ENGR 2541 - Mechanics of Materials

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Contents

1 Types of Stress

Mechanics of materials provides a means to analyze the effects of *stresses* and *deformations*. Statics covered finding a balance of forces, including internal forces such as *shear*, *bending*, and *tension/compression*. Finding these internal forces is imperative to be able to determine the integrity of a structure. In addition to the internal forces, the integrity of a structure is also partially determined by the *dimensions* and *materials* of that structure.

1.1 Axial Stress

1.1.1 Overview

Axial Stress **1.1**

Internal stress experienced by a member due to forces applied along its axis. Specifically, stress is a force per area, where the area is the cross section normal to the axis.

In the analysis of a rod, for example, the ability for that rod to withstand the internal forces (its structural integrity) is determined by the cross-sectional area and the material of the rod.

(a) Resultant Force (b) Elementary Forces

Figure 1: Axial force is the resultant of distributed elementary forces

1.1.2 Analyzing Axial Stress

In Figure 1, the *stress* being experiences by the member is the **force per unit area**, denoted by the Greek letter sigma (σ).

Axial Stress

Stress can be calculated by dividing the total axial force by the cross-sectional area:

$$\sigma = \frac{P}{A} \quad \textbf{1.1}$$

This formula gives the *average* axial stress over the cross section of a member. Although in reality the distribution of the stress throughout the member varies, stress can be assumed to be *uniform* throughout the cross section.

(a) Tensile Stress (b) Compressive Stress

Figure 2: Tensile vs. Compressive Force

By convention, a positive force indicates **tensile stress** while a negative force indicates **compressive stress**.

The cross section, as seen in Figure 3, is perpendicular to the axial forces. The corresponding stress in the member in described as axial/normal stress.

Thus, the formula of $\sigma = \frac{P}{A}$ gives the normal stress of an member under axial loading.

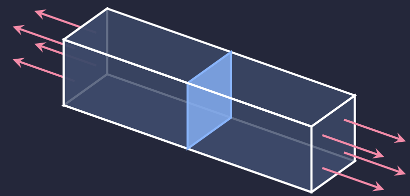


Figure 3: Normal Stress

1.1.3 Stress Points

The formula of $\sigma = \frac{P}{A}$ is only useful for *averages* or *ranges*. This can calculate the average value of the stress over the entire cross section. However, what about calculating at specific points?

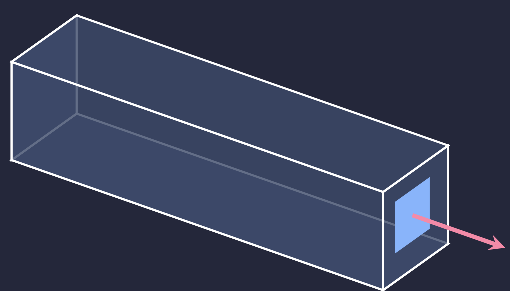


Figure 4: Stress Points

In Figure 4, to find the stress of the highlighted area, the working equation can still be applied, just now with a smaller area. To find the stress at a single point, rather than just a smaller area, stress must be calculated as the area approaches zero.

Stress at a Single Point **1.2**

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

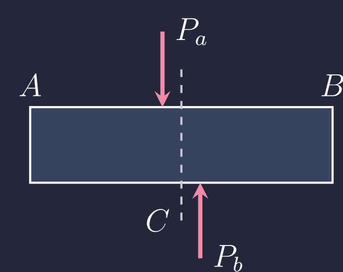
1.2 Shearing Stress

1.2.1 Overview

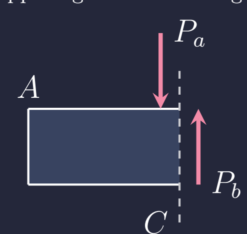
In Section 1.1, the forces creating stress acted perpendicular to the face being analyzed (see Figure 3). This created axial stress. In the case of transverse forces acting parallel to the face, shearing stress is created.

Shear Stress **1.2**

Internal stresses caused by transverse forces acting parallel to a plane.



