MATH 2070 - Differential Equations

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Contents

1 Introduction to Differential Equations

1.1 What is a Differential Equation

Differential equations are foundational to studying engineering and physics. A very basic example of a differential equation is:

$$\frac{dx}{dt} + x = 2\cos(t) \tag{1}$$

Here x is the *dependent variable* and t is the *independent variable*. To solve (1) is to find x in terms of t such that the equation still holds when everything (x, t, and $\frac{dx}{dt}$) is plugged in. Consider

$$x = x(t) = \cos(t) + \sin(t)$$

as the solution for (1). Plugging this in as appropriate will verify this solution:

$$\begin{aligned}
\big(-\sin(t) + \cos(t)\big) + \big(\cos(t) + \sin(t)\big) &= 2\cos(t) \\
-\sin(t) + \sin(t) + \cos(t) + \cos(t) &= 2\cos(t) \\
\cos(t) + \cos(t) &= 2\cos(t) \\
2\cos(t) &= 2\cos(t)
\end{aligned}$$

Clearly this equality holds and a **particular solution** has been found for (1).

Particular Solution **1.1**

Some solution for a given differential equation.

However, multiple solutions can exist; now consider

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - e^{-t}$$

as a solution. To verify:

$$\begin{aligned}
\big(-\sin(t) + \cos(t) - e^{-t}\big) + \big(\cos(t) + \sin(t) + e^{-t}\big) &= 2\cos(t) \\
-\sin(t) + \sin(t) + \cos(t) + \cos(t) - e^{-t} + e^{-t} &= 2\cos(t) \\
\cos(t) + \cos(t) &= 2\cos(t) \\
2\cos(t) &= 2\cos(t)
\end{aligned}$$

Several solutions can exist for a given differential equation. For (1), the family of solutions exists in the form:

$$\frac{dx}{dt} = -\sin(t) + \cos(t) - Ce^{-t}$$

where C is some constant. This is called the **General Solution** for the differential equation.

General Solution **1.2**

The entire family of solutions for a given differential equation. A general form of the solution that can be adapted to different specifications.

Each value of C gives a different solution, so really there are infinite solutions for (1).

Differential Equation **1.3**

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation (DE).

1.2 Four Fundamental Equations

There exist four equations that are each very common and have solutions that can be memorized. The **first** among them is:

$$\frac{dy}{dx} = ky \tag{2}$$

For some constant $k > 0$, the general solution to (2) is:

$$y(x) = Ce^{kx}$$

The **second** among the four fundamental equations is:

$$\frac{dy}{dx} = -ky \tag{3}$$

For some constant $k > 0$, the general solution to (3) is:

$$y(x) = Ce^{-kx}$$

The **third** is a second order derivative (see 1.3.2):

$$\frac{d^2y}{dx^2} = -k^2y \tag{4}$$

For some constant $k > 0$, the general solution for (4) is:

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

Since (3) is of the second order, there are two constants in the solution.

Lastly, the **fourth** fundamental equation is:

$$\frac{d^2y}{dx^2} = k^2y \tag{5}$$

For some constant $k > 0$, the general solution for (5) is:

$$y(x) = C_1e^{kx} + C_2e^{-kx}$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(-kx)$$

Where:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

1.3 Classification of Differential Equations

There exist several types of differential equations. Consequently, they are classified according to **type**, **order**, and **linearity**.

1.3.1 Classification by Type

If a given differential equation 1) includes only ordinary derivatives of a number of unknown functions, and 2) those derivatives are all with respect to the same independent variable then it is classified as an **Ordinary Differential Equation (ODE)**.

Ordinary Differential Equation	**1.4**
Equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.	

$$\frac{dy}{dx} + 5y = e^x \; ; \frac{dy}{dx} + \frac{dr}{dx} = 14x \; ; \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} = 0 \tag{6}$$

Each equation in (6) is an example of an ODE. Notice that each equation contains only functions derived with respect to the same variable.

