y = \big((6At + 2B)e^t &+ 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t\big) \\ &- 2\big((3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t\big) + \big((At^3 + Bt^2)e^t\big) \end{aligned}$$

Which can be simplified into:

$$(6At + 2B)e^t = RHS = 2te^t \to A = \frac{1}{6},\ B = 0$$

6.2.3 Finding the Final Ansatz

Trying the first, then second, then third, etc., ansatz consecutively isn't the *wrong* way to approach finding the particular solution, but it's cumbersome and tedious. Can the ODE be analyzed in such a way that will determine which ansatz (1^{st}, 2^{nd}, 3^{rd}, etc) will work?

Consider the two previous examples:

$$y'' - y' - 2y = e^{2t}$$

1. This ODE was determined to need the second ansatz
2. The auxiliary equation is $r^2 - r - 2r$, which has roots $r_1 = 2$, $r_2 = -1$
3. The exponential coefficient $(\alpha + \beta i)$ of $g(t)$ is $2 + 0i$

$$y'' - 2y' + y = 2te^{t}$$

1. This ODE was determined to need the third ansatz
2. The auxiliary equation is $r^2 - 2r + r$, which has roots $r_1 = r_2 = 1$
3. The exponential coefficient $(\alpha + \beta i)$ of $g(t)$ is $1 + 0i$

Based on these two examples, it can be seen that, for the ODE:

$$ay'' + by' + cy = g(t)$$

where:

$$g(t) = e^{\alpha t}\cos(\beta t) \cdot (\text{polynomial}) + e^{\alpha t}\sin(\beta t) \cdot (\text{polynomial})$$

the first m ansatz fail, where m is the number of times the exponential coefficient $(\alpha + \beta i)$ appears as a root for the auxiliary equation. Thus, the first ansatz must be modified by t^m.

6.2.4 Principle of Superposition

In all $g(t)$ seen thus far, all exponential coefficients have been the same. However, this will not always be the case. What should be done in $g(t)$ is a sum of functions with different exponential coefficients (different α and β)?

In such cases, the principle of superposition can be applied:

Principle of Superposition **6.2**

If Y_1 is a particular solution of:

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and if Y_2 is a particular solution of:

$$y'' + p(t)y' + q(t)y = g_2(t)$$

then $Y_1 + Y_2$ is a particular solution of:

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$$

Essentially, individual particular solutions can be found for the sum components of $g(t)$, and then can be added together to find the solution to $g(t)$.

7 Higher Order Linear ODE

7.1 General Theory

7.1.1 Homogeneous Case

For homogeneous higher order linear ODEs:

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0$$

The fundamental set of solutions is a set of solutions $(y_1, \ldots, y_n)$ that are:

- solutions to the ODE
- linearly independent (Subsection 4.3)
- can be summed, through the principle of superposition, to find the general solution:

$$y = C_1y_1 + \ldots + C_ny_n$$

7.1.2 Non-Homogeneous Case

For non-homogeneous higher order linear ODEs:

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

The structure of the general solution is $y = y_c + Y$:

- y_c is the complementary solution (the general solution to the corresponding homogeneous ODE)
- Y is a particular solution to the non-homogeneous ODE

Solving these higher-order linear ODEs is incredibly similar to the second order ODEs in Section 6.

7.2 Constant Coefficients

7.2.1 Homogeneous Case

For the homogeneous ODE

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0$$

the auxiliary equation can be formulated as

$$a_nr^n + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$$

where r is a solution to the auxiliary equation. If r is:

- a real *single* root, then it contributes *one* function e^{rt} to the fundamental set of solutions
- a real root *repeated m times*, then it contributes m functions

  $$e^{rt}, te^{rt}, \ldots, t^{m-1}e^{rt}$$

  to the fundamental set of solutions

- a complex ($r = \alpha + \beta i$) *single* root, then together with its conjugate ($r = \alpha - \beta i$), two functions are contributed the to fundamental set of solutions

$$e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)$$

- a complex ($r = \alpha + \beta i$) root *repeated m times*, then together with its conjugate ($r = \alpha - \beta i$), $2m$ functions are contributed the to fundamental set of solutions

$$e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t),\ te^{\alpha t}\cos(\beta t),\ te^{\alpha t}\sin(\beta t),\ \ldots\ ,\ t^{m-1}e^{\alpha t}\cos(\beta t),\ t^{m-1}e^{\alpha t}\sin(\beta t)$$

7.2.2 Non-Homogeneous Case

For the ODE:

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

To find the complementary solution to a non-homogeneous ODE, the same exact steps as in Subsubsection 7.2.1 can be taken with the associated homogeneous ODE.

