MATH 2070 - Differential Equations

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1 Introduction to Differential Equations

1.1 What is a Differential Equation

Differential equations are foundational to studying engineering and physics. A very basic example of a differential equation is:

$$\frac{dx}{dt} + x = 2\cos(t) \tag{1}$$

Here x is the *dependent variable* and t is the *independent variable*. To solve (1) is to find x in terms of t such that the equation still holds when everything (x, t, and $\frac{dx}{dt}$) is plugged in. Consider

$$x = x(t) = \cos(t) + \sin(t)$$

as the solution for (1). Plugging this in as appropriate will verify this solution:

$$\begin{aligned}\big(-\sin(t) + \cos(t)\big) + \big(\cos(t) + \sin(t)\big) &= 2\cos(t)\\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) &= 2\cos(t)\\ \cos(t) + \cos(t) &= 2\cos(t)\\ 2\cos(t) &= 2\cos(t)\end{aligned}$$

Clearly this equality holds and a **particular solution** has been found for (1).

Particular Solution **1.1**

Some solution for a given differential equation.

However, multiple solutions can exist; now consider

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - e^{-t}$$

as a solution. To verify:

$$\begin{aligned}\big(-\sin(t) + \cos(t) - e^{-t}\big) + \big(\cos(t) + \sin(t) + e^{-t}\big) &= 2\cos(t)\\ -\sin(t) + \sin(t) + \cos(t) + \cos(t) - e^{-t} + e^{-t} &= 2\cos(t)\\ \cos(t) + \cos(t) &= 2\cos(t)\\ 2\cos(t) &= 2\cos(t)\end{aligned}$$

Several solutions can exist for a given differential equation. For (1), the family of solutions exists in the form:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - Ce^{-t}$$

where C is some constant. This is called the **General Solution** for the differential equation.

General Solution **1.2**

The entire family of solutions for a given differential equation. A general form of the solution that can be adapted to different specifications.

Each value of C gives a different solution, so really there are infinite solutions for (1).

Differential Equation **1.3**

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation (DE).

1.2 Four Fundamental Equations

There exist four equations that are each very common and have solutions that can be memorized. The **first** among them is:

$$\frac{dy}{dx} = ky \tag{2}$$

For some constant $k > 0$, the general solution to (2) is:

$$y(x) = Ce^{kx}$$

The **second** among the four fundamental equations is:

$$\frac{dy}{dx} = -ky \tag{3}$$

For some constant $k > 0$, the general solution to (3) is:

$$y(x) = Ce^{-kx}$$

The **third** is a second order derivative (see 1.3.2):

$$\frac{d^2y}{dx^2} = -k^2y \tag{4}$$

For some constant $k > 0$, the general solution for (4) is:

$$y(x) = C_1\cos(kx) + C_2\sin(kx)$$

Since (3) is of the second order, there are two constants in the solution.

Lastly, the **fourth** fundamental equation is:

$$\frac{d^2y}{dx^2} = k^2y \tag{5}$$

For some constant $k > 0$, the general solution for (5) is:

$$y(x) = C_1e^{kx} + C_2e^{-kx}$$

or

$$y(x) = D_1\cosh(kx) + D_2\sinh(-kx)$$

Where:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

1.3 Classification of Differential Equations

There exist several types of differential equations. Consequently, they are classified according to **type**, **order**, and **linearity**.

1.3.1 Classification by Type

If a given differential equation 1) includes only ordinary derivatives of a number of unknown functions, and 2) those derivatives are all with respect to the same independent variable then it is classified as an **Ordinary Differential Equation (ODE)**.

Ordinary Differential Equation **1.4**

Equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.

$$\frac{dy}{dx} + 5y = e^x \; ; \frac{dy}{dx} + \frac{dr}{dx} = 14x \; ; \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} = 0 \tag{6}$$

Each equation in (6) is an example of an ODE. Notice that each equation contains only functions derived with respect to the same variable.

A **Partial Differential Equation (PDE)** differs in that it contains derivatives of functions with respect to multiple independent variables.

$$\frac{\delta y}{\delta x} + 5\frac{\delta r}{\delta t} = \ln(x) \; ; \frac{\delta y}{\delta x} + \frac{\delta r}{\delta t} = 14x \; ; \frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 b}{\delta x^2} \tag{7}$$

(7) are examples of partial differential equations. The Greek letter delta (δ) is used to denote a partial derivative. Thus, $\frac{\delta y}{\delta x}$ is the partial derivative of the function y with respect to x.

Partial Differential Equation **1.5**

Equations that depend on partial derivatives of several variables. That is, there are several independent variables.

1.3.2 Classification by Order

The **order** of a differential equation is the highest order among derivatives it contains. For example, (8) is a third-order differential equation because $\frac{d^3u}{dx^3}$ is a third derivative and is the highest derivative.

$$\frac{dy}{dx} - \frac{d^2r}{dx^2} = \frac{d^3u}{dx^3} \tag{8}$$

1.3.3 Classification by Linearity

An equation is linear if the dependent variable (or variables) and their derivatives appear linearly, that is, only as first powers, they are not multiplied together, and no other functions of the dependent variables appear.

$$e^x\frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} + x^2y = \frac{1}{x} \tag{9}$$

(9) is a linear differential equation because its *dependent* variable (y) only appears linearly. It does not matter that the independent variable (x) appears non-linearly. Conversely, (10) is non-linear because y is squared.

$$\frac{dy}{dx} = y^2 \tag{10}$$

Similarly, (11) is also non-linear because Θ appears inside a sin function.

$$\frac{d^2\Theta}{dx^2} + \sin(\Theta) = 0 \tag{11}$$

2 First Order Equations

A first order ODE is an equation in the form of:

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y) \tag{1}$$

There is no strict process to find a solution to these equations. Thus, a lot of this class is spent on the various different ways to find solutions. To see one of the simpler ways, consider an equation where f is a function of only x:

$$y' = f(x)$$

Integrating both sides with respect to x, the equation becomes:

$$\begin{aligned} \int y'\,dx &= \int f(x)\,dx + C \\ y(x) &= \int f(x)\,dx + C \end{aligned}$$

This $y(x)$ is the general solution to (1). Thus, to solve a differential equation with just a single dependent variable x, finding the antiderivative of $f(x)$ is sufficient to find the general solution.