A **Partial Differential Equation (PDE)** differs in that it contains derivatives of functions with respect to multiple independent variables.

$$\frac{\delta y}{\delta x} + 5\frac{\delta r}{\delta t} = \ln(x) \; ; \frac{\delta y}{\delta x} + \frac{\delta r}{\delta t} = 14x \; ; \frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 b}{\delta x^2} \tag{7}$$

(7) are examples of partial differential equations. The Greek letter delta (δ) is used to denote a partial derivative. Thus, $\frac{\delta y}{\delta x}$ is the partial derivative of the function y with respect to x.

Partial Differential Equation	**1.5**
Equations that depend on partial derivatives of several variables. That is, there are several independent variables.	

1.3.2 Classification by Order

The **order** of a differential equation is the highest order among derivatives it contains. For example, (8) is a third-order differential equation because $\frac{d^3u}{dx^3}$ is a third derivative and is the highest derivative.

$$\frac{dy}{dx} - \frac{d^2r}{dx^2} = \frac{d^3u}{dx^3} \tag{8}$$

1.3.3 Classification by Linearity

An equation is linear if the dependent variable (or variables) and their derivatives appear linearly, that is, only as first powers, they are not multiplied together, and no other functions of the dependent variables appear.

$$e^x \frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} + x^2 y = \frac{1}{x} \tag{9}$$

(9) is a linear differential equation because its *dependent* variable (y) only appears linearly. It does not matter that the independent variable (x) appears non-linearly. Conversely, (10) is non-linear because y is squared.

$$\frac{dy}{dx} = y^2 \tag{10}$$

Similarly, (11) is also non-linear because Θ appears inside a sin function.

$$\frac{d^2\Theta}{dx^2} + \sin(\Theta) = 0 \tag{11}$$

2 Ordinary Differential Equations

Ordinary differential equations (ODE) are differential equations with just a single input, generally thought of as time (t).

Consider the relationships between position (x), velocity (v), and acceleration (a). Velocity is the derivative of position and acceleration the derivative of velocity.

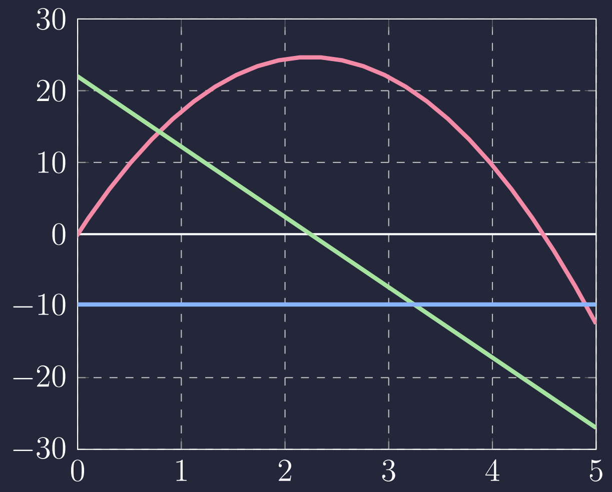


Figure 1: Position, Velocity, and Acceleration

If only the acceleration of an object across time is known ($g = 9.8\frac{m}{s^2}$), and nothing else about that object is known, then differential equations can be used to solve for the object's velocity and acceleration.

$$y''(t) = -g$$

$$\frac{d(?)}{dt}(t) = -g$$

Based on this, if a function can be found to have a derivative of $-g$ then the velocity of the object can be said to have a velocity equal to that function. In this example, it is as simple as integrating the function for acceleration:

$$\frac{d(-gt + v_0)}{dt}(t) = -g$$

Going one step further and integrating the velocity will yield the position of the object.

$$\frac{d\left(-\frac{1}{2}gt^2 + v_0t + x_0\right)}{dt}(t) = -gt + v_0$$

The last thing to consider with this example are the initial conditions of the differential equation. v_0 and x_0 are not known quantities, however, if they were to be specified then the differential would have an initial condition to satisfy.

