MATH 2070 - Differential Equations

Ethan Anthony

Contents

1 Introduction to Differential Equations

1.1 What is a Differential Equation

Differential Equation **1.1**

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation (DE).

Differential equations are foundational to studying engineering and physics. A very basic example of a differential equation is:

$$\frac{dx}{dt} + x = 2\cos(t) \tag{1}$$

Here x is the *dependent variable* and t is the *independent variable*. To solve (1) is to find x in terms of t such that the equation still holds when everything (x, t, and $\frac{dx}{dt}$) is plugged in. Consider

$$x = x(t) = \cos(t) + \sin(t)$$

as the solution for (1). Plugging this in as appropriate will verify this solution:

$$\begin{aligned}\big(-\sin(t) + \cos(t)\big) + \big(\cos(t) + \sin(t)\big) &= 2\cos(t)\\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) &= 2\cos(t)\\ \cos(t) + \cos(t) &= 2\cos(t)\\ 2\cos(t) &= 2\cos(t)\end{aligned}$$

Clearly this equality holds and a **particular solution** has been found for (1).

Particular Solution **1.2**

Some solution for a given differential equation.

1.2 Four Fundamental Equations

There exist four equations that are each very common and have solutions that can be memorized. The **first** among them is:

$$\frac{dy}{dx} = ky \tag{2}$$

For some constant $k > 0$, the general solution to (2) is:

$$y(x) = Ce^{kx}$$

The **second** among the four fundamental equations is:

$$\frac{dy}{dx} = -ky \tag{3}$$

For some constant $k > 0$, the general solution to (3) is:

$$y(x) = Ce^{-kx}$$

The **third** is a second order derivative (see 1.3.2):

$$\frac{d^2y}{dx^2} = -k^2y \tag{4}$$

For some constant $k > 0$, the general solution for (4) is:

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

Since (3) is of the second order, there are two constants in the solution.

Lastly, the **fourth** fundamental equation is:

$$\frac{d^2y}{dx^2} = k^2y \tag{5}$$

For some constant $k > 0$, the general solution for (5) is:

$$y(x) = C_1e^{kx} + C_2e^{-kx}$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(-kx)$$

Where:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

1.3 Classification of Differential Equations

There exist several types of differential equations. Consequently, they are classified according to **type**, **order**, and **linearity**.

1.3.1 Classification by Type

If a given differential equation 1) includes only ordinary derivatives of a number of unknown functions, and 2) those derivatives are all with respect to the same independent variable then it is classified as an **Ordinary Differential Equation (ODE)**.

Ordinary Differential Equation **1.3**

Equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.

$$\frac{dy}{dx} + 5y = e^x \; ; \frac{dy}{dx} + \frac{dr}{dx} = 14x \; ; \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} = 0 \tag{6}$$

Each equation in (6) is an example of an ODE. Notice that each equation contains only functions derived with respect to the same variable.

A **Partial Differential Equation (PDE)** differs in that it contains derivatives of functions with respect to multiple independent variables.

$$\frac{\delta y}{\delta x} + 5\frac{\delta r}{\delta t} = \ln(x) \; ; \frac{\delta y}{\delta x} + \frac{\delta r}{\delta t} = 14x \; ; \frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 b}{\delta x^2} \tag{7}$$

(7) are examples of partial differential equations. The Greek letter delta (δ) is used to denote a partial derivative. Thus, $\frac{\delta y}{\delta x}$ is the partial derivative of the function y with respect to x.

Partial Differential Equation **1.4**

Equations that depend on partial derivatives of several variables. That is, there are several independent variables.

1.3.2 Classification by Order

The **order** of a differential equation is the highest order among derivatives it contains. For example, (8) is a third-order differential equation because $\frac{d^3u}{dx^3}$ is a third derivative and is the highest derivative.

$$\frac{dy}{dx} - \frac{d^2r}{dx^2} = \frac{d^3u}{dx^3} \tag{8}$$

1.3.3 Classification by Linearity

An equation is linear if the dependent variable (or variables) and their derivatives appear linearly, that is, only as first powers, they are not multiplied together, and no other functions of the dependent variables appear.

$$e^x \frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} + x^2y = \frac{1}{x} \tag{9}$$

(9) is a linear differential equation because its *dependent* variable (y) only appears linearly. It does not matter that the independent variable (x) appears non-linearly. Conversely, (10) is non-linear because y is squared.

$$\frac{dy}{dx} = y^2 \tag{10}$$

Similarly, (11) is also non-linear because θ appears inside a sin function.

$$\frac{d^2\theta}{dx^2} + \sin(\theta) = 0 \tag{11}$$

2 Ordinary Differential Equations

Ordinary differential equations (ODE) are differential equations with just a single input, generally thought of as time (t).

Consider the relationships between position (x), velocity (v), and acceleration (a). Velocity is the derivative of position and acceleration the derivative of velocity.

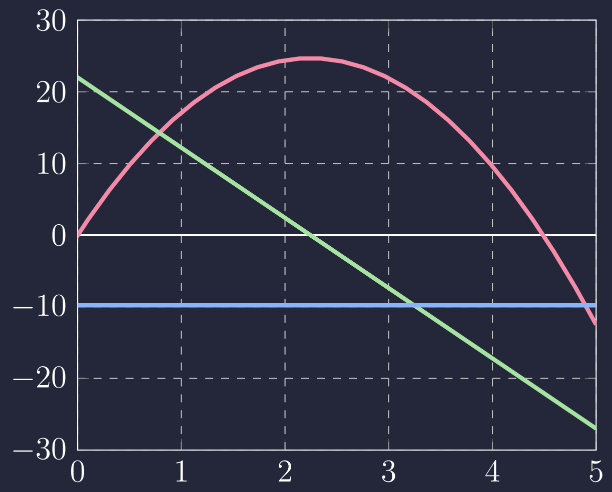


Figure 1: Position, Velocity, and Acceleration

If only the acceleration of an object across time is known ($g = 9.8\frac{m}{s^2}$), and nothing else about that object is known, then differential equations can be used to solve for the object's velocity and acceleration.

$$y''(t) = -g$$
$$\frac{d(?)}{dt}(t) = -g$$

Based on this, if a function can be found to have a derivative of $-g$ then the velocity of the object can be said to have a velocity equal to that function. In this example, it is as simple as integrating the function for acceleration:

$$\frac{d(-gt + v_0)}{dt}(t) = -g$$

Going one step further and integrating the velocity will yield the position of the object.

$$\frac{d\left(-\frac{1}{2}gt^2 + v_0 t + x_0\right)}{dt}(t) = -gt + v_0$$

The last thing to consider with this example are the initial conditions of the differential equation. v_0 and x_0 are not known quantities, however, if they were to be specified then the differential would have an initial condition to satisfy.