(a) Opposing Forces Creating Shear



(b) Opposing Forces Creating Shear

Figure 5: Shear Stress

In Figure 5, a member has two opposing forces, P_a and P_b, acting upon it. These two forces create internal shear in the section between them as seen in Figure 5b.

By analyzing the section at C in the member, it can be seen that the internal force created by the two forces in the section is equal to P_b. This resultant force is called a **shear force** (P). Dividing P by the area (A) results in the **average shearing stress** in the section.

Average Shearing Stress **1.3**

$$\tau_{avg} = \frac{P}{A}$$

The average shearing stress cannot be assumed to be uniform throughout the section. Though not the full story, the shearing stress is generally distributed throughout the section such that it is zero at the surface and greater than the average near the center.

1.2.2 Single and Double Shear

The member in Figure 5 is said to be in **single shear** because the shear stress is entirely on a single plane. If forces were to be applied to a member as in Figure 6, there are now two sections and thus the member is said to be in **double shear**.

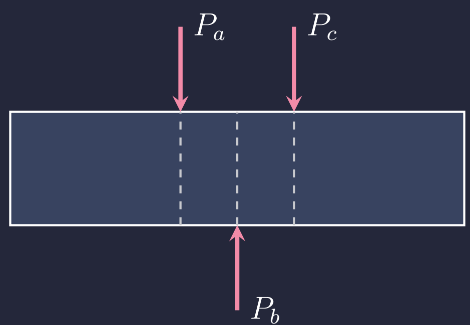


Figure 6: Double Shear

In a case of double shear, the shearing force is divided in half between the planes one between P_a and P_b, and one between P_b and P_c. Thus, to find the shear stress on either of these planes, only half the force will be used.

Double Shear **1.4**

$$\tau = \frac{P}{2A}$$

Since there are two faces of shear stress, they will add together to $\frac{P}{A}$.

1.2.3 Bearing Stress

Often, rather than singular forces creating axial stress, the surfaces of multiple objects will apply distributed forces to each other, thus creating **bearing stress**.

Bearing Stress **1.3**

Similar to axial stress, bearing stress is the result of two members interfacing with each other and applying force along an axis.

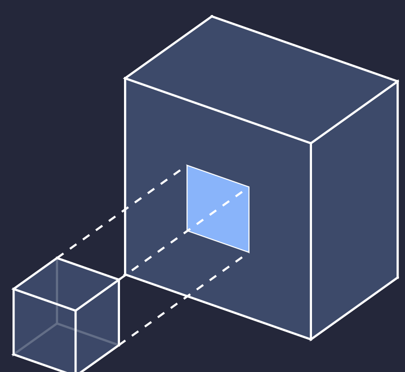


Figure 7: Bearing Shear

In Figure 7, two objects are applying reciprocal forces to each other. Throughout the area the two objects interface over, the distribution of the force is not even. So, in practice, the average nominal value of the stress (σ_b) called the **bearing stress** is used.

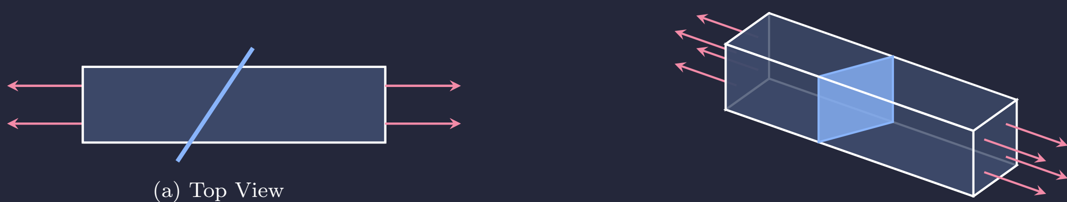
This value is obtained by dividing the load P by the area of the surface of contact.

Bearing Stress

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{1.5}$$

1.3 Stress on an Oblique Plane

Thus far, only stresses perpendicular and parallel to planes have been considered. However, what if an object such as the one in Figure 8 were to be considered where the plane is at an angle?



(a) Top View

(b) Normal View

Figure 8: Oblique Plane

In this situation, the "axial" force applied to the member is no longer just an axial force. There is a component of the force perpendicular to the plane just as there is one parallel. This can be seen in Figure 9, where the **axial stress** to the plane is purple and the **shear stress** to the plane is yellow.