2.1 First Order Linear ODE

Before approaching First Order ODEs in Subsection 2.2, First Order Linear ODEs follow a simple pattern for their solutions:

If a differential equation is given in **standard form**:

$$y' + p(t)y = g(t)$$

Where $p(t)$ and $g(t)$ are arbitrary functions of t, then the general solution can be expressed as:

$$y(t) = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)}$$

Where the **integrating factor** ($\mu(t)$) is:

$$\mu(t) = e^{\int p(t)\,dt}$$

Example **2.1**

$$y' + 2y = e^{3t},\ y(0) = 3$$

This ODE is already in standard form, so the integrating factor can be written as:

$$\begin{aligned} \mu(t) &= e^{\int 2\,dt} \\ &= e^{2t} \end{aligned}$$

And thus the general solutions is:

$$\begin{aligned} y(t) &= \frac{\int e^{2t} \cdot e^{3t}\,dt + C}{e^{2t}} \\ &= \frac{\int e^{5t}\,dt + C}{e^{2t}} \\ &= \frac{\frac{1}{5}e^{5t} + C}{e^{2t}} \\ &= \frac{1}{5}\frac{e^{5t}}{e^{2t}} + \frac{C}{e^{2t}} \\ &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}} \end{aligned}$$

Using the initial condition to find the specific solution:

$$\begin{aligned} y(t) &= \frac{1}{5} \cdot e^{3t} + \frac{C}{e^{2t}} \\ y(0) &= \frac{1}{5} \cdot e^{3\cdot 0} + \frac{C}{e^{2\cdot 0}} \\ 3 &= \frac{1}{5} + C \\ \frac{14}{5} &= C \end{aligned}$$

Thus, the solution of the IVP is:

$$y(t) = \frac{1}{5} \cdot e^{3t} + \frac{14}{15} \cdot e^{-2t}$$

2.2 First Order ODE

A first order ODE is an equation in the form of:

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y) \tag{1}$$

There is no strict process to find a solution to these equations. Thus, a lot of this class is spent on the various different ways to find solutions. To see one of the simpler ways, consider an equation where f is a function of only x:

$$y' = f(x)$$

Integrating both sides with respect to x, the equation becomes:

$$\int y'\,dx = \int f(x)\,dx + C$$
$$y(x) = \int f(x)\,dx + C$$

This $y(x)$ is the general solution to (1). Thus, to solve a differential equation with just a single dependent variable x, finding the antiderivative of $f(x)$ is sufficient to find the general solution.

Example **2.2**

Find the general solution for:

$$y' = 3x^2$$

Integrating each side gives:

$$\int y'\,dx = \int 3x^2\,dx + C$$
$$y(x) = x^3 + C$$

Thus, the general solution for $y' = 3x^2$ is $y(x) = x^3 + C$.

Generally, there will also be a condition that the solution to a differential equation must satisfy. In general terms, this might look like:

$$y' = f(x),\ y(x_0) = y_0 \tag{2}$$

Leaving the solution to (2) as:

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0$$

Verifying the solution, first y' is computed based on our solution.

$$\frac{d}{dx}\big(y(x)\big) = \frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt + y_0\right)$$
$$y' = f(x)$$

Second, to verify that the initial condition is satisfied:

$$\begin{aligned} y(x) &= \int_{x_0}^{x} f(t)\,dt + y_0 \\ y(x_0) &= \int_{x_0}^{x_0} f(t)\,dt + y_0 \\ y(x_0) &= 0 + y_0 \\ y(x_0) &= y_0 \end{aligned}$$

This confirms the initial condition, and thus it can be seen that the solution to the differential equation with an initial condition has been found.

Example **2.3**

Solve

$$y' = e^{-x^2}, \;\; y(0) = 1$$

First, finding the solution, both sides can be integrated:

$$y(x) = \int_{0}^{x} e^{-t^2}\,dt + 1$$

And to verify the solution:

$$\begin{aligned} \frac{d}{dx}\big(y(x)\big) &= \frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt + 1\right) \\ y' &= e^{-x^2} \end{aligned}$$

$$\begin{aligned} y(0) &= \int_0^x e^{t^2}\,dt + 1 \\ y(0) &= 0 + 1 \\ y(0) &= 1 \end{aligned}$$

The solution passes both verification tests, and so it can be safely said that the solution has been found.