2.1 First Order ODE

Based on the classification in Subsubsection **??**, a first order ODE is a differential equation that whose highest order derivative is a first order derivative and all derivatives are with respect to the same variable.

First Order ODE **2.1**

A differential equation of the first order, with all derivatives being with respect to a single variable (usually x or t).

Previously, in Subsection 1.1, a particular solution was found for the differential equation of:

$$\frac{dx}{dt} + x = 2\cos(t)$$

However, this being a first order ODE, it has more than just a single particular solution. Consider:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - e^{-t}$$

as a solution to (2.1). To verify:

$$\begin{aligned}\big(-\sin(t) + \cos(t) - e^{-t}\big) + \big(\cos(t) + \sin(t) + e^{-t}\big) &= 2\cos(t)\\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) - e^{-t} + e^{-t} &= 2\cos(t)\\ \cos(t) + \cos(t) &= 2\cos(t)\\ 2\cos(t) &= 2\cos(t)\end{aligned}$$

As can be seen, there exists more than just the single particular solution. In fact, for (1), the entire family of solutions exists in the form:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - Ce^{-t}$$

where C is some constant. This is called a **One-Parameter Family of Solutions** for the differential equation.

One-Parameter Family of Solutions **2.2**

A one-parameter family of solutions is a solution to a differential equation containing a single constant C. This constant is arbitrary, and thus the family of solutions it represents consists of all values of C.

If a one-parameter family of solutions contains every possible solution to the differential equation, then it is referred to as the **General Solution**.

General Solution **2.3**

The entire family of solutions for a given differential equation. A general form of the solution that can be adapted to different specifications.

Each value of C gives a different solution, so really there are infinite solutions for (1).

2.1.1 First Order Linear ODE

First Order Linear ODEs follow a simple pattern for their solutions. However, before getting to that, what is a first order linear ODE?

First Order Linear ODE **2.4**

A differential equation that 1) doesn't contain a derivative beyond the first order, 2) contains only derivatives with respect to a single variable, and 3) has its dependent variable appear linearly (not part of a sin, cos, square, etc.).

If a differential equation is given in **standard form**:

$$y' + p(t)y = g(t)$$

Where $p(t)$ and $g(t)$ are arbitrary functions of t, then the general solution can be expressed as:

$$y(t) = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)}$$

Where the **integrating factor** ($\mu(t)$) is:

$$\mu(t) = e^{\int p(t)\,dt}$$

Example **2.1**

$$y' + 2y = e^{3t},\ y(0) = 3$$

This ODE is already in standard form, so the integrating factor can be written as:

$$\begin{aligned}\mu(t) &= e^{\int 2\,dt} \\ &= e^{2t}\end{aligned}$$

And thus the general solutions is:

$$\begin{aligned}y(t) &= \frac{\int e^{2t} \cdot e^{3t}\,dt + C}{e^{2t}} \\ &= \frac{\int e^{5t}\,dt + C}{e^{2t}} \\ &= \frac{\frac{1}{5}e^{5t} + C}{e^{2t}} \\ &= \frac{1}{5}\frac{e^{5t}}{e^{2t}} + \frac{C}{e^{2t}} \\ &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}}\end{aligned}$$

Using the initial condition to find the specific solution:

$$\begin{aligned}y(t) &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}} \\ y(0) &= \frac{1}{5} \cdot e^{3\cdot 0} + \frac{C}{e^{2\cdot 0}} \\ 3 &= \frac{1}{5} + C \\ \frac{14}{5} &= C\end{aligned}$$

Thus, the solution of the IVP is:

$$y(t) = \frac{1}{5} \cdot e^{3t} + \frac{14}{15} \cdot e^{-2t}$$

2.1.2 First Order ODE

A first order ODE is an equation in the form of:

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y) \tag{1}$$

There is no strict process to find a solution to these equations. Thus, a lot of this class is spent on the various different ways to find solutions. To see one of the simpler ways, consider an equation where f is a function of only x:

$$y' = f(x)$$

Integrating both sides with respect to x, the equation becomes:

$$\int y'\,dx = \int f(x)\,dx + C$$
$$y(x) = \int f(x)\,dx + C$$

This $y(x)$ is the general solution to (1). Thus, to solve a differential equation with just a single dependent variable x, finding the antiderivative of $f(x)$ is sufficient to find the general solution.

Example **2.2**

Find the general solution for:

$$y' = 3x^2$$

Integrating each side gives:

$$\int y'\,dx = \int 3x^2\,dx + C$$
$$y(x) = x^3 + C$$

Thus, the general solution for $y' = 3x^2$ is $y(x) = x^3 + C$.

2.1.3 Initial Value Problem

Generally, there also might be a condition that the solution to a differential equation must satisfy. In general terms, this might look like:

$$y' = f(x),\ y(x_0) = y_0 \tag{2}$$

Leaving the solution to (2) as:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$

Verifying the solution, first y' is computed based on our solution.

$$\frac{d}{dx}\big(y(x)\big) = \frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt + y_0\right)$$
$$y' = f(x)$$

Second, to verify that the initial condition is satisfied:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$
$$y(x_0) = \int_{x_0}^{x_0} f(t)\,dt + y_0$$
$$y(x_0) = 0 + y_0$$
$$y(x_0) = y_0$$

This confirms the initial condition, and thus it can be seen that the solution to the differential equation with an initial condition has been found.