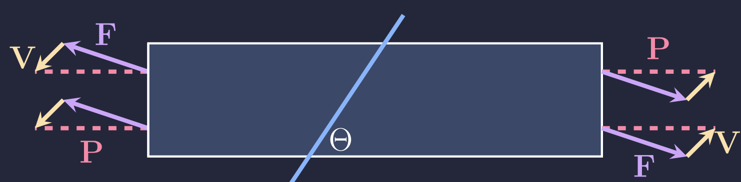


Figure 9: Components of the Applied Force

This begins to illustrate the idea that shear and axial stresses aren't very different, and largely differ based on the perspective they are analyzed from.

To calculate the stresses created by the forces exerted on this member, it's just a matter of calculating the vector components of the force and using them where appropriate.

2 Design Considerations

Determining the stresses in a body, in and of itself, serves no purpose. However, these methods of analysis can be used to inform design decisions and create sturdy structures and machines. An important family of information to an engineer is how a certain material will behave under various kinds of loads.

2.1 Determining the Strength of Materials

Properties, such as the **ultimate load**, of materials can be empirically tested by pushing that material to its limits and taking measurements. To find the ultimate load, a member is put under tensile stress until it cannot go any further.

Ultimate Load **2.1**

The largest axial force that may be applied to a material before it breaks, deforms, or begins to carry less load. Ultimate load is denoted by P_U.

Since axial stress is uniformly distributed, the ultimate load divided by the cross-sectional area of the member gives the **ultimate normal stress** of the material.

Ultimate Normal Stress

$$\sigma_U = \frac{P_U}{A} \tag{2.1}$$

Also called the *Ultimate Strength in Tension*, this measures the maximum axial stress a material can undergo before failing.

Other tests can be performed to determine the **ultimate shearing stress** of a material. The same idea as prior is applied, however this time the load is applied is shear rather than axially.

Ultimate Shearing Stress

$$\tau_U = \frac{P_U}{A} \tag{2.2}$$

Also called the *Ultimate Strength in Shear*, this measures the maximum shear stress a material can undergo before failing.

2.2 Factor of Safety

Though a material can theoretically sustain stresses up to its ultimate load, in practice this is not the case. Since loads are unpredictable and vary, the maximum load a component is designed to sustain should only be a percentage of its ultimate load. This concept is reflected in the idea of a **factor of safety**.

Factor of Safety

$$\text{Factor of Safety} = \text{F.S.} = \frac{\text{ultimate load}}{\text{allowable load}} = \frac{\text{ultimate stress}}{\text{allowable stress}} \qquad \textbf{2.3}$$

The factor of safety is calculated by finding the ratio between the ultimate load/stress and the allowable load/stress.

The relationship between the F.S. as calculated by loads vs. stresses only holds up when a linear relationship exists between the load and the stress. However, most materials experience non-linear a relationship between these two values as the material approaches its ultimate load.

The F.S. to use in a design is a subjective choice, but is based on many factors such as:

1. Variations that may occur in the properties of the member
2. The number of loadings expected during the life of the structure or machine
3. The type of loadings planned for in the design or that may occur in the future
4. The type of failure (brittle and ductile materials will fail differently)
5. Uncertainty due to methods of analysis
6. Deterioration that may occur in the future because of poor maintenance or unpreventable natural causes
7. The importance of a given member in the integrity of the entire structure

3 Deformation

3.1 Normal Strain

When studying the mechanics of materials, analyzing them entirely in static equilibrium would yield little to no information about how a material changes when ample load is applied. Consider the beam in Figure 10. After having endured an axial load, the beam deforms be elongating.

In this case, the elongation undergone by the beam is considered the **strain** endured by the beam. In other situations, strain can look different, such as the bending or compression of a material.