Using the same method as before, equations of the form in (3) can be solved as well.

$$y' = f(y) \;\; \text{or} \;\; \frac{dy}{dx} = f(y) \tag{3}$$

(3) can be rewritten using the inverse function theorem from calculus to switch the roles of x and y to get:

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

Finally, at this point both sides can be integrated with respect to y to get:

$$x(y) = \int \frac{1}{f(y)}\,dy + C$$

From here, it is just a matter of solving for y.

Example **2.4**

In Subsection 1.2, the claim was made that the solution for $y' = ky$ is $y = Ce^{kx}$ for $k > 0$. To show this, the method of integration can be used:

$$\frac{dy}{dx} = ky \to \frac{dx}{dy} = \frac{1}{ky}$$
$$x(y) = \int \frac{1}{ky}\, dy$$
$$x(y) = \frac{1}{k} \ln|y| + D$$

Now solving for y:

$$x(y) = \frac{1}{k} \ln|y| + D$$

Example **2.5**

$$y' + y = \cos(2t)$$

This is already in standard form, so we can go on to calculate the integrating factor:

$$\mu(t) = e^{\int 1\, dt} = e^t$$

Thus, the general solution will be:

$$y = \frac{\int e^t \cdot \cos(2t)\, dt}{e^t}$$

Integrating the numerator:

$$\int e^t \cdot \cos(2t)\, dt$$
$$\vdots$$
$$\int e^t \cdot \cos(2t)\, dt = \frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C$$

Thus:

$$y = \frac{\frac{1}{5}\left(e^t \cos(2t) + 2e^t \sin(2t)\right) + C}{e^t}$$

or:

$$y = \frac{1}{5}\left(\cos(2t) + 2\sin(2t)\right) + Ce^{-t}$$

2.3 Separable ODE

The basic form of a separable ODE is:

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

i.e., the derivative is a product of two functions, one depending on x and the other on y.

This type of ODE is solved first by separating the variables:

$$\frac{dy}{f(y)} = g(x)dx$$

Now that there is a single type of variable on each side (only x or only y), both sides can be integrated.

Example **2.6**

$$y' = xy$$

This equation can be rewritten as:

$$\frac{dy}{dx} = x \cdot y$$

Here, it can be easily seen that this equation is separable:

$$\frac{dy}{y} = x\ dx$$

$$\int \frac{1}{y}\, dy = \int x\, dx + C$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$e^{\ln|y|} = e^{\frac{x^2}{2}+C}$$

$$|y| = De^{\frac{x^2}{2}}$$

Where $D > 0$. Since $y = 0$ is a solution as well, this can be simplified to:

$$y = De^{\frac{x^2}{2}}$$

Principle: If an ODE is given without initial values, a one-parameter family of solutions is sufficient. However, if an initial value is given (IVP), an explicit solution might be possible with an interval of existence.

2.4 Implicit and Explicit Solutions

In general, anytime an ODE is given *without* initial values, then a single-parameter family of *implicit* solutions is sufficient.

If given an IVP, however, then finding an *explicit* solution and an *interval of existence* should be attempted. In some situations, this won't be possible, but it should be attempted at least.

2.4.1 Implicit Solutions

Sometimes a wall is reached even if the integration is possible. Consider:

$$y' = \frac{xy}{y^2+1} \tag{4}$$

Using the technique described in Subsection 2.3, this can be separated into:

$$\begin{aligned}
y' &= \frac{xy}{y^2+1} \\
\frac{dy}{dx} &= \frac{xy}{y^2+1} \\
\frac{y^2+1}{y} dy &= x\ dx \\
\left(y + \frac{1}{y}\right) dy &= x\ dx
\end{aligned}$$

Integrating both sides gives:

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C$$

The integration of this equation is quite simple. However, try to solve for y and see how difficult that will be. Though solving for y itself is too difficult, this form is still a solution and can still be verified.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{y^2}{2} + \ln|y|\right) &= \frac{d}{dx}\left(\frac{x^2}{2} + C\right) \\
y'\left(y + \frac{1}{y}\right) &= x \\
y \cdot \left(y'\left(y + \frac{1}{y}\right)\right) &= y \cdot x \\
y'\left(y^2+1\right) &= y \cdot x \\
y' &= \frac{xy}{y^2+1}
\end{aligned}$$

Producing the exact same equation in (4), thus verifying the solution.

