Circuits II - ENEE 2022

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Contents

1 Capacitors and Inductors

Capacitors and inductors **store** energy rather than dissipate it. Because of this, these circuit elements are useful for creating circuits beyond the simple ones possible with just resistors.

1.1 Capacitors

A capacitor is a passive element designed to store energy in its electric field. It's most basic construction is two conductive plates separated by an insulator. See Figure 1.

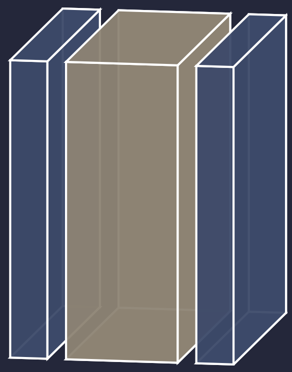


Figure 1: Capacitor

Capacitor **1.1**

A capacitor consists of two conducting plates separated by an insulator (dielectric).

When a voltage source is connected to the capacitor, a charge differential is built up between the plates with one plate accumulating a negative charge and the other a positive charge.

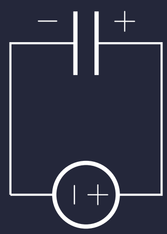


Figure 2: Capacitor Connected to Voltage Source

The amount of charge accumulated (q) is proportional to the voltage (v) supplied as well as the capacitance (C) of the capacitor:

Charge in a Capacitor **1.1**

$$q = Cv$$

1.1.1 Capacitance

Capacitance **1.2**

Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates. The capacitance of a capacitor is measured in Farads (F) where $1\,\mathrm{F} = \frac{1\,\mathrm{C}}{1\,\mathrm{V}}$.

The capacitance of a capacitor depends on the physical properties of the capacitors: the area (A) of the plates, distance (d) between the plates, and the permittivity of the dielectric material (ϵ).

Capacitance **1.2**

$$C = \frac{\epsilon A}{d}$$

1.1.2 Types of Capacitors

Capacitors are generally described by 1) the dielectric material and 2) whether they are fixed or variable. The dielectric material used in a capacitor is outside the scope of this course. However, whether a capacitor is variable is important to how the capacitor functions in a circuit.

(a) Fixed Value Capacitor (b) Variable Capacitor

Figure 3: Types of Capacitors

Similar to resistors, a variable capacitor's capacitance can be changed in real time, usually by turning a knob. It functions the same way a potentiometer does, just with capacitance rather than resistance.

1.1.3 Voltage and Current Across a Capacitor

Recall the formula from Subsection 1.1:

$$q = Cv$$

Additionally, recall that current is nothing more that the movement of charge (in the form of electrons) over time, and thus:

$$i = \frac{dq}{dt}$$

Thus, to find the relationship between current and voltage, simply derive both sides of the formula:

$$\frac{d}{dt}(q) = \frac{d}{dt}(Cv) \rightarrow i = C\frac{dv}{dt}$$

Note that C is simply a constant coefficient of v and thus remains as a coefficient of $\frac{dv}{dt}$. And so the relationship between current and voltage across a capacitor can be modeled as:

Current-Voltage Relationship of a Capacitor	
$$i = C\frac{dv}{dt}$$	**1.3**

Plotting this on a graph (Figure 4), the slope of the line is the value of the capacitance of the capacitor.

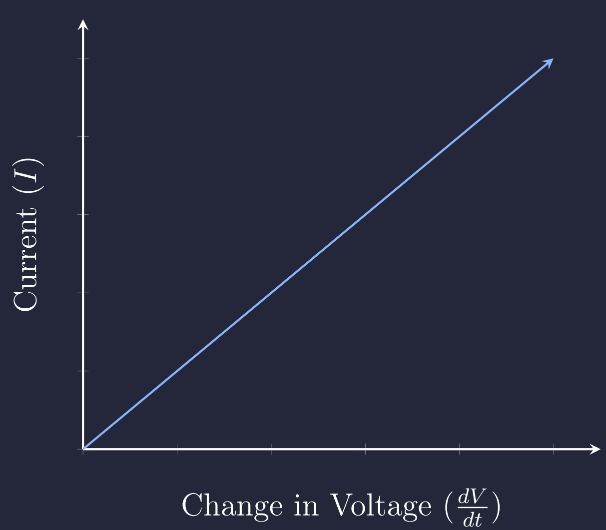


Figure 4: Current-Voltage Relationship of a Capacitor

The relationship between the current and voltage over a capacitor can also be modeled in the other direction. That is to say, rather than expressing current in terms of voltage, voltage can be expressed in terms of current:

$$\left[i = C\frac{dv}{dt}\right] \rightarrow \left[\frac{1}{C}i = \frac{dv}{dt}\right] \rightarrow \left[\frac{1}{C}\int i\,dt = \int \frac{dv}{dt}\right] \rightarrow \left[v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau\right]$$

Which can be alternatively expressed as:

Voltage-Current Relationship of a Capacitor	
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$	**1.4**

Where $v(t_0)$ is the voltage across the capacitor at time t_0. Here, it can be seen that the capacitor has a memory of sorts. At any given time t, the voltage across the capacitor will depend in part on the initial voltage $v(t_0)$.