Similarly, the particular solution can be found just as it was in Subsection 6.2.

8 Laplace Transform

The general notation of a Laplace Transform is:

$$\mathcal{L}\{f(t)\} = F(s)$$

where $f(t)$ is an input function that will be transformed into some other function: $F(s)$. Notice, the variable each function depends on is different: $t \to s$.

The actual formula for computing the Laplace Transform is:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

8.1 Linearity of Laplace Transform

For some real numbers α and β, consider the following Laplace Transform:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\}$$

By the formula of the Laplace Transform in Section 8:

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty \big(\alpha f(t) + \beta g(t)\big) e^{-st}\, dt \\
&= \int_0^\infty \alpha f(t) e^{-st} + \beta g(t) e^{-st}\, dt \\
&= \int_0^\infty \alpha f(t) e^{-st}\, dt + \int_0^\infty \beta g(t) e^{-st}\, dt \\
&= \alpha \int_0^\infty f(t) e^{-st}\, dt + \beta \int_0^\infty g(t) e^{-st}\, dt \\
&= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}
\end{aligned}$$

Thus, it can be seen that, through the linearity of integrals, so too is the Laplace Transformation linear. In short:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

8.2 Basic Laplace Transform Formulas

For **power functions**, the Laplace Transform is:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ s > 0$$

This formula, in tandem with the linearity of Laplace Transforms, can be used to find the Laplace Transform of any polynomial function.

For any **exponential function**:

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s - \alpha},\ s > \alpha$$

For **sine** and **cosine**:

$$\mathcal{L}\{\cos(\beta t)\} = \frac{s}{s^2 + \beta^2},\ s > 0$$

$$\mathcal{L}\{\sin(\beta t)\} = \frac{\beta}{s^2 + \beta^2},\ s > 0$$

For **hyperbolic** functions:

$$\mathcal{L}\Big\{ \cosh(kt) \Big\} = \frac{s}{s^2 - k^2},\ s > k$$
$$\mathcal{L}\Big\{ \sinh(kt) \Big\} = \frac{k}{s^2 - k^2},\ s > k$$

8.3 Existence of the Laplace Transform

Prior to defining the existence of $\mathcal{L}\Big\{ f(t) \Big\}$, **piecewise continuity** and **exponential order** must be understood.

Piecewise Continuity **8.1**

A function is said to be piecewise continuous over some interval $[0, \infty)$ if, for any subinterval $[a, b]$, there is a finite number of discontinuities.

These discontinuities can be either hole or jump discontinuities since those exist at finite points. Consider the function:

$$f(t) = \frac{t^2}{t-2}$$

This function is *piecewise* continuous over $[0, \infty)$ since there only exists a discontinuity at $t = 2$, which is a finite number of discontinuities (1). Now, consider the function:

$$f(t) = \sqrt{t - 10}$$

Since this function is undefined when $t < 10$, the function is discontinuous at infinitely many numbers over the range $(0, 10)$. Thus, this function is *not* piecewise continuous.

Exponential Order **8.2**

A function is said to be of exponential order if there exist constants c, $M > 0$, and $T > 0$ such that:

$$\forall t > T,\ \Big| f(t) \Big| \leq Me^{ct}$$

In other words, $\Big|$f(t)$\Big|$ must not grow faster that some generic exponential function with some positive coefficient.

Polynomial functions, exponential functions of the form $e^{\alpha t}$, cosine, and sine functions are all of exponential order. However, consider the function:

$$f(t) = e^{t^2}$$

This function is *not* of exponential order since, for every possible $c > 0$:

$$\left| \frac{e^{t^2}}{e^{ct}} \right| = e^{t^2 - ct} = e^{t(t-c)} \to \infty$$

and thus e^{t^2} grows faster than any possible multiple of e^{ct} and is shown to *not* be of exponential order.

With these two ideas covered, $\mathcal{L}\Big\{ f(t) \Big\}$ exists for $s > c$ given that $f(t)$ is piecewise continuous over $[0, \infty)$ and is of exponential order. These two conditions are sufficient, but not necessary.