Example **2.1**

Find the general solution for:

$$y' = 3x^2$$

Integrating each side gives:

$$\begin{aligned} \int y'\,dx &= \int 3x^2\,dx + C \\ y(x) &= x^3 + C \end{aligned}$$

Thus, the general solution for $y' = 3x^2$ is $y(x) = x^3 + C$.

Generally, there will also be a condition that the solution to a differential equation must satisfy. In general terms, this might look like:

$$y' = f(x),\ y(x_0) = y_0 \tag{2}$$

Leaving the solution to (2) as:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$

Verifying the solution, first y' is computed based on our solution.

$$\begin{aligned} \frac{d}{dx}\big(y(x)\big) &= \frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt + y_0\right) \\ y' &= f(x) \end{aligned}$$

Second, to verify that the initial condition is satisfied:

$$\begin{aligned} y(x) &= \int_{x_0}^{x} f(t)\,dt + y_0 \\ y(x_0) &= \int_{x_0}^{x_0} f(t)\,dt + y_0 \\ y(x_0) &= 0 + y_0 \\ y(x_0) &= y_0 \end{aligned}$$

This confirms the initial condition, and thus it can be seen that the solution to the differential equation with an initial condition has been found.

Example **2.2**

Solve

$$y' = e^{-x^2},\ \ y(0) = 1$$

First, finding the solution, both sides can be integrated:

$$y(x) = \int_{0}^{x} e^{-t^2}\,dt + 1$$

And to verify the solution:

$$\begin{aligned} \frac{d}{dx}\big(y(x)\big) &= \frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt + 1\right) \\ y' &= e^{-x^2} \end{aligned}$$

$$\begin{aligned} y(0) &= \int_0^x e^{t^2}\,dt + 1 \\ y(0) &= 0 + 1 \\ y(0) &= 1 \end{aligned}$$

The solution passes both verification tests, and so it can be safely said that the solution has been found.

Using the same method as before, equations of the form in (3) can be solved as well.

$$y' = f(y) \ \text{ or } \ \frac{dy}{dx} = f(y) \tag{3}$$

(3) can be rewritten using the inverse function theorem from calculus to switch the roles of x and y to get:

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

Finally, at this point both sides can be integrated with respect to y to get:

$$x(y) = \int \frac{1}{f(y)}\,dy + C$$

From here, it is just a matter of solving for y.

Example **2.3**

In Subsection 1.2, the claim was made that the solution for $y' = ky$ is $y = Ce^{kx}$ for $k > 0$. To show this, the method of integration can be used:

$$\frac{dy}{dx} = ky \to \frac{dx}{dy} = \frac{1}{ky}$$
$$x(y) = \int \frac{1}{ky}\,dy$$
$$x(y) = \frac{1}{k}\ln|y| + D$$

Now solving for y:

$$x(y) = \frac{1}{k}\ln|y| + D$$

Example **2.4**

$$y' + y = \cos(2t)$$

This is already in standard form, so we can go on to calculate the integrating factor:

$$\mu(t) = e^{\int 1\,dt} = e^t$$

Thus, the general solution will be:

$$y = \frac{\int e^t \cdot \cos(2t)\,dt}{e^t}$$

Integrating the numerator:

$$\int e^t \cdot \cos(2t)\,dt$$
$$\vdots$$
$$\int e^t \cdot \cos(2t)\,dt = \frac{1}{5}\left(e^t\cos(2t) + 2e^t\sin(2t)\right) + C$$

Thus:

$$y = \frac{\frac{1}{5}\left(e^t\cos(2t) + 2e^t\sin(2t)\right) + C}{e^t}$$

or:

$$y = \frac{1}{5}\left(\cos(2t) + 2\sin(2t)\right) + Ce^{-t}$$

2.1 Separable ODE

The basic form of a separable ODE is:

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

i.e., the derivative is a product of two functions, one depending on x and the other on y.

This type of ODE is solved first by separating the variables:

$$\frac{dy}{f(y)} = g(x)dx$$

Now that there is a single type of variable on each side (only x or only y), both sides can be integrated.

Example **2.5**

$$\frac{dy}{dx} = \frac{e^x - x}{e^{-y} + y}$$

First, move to get one variable on each side:

$$e^{-y} + ydy = e^x - xdx$$

Then integrate:

$$\int e^{-y} + y\, dy = \int e^x - x\, dx$$

$$\vdots$$

$$-e^{-y} + \frac{y^2}{2} = e^x - \frac{x^2}{2} + C$$

From here, it's impossible to solve y from x universally. In this situation, a Taylor Series would be necessary...

Principle: If an ODE is given without initial values, a one-parameter family of solutions is sufficient. However, if an initial value is given (IVP), an explicit solution might be possible with an interval of existance.