1.1.4 Power Delivered to a Capacitor

Knowing that the power across some component in a circuit is:

$$p = vi$$

the instantaneous power delivered to a capacitor can be modeled as:

$$p = v \cdot C\frac{dv}{dt}$$

Since the energy stored in a capacitor will simply be the accumulation of the energy delivered, integrating the energy delivered over time will give the energy stored:

$$w = \int_{-inf}^{t} p(\tau)\,d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}\,d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv = \left[\frac{1}{2}Cv^2\right]_{v(-\infty)}^{v(t)}$$

Note that $v(-\infty) = 0$ since the capacitor is uncharged at $t = -\infty$:

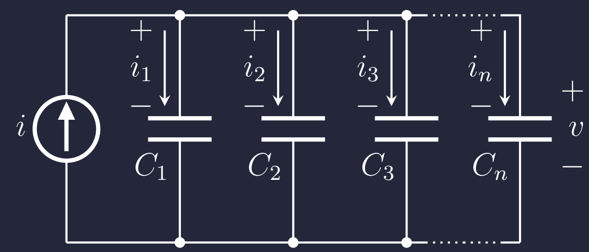
$$\left[\frac{1}{2}Cv^2\right]_{v(-\infty)}^{v(t)} = \frac{1}{2}Cv^2$$

Thus, showing the energy stored in a capacitor:

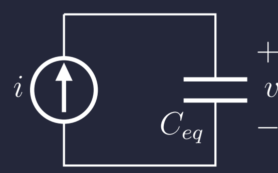
Energy Stored in a Capacitor	
$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}$$	**1.5**

1.1.5 Capacitors in Series and Parallel

Simplifying circuits is a very powerful skill to have to analyze circuits. When dealing with capacitors, how can they be simplified when in series or in parallel. First, capacitors in parallel.



(a) Original Circuit



(b) Simplified Circuit

Figure 5: Capacitors in Parallel

Consider the circuit in Figure 5a. How can the capacitors in the circuit be summed to find some capacitance value C_{eq} such that the circuit in Figure 5b is an equivalent circuit?

Applying KCL to the first circuit:

$$i = i_1 + i_2 + i_3 + \ldots + i_n$$

Then, each current can be rewritten according to $i = C\frac{dv}{dt}$ for capacitors:

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \ldots + C_n\frac{dv}{dt}$$

Since all the capacitors are in parallel, the voltages across them all are equal, thus:

$$i = \frac{dv}{dt}\left(C_1 + C_2 + C_3 + \ldots + C_n\right) = \frac{dv}{dt}\left(\sum_{n=1}^{N} C_n\right) = C_{eq}\frac{dv}{dt}$$

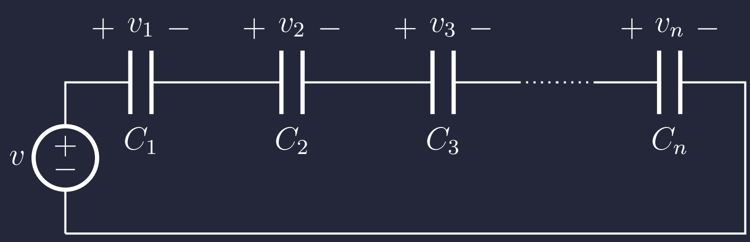
where:

$$C_{eq} = \sum_{n=1}^{N} C_n$$

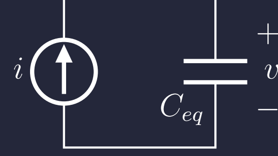
Capacitors in Parallel

$$C_{eq} = C_1 + C_2 + C_3 + \ldots + C_n = \sum_{n=1}^{N} C_n \qquad \textbf{1.6}$$

How can capacitors be simplified when in series?



(a) Original Circuit



(b) Simplified Circuit

Figure 6: Capacitors in Series

Consider the circuit in Figure 6a. By applying KVL to the circuit, the voltages can be equated as:

$$v = v_1 + v_2 + v_3 + \ldots + v_n$$

Then, each voltage can be rewritten according to $v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0)$:

$$v = \frac{1}{C_1}\int_{t_0}^{t} t(\tau)\, d\tau + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} t(\tau)\, d\tau + v_2(t_0) + \frac{1}{C_3}\int_{t_0}^{t} t(\tau)\, d\tau + v_3(t_0) + \ldots + \frac{1}{C_4}\int_{t_0}^{t} t(\tau)\, d\tau + v_4(t_0)$$

Since the capacitors are in series, the current running through each must be the same. Thus:

$$v = \left(\sum_{n=1}^{N} C_n^{-1}\right)\int_{t_0}^{t} i(\tau)\, d\tau + \left[v_1(t_0) + v_2(t_0) + v_3(t_0) + \ldots + v_n(t_0)\right] = \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0)$$

where:

$$\frac{1}{C_{eq}} = \sum_{n=1}^{N} C_n^{-1}$$

Capacitors in Series

$$C_{eq} = \left(C_1^{-1} + C_2^{-1} + C_3^{-1} + \ldots + C_n^{-1}\right)^{-1} = \left(\sum_{n=1}^{N} C_n^{-1}\right)^{-1} \quad \textbf{1.7}$$