Since these solutions are implicit, they might not be able to be graphed as a valid function. In those cases, other information, such as an initial condition, can be used to further inform the appropriate solution.

2.5 Intervals of Existence

Existence and Uniqueness Theorem **2.1**

Consider:

$$y' + p(t)y = g(t) + C,\ y(t_0) = y_0$$

where the ODE is *linear* and in its *standard form*. Assuming that:

1. Both $p(t)$ and $g(t)$ are continuous over the open interval (a, b)
2. The open interval (a, b) contains t_0

Then there exists a unique function $y = y(t)$ over the interval (a, b) that solves the IVP.

Based on this theorem, it can be asserted that there exists a unique solution over an interval (a,b) provided that this interval contains t_0 from the IVP and both functions $p(t)$ and $g(t)$ are continuous over this interval.

Example **2.7**

$$ty' + (t-1)y = -e^{-t},\ y\left(\ln(2)\right) = \frac{1}{2}$$

To use the theorem, first the equation must be expressed in its standard form:

$$y' + \frac{t-1}{t} \cdot y = -\frac{e^{-t}}{t}$$

Where:

$$p(t) = \frac{t-1}{t}$$
$$g(t) = -\frac{e^{-t}}{t}$$

Since, for both of these functions, the only point of discontinuity is $t = 0$, then it is known that the only two possible intervals of existence are either $(-\infty, 0)$ or $(0, \infty)$.

Furthermore, since the initial condition is given as $y\left(\ln(2)\right) = \frac{1}{2}$, $t_0 = \ln(2)$. Thus, the interval of existence for the solution of this ODE would be the interval of $(0, \infty)$.

In some differential equations, a solution that is found is only valid over a certain interval. Consider (5):

$$\frac{dy}{dx} = \frac{4}{(x-1)^{\frac{2}{3}}},\ y(0) = -10 \tag{5}$$

This differential equation can be solved pretty straightforwardly:

$$y = \int \frac{4}{(x-1)^{2/3}},\ dx$$
$$y = 12(x-1)^{\frac{1}{3}} + C$$

Then using the initial condition to find the value of C:

$$-10 = 12(0-1)^{\frac{1}{3}} + C$$
$$-10 = 12(-1)^{\frac{1}{3}} + C$$
$$-10 = 12(-1) + C$$
$$-10 = -12(-1) + C$$
$$2 = C$$

And finally the solution to (5):

$$y = 12(x-1)^{\frac{1}{3}} + 2$$

Now that the solution has been found, over what interval is this solution valid? This question is the crux of what the interval of existence means for a differential equation.

Looking at the solution: $y = 12(x-1)^{\frac{1}{3}} + 2$, it can be seen that the solution is infinitely continuous, or continuous over the interval of $(-\infty, \infty)$.

However, the original equation given: $\frac{dy}{dx} = \frac{4}{(x-1)^{\frac{2}{3}}}$ is differentiable over the interval $(-\infty, 1) \cup (1, \infty)$.

Since there is then a discontinuity (caused by the derivative being non-differentiable at a point) at 1, the interval of existence will only be the portion of continuity that contains the initial condition of $y(0) = -10$.

Thus, the interval of existence of (5) is $(-\infty, 1)$. Note that, with different initial conditions, the interval of existence could be different even with the same differential equation.