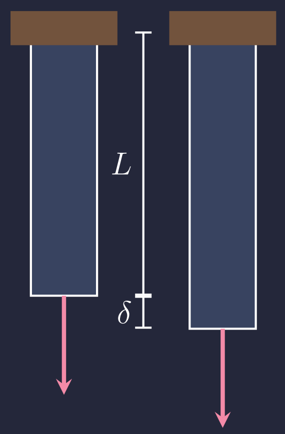


Figure 10: Deformation Due to Axial Stress

> **Strain** **3.1**
>
> Deformation experienced by an object as the result of stresses exceeding the materials ability to maintain shape.

In the case of Figure 10, the strain is specifically **normal strain**.

> **Normal Strain** **3.2**
>
> Strain specifically relating to axial/normal stress. It is specifically defined as the *deformation per unit length*. Normal strain is denoted by the Greek letter epsilon (ϵ).

Since normal strain is the deformation per unit length, to find the value of ϵ, one must divide the deformation by the *original* length of the member.

Normal Strain

$$\epsilon = \frac{\delta}{L} \quad (3.1)$$

Though indirectly caused by an axial load, the factors in determining the amount of normal strain experienced by a material are: the materials inherent properties and the axial stress on the member.

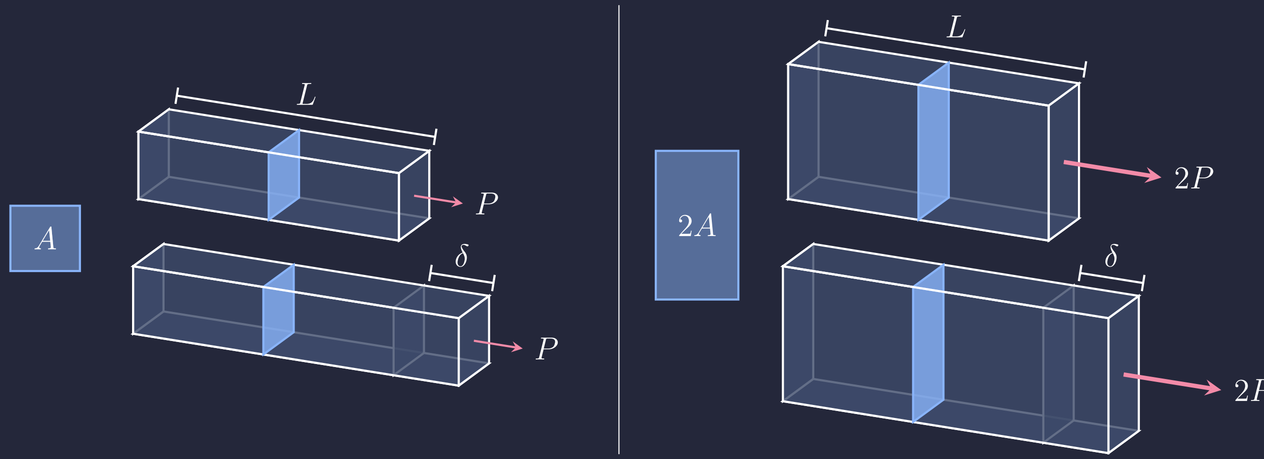


Figure 11: Strain For Two Rods of Different Cross Sections

Assuming the same material, two members of different dimensions, but undergoing the same stress, will strain the same amount. In Figure 11, this exact situation is visualized. Since stress is just a ratio of force to area, a member double the area experiencing an axial load double of another member will strain the same.