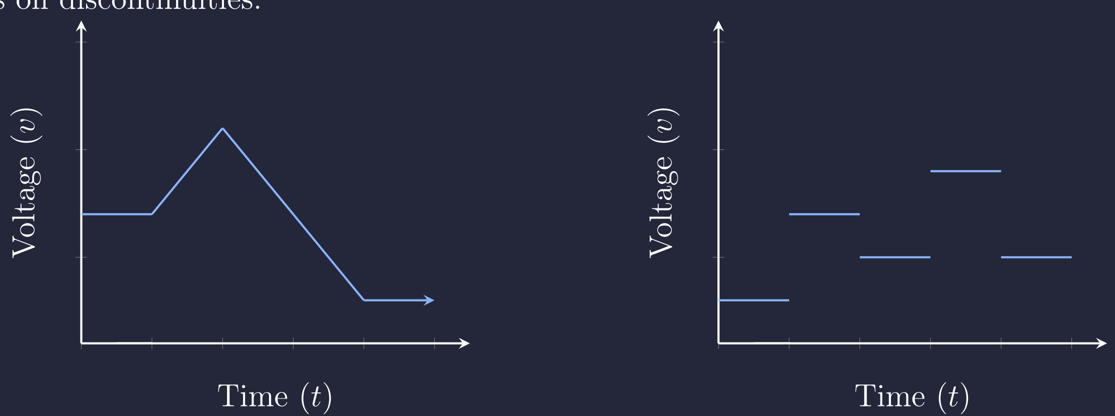
1.1.6 Important Properties of a Capacitor

Returning to the previous equation:

$$i = C\frac{dv}{dt}$$

a few important properties can be extrapolated from this. Firstly, anytime the voltage across a capacitor is constant (unchanging), the current across that capacitor will be zero. Thus, **a capacitor is an open circuit under DC conditions**.

Now, consider the graphs in Figure 7. Both plot voltage over time. However, in Figure 7a, the voltage is continuous meaning that $\forall t, \frac{dv}{dt} \in \mathbb{R}$. However, the same cannot be said for Figure 7b since there are points on discontinuities.



(a) Continuous Voltage

(b) Discontinuous Voltage

Figure 7: Voltage Across a Capacitor

Based on the equation:

$$i = C\frac{dv}{dt}$$

for Figure 7b to be a valid measurement of voltage across a capacitor, the current across that capacitor must be infinite. Clearly, this is not realistic. Thus, another important property of capacitors is that **voltage across a capacitor cannot change abruptly**.

Lastly, like all components that will be studied theoretically in this course, it is to be assumed that capacitors are **ideal**, meaning that **it does not dissipate energy**.

1.1.7 Summary of Capacitors

In short, a capacitor accumulates a differential of charge between two plates to store energy. The charge is:

$$q = Cv$$

where C is the capacitance of the capacitor, proportional to the area of the plates (A), distance between the plates (d), and the permittivity of the dielectric (ϵ):

$$C = \frac{\epsilon A}{d}$$

The relationship between the voltage and current across a capacitor can be expressed as:

$$i = C\frac{dv}{dt} \quad \text{or} \quad v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$

The energy stored in a capacitor is:

$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}$$

When adding capacitors in **series**:

$$C_{eq} = \left(\sum_{n=1}^{N} C_n^{-1}\right)^{-1}$$

When adding capacitors in **parallel**:

$$C_{eq} = \sum_{n=1}^{N} C_n$$

Lastly, capacitors:

1. Function as open circuits under DC conditions

$$\frac{dv}{dt} = 0 \to i = C\frac{dv}{dt} \to i = C \cdot 0 \to i = 0$$

2. Voltage across a capacitor cannot change abruptly

$$\frac{dv}{dt} = \infty \to i = \infty \to \text{not possible}$$

3. An ideal capacitor does **not** dissipate energy

1.2 Inductors

An inductor is a passive element designed to store energy. This effect is generally achieved by coiling wire into a cylindrical shape.

Inductor **1.3**

An inductor is a passive, two-terminal, electrical component that stores energy in a magnetic field when an electric current flows through it.

When current passes through an inductor, the voltage across the inductor is directly proportional to the **rate of change** of the current. Thus:

Voltage Across an Inductor **1.8**

$$v = L\frac{di}{dt}$$

where v is the voltage across the inductor, L is the inductance of the inductor, and $\frac{di}{dt}$ is the rate of change of the current flowing through the inductor.

1.2.1 Inductance

Inductance **1.4**

Inductance is a property that describes how much an inductor opposes a change in current flowing through it. It is measured in Henrys (H).

The inductance of an inductor, similar to the capacitance of a capacitor, is determined by the physical properties of the inductor. Specifically, it is dependent on the cross-sectional area (A), number of turns/coils (N), length of the wire (l), and the permeability of the core (μ).

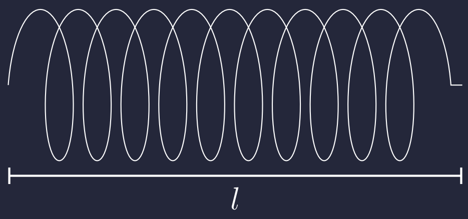


Figure 8: Physical Properties of an Inductor

Inductance **1.9**

$$L = \frac{N^2 \mu A}{l}$$

1.2.2 Types of Inductors

Similar to capacitors, various types of inductors exist. They are described in terms of 1) their core (the material the coil is wrapped around), and 2) whether they are fixed value of

(a) Fixed Value Inductor

(b) Variable Inductor

Figure 9: Types of Inductors

variable. The core of the inductor is outside the scope of this course, but whether an inductor is fixed or variable still may be relevant.