2.6 Single Solutions

Consider

$$y' = xy^3\left(1+x^2\right)^{-\frac{1}{2}} \tag{6}$$

(6) is a separable ODE that can be reorganized as:

$$y' = y^3 \cdot \frac{x}{\sqrt{1+x^2}}$$

And subsequently separated into:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}}\,dx$$

And solved:

$$\begin{aligned}\frac{dy}{y^3} &= \frac{x}{\sqrt{1+x^2}}\,dx\\ \int \frac{1}{y^3}\,dy &= \int \frac{x}{\sqrt{1+x^2}}\,dx\\ -\frac{1}{2y^2} &= \sqrt{1+x^2} + C\end{aligned}$$

It would now be tempting to say that $-\frac{1}{2y^2} = \sqrt{1+x^2} + C$ is a general solution to (6). However, $y = 0$ is also a solution to (6)...

The true answer then would be to say that $-\frac{1}{2y^2} = \sqrt{1+x^2} + C$ is a *family* of solutions rather than the general solution and that $y = 0$ is a singular solution.

2.7 Autonomous ODE

$$y' = f(y)$$

Such that the right hand side (RHS) does not involve t. In other words, it is a function purely in terms of y.

The equilibrium solution:

$$y = y_0$$

Such that:

$$f(y_0) = 0$$

Example **2.8**

$$y' = y^3 - y$$

Clearly, the RHS is purely in terms of y, so this would be able to be solved as an autonomous ODE. So, taking the RHS and solving for 0:

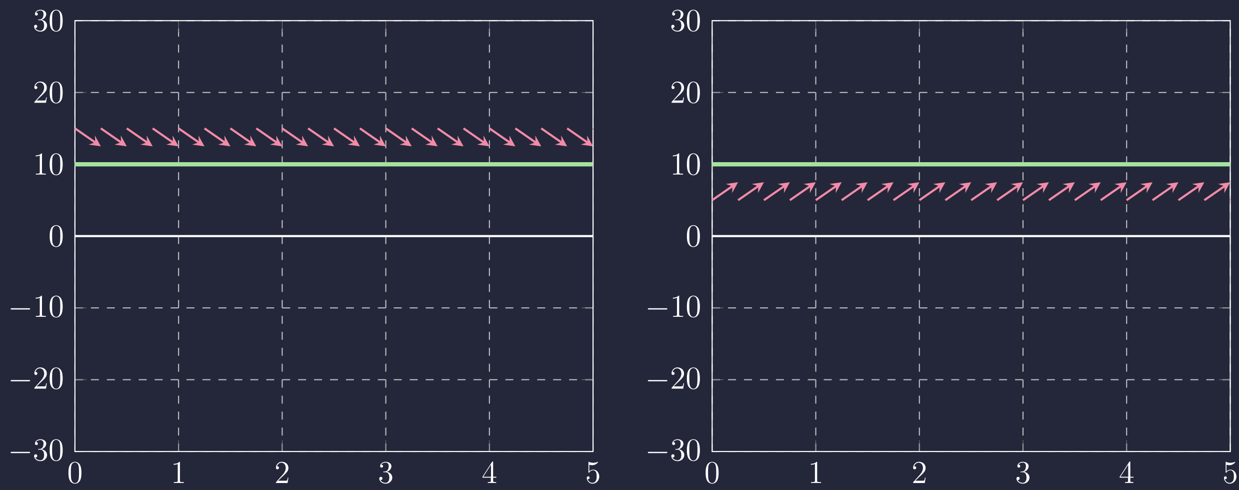
$$\begin{aligned} y^3 - y &= 0 \\ y\left(y^2 - 1\right) &= 0 \\ y\,(y-1)\,(y+1) &= 0 \\ y = 0;\ y = 1;\ y &= -1 \end{aligned}$$

Thus, there are three equilibrium solutions.

2.7.1 Stability of an Equilibrium Solution

Stability **2.2**

Let $y = y_0$ be an equilibrium solution of $y' = f(y)$. $y = y_0$ is *stable from above* if, for every $y > y_0$ near y_0, $f(y) < 0$.



(a) Stable From Above

(b) Stable From Below

Figure 3: Stability

Using Figure 3a to visualize this, the solution is stable from above because, if you were to deviate from the solution, the direction field above will guide you back down to the solution. Figure 3b is stable from below for the same reasons.