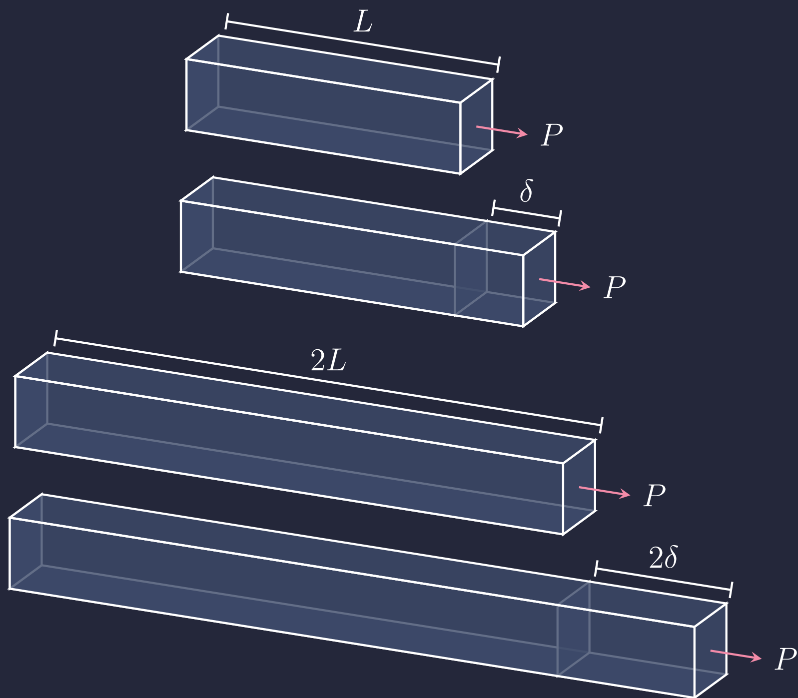


Figure 12: Strain For Two Rods of Different Lengths

Similarly, since strain is a ratio between the original length and the final length, a body double the length of another will experience the same strain assuming all else is the same. See Figure 12.

Strain so far has only been considered for uniform bodies: the same material, same cross-section, etc, and only for members loaded axially on their ends. What happens when these conditions aren't met? Conveniently, strain can be added for subsections of a member, and summed together to find the total strain:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

3.2 Stress Strain Diagrams

Having seen that strain is independent of the dimensions of a material, and rather depends on the stress experienced by the material, it follows that plotting strain as a function of stress would result in a diagram generally applicable to a specific material. This curve, called a **stress-strain diagram** characterizes the properties of a material.

Though all materials behave differently when examined through a stress-strain diagram, there are two broad categories of materials: *brittle* and *ductile*.

Ductile materials are able to yield without necessarily failing entirely. Their elongation initially increases linearly with stress until a some value σ_Y where suddenly undergoes a large deformation with a relatively small increase in stress.

Measuring Ductility

$$\text{Percent Elongation} = 100 \cdot \frac{\text{Length at Failure} - \text{Initial Length}}{\text{Initial Length}} = 100 \cdot \frac{L_B - L_0}{L_0} \quad \textbf{3.2}$$

$$\text{Percent Reduction in Area} = 100 \cdot \frac{\text{Initial Area} - \text{Area at Failure}}{\text{Initial Area}} = 100 \cdot \frac{A_0 - A_B}{A_0}$$

Brittle materials, on the other hand, experience a very small amount of yield, after which they tend to fail suddenly. There is a distinct lack of necking in brittle materials.

3.3 True vs. Engineering Stress and Strain

When an object undergoes elongation, the cross sectional area of that object will change. Thus introduces a question: should the original or current cross-section be used as the area when calculating stress? Rather than having an answer, there are just two types of stress which each reflect one of the options: **engineering stress** and **true stress**.

True vs. Engineering Stress **3.3**

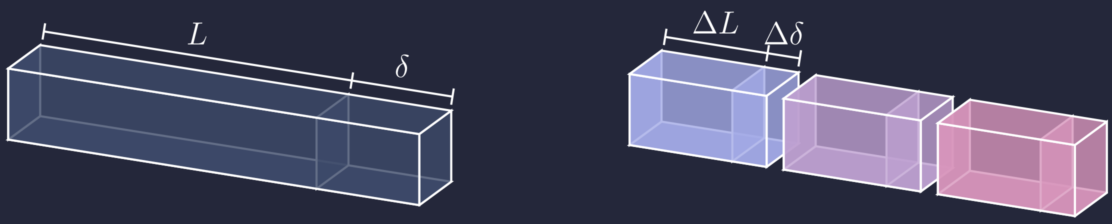
Engineering stress is stress experienced by an object calculated using the original cross-section of the body.

$$\sigma_e = \frac{P}{A_{initial}}$$

True stress is stress as calculated by using the current cross-section of the body.

$$\sigma_t = \frac{P}{A_{current}}$$

A similar situation occurs when measuring the *strain* of a body. Using the formula for strain as introduced in 3.1 produces the **engineering strain** experienced by a body. However, if rather than measuring a single value for the length and deformation of the body, one were to subdivide the body into several subsections and measure the deformation for each subsection, something closer to the **true strain** would be produced. This is seen in Figure 13b.