Example **2.3**

Solve

$$y' = e^{-x^2}, \ y(0) = 1$$

First, finding the solution, both sides can be integrated:

$$y(x) = \int_0^x e^{-t^2}\,dt + 1$$

And to verify the solution:

$$\frac{d}{dx}\big(y(x)\big) = \frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt + 1\right)$$
$$y' = e^{-x^2}$$

$$y(0) = \int_0^x e^{t^2}\,dt + 1$$
$$y(0) = 0 + 1$$
$$y(0) = 1$$

The solution passes both verification tests, and so it can be safely said that the solution has been found.

Using the same method as before, equations of the form in (3) can be solved as well.

$$y' = f(y) \quad \text{or} \quad \frac{dy}{dx} = f(y) \tag{3}$$

(3) can be rewritten using the inverse function theorem from calculus to switch the roles of x and y to get:

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

Finally, at this point both sides can be integrated with respect to y to get:

$$x(y) = \int \frac{1}{f(y)}\,dy + C$$

From here, it is just a matter of solving for y.

Example **2.4**

In Subsection 1.2, the claim was made that the solution for $y' = ky$ is $y = Ce^{kx}$ for $k > 0$. To show this, the method of integration can be used:

$$\frac{dy}{dx} = ky \to \frac{dx}{dy} = \frac{1}{ky}$$
$$x(y) = \int \frac{1}{ky}\, dy$$
$$x(y) = \frac{1}{k} \ln |y| + D$$

Now solving for y:

$$x(y) = \frac{1}{k} \ln |y| + D$$

Example **2.5**

$$y' + y = \cos(2t)$$

This is already in standard form, so we can go on to calculate the integrating factor:

$$\mu(t) = e^{\int 1\, dt} = e^t$$

Thus, the general solution will be:

$$y = \frac{\int e^t \cdot \cos(2t)\, dt}{e^t}$$

Integrating the numerator:

$$\int e^t \cdot \cos(2t)\, dt$$
$$\vdots$$
$$\int e^t \cdot \cos(2t)\, dt = \frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C$$

Thus:

$$y = \frac{\frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C}{e^t}$$

or:

$$y = \frac{1}{5}\left(\cos(2t) + 2\sin(2t)\right) + Ce^{-t}$$

2.2 Non-Linear ODE

Generally, non-linear ODEs are quite difficult to solve. For some of them, there are techniques that can be used to simplify the process.

2.2.1 Separable ODE

The basic form of a separable ODE is:

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

i.e., the derivative is a product of two functions, one depending on x and the other on y.

This type of ODE is solved first by separating the variables. The LHS is reserved for y and the RHS for x:

$$\frac{dy}{f(y)} = g(x)dx$$

Now that there is a single type of variable on each side (only x or only y), both sides can be integrated. This will yield a one-parameter family of *implicit solutions* (see Subsection 2.3).

Example **2.6**

$$y' = xy$$

This equation can be rewritten as:

$$\frac{dy}{dx} = x \cdot y$$

Here, it can be easily seen that this equation is separable:

$$\frac{dy}{y} = x\ dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx + C$$

$$\ln|y| = \frac{x^2}{2} + C$$

This here is the implicit solution to the ODE. It won't always be the case, but here algebra can be used to find the explicit solution:

$$e^{\ln|y|} = e^{\frac{x^2}{2}+C}$$

$$y = e^{\frac{x^2}{2}} \cdot e^C$$

$$y = De^{\frac{x^2}{2}}$$

Where $D > 0$. Since $y = 0$ is a solution as well, this can be simplified to:

$$y = De^{\frac{x^2}{2}}$$

Principle: If an ODE is given without initial values, a one-parameter family of solutions is sufficient. However, if an initial value is given (IVP), an explicit solution might be possible with an interval of existence.

Example **2.7**

$$\frac{dy}{dx} = \frac{e^x - x}{e^{-y} + y}$$

First, separating the variables to each side:

$$\frac{dy}{dx} = (e^x - x) \cdot \frac{1}{e^{-y} + y}$$
$$\left(e^{-y} + y\right) dy = (e^x - x)\, dx$$

Then integrating each side appropriately:

$$\int e^{-y} + y\, dy = \int e^x - x\, dx$$
$$-e^{-y} + \frac{1}{2}y^2 = e^x - \frac{1}{2}x^2 + C$$

And thus, a one-parameter family of implicit solutions has been found.

2.2.2 Singular Solutions

Consider:

$$y' = xy^3 \left(1 + x^2\right)^{-\frac{1}{2}} \tag{4}$$

(4) is a separable ODE that can be reorganized as:

$$y' = y^3 \cdot \frac{x}{\sqrt{1 + x^2}}$$

And subsequently separated into:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1 + x^2}}\, dx$$

And solved:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1 + x^2}}\, dx$$
$$\int \frac{1}{y^3}\, dy = \int \frac{x}{\sqrt{1 + x^2}}\, dx$$
$$-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$$

It would now be tempting to say that $-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$ is a general solution to (4). However, $y = 0$ is also a solution to (4)...

The true answer then would be to say that $-\frac{1}{2y^2} = \sqrt{1 + x^2} + C$ is a *family* of solutions rather than the general solution and that $y = 0$ is a singular solution.

2.3 Implicit and Explicit Solutions

In general, anytime an ODE is given *without* initial values, then a single-parameter family of *implicit* solutions is sufficient.

If given an IVP, however, then finding an *explicit* solution and an *interval of existence* should be attempted. In some situations, this won't be possible, but it should be attempted at least.