1.2.3 Voltage and Current Across an Inductor

Recall the formula from Subsection 1.2:

$$v = L\frac{di}{dt}$$

using this, the current-voltage relationship can be derived:

$$\left[v = L\frac{di}{dt}\right] \rightarrow \left[di = \frac{1}{L}v\,dt\right] \rightarrow \left[i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau\right]$$

which can be rewritten to give the general relationship between the current flowing through an inductor and the voltage across it:

Current-Voltage Relationship of an Inductor	
$$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$$	**1.10**

1.2.4 Energy Stored in an Inductor

Again, returning to the previous equation from Subsection 1.2:

$$v = L\frac{di}{dt}$$

the power delivered to an inductor can be shown to be:

$$p = vi = L\frac{di}{dt}i$$

and then by integrating both sides, the accumulated energy stored in the inductor is:

$$w = \int_{-\infty}^{t} p(\tau)\,d\tau = L\int_{-\infty}^{t} i\frac{di}{d\tau}\,d\tau = L\int_{-\infty}^{t} i\,di = \left[\frac{1}{2}Li(\tau)^2\right]_{-\infty}^{t}$$

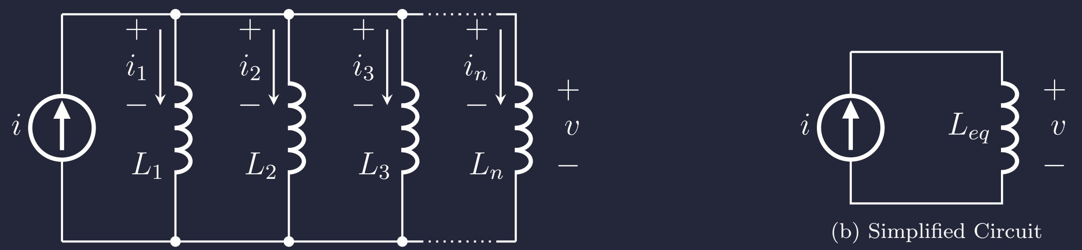
Since $i(-\infty) = 0$:

Energy Stored in an Inductor	
$$w = \frac{1}{2}Li^2$$	**1.11**

1.2.5 Inductors in Series and Parallel

Consider the circuit in Figure 10a. Using KCL, the current in the circuits can be equated as:

$$i = i_1 + i_2 + i_3 + \ldots + i_n$$



(a) Original Circuit

(b) Simplified Circuit

Figure 10: Inductors in Parallel

Then, using the current-voltage relationship of an inductor:

$$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$$

the equation can be rewritten as:

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) + \frac{1}{L_3}\int_{t_0}^{t} v\,dt + i_3(t_0) + \ldots + \frac{1}{L_n}\int_{t_0}^{t} v\,dt + i_n(t_0)$$

Since the voltage v across all inductors is the same (all in parallel), the equation can be simplified into:

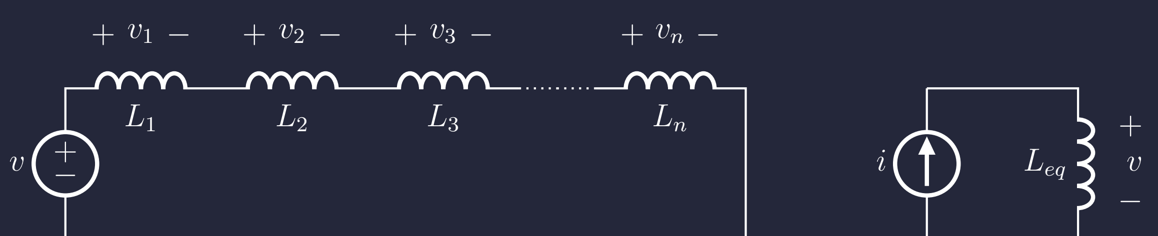
$$i = \left(\sum_{n=1}^{N}\frac{1}{L_n}\right)\int_{t_0}^{t} v\,dt + \left(\sum_{n=1}^{N} i_n(t_0)\right) = \left(\frac{1}{L_{eq}}\right)\int_{t_0}^{t} v\,dt + i(t_0)$$

thus showing that:

Inductors in Parallel

$$L_{eq} = \left(L_1^{-1} + L_2^{-1} + L_3^{-1} + \ldots + L_n^{-1}\right)^{-1} = \left(\sum_{n=1}^{N} L_n^{-1}\right)^{-1} \qquad \textbf{1.12}$$

Now, consider the circuit in Figure 11:



(a) Original Circuit

(b) Simplified Circuit

Figure 11: Inductors in Series

Using KVL in the circuit, the following equation can be established:

$$v = v_1 + v_2 + v_3 + \ldots + v_n$$

then, by using the formula for the voltage across an inductor from Subsection 1.2:

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \ldots + L_n\frac{di}{dt}$$

since all inductors are in series, the current flowing through each is the same. Thus, the equation can be rewritten to:

$$v = (L_1 + L_2 + L_3 + \ldots + L_n)\frac{di}{dt} = \left(\sum_{n=1}^{N} L_n\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

thus showing that:

Inductors in Series	
$$L_{eq} = (L_1 + L_2 + L_3 + \ldots + L_n) = \sum_{n=1}^{N} L_n$$	**1.13**

1.2.6 Important Properties of a Inductor

Returning once more to the voltage equation for a capacitor from Subsection 1.2:

$$v = L\frac{di}{dt}$$

When there is no current flowing through an inductor, it acts as a short circuit since there is no voltage drop across it. In other words, **an inductor is a short circuit in DC conditions**.