8.4 Long Term Behavior of the Laplace Transform

Continuing with piecewise continuity and exponential order, it can be said that if a function $f(t)$ is both piecewise continuous and of exponential order, then the function $F(s) = \mathcal{L}\big\{f(t)\big\}$ satisfies:

$$\lim_{s\to\infty} F(s) = 0$$

8.5 Inverse Laplace Transform

Just as any other inverse function, the inverse Laplace Transform will essentially reverse the process of the Laplace Transformation. If $F(s)$ represents the Laplace Transform of $f(t)$, then $f(t)$ is the inverse Laplace Transform of $F(s)$.

$$\mathcal{L}\big\{f(t)\big\} = F(s) \to \mathcal{L}^{-1}\big\{F(s)\big\} = f(t)$$

8.5.1 Linearity of the Inverse Laplace Transform

The inverse Laplace Transform maintains that same linearity as the Laplace Transform does. As such:

$$\mathcal{L}^{-1}\big\{\alpha F(s) + \beta G(s)\big\} = \alpha\mathcal{L}^{-1}\big\{F(s)\big\} + \beta\mathcal{L}^{-1}\big\{G(s)\big\}$$

8.5.2 Basic Inverse Laplace Transform Formulas

Furthermore, just as there are fundamental Laplace Transforms that should be memorized and immediately recognized, so too are the inverse Laplace Transforms to be memorized (they're the same ones, just in reverse).

For **power functions**:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

For **exponential functions**:

$$\forall\alpha \in \mathbb{R},\ \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

For **sine** and **cosine**:

$$\forall\beta \in \mathbb{R},\ \beta > 0 \to \mathcal{L}^{-1}\left\{\frac{s}{s^2+\beta^2}\right\} = \cos(\beta t)$$
$$\forall\beta \in \mathbb{R},\ \beta > 0 \to \mathcal{L}^{-1}\left\{\frac{1}{s^2+\beta^2}\right\} = \frac{1}{\beta}\sin(\beta t)$$

For **hyperbolic functions**:

$$\forall k \in \mathbb{R},\ k > 0 \to \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$$
$$\forall k \in \mathbb{R},\ k > 0 \to \mathcal{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$$

8.6 Laplace Transform of Derivatives

If $f(t)$ is continuous on the interval $[0, \infty)$, of exponential order, and if $f'(t)$ is piecewise continuous on $[0, \infty)$, then:

$$\mathcal{L}\big\{f'(t)\big\} = s\mathcal{L}\big\{f(t)\big\} - f(0) = sF(s) - f(0)$$

To prove this, consider the following Laplace Transform:

$$\mathcal{L}\big\{f'(t)\big\} = \int_0^\infty f'(t)e^{-st}\,dt$$

Through integration by parts, this can be written as:

$$\begin{aligned}
\int_0^\infty f'(t)e^{-st}\,dt &= \big[f(t)e^{-st}\big]_0^\infty - \int_0^\infty -s\cdot f(t)e^{-st}\,dt \\
&= \big[f(t)e^{-st}\big]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\
&= \left(\lim_{b\to\infty} f(b)e^{-sb} - f(0)e^0\right) + sF(s)
\end{aligned}$$

Under the assumption that $f(t)$ is of exponential order, for a sufficiently large value of s:

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

thus showing that:

$$\mathcal{L}\big\{f'(t)\big\} = sF(s) - f(0)$$

This process of proof can be used to prove:

$$\begin{aligned}
\mathcal{L}\big\{f''(t)\big\} &= s^2F(s) - sf(0) - f'(0) \\
\mathcal{L}\big\{f'''(t)\big\} &= s^3F(s) - s^2f(0) - sf'(0) - f''(0) \\
&\vdots \\
\mathcal{L}\big\{f^{(n)}(t)\big\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)
\end{aligned}$$

Provided that, for the Laplace Transform of some derivative $f^n(t)$, $f^n(t)$ is piecewise continuous and all previous functions $\big[f(t),\ f'(t), \ldots\big]$ are continuous over $[0, \infty)$ and of exponential order. The Laplace of the n^{th} derivative can also be expressed as:

$$\mathcal{L}\big\{f^{(n)}(t)\big\} = s^nF(s) - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0)$$

8.7 First Translation Theorem

First Translation Theorem **8.3**

If $\mathcal{L}\big\{f(t)\big\} = F(s)$, then

$$\mathcal{L}\big\{e^{at}f(t)\big\} = F(s-a)$$

or

$$\mathcal{L}\big\{e^{at}f(t)\big\} = \mathcal{L}\big\{f(t)\big\}\Big|_{s\mapsto s-a}$$

Consider the Laplace Transform of:

$$\mathcal{L}\left\{e^{at}f(t)\right\}$$

This can be rewritten as:

$$\int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a)$$

What this shows is that, when a function $f(t)$ is multiplied by an exponential function of e^{at}, the resulting Laplace Transform will be whatever the Laplace Transform of $f(t)$ is, with $s \mapsto s-a$.