2.3.1 Implicit Solutions

Sometimes a wall is reached even if the integration is possible. Consider:

$$y' = \frac{xy}{y^2+1} \tag{5}$$

Using the technique described in Subsection 2.2.1, this can be separated into:

$$\begin{aligned} y' &= \frac{xy}{y^2+1} \\ \frac{dy}{dx} &= \frac{xy}{y^2+1} \\ \frac{y^2+1}{y} dy &= x\ dx \\ \left(y + \frac{1}{y}\right) dy &= x\ dx \end{aligned}$$

Integrating both sides gives:

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C$$

The integration of this equation is quite simple. However, try to solve for y and see how difficult that will be. Though solving for y itself is too difficult, this form is still a solution and can still be verified.

$$\begin{aligned} \frac{d}{dx}\left(\frac{y^2}{2} + \ln|y|\right) &= \frac{d}{dx}\left(\frac{x^2}{2} + C\right) \\ y'\left(y + \frac{1}{y}\right) &= x \\ y \cdot \left(y'\left(y + \frac{1}{y}\right)\right) &= y \cdot x \\ y'\left(y^2+1\right) &= y \cdot x \\ y' &= \frac{xy}{y^2+1} \end{aligned}$$

Producing the exact same equation in (5), thus verifying the solution.

Since these solutions are implicit, they might not be able to be graphed as a valid function. In those cases, other information, such as an initial condition, can be used to further inform the appropriate solution.

2.4 Pathological and Reasonable IVP

2.4.1 Pathological IVP

Consider:

$$\frac{dy}{dx} = x\sqrt{y},\ y(0) = 0 \tag{6}$$

Solving this IVP, it can be seen that both:

$$y = \frac{1}{16}x^2 \quad \text{and} \quad y = 0$$

are both solutions to (6). This would then be referred to as a **Pathological IVP**.

Pathological IVP **2.5**

An IVP with zero solutions, more than one solution, or infinitely many solutions.

Example **2.8**

$$ty' + (t-1)y = -e^{-t},\ y(0) = 1$$

First, putting the ODE into standard form:

$$y' + \frac{t-1}{t}y = \frac{-e^{-t}}{t}$$

Thus:

$$p(t) = \frac{t-1}{t} \qquad g(t) = \frac{-e^{-t}}{t} \qquad \mu(t) = \frac{e^t}{t}$$

Giving the general solution as:

$$y = \frac{\int \frac{e^t}{t} \cdot \frac{-e^{-t}}{t}\, dt}{\frac{e^t}{t}} = \frac{-\int \frac{1}{t^2}\, dt}{\frac{e^t}{t}} = \frac{\frac{1}{t} + C}{\frac{e^t}{t}} = e^{-t} + Cte^{-t}$$

Now to solve the IVP, plugging in the values of $y(0)$ and $t = 0$:

$$y = e^{-t} + Cte^{-t} \quad \Rightarrow \quad 1 = e^0 + C \cdot 0 \cdot e^0 = 1 + 0$$

The disappearance of C (due to being multiplied by 0) reveals that there are infinitely many solutions to this IVP, thus showing its pathological nature.

Example **2.9**

$$ty' + (t-1)y = -e^{-t},\ y(0) = 0$$

By taking the same IVP as previous, but changing the initial condition from $y(0) = 1$ to $y(0) = 0$, the behavior changes:

$$y = e^{-t} + Cte^{-t} \quad \Rightarrow \quad 0 = e^0 + C \cdot 0 \cdot e^0 = 1 + 0$$

Clearly, $0 \neq 1$, showing that this IVP has zero solutions. Again, this is a pathological IVP.

2.4.2 Reasonable IVP

It might be tempting to say that any IVP that is *not* pathological is then reasonable. While exclusivity exists between the two, the proper method to determine if an IVP is reasonably posed is by using the **Existence and Uniqueness Theorem**.

Existence and Uniqueness Theorem **2.6**

Consider:

$$y' + p(t)y = g(t) + C,\ y(t_0) = y_0$$

where the ODE is *linear* and in its *standard form.* Assuming that:

1. Both $p(t)$ and $g(t)$ are continuous over the open interval (a, b)
2. The open interval (a, b) contains t_0

Then there exists a unique function $y = y(t)$ over the interval (a, b) that solves the IVP.

Notice that, to determine if a unique solutions exists based on the existence and uniqueness theorem, the IVP need not be solved.

Example **2.10**

$$ty' + (t-1)y = -e^{-t},\ y\left(\ln|2|\right) = \frac{1}{2}$$

Putting into standard form:

$$y' + \frac{t-1}{t}y = \frac{-e^{-t}}{t}$$

From here, $p(t)$ and $g(t)$ can be seen to be:

$$p(t) = \frac{t-1}{t} \qquad g(t) = \frac{-e^{-t}}{t}$$

Both $p(t)$ and $g(t)$ are continuous everywhere except for at $t = 0$, where both of their denominators will equal 0. This says that there are two open intervals over which there could possbily be a single unique solution: $(-\infty, 0)$ or $(0, \infty)$.

Because the initial value is $y\left(\ln|2|\right) = \frac{1}{2}$, the second interval is chosen as $t_0 = \ln|2|$ lays on the interval $(0, \infty)$.

Based on this theorem, it can be asserted that there exists a unique solution over an interval (a, b) provided that this interval contains t_0 from the IVP and both functions $p(t)$ and $g(t)$ are continuous over this interval.

2.5 Intervals of Existence

2.5.1 Linear IVPs

Based on the existence and uniqueness theorem as outlined in Subsubsection 2.4.2, the general steps for finding the interval of existence is:

1. Find the standard form
$$y' + p(t)y = g(t),\ y(t_0) = y_0$$
2. Locate the singular points (where $p(t)$ or $g(t)$ are not continuous)
3. Based on the singular points, find the open intervals over which both $p(t)$ and $g(t)$ are continuous
4. Pick the interval that contains t_0

Example **2.11**

$$ty' + (t-1)y = -e^{-t},\ y\left(\ln(2)\right) = \frac{1}{2}$$

To use the theorem, first the equation must be expressed in its standard form:

$$y' + \frac{t-1}{t} \cdot y = -\frac{e^{-t}}{t}$$

Where:

$$p(t) = \frac{t-1}{t}$$
$$g(t) = -\frac{e^{-t}}{t}$$

Since, for both of these functions, the only point of discontinuity is $t = 0$, then it is known that the only two possible intervals of existence are either $(-\infty, 0)$ or $(0, \infty)$.