Furthermore, since the rate of change of the current flowing through it determines the voltage, an inductor is naturally resistant to instantaneous changes in current. This is because an instantaneous change in current would require an infinite voltage. Thus, **current flowing through an inductor cannot change instantaneously**.

Lastly, an **ideal inductor does not dissipate energy**. The energy stored in it remains indefinitely to be used later. The inductor only draws power when "charging" and only delivers power when returning the previously drawn energy.

1.2.7 Summary of Inductors

In summary, an inductor stores energy in its coils from a circuit and can provide a voltage. The voltage in an inductor is:

$$v = L\frac{di}{dt}$$

where L is the inductance of the inductor, proportional to the area of the cross-section of the coil (A), number of coils (N), length of the coil (l), and the permeability of the core (μ):

$$L = \frac{N^2 \mu A}{l}$$

The relationship between the voltage and current across an inductor can be expressed as:

$$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau + i(t_0)$$

The energy stored in an inductor is:

$$w = \frac{1}{2}Li^2$$

When adding inductors in **series**:

$$L_{eq} = \sum_{n=1}^{N} L_n$$

When adding inductors in **parallel**:

$$L_{eq} = \left(\sum_{n=1}^{N} L_n^{-1} \right)^{-1}$$

Lastly, inductors

1. Function as short circuits under DC conditions

$$\frac{di}{dt} = 0 \to v = L\frac{di}{dt} \to v = L \cdot 0 \to v = 0$$

2. Current across an inductor cannot change abruptly

$$\frac{di}{dt} = \infty \to v = \infty \to \text{not possible}$$

3. An ideal inductor does **not** dissipate energy

1.3 Applications

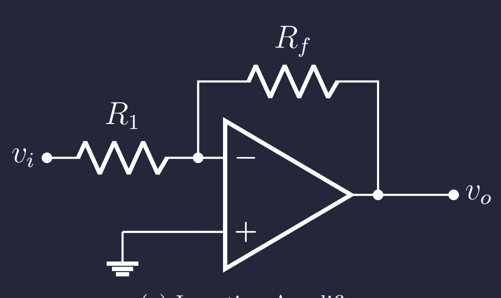
While capacitors and inductors have different applications, they share some properties that make them both very useful:

1. Both can store energy, making them useful as quick sources of high voltage/current
2. Capacitors and inductors resist instantaneous changes in voltage and current respectively making them useful for smoothing out circuits

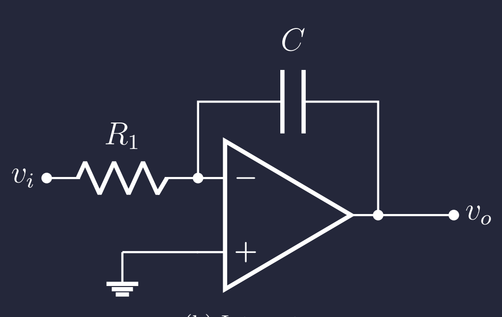
Capacitors and inductors have countless applications. However, there are two fundamental applications that be implemented with the help of op-amps: integrators and differentiators.

1.3.1 Integrator

Recall the configuration of a simple inverting amplifier as shown in Figure 12a. An integrator circuit is incredibly similar, with the only difference being that the resistor R_f is replaced with a capacitor. With this simple change, an integrator circuit is constructed.



(a) Inverting Amplifier



(b) Integrator

Figure 12: From Inverting Amplifier to Integrator

Integrator **1.5**

An integrator is an op-amp circuit whose output is proportional to the integral of the input signal.

But how does this circuit integrate the input voltage? Consider the circuit in Figure 13.

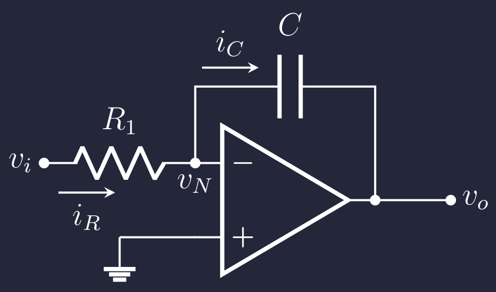


Figure 13: Integrator

Applying KCL at v_N, the following equation can be created. Note that the current flowing from v_N is zero:

$$i_R = i_C$$

The currents across R_1 and C respectively are:

$$i_R = \frac{v_i - v_N}{R_1}; \quad i_C = -C\frac{dv_o}{dt}$$

Thus, the previous equation can be rewritten as:

$$\left[\frac{v_i - v_N}{R_1} = -C\frac{dv_o}{dt}\right] \to \left[-\frac{v_i - 0}{CR_1}dt = dv_o\right] \to \left[-\frac{1}{CR_1}v_i dt = dv_o\right]$$

Then, by integrating both sides:

$$\left[\int_0^t dv_o = -\frac{1}{CR_1}\int_0^t v_i\,dt\right] \to \left[v_o(t) - v_o(0) = -\frac{1}{CR_1}\int_0^t v_i\,dt\right]$$

Assuming that the capacitor is fully discharged at $t = 0$, then $v_o(0) = 0$ and the equation for the integrator circuit is:

Integrator Voltage **1.14**

$$v_o(t) = -\frac{1}{CR_1}\int_0^t v_i\,dt$$

While the circuit in Figure 13 is a functional integrating circuit, it is not fully practical. As described in Subsubsection 1.1.6, a capacitor functions as an open circuit under DC conditions. So, if some DC voltage were to be applied at v_i, the feedback through the capacitor would become open and the op-amp would function as a comparator. To remedy this, it is common practice to connect a resistor in parallel with the capacitor as in Figure 14.

