MATH 2070 - Differential Equations

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1 Second Order ODEs

Similar to first order ODe's, second order ODE's also have a standard form:

$$y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous **1.1**

If the RHS of a second order ODE in standard form is equal to zero ($g(t) = 0$), then the ODE is said to be homogeneous. Otherwise ,the ODE is non-homogeneous.

Rather than focus first on how to solve these second order ODEs, first a little bit of theory about them will be covered.

1.1 Theory

1.1.1 Existence and Uniqueness Theorem

Given an IVP in the standard form with:

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0$$

and given that $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval (a, b), and $t\exists(a, b)$, then the IVP has a unique solution on the interval (a, b).

1.1.2 Wronskian for Linear Independence

If y_a, y_b are solutions of $y'' + p(t)y' + q(t)y = 0$ on an interval where the existence and uniqueness theorem (1.1.1) holds, then y_a, y_b are linearly independent if and only if:

$$W(y_a, y_b) = \begin{vmatrix} y_a(t) & y_b(t) \\ y'_a(t) & y'_b(t) \end{vmatrix} = (y_a(t) \cdot y'_b(t)) - (y_b(t) \cdot y'_a(t)) = 0$$

on the interval.

1.1.3

Consider the ODE:

$$ay'' + by' + cy = 0 \tag{1}$$

where a, b, and c are constants.

Idea: Try $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Using this, (1) becomes:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$\left(ar^2 + br + c\right) e^{rt} = 0$$

e^{rt} will never be 0, so to solve this, use the quadratic equation to solve for r:

$$ar^2 + br + c = 0$$

This equation is referred to as the **auxiliary equation**. If the auxiliary equation has two distinct real roots $r_a \neq r_b$, then there are two solutions:

$$y_a = e^{r_a t} \quad \text{and} \quad y_b = e^{r_b t}$$

$$W(y_a, y_b) = \begin{vmatrix} e^{r_a t} & e^{r_b t} \\ r_a e^{r_a t} & r_b e^{r_b t} \end{vmatrix} = \left(e^{r_a t} \cdot r_b e^{r_b t}\right) - \left(e^{r_b t} \cdot r_a e^{r_a t}\right) \neq 0 (\text{since } r_a \neq r_b)$$

Example 1.1

$$y'' - 5y' + 6y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ (r-2)(r-3) &= 0 \\ r_a = 2, r_b &= 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2t} + C_b e^{3t}$$

Example 1.2

$$2y'' - 7y' + 3y = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} 2r^2 - 7r + 3 &= 0 \\ (2r-1)(r-3) &= 0 \\ r_a = \frac{1}{2}, r_b &= 3 \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{\frac{1}{2}t} + C_b e^{3t}$$

Example **1.3**

$$y'' - 4y' - 6y = 0, y(0) = 1, y'(0) = 0$$

Thus, the auxiliary equation is:

$$\begin{aligned} r^2 - 4r - 6 &= 0 \\ r^2 - 4r &= 6 \\ r^2 - 4r &= 6 \\ r^2 - 4r + 4 &= 10 \\ (r-2)^2 &= 10 \\ r - 2 &= \pm\sqrt{10} \\ r &= 2 \pm \sqrt{10} \end{aligned}$$

Thus, the general solution would be:

$$y = C_a e^{2+\sqrt{10}} + C_b e^{2-\sqrt{10}}$$

To solve for the initial conditions, first find y':

$$y' = (2+\sqrt{10})C_a e^{(2+\sqrt{10})t} + (2-\sqrt{10})C_b e^{(2-\sqrt{10})t}$$

Then create a system of equations based on the initial conditions:

$$y(0) = 1 \Rightarrow C_a + C_b = 1$$
$$y'(0) = 0 \Rightarrow C_a(2+\sqrt{10}) + C_b(2-\sqrt{10}) = 0$$

Solving the system of equations, the solution is found:

$$y = \frac{5-\sqrt{10}}{10} e^{(2+\sqrt{10})t} + \frac{5+\sqrt{10}}{10} e^{(2-\sqrt{10})t}$$