Furthermore, since the initial condition is given as $y\left(\ln(2)\right) = \frac{1}{2}$, $t_0 = \ln(2)$. Thus, the interval of existence for the solution of this ODE would be the interval of $(0, \infty)$.

2.5.2 Non-Linear IVPs

Consider:

$$y' = f(t, y),\ y(t_0) = y_0$$

Assume that:

1. The function $f(t, y)$ is continuous **near** (t_0, y_0)
2. The function $\frac{\delta f}{\delta y}(t, y)$ is continuous **near** (t_0, y_0)

Then a unique solution $y = y(t)$ exists **near** $t = t_0$, i.e., there exists a small number $\epsilon > 0$ such that a solution exists on $(t_0 - \epsilon, t_0 + \epsilon)$.

Example **2.12**

$$y' = y^{\frac{1}{3}},\ y(0) = 1$$

First, find the behavior of $f(t, y)$:

$$f(t, y) = y^{\frac{1}{3}} \text{ is continuous everywhere.}$$

Then, find the behavior of $\frac{\delta f}{\delta y}(t, y)$:

$$\frac{\delta f}{\delta y}(t, y) = \frac{d}{dx}\left(y^{\frac{1}{3}}\right) = \frac{1}{3}y^{-\frac{2}{3}} \text{ is continuous everywhere when } y \neq 0.$$

Thus, since the IVP is $y(0) = 1 \Rightarrow (t_0, y_0) = (0, 1)$, the IVP has a unique solution near $t = 0$. The discontinuity when $y = 0$ is irrelevant because it is not near $y_0 = 1$.

Though still very helpful, this theorem is not as powerful as the linear version as it only concludes local existence rather than an interval of existence. However, it can still be used to determine whether an IVP is reasonably or pathologically formulated.

2.6 Autonomous ODE

$$y' = f(y)$$

Such that the right hand side (RHS) does not involve t. In other words, it is a function purely in terms of y.

The equilibrium solution:

$$y = y_0$$

Such that:

$$f(y_0) = 0$$

This means that the equilibrium solutions must be a value such that $y'(y_0) = 0$.

Example **2.13**

$$y' = y^3 - y$$

Clearly, the RHS is purely in terms of y, so this would be able to be solved as an autonomous ODE. So, taking the RHS and solving for 0:

$$\begin{aligned} y^3 - y &= 0 \\ y\left(y^2 - 1\right) &= 0 \\ y\left(y-1\right)\left(y+1\right) &= 0 \\ y = 0;\ y = 1;\ y &= -1 \end{aligned}$$

Thus, there are three equilibrium solutions.

$$@y = 0$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow 0\left(0-1\right)\left(0+1\right) = 0 \cdot -1 \cdot 1 = 0$$

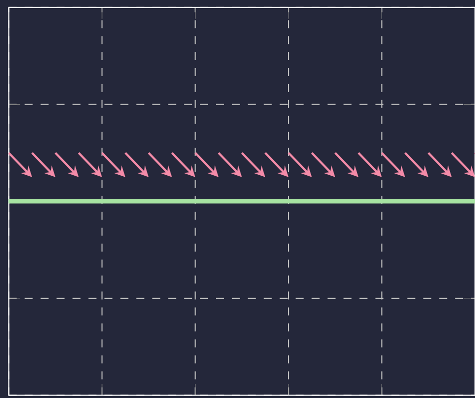
$$@y = 1$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow 1\left(1-1\right)\left(1+1\right) = 1 \cdot 0 \cdot 2 = 0$$

$$@y = -1$$
$$y' = y\left(y-1\right)\left(y+1\right) \Rightarrow -1\left(-1-1\right)\left(-1+1\right) = -1 \cdot -2 \cdot 0 = 0$$

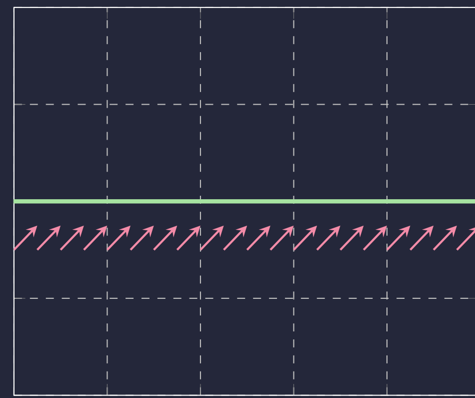
2.6.1 Stability of an Equilibrium Solution

Stability **2.7**

Let $y = y_0$ be an equilibrium solution of $y' = f(y)$. $y = y_0$ is *stable from above* if, for every $y > y_0$ near y_0, $f(y) < 0$.



(a) Stable From Above

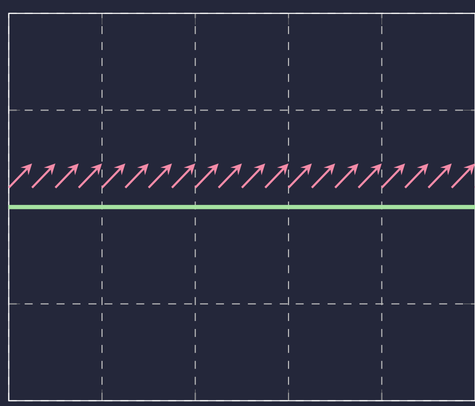


(b) Stable From Below

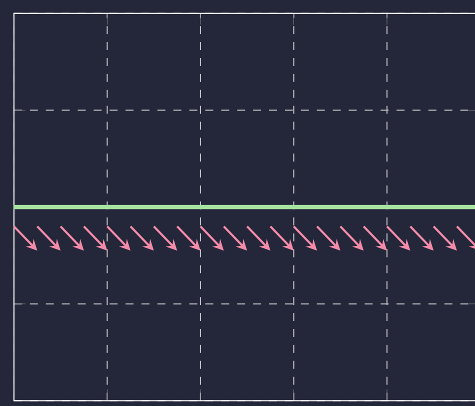
Figure 3: Stability

Using Figure 3a as a visual, the solution is stable from above because, if you were to deviate from the solution, the direction field above will guide you back down to the solution. Figure 3b is stable from below for the same reasons.