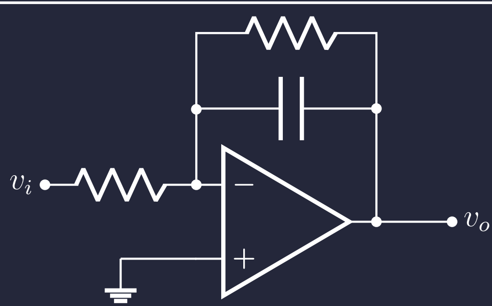


Figure 14: Practical Integrator

1.3.2 Differentiator

To construct a differentiator, simply swapping the resistor and capacitor from the integrator circuit is sufficient. Thus, Figure 15 shows the basic construction of a differentiator.

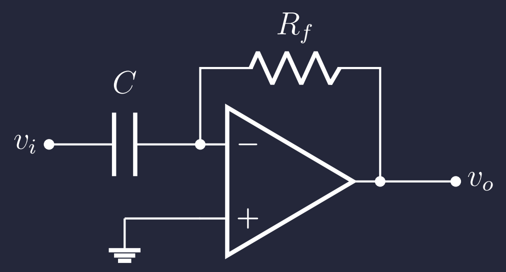


Figure 15: Differentiator

Differentiator **1.6**

An differentiator is an op-amp circuit whose output is proportional to the rate of change (derivative) of the input signal.

But how does this circuit differentiate the input voltage? Consider the circuit in Figure 16.

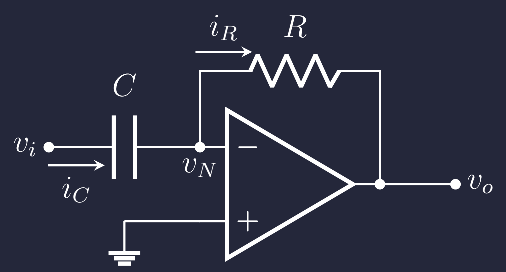


Figure 16: Differentiator

Applying KCL at v_N:

$$i_R = i_C$$

The currents i_R and i_C are:

$$i_R = \frac{0 - v_o}{R}; \quad i_C = C\frac{dv_i}{dt}$$

Meaning that the previous equation can be rewritten as:

$$\left[\frac{0 - v_o}{R} = C\frac{dv_i}{dt}\right] \rightarrow \left[v_o = -RC\frac{dv_i}{dt}\right]$$

Thus, the output is shown to be the derivative of the input amplified by $-RC$:

Differentiator Output Signal **1.15**

$$v_o = -RC\frac{dv_i}{dt}$$

2 First Order Circuits

Now having the tools to analyze capacitors and inductors, circuits beyond simple resistance-based ones can be analyzed. Previously with circuit consisting of only sources and resistors, applying Kirchoff's Laws resulted in algebraic expressions that represented the circuit.

With capacitors and inductors, applying these same laws will yield differential equations that represent the circuit. Considering that the equation for the current across a capacitor is:

$$i = C\frac{dv}{dt}$$

and the voltage across an inductor is:

$$v = L\frac{di}{dt}$$

any set of equations involving one of these equalities would contain derivatives of the first order at most. Thus, circuits involving a single capacitor or inductor are **First Order Circuits**.

First Order Circuit **2.1**

A First Order Circuit is a circuit that is characterized by a first order differential equation.

2.1 RC Circuits

An **RC Circuit** is a foundational circuit both for its simplicity as well as how widely it is used in practice. The basic configuration of the circuit is seen in Figure 17.

RC Circuit **2.2**

An RC Circuit is a circuit comprising of a resistor (R) and a capacitor (C).

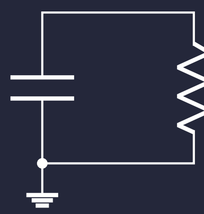


Figure 17: RC Circuit

2.1.1 RC Circuit Without External Source

Consider the circuit in Figure 18 where there is no independent voltage source connected in the circuit. There is only a capacitor and a resistor connected in series. Additionally, assume that the capacitor is initially charged (which is where the energy in the circuit will come from).