(a) Engineering Strain

(b) True(ish) Strain

Figure 13: Engineering vs. True Strain

Taking this subdividing to its limit, the **True Strain** will be a perfectly continuous measurement of the change in deformation over some change in length:

$$\epsilon_{true} = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}$$

3.4 Hooke's Law and the Modulus of Elasticity

Generally, structures are designed to keep any deformations within the linear portion of the stress-strain diagram. Given that some body is kept within that range, the stress (σ) is directly proportional to the strain (ϵ):

$$\sigma = E\epsilon$$

Where E is the **modulus of elasticity** of the material.

Modulus of Elasticity	**3.4**
An inherent value to a material that reflects the proportionality between the stress and strain experienced by a body.	

This linear relationship, governed by a constant coefficient (E) is known as **Hooke's Law**. This law only applies until the material reaches its **proportional limit**.

Proportional Limit	**3.5**
The upper bound of the range over which Hooke's Law applies to some material.	

3.5 Repeated Loadings and Fatigue

So far, only single instances of loading have been considered for the materials. What happens after thousands, or millions, of loads have been repeatedly applied to a material? In these cases, the material is said to have undergone **fatigue**.

Fatigue	**3.6**
The result of many repeated loadings, fatigue is a general term for the effect that these repeated loadings have on a material.	

Failures due to fatigue are generally more brittle than they are ductile. Since fatigue not only increases the chances of a material failing, but also makes it such that there is less of a yield period, it is imperative to consider fatigue when designing structures and machines that are expected to support time-varying loads.

To measure the fatigue of a material, a stress to loading curve can be made. This measures the number of repeated loads required to cause failure at different load-sizes for a given material. A generic one can be seen in Figure 14, illustrating the typical shape of such a curve.

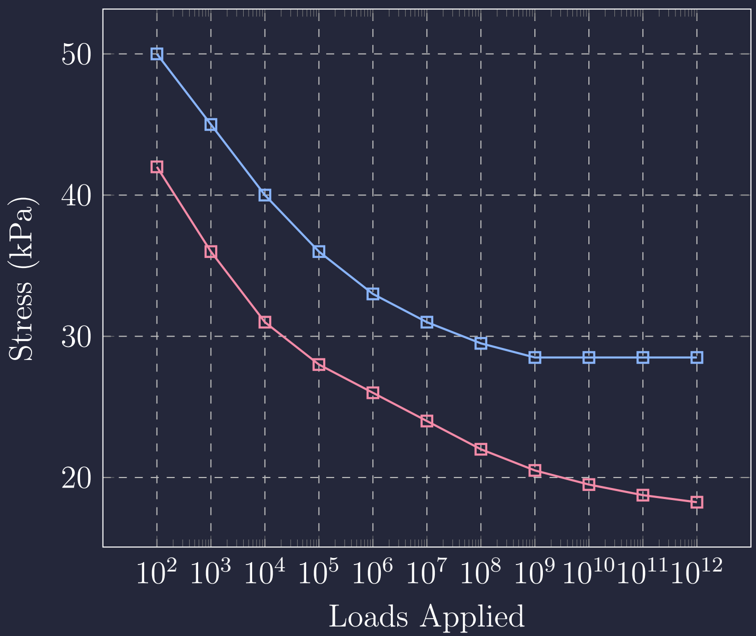


Figure 14: Stress to Loading Curve

The flat part at the end of the blue material in Figure 14 marks the **endurance limit** of the material. This is where, regardless of the number of time a load is applied, the material will never fail. Any force value below the endurance limit on the y-axis (stress) will never cause a material to fail.

However, for non-ferrous materials, there is no endurance limit. Rather, the curve just continues to go down. This is seen in the red curve in Figure 14. This behavior is caused by microscopic failure after each consecutive loading, eventually compounding into total failure.