It follows then, that if the direction field around the solution points *away* from the solution, then the solution is considered to be *un*stable. See this in Figure 5.



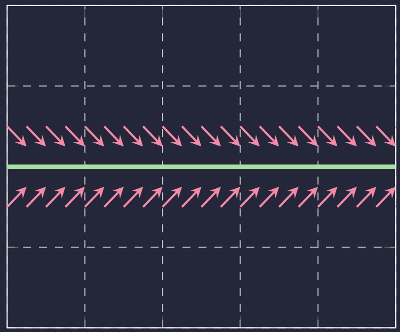
(a) Unstable From Above



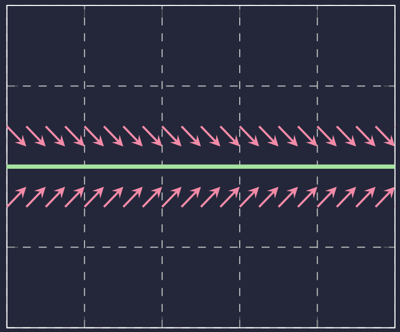
(b) Unstable From Below

Figure 4: Instability

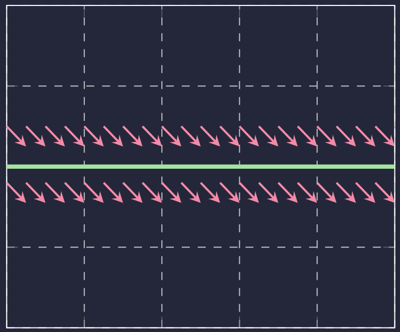
When considering a solutions behavior from both sides (top and bottom), it can be said to be either stable, unstable, or semistable.



(a) Stable



(b) Unstable



(c) Semistable

Figure 5: Instability

Example **2.14**

Falling object from a great height, subject to acceleration due to gravity (mg) and air resistance (kv^2). Newton's second law:

$$m \cdot \frac{dv}{dt} = mg - kv^2;\ v(0) = 0$$

This ODE is autonomous since the RHS has no t involved, so the equilibrium solution will be obtained by setting the RHS = 0.

$$mg - kv^2 = 0$$
$$\vdots$$
$$v = \pm\sqrt{\frac{mg}{k}}$$

Finding the stability of this solution:
Based on this model, the terminal velocity of the falling object will be:

$$v = \sqrt{\frac{mg}{k}}$$

This ODE can also be solved by finding the explicit solution, first by rearranging the equation:

$$m \cdot \frac{dv}{dt} = mg - kv^2$$
$$\vdots$$
$$\frac{m}{mg - kv^2} dv = dt$$

Then integrate both sides. Keep in mind that g, k, and m are constants.

$$\int \frac{m}{mg - kv^2}\, dv = \int\, dt$$
$$\frac{m}{k}\int \frac{m}{\frac{mg}{k} - v^2}\, dv = \int\, dt$$
$$\frac{m}{k}\int \frac{m}{\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)}\, dv = \int\, dt$$

2.7 Second Order ODEs

Similar to first order ODe's, second order ODE's also have a standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous **2.8**

If the RHS of a second order ODE in standard form is equal to zero ($g(t) = 0$), then the ODE is said to be homogeneous. Otherwise ,the ODE is non-homogeneous.

Rather than focus first on how to solve these second order ODEs, first a little bit of theory about them will be covered.

2.7.1 Existence and Uniqueness Theorem

Given an IVP in the standard form with:

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0$$

and given that $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval (a, b), and $t\exists(a, b)$, then the IVP has a unique solution on the interval (a, b).

2.7.2 Wronskian for Linear Independence

If y_a, y_b are solutions of $y'' + p(t)y' + q(t)y = 0$ on an interval where the existence and uniqueness theorem (2.7.1) holds, then y_a, y_b are linearlly independent if and only if:

$$W(y_a, y_b) = \begin{vmatrix} y_a(t) & y_b(t) \\ y'_a(t) & y'_b(t) \end{vmatrix} = (y_a(t) \cdot y'_b(t)) - (y_b(t) \cdot y'_a(t)) = 0$$

on the interval.

2.7.3

Consider the ODE:

$$ay'' + by' + cy = 0 \tag{7}$$

where a, b, and c are constants.

Idea: Try $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Using this, (7) becomes:

$$\begin{aligned} ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ \left(ar^2 + br + c\right) e^{rt} &= 0 \end{aligned}$$

e^{rt} will never be 0, so to solve this, use the quadratic equation to solve for r:

$$ar^2 + br + c = 0$$

This equation is referred to as the **auxiliary equation**. If the auxiliary equation has two distinct real roots $r_a \neq r_b$, then there are two solutions:

$$y_a = e^{r_at} \quad \text{and} \quad y_b = e^{r_bt}$$

$$W(y_a, y_b) = \begin{vmatrix} e^{r_at} & e^{r_bt} \\ r_ae^{r_at} & r_be^{r_bt} \end{vmatrix} = \left(e^{r_at} \cdot r_be^{r_bt}\right) - \left(e^{r_bt} \cdot r_ae^{r_at}\right) \neq 0 (\text{since } r_a \neq r_b)$$