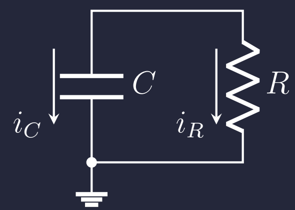


Figure 18: Source Free RC Circuit

Applying KCL at the ground node of the circuit, it can be seen that:

$$i_C + i_R = 0$$

By definition:

$$i_C = C\frac{dv}{dt} \quad \text{and} \quad i_R = \frac{v}{R}$$

Thus, the KCL equation can be expanded to:

$$\left[C\frac{dv}{dt} + \frac{v}{R} = 0\right] \to \left[\frac{dv}{dt} + \frac{v}{CR} = 0\right]$$

Here it should be apparent that this is a first order differential equation. To solve it, first rearrange it:

$$\left[\frac{dv}{dt} + \frac{v}{CR} = 0\right] \to \left[\frac{dv}{dt} = -\frac{v}{CR}\right] \to \left[\frac{1}{v}dv = -\frac{1}{CR}dt\right]$$

Then, integrating both sides:

$$\left[\frac{1}{v}dv = -\frac{1}{CR}dt\right] \to \int \to \left[\ln|v| = -\frac{1}{CR}\cdot t + C\right]$$

Imagine the integration constant C as being $\ln|A|$. In other words, set $C = \ln|A|$:

$$\left[\ln|v| = -\frac{1}{CR}\cdot t + C\right] \to \left[\ln|v| = -\frac{1}{CR}\cdot t + \ln|A|\right]$$

$$\to \left[\ln|v| - \ln|A| = -\frac{1}{CR}\cdot t\right] \to \left[\ln\left|\frac{v}{A}\right| = -\frac{1}{CR}\cdot t\right]$$

Then, exponentiating both sides:

$$\left[\ln\left|\frac{v}{A}\right| = -\frac{1}{CR}\cdot t\right] \to e^x \to \left[e^{\ln\left|\frac{v}{A}\right|} = e^{\frac{-t}{CR}}\right] \to \left[\frac{v}{A} = e^{\frac{-t}{CR}}\right] \to \left[v = Ae^{\frac{-t}{CR}}\right]$$

Considering that the capacitor in the circuit was initially charged, how will that initial charge be represented in the characteristic equation? Recall that A was the integration constant, so based on the initial conditions: $A = v(0) = V_i$. Thus:

$$\left[v = Ae^{\frac{-t}{CR}}\right] \to \left[v = V_i e^{\frac{-t}{CR}}\right]$$

This equations is the characteristic equation of a simple, source-free, RC Circuit. Notice the behavior of the equation; starting at $t = 0$, the voltage in the circuit is initially V_i and decays exponentially over time.

Natural Response of an RC Circuit	2.1
$v = V_i e^{-\frac{t}{CR}} = V_i e^{-\frac{t}{\tau}}$	

The rate at which the voltage decay is important. It's known that the decay is exponential, but the decay can be described more specifically than just "exponentially". For some exponential function:

$$e^{\frac{t}{\tau}}$$

the function will decay by a factor of $\frac{1}{e}$ every τ seconds (assuming t is measured in seconds). This value τ is called the **time constant**. So, with the function for the RC Circuit:

$$V_i e^{-\frac{t}{CR}}$$

it can be seen that $\tau = CR$.

The voltage of the circuit can be graphed over time to visually see the behavior of the circuit.

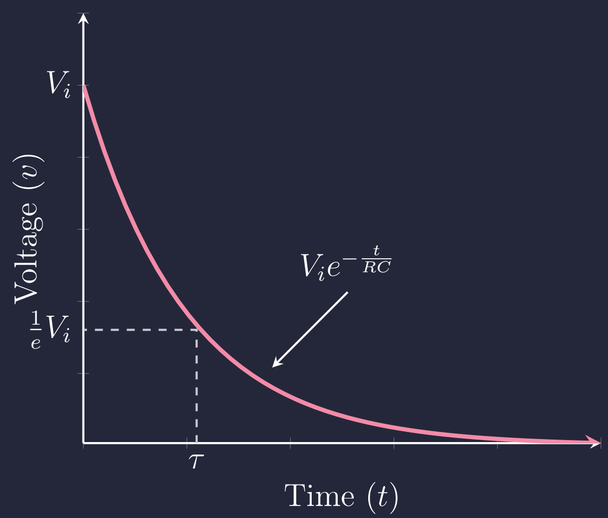


Figure 19: Voltage Response of the RC Circuit

Since all the behavior of the circuit discussed so far is the behavior of the circuit without external interference (some voltage or current source), this is considered the **natural response** of the circuit.

> **Natural Response** **2.3**
>
> The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

2.1.2 Power and Energy

Considering the circuit from the perspective of the resistor, by applying Ohm's Law to the voltage function, a similar function to model the current through the circuit can also be derived:

$$i_R(t) = \frac{v(t)}{R} = \frac{V_i}{R} e^{-\frac{t}{RC}}$$

Knowing the $p = vi$, the power dissipated by the resistor is:

$$p = vi_R = V_i e^{-\frac{t}{RC}} \cdot \frac{V_i}{R} e^{-\frac{t}{RC}} = \frac{V_i^2}{R} e^{-\frac{2t}{RC}}$$

and integrating the power dissipated gives the total energy absorbed by the resistor:

$$w = \frac{V_i^2}{R} \int_0^t e^{-\frac{2\lambda}{RC}}\, d\lambda = -\frac{RCV_i^2}{2R} \left[e^{-\frac{2\lambda}{RC}} \right]_0^t = -\frac{CV_i^2}{2} \left(e^{-\frac{2(0)}{RC}} - e^{-\frac{2(t)}{RC}} \right) = -\frac{CV_i^2}{2} \left(1 - e^{-\frac{2t}{RC}} \right)$$

Notice that as $t \to \infty$, $w_R \to \frac{1}{2}CV_i^2$ which is the same as the initial energy stored in the capacitor. This drives home the point that, throughout the life of the circuit, the energy initially stored in the capacitor is being dissipated by the resistor.