The First Translation Theorem can be used to transform a function with an exponential factor.

Example **8.1**

Evaluate $\mathcal{L}\left\{e^{5t}t^3\right\}$:

$$\mathcal{L}\left\{e^{5t}t^3\right\} = \mathcal{L}\left\{t^3\right\}\Big|_{s\mapsto s-5} = \frac{3!}{s^4}\Big|_{s\mapsto s-5} = \frac{6}{(s-5)^4}$$

Evaluate $\mathcal{L}\left\{e^{-3t}sin(6t)\right\}$:

$$\mathcal{L}\left\{e^{-3t}sin(6t)\right\} = \mathcal{L}\left\{sin(6t)\right\}\Big|_{s\mapsto s+3} = \frac{6}{s^2+6^2}\Big|_{s\mapsto s+3} = \frac{6}{(s+3)^2+36}$$

It can also be used in reverse to inverse Laplace Transform a fraction with repeated linear factors:

Example **8.2**

Evaluate $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$:

First, the fraction is broken into partial fractions:

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2} \to \ldots \to A=2,\ B=11 \to \frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

Therefore:

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{(s-3)^2}\right\}$$

Then, by applying the First Translation Theorem:

$$\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}\Big|_{s\mapsto s-3} + \mathcal{L}^{-1}\left\{\frac{11}{s^2}\right\}\Big|_{s\mapsto s-3} = e^{3t}\mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + e^{3t}\mathcal{L}^{-1}\left\{\frac{11}{s^2}\right\} = e^{3t}\cdot 2\cdot 1 + e^{3t}\cdot 11\cdot t$$

Thus:

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{3t} + 11te^{3t}$$

8.7.1 Application on Fraction With an Irreducible Quadratic Denominator

The process in the previous example can be generalized into a process that will work with any fraction in the form of (under the assumption that the denominator is irreducible/has no real roots):

$$\frac{As+B}{as^2+bs+c}$$

where the general process is completing the square of the denominator to find α (the number to shift by) and β (the trigonometric coefficient), process the numerator to get it into the form of $s-\alpha+C$, then separating the numerator such that $s-\alpha$ is one fraction and C is another. From there, each fraction can be inverse Laplace Transformed on its own due to the linearity of the inverse Laplace Transform:

1) Complete the Square of the Denominator:

$$as^2+bs+c=a\left(s^2+\frac{b}{a}s+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)=a\left(\left(s+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a^2}\right)$$

During this process, it is assumed that the denominator is *irreducible*, meaning that it has no real roots. Because of this, it can be stated that $-b^2+4ac>0$. Thus:

$$\alpha=-\frac{b}{2a},\ \beta=\frac{\sqrt{-b^2+4ac}}{2a}$$

2) Process the Numerator to Find $s-\alpha$

$$As+B=A(s-\alpha+\alpha)+B=A(s-\alpha)+(A\alpha+B)$$

3) Putting it All Together

$$\frac{As+B}{as^2+bs+c}=\frac{A(s-\alpha)}{a\big[(s-\alpha)^2+\beta^2\big]}+\frac{A\alpha+B}{a\big[(s-\alpha)^2+\beta^2\big]}=\frac{A}{a}\cdot\frac{(s-\alpha)}{(s-\alpha)^2+\beta^2}+\frac{A\alpha+B}{a}\cdot\frac{1}{(s-\alpha)^2+\beta^2}$$

4) Inverse Laplace Transform the Fractions

$$\frac{A}{a}\cdot\mathcal{L}^{-1}\left\{\frac{(s-\alpha)}{(s-\alpha)^2+\beta^2}\right\}+\frac{A\alpha+B}{a}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)^2+\beta^2}\right\}$$

$$=\frac{A}{a}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+\beta^2}\right\}\Big|_{s\mapsto s-\alpha}+\frac{A\alpha+B}{a}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+\beta^2}\right\}\Big|_{s\mapsto s-\alpha}$$

$$=\frac{A}{a}e^{\alpha t}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+\beta^2}\right\}+\frac{A\alpha+B}{a}e^{\alpha t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+\beta^2}\right\}$$

$$=\frac{A}{a}e^{\alpha t}\cos(\beta t)+\frac{A\alpha+B}{a}\cdot\frac{1}{\beta}e^{\alpha t}\sin(\beta t)$$