Example **2.15**

$$y'' - 5y' + 6y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ (r-2)(r-3) &= 0 \\ r_a &= 2, r_b = 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2t} + C_b e^{3t}$$

Example **2.16**

$$2y'' - 7y' + 3y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} 2r^2 - 7r + 3 &= 0 \\ (2r-1)(r-3) &= 0 \\ r_a &= \frac{1}{2}, r_b = 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{\frac{1}{2}t} + C_b e^{3t}$$

Example **2.17**

$$y'' - 4y' - 6y = 0, y(0) = 1, y'(0) = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 4r - 6 &= 0 \\ r^2 - 4r &= 6 \\ r^2 - 4r &= 6 \\ r^2 - 4r + 4 &= 10 \\ (r-2)^2 &= 10 \\ r - 2 &= \pm\sqrt{10} \\ r &= 2 \pm \sqrt{10} \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2+\sqrt{10}} + C_b e^{2-\sqrt{10}}$$

To solve for the initial conditions, first find y':

$$y' = (2+\sqrt{10})C_a e^{(2+\sqrt{10})t} + (2-\sqrt{10})C_b e^{(2-\sqrt{10})t}$$

Then create a system of equations based on the initial conditions:

$$\begin{gathered} y(0) = 1 \Rightarrow C_a + C_b = 1 \\ y'(0) = 0 \Rightarrow C_a(2+\sqrt{10}) + C_b(2-\sqrt{10}) = 0 \end{gathered}$$

Solving the system of equations, the solution is found:

$$y = \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} + \frac{5+\sqrt{10}}{10} e^{(2-\sqrt{10})t}$$

Analyzing the long term behavior:

$$\begin{gathered} \lim_{t\to\infty} \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} = +\infty \\ \lim_{t\to\infty} \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} = +\infty \end{gathered}$$

3 Complex Numbers

Complex numbers are of the form:

$$a + bi$$

where a is the *real* part of the number, and b is the imaginary part.

3.1 Multiplication

$$(a + bi)(c + di)$$
$$\Rightarrow$$
$$ac + adi + cbi + bdi^2$$
$$\Rightarrow$$
$$(ac - bd) + i(ad + cb)$$

3.2 Geometric Interpretation

$a + bi$ may be identified with (a, b) in the complex plane where the real part (a) is represented by the x coordinate and the imaginary part (b) by the y coordinate.

3.3 Euler's Formula

Euler's Formula provides a way to represent a complex number in terms of sin and cos:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{or}$$
$$re^{i\theta} = r\cos(\theta) + ri\sin(\theta),\ r \geq 0$$

As can be seen, Euler's Formula is simply just the polar coordinate transformation.

3.4 Powers and Roots

3.4.1 Powers

To square $re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$ using the previous description, it results in:

$$r^2 e^{2i\theta} = r^2 \left(\cos(2\theta) + i\sin(2\theta)\right)$$

Generally, for any integer n, the n^{th} power of $re^{i\theta}$ is simply:

$$r^n e^{ni\theta} = r^n \left(\cos(n\theta) + i\sin(n\theta)\right)$$

Thus,

$$\begin{aligned}
(1+i)^4 &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 \\
&= \left(\sqrt{2}\right)^4 \left(e^{i\frac{\pi}{4}}\right)^4 \\
&= 4e^{i\pi} \\
&= -4
\end{aligned}$$

3.4.2 Roots

The roots are somewhat more complicated. Generally, for the n^{th} root $\sqrt{z^{\frac{1}{n}}}$ has n different candidates. Let:

$$z = Re^{i\theta},\ w = \sqrt{z^{\frac{1}{n}}}$$

means that:

$$w^n = z$$

If

$$w = re^{i\alpha}$$

then,

$$r^n e^{in\alpha} = Re^{i\theta}$$

The amplitude r is uniquely $R^{\frac{1}{n}}$, however, the phase α is *not* unique. This is because $e^{in\alpha} = e^{i\theta}$ means that:

$$n\alpha = \theta + 2k\pi, k = 0, \pm 1, \pm 2, \ldots$$
$$\alpha = \frac{\theta + 2k\pi}{n}, k = 0, \pm 1, \pm 2, \ldots$$

Example **3.1**

$$1^{\frac{1}{3}} = \left(e^{i\cdot 2k\pi}\right)^{\frac{1}{3}} = e^{i\frac{2k\pi}{3}}$$

Depending on the value of k, there are *three* different possibilites for e:

$$\text{If } k = 0, 3, 6, \ldots \Rightarrow e^{i0} = 1$$
$$\text{If } k = 1, 4, 7, \ldots \Rightarrow e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$\text{If } k = 2, 5, 8, \ldots \Rightarrow e^{\frac{3\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

4 Non-Homogeneous ODE

Non-Homogeneous ODE's are of the form:

$$y'' + p(t)y' + g(t)y = g(t)$$

These ODEs behave differently to previous ones. As such, the principle of superposition would now say:

Principle of Superposition (Non-Homogeneous) **4.1**

If Y_1, Y_2 are two solutions of $y'' + p(t)y' + g(t)y = g(t)$, then $Y_1 - Y_2$ is a solution of $y'' + p(t)y' + g(t)y = 0$.

The proof of this is as follows:

$$y_1'' + p(t)y_1' + g(t)y_1 = g(t)$$
$$y_2'' + p(t)y_2' + g(t)y_2 = g(t)$$

Then:

$$Y_1 - Y_2 = y_1'' - y_2'' + p(t)\left(y_1' - y_2'\right) + g(t)\left(y_1 - y_2\right) = g(t) - g(t) = 0$$

The general solutions of $y'' + p(t)y' + g(t)y = g(t)$ is of the form:

$$\begin{aligned} y &= y_c + Y \\ &= C_1y_1 + C_2y_2 + Y \end{aligned}$$

where $y_c = C_1y_1 + C_2y_2$ is the general solution of the *homogeneous* ODE $y'' + p(t)y' + g(t)y = 0$, and Y is a part of the *non-homogeneous* $y'' + p(t)y' + g(t)y = 0$.