It follows then, that if the direction field around the solution points *away* from the solution, then the solution is considered to be *un*stable. Lastly, a solution is *semi*stable if it is stable from one direction and unstable from the other.

Example **2.9**

$$\frac{dP}{dt} = R - \frac{P}{V} \cdot W;\ P(0) = 0$$

This is an autonomous ODE since the RHS *does not* depend on t. Thus, setting the RHS $= 0$:

$$R - \frac{P}{V} \cdot W = 0$$

$$\vdots$$

$$P = \frac{VR}{W}$$

Finding the stability of this solution:

Example **2.10**

Falling object from a great height, subject to acceleration due to gravity (mg) and air resistance (kv^2). Newton's second law:

$$m \cdot \frac{dv}{dt} = mg - kv^2;\ v(0) = 0$$

This ODE is autonomous since the RHS has no t involved, so the equilibrium solution will be obtained by setting the RHS $= 0$.

$$mg - kv^2 = 0$$

$$\vdots$$

$$v = \pm\sqrt{\frac{mg}{k}}$$

Finding the stability of this solution:
Based on this model, the terminal velocity of the falling object will be:

$$v = \sqrt{\frac{mg}{k}}$$

Example **2.11**

3 Second Order ODEs

Similar to first order ODe's, second order ODE's also have a standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous **3.1**

If the RHS of a second order ODE in standard form is equal to zero ($g(t) = 0$), then the ODE is said to be homogeneous. Otherwise ,the ODE is non-homogeneous.

Rather than focus first on how to solve these second order ODEs, first a little bit of theory about them will be covered.

3.1 Theory

3.1.1 Existence and Uniqueness Theorem

Given an IVP in the standard form with:

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'$$

and given that $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval (a, b), and $t \exists (a, b)$, then the IVP has a unique solution on the interval (a, b).

3.1.2 Wronskian for Linear Independence

If y_a, y_b are solutions of $y'' + p(t)y' + q(t)y = 0$ on an interval where the existence and uniqueness theorem (3.1.1) holds, then y_a, y_b are linearly independent if and only if:

$$W(y_a, y_b) = \begin{vmatrix} y_a(t) & y_b(t) \\ y_a'(t) & y_b'(t) \end{vmatrix} = (y_a(t) \cdot y_b'(t)) - (y_b(t) \cdot y_a'(t)) = 0$$

on the interval.

3.1.3

Consider the ODE:

$$ay'' + by' + cy = 0 \tag{7}$$

where a, b, and c are constants.

Idea: Try $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. Using this, (7) becomes:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$\left(ar^2 + br + c\right) e^{rt} = 0$$

e^{rt} will never be 0, so to solve this, use the quadratic equation to solve for r:

$$ar^2 + br + c = 0$$

This equation is referred to as the **auxiliary equation**. If the auxiliary equation has two distinct real roots $r_a \neq r_b$, then there are two solutions:

$$y_a = e^{r_a t} \quad \text{and} \quad y_b = e^{r_b t}$$

$$W(y_a, y_b) = \begin{vmatrix} e^{r_a t} & e^{r_b t} \\ r_a e^{r_a t} & r_b e^{r_b t} \end{vmatrix} = \left(e^{r_a t} \cdot r_b e^{r_b t}\right) - \left(e^{r_b t} \cdot r_a e^{r_a t}\right) \neq 0 (\text{since } r_a \neq r_b)$$

Example **3.1**

$$y'' - 5y' + 6y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ (r-2)(r-3) &= 0 \\ r_a = 2, r_b &= 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2t} + C_b e^{3t}$$

Example **3.2**

$$2y'' - 7y' + 3y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} 2r^2 - 7r + 3 &= 0 \\ (2r-1)(r-3) &= 0 \\ r_a = \frac{1}{2}, r_b &= 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{\frac{1}{2}t} + C_b e^{3t}$$