8.8 Unit Step Function

The unit step function is defined as:

$$\mathcal{U}(t-a)=\begin{cases}0,\ 0\le t<a\\1,\ a\le t\end{cases}$$

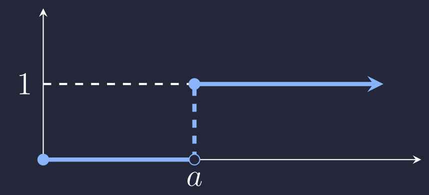


Figure 8: Unit Step Function

Any piecewise function can be expressed as a sum of functions with unit step functions. Consider the following generic piecewise function:

$$f(t)=\begin{cases}f_1(t),\ 0\le t<a\\f_2(t),\ t\le a\end{cases}$$

This function can be expressed as a sum of functions using the unit step function as follows:

$$f(t) = f_1(t) + \big(f_2(t) - f_1(t)\big)\mathcal{U}(t-a)$$

The general concept is that, prior to t reaching a, $\mathcal{U}(t-a) = 0$, so the $f_2(t) - f_1(t)$ has a coefficient of 0, and thus has no effect on the value of $f(t)$. However, upon t reaching a, $\mathcal{U}(t-a) = 1$, and thus the value of $f(t)$ is modified by the difference between $f_1(t)$ and $f_2(t)$.

In other words, once the unit step function's a is reached, the value of $f(t)$ changes from $f_1(t)$ to $f_2(t)$ by adding the difference between the two sub-functions. This "transformation" of a piecewise function can be done an arbitrary number of times:

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t), & a \le t < b \\ f_3(t), & b \le t < c \\ f_4(t), & c \le t < d \\ f_5(t), & d \le t \end{cases}$$

can be equivalently expressed as:

$$f_1(t) + \big(f_2(t) - f_1(t)\big)\mathcal{U}(t-a) + \big(f_3(t) - f_2(t)\big)\mathcal{U}(t-b) + \big(f_4(t) - f_3(t)\big)\mathcal{U}(t-c) + \big(f_5(t) - f_4(t)\big)\mathcal{U}(t-d)$$

Furthermore, a function given in terms of unit step functions can be translated back into a piecewise function:

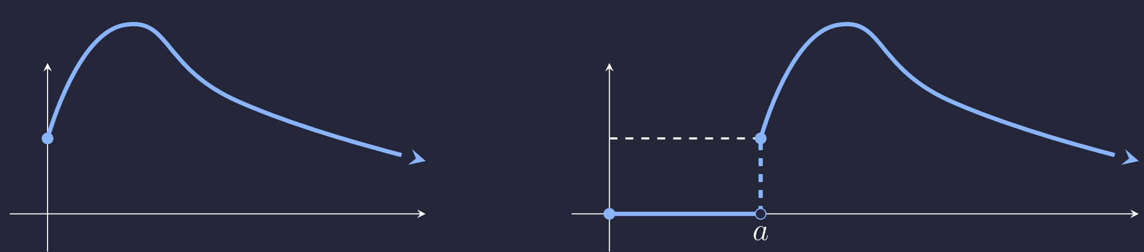
$$f(t) = g_1(t) + g_2(t)\mathcal{U}(t-a) + g_3(t)\mathcal{U}(t-b)$$

becomes

$$f(t) = \begin{cases} g_1(t), & 0 \le t < a \\ g_1 + g_2(t), & a \le t < b \\ g_1 + g_2 + g_3(t), & b \le t \end{cases}$$

8.8.1 Second Translation Theorem

Considering some function shifted by some value a with the unit step function:



(a) $y = f(t)$

(b) $y = f(t-a)\mathcal{U}(t-a)$

the **Second Translation Theorem** is:

Second Translation Theorem **8.4**

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

The proof of which is as follows. Considering some function $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}$, the Laplace Transform of the function is:

$$\int_0^\infty f(t-a)\mathcal{U}(t-a)e^{-st}\,dt = \int_a^\infty f(t-a)\mathcal{U}(t-a)e^{-st}\,dt$$

this equivalency is valid since, up until $t = a$, the value of the entire function is 0 since the unit step function has yet to be "activated". Then, mapping $(t-a) \mapsto T$:

$$\int_0^\infty f(T)\mathcal{U}(T)e^{-s(T+a)}\,dt = \int_0^\infty f(T)\mathcal{U}(T)e^{-sT-sa}\,dT = e^{-sa}\int_0^\infty f(T)\cdot 1\cdot e^{-sT}\,dT = e^{-as}F(s)$$

Thus:

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s) \quad \text{or} \quad \mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$