The proof of this starts with verifying that $y_c + Y$ satisfies the ODE:

$$\begin{aligned} \text{LHS} &= y_c'' + Y'' + p(t)\left(y_c' + Y'\right) + g(t)\left(y_c + Y\right) \\ &= y_c'' + p(t)y_c' + g(t)y_c + Y'' + p(t)Y' + g(t)Y \\ &= g(t) = \text{RHS} \end{aligned}$$

Since:

$$y_c'' + p(t)y_c' + g(t)y_c = 0$$
$$Y'' + p(t)Y' + g(t)Y = g(t)$$

Any solution of the non-homogeneous ODE is of the form $y_c + Y$ for some C_1, C_2.

Set y to be any solution of $y'' + p(t)y' + g(t)y = g(t)$.

Principle $\Rightarrow$ $y - Y$ is a solutions of $y'' + p(t)y' + g(t)y = 0 \Rightarrow y - Y = C_1y_1 + C_2y_2$ for some C_1, C_2.

The most general way to obtain Y is a **Variation of Parameters**.

Knowing that $y_c = C_1y_1 + C_2y_2$, then C_1, C_2 are varriated into function such that:

$$Y = u_1y_1 + u_2y_2$$

Then plugging Y into the non-homogeneous ODE:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(t) \end{cases} \quad \Rightarrow u_1' = \frac{-y_2 \cdot g}{W(y_1, y_2)}, u_1' = \frac{y_1 \cdot g}{W(y_1, y_2)}$$

$$Y = y_1 \cdot \int \frac{-y_2 \cdot g}{W(y_1, y_2)}\, dt + y_2 \cdot \int \frac{y_1 \cdot g}{W(y_1, y_2)}\, dt$$

As can probably be seen, higher order generalization is very complicated. Thus introduces the **Method of Undetermined Coefficients**.

4.1 Method of Undetermined Coefficients

The method makes it easy ot generalize higher order ODEs, however it only works for:

$$ay'' + by' + c = g(t)$$

Where $g(t)$ is a product of exponential polynomials, sin, and cos functions. However, these limitations aren't very relevant for engineering applications.

Example **4.1**

$$y'' - 2y' - 3y = 3$$

RHS is a constant function, so try a constant function.
First, using the complementary equation, namely:

$$y'' - 2y' - 3y = 0$$

The auxiliarry equation can be formulated as:

$$r^2 - 2r - 3$$

Giving roots of $r_1 = 3$, $r_2 = -1$, thus giving the general solution as:

$$y_c = C_1 e^{3t} + C_2 e^{-t}$$

Set $Y = A$, and plug in the ODE:

$$Y'' - 2Y' - 3Y = 0 - 0 - 3A = 3 \Rightarrow A = -1$$

Giving $Y = -1$ as a particular solution, so then the general solutions is:

$$y = c_1 e^{3t} + C_2 e^{-t} - 1$$

It should be noted that if the RHS is apolynomial of degree n, Y should be set as a generic polynomial of degree n. Even when $g(t) = t$, Y should still start from degree n, and work all the way down to the constant.

Example **4.2**

$$y'' - y' - 2y = t^2 + 1$$

Since the RHS is a polynomial function of degree 2, so to solve, first try a polynomial function of degree 2.
By inspection, the complimentary solution of $y'' - y' - 2y = 0$ is:

$$y_c = C_1 e^{-t} + C_2 e^{2t}$$

Set:

$$\begin{aligned} Y &= At^2 + Bt + C \\ Y' &= 2At + B \\ Y'' &= 2A \end{aligned}$$

Thus:

$$\begin{aligned} Y'' - Y' - 2Y &= 2A - (2At + B) - 2\left(at^2 + Bt + c\right) \\ &= 2At^2 + (-2A - 2B)t + 2A - B - 2C \\ &= t^2 + 1 \end{aligned}$$

Thus, setting $t^2 + 1$ and $2At^2 + (-2A - 2B)t + 2A - B - 2C$ equal to each other, it can be found that:

$$\begin{aligned} -2A &= 1 \\ -2A - 2B &= 0 \\ 2A - B - 2C &= 1 \end{aligned}$$

Then solving for each value of A, B, and C:

$$\begin{aligned} A &= -\frac{1}{2} \\ B &= -A = \frac{1}{2} \\ C &= \frac{2A - B - 1}{2} = -\frac{5}{4} \end{aligned}$$

So:

$$Y = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4}$$

Giving a general solution of:

$$y = C_1 e^{-t} + C_2 e^{2t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4}$$

Example **4.3**

$$y'' - y' - 2y = t^3$$

$$\begin{aligned}
Y &= At^3 - Bt^2 + Ct + D \\
Y' &= 3At^2 - 2Bt + C \\
Y'' &= 6At^1 - 2B
\end{aligned}$$

Thus:

$$Y'' - Y' - 2Y = -2A^3 + (-3A - 2B)\, t^2 + (6A - 2B - 2C)t + (2B - C - 2D)$$

Comparing $Y = At^3 - Bt^2 + Ct + D$ with $-2A^3 + (-3A - 2B)\, t^2 + (6A - 2B - 2C)t + (2B - C - 2D)$, a system of equations can be found as:

$$\begin{aligned}
-2A &= 1 \\
-3A - 2B &= 0 \\
6A - 2B - 2C &= 0 \\
2B - C - 2D &= 0
\end{aligned}$$

Thus:

$$\begin{aligned}
A &= -\frac{1}{2} \\
B &= -\frac{3}{2}A = \frac{3}{4} \\
C &= \frac{1}{2}(6A - 2B) = -\frac{9}{4} \\
D &= \frac{1}{2}(2B - C) = \frac{15}{8}
\end{aligned}$$

Thus:

$$Y = -\frac{1}{2}t^3 + \frac{3}{4}t^2 + -\frac{9}{4}t + \frac{15}{8}$$