Example **3.3**

$$y'' - 4y' - 6y = 0, y(0) = 1, y'(0) = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 4r - 6 &= 0 \\ r^2 - 4r &= 6 \\ r^2 - 4r &= 6 \\ r^2 - 4r + 4 &= 10 \\ (r-2)^2 &= 10 \\ r - 2 &= \pm\sqrt{10} \\ r &= 2 \pm \sqrt{10} \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2+\sqrt{10}} + C_b e^{2-\sqrt{10}}$$

To solve for the initial conditions, first find y':

$$y' = (2+\sqrt{10})C_a e^{(2+\sqrt{10})t} + (2-\sqrt{10})C_b e^{(2-\sqrt{10})t}$$

Then create a system of equations based on the initial conditions:

$$\begin{gathered} y(0) = 1 \Rightarrow C_a + C_b = 1 \\ y'(0) = 0 \Rightarrow C_a(2+\sqrt{10}) + C_b(2-\sqrt{10}) = 0 \end{gathered}$$

Solving the system of equations, the solution is found:

$$y = \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} + \frac{5+\sqrt{10}}{10} e^{(2-\sqrt{10})t}$$

Analyzing the long term behavior:

$$\begin{gathered} \lim_{t\to\infty} \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} = +\infty \\ \lim_{t\to\infty} \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} = +\infty \end{gathered}$$

4 Complex Numbers

Complex numbers are of the form:

$$a + bi$$

where a is the *real* part of the number, and b is the imaginary part.

4.1 Multiplication

$$(a + bi)(c + di)$$
$$\Rightarrow$$
$$ac + adi + cbi + bdi^2$$
$$\Rightarrow$$
$$(ac - bd) + i(ad + cb)$$

4.2 Geometric Interpretation

$a + bi$ may be identified with (a, b) in the complex plane where the real part (a) is represented by the x coordinate and the imaginary part (b) by the y coordinate.

4.3 Euler's Formula

Euler's Formula provides a way to represent a complex number in terms of sin and cos:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{or}$$
$$re^{i\theta} = r\cos(\theta) + ri\sin(\theta),\ r \geq 0$$

As can be seen, Euler's Formula is simply just the polar coordinate transformation.

4.4 Powers and Roots

4.4.1 Powers

To square $re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$ using the previous description, it results in:

$$r^2e^{2i\theta} = r^2\left(\cos(2\theta) + i\sin(2\theta)\right)$$

Generally, for any integer n, the n^{th} power of $re^{i\theta}$ is simply:

$$r^ne^{ni\theta} = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Thus,

$$\begin{aligned}(1+i)^4 &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 \\ &= \left(\sqrt{2}\right)^4\left(e^{i\frac{\pi}{4}}\right)^4 \\ &= 4e^{i\pi} \\ &= -4\end{aligned}$$

4.4.2 Roots

The roots are somewhat more complicated. Generally, for the n^{th} root $\sqrt{z^{\frac{1}{n}}}$ has n different candidates. Let:

$$z = Re^{i\theta},\ w = \sqrt{z^{\frac{1}{n}}}$$

means that:

$$w^n = z$$

If

$$w = re^{i\alpha}$$

then,

$$r^n e^{in\alpha} = Re^{i\theta}$$

The amplitude r is uniquely $R^{\frac{1}{n}}$, however, the phase α is *not* unique. This is because $e^{in\alpha} = e^{i\theta}$ means that:

$$n\alpha = \theta + 2k\pi, k = 0, \pm 1, \pm 2, \ldots$$
$$\alpha = \frac{\theta + 2k\pi}{n}, k = 0, \pm 1, \pm 2, \ldots$$

Example **4.1**

$$1^{\frac{1}{3}} = \left(e^{i \cdot 2k\pi}\right)^{\frac{1}{3}} = e^{i\frac{2k\pi}{3}}$$

Depending on the value of k, there are *three* different possibilites for e:

$$\text{If } k = 0, 3, 6, \ldots \Rightarrow e^{i0} = 1$$
$$\text{If } k = 1, 4, 7, \ldots \Rightarrow e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$\text{If } k = 2, 5, 8, \ldots \Rightarrow e^{\frac{3\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$