Energy in a Capacitor $$w_C = \frac{1}{2}CV_i^2$$	**2.2**

2.1.3 RC Circuit with External Voltage Source

In many cases, an RC Circuit won't exist without any external source supplying voltage to it. When an external source is applied at some time, the source can be modeled as a **step function**.

Step Function **2.4**

A step function is a function that is low (zero) by default until some time α where it jumps to some high value. The *unit* step function starts at zero and goes to one. Expressed with notation, it is:

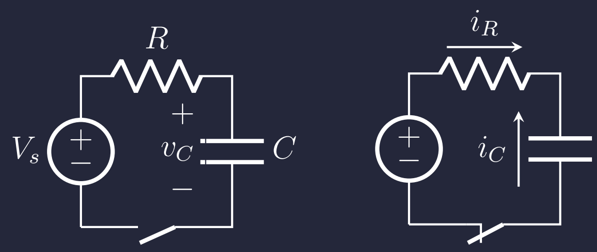
$$\mathcal{U}(t-\alpha) = \begin{cases} 0, & t < \alpha \\ 1, & \alpha \leq 0 \end{cases}$$

Consider the circuit in Figure 20. Applying KCL to the circuit, the following equation can be found:

$$i_C + i_R = 0$$

or, through Ohm's law, it can be rewritten as:

$$\left[C\frac{dv}{dt} + \frac{v_C - V_s\mathcal{U}(t)}{R} = 0\right] \rightarrow \left[\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s\mathcal{U}(t)}{RC}\right]$$



(a) $t < 0$ (b) $t \geq 0$

Figure 20: RC Circuit with a Step Input

Considering only time after the switch has been closed, $\mathcal{U}(t) = 1$:

$$\left[\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s\mathcal{U}(t)}{RC}\right] = \left[\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}\right]$$

then, further rearranging the equation and integrating:

$$\left[\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}\right] \rightarrow \left[\frac{1}{v - V_s}dv = -\frac{1}{RC}dt\right] \rightarrow \int \rightarrow \left[\ln|v - V_s| = -\frac{t}{RC}\right]$$

What bounds are being integrated over? For dt, it would simply be from $t = 0$ to $t = t$. For dv, the initial condition is whatever charge the capacitor had *at* $t = 0$: $v(0) = V_0$. The final condition is the charge of the capacitor at some time t: $v(t)$. Thus:

$$\Big[\ln|v - V_s|\Big]_{V_0}^{v(t)} = \left[-\frac{t}{RC}\right]_0^t$$

which can be rearranged as:

$$\left[\ln|v(t) - V_s| - \ln|V_0 - V_s| = -\frac{t}{RC} + \frac{0}{RC}\right] \rightarrow \left[\ln\left|\frac{v(t) - V_s}{V_0 - V_s}\right| = -\frac{t}{RC}\right]$$

$$\rightarrow e^x \rightarrow \left[\frac{v(t) - V_s}{V_0 - V_s} = e^{-\frac{t}{RC}}\right] \rightarrow \left[v(t) - V_s = (V_0 - V_s)e^{-\frac{t}{RC}}\right] \rightarrow \left[v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{RC}}\right]$$

Recall that this is all during the interval of $t \geq 0$, while before $t = 0$, the capacitor had some arbitrary charge of V_0. Thus:

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{RC}}, & 0 \leq t \end{cases}$$

This equation models the **complete response** of the RC Circuit.

Complete Response of an RL Circuit **2.3**

$$\begin{aligned} v(t) &= v(\infty) + \big[v(0) - v(\infty)\big]e^{-\frac{t}{\tau}} \\ &= V_i e^{-\frac{t}{\tau}} + V_s\left(1 - e^{-\frac{t}{\tau}}\right) \\ &= \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & 0 \leq t \end{cases} \end{aligned}$$

This is because it incorporates both the *natural* and *forced* responses of the circuit. The natural response, as described earlier, is the behavior of the circuit due to the initial charge in the capacitor. The forced response is the behavior due to the external source applied to the circuit.

Forced Response **2.5**

The forced response of a circuit refers to the behavior of the circuit as caused by the application of some external voltage or current source.

A more conceptual way to view the equation is in terms of the natural response (v_n) and forced response (v_f) directly:

$$v(t) = v_n + v_f$$

where:

$$v_n = V_0 e^{-\frac{t}{RC}};\ v_f = V_s\left(1 - e^{-\frac{t}{RC}}\right)$$

Furthermore, a practical way to view the equation is:

$$v(t) = v(\infty) + [v(0) - v(\infty)]\, e^{-\frac{t}{RC}}$$

where:

$$v(\infty) = \text{final steady-state voltage}; v(0) = \text{initial voltage}$$

which makes solving the circuit as straightforward as finding:

1. $v(0)$: the initial voltage across the capacitor
2. $v(\infty)$: the final voltage across the capacitor
3. $\tau = RC$: the time constant of the circuit

2.2 RL Circuits

An **RL Circuit** occupies a similar space in terms of its simplicity and popularity as the RC Circuit does. Functionally, the only difference is that the RL Circuit uses an inductor where the RC Circuit uses a capacitor.

RL Circuit **2.6**

An RL Circuit is a circuit comprising of a resistor (R) and an inductor (L).

2.2.1 RL Circuit Without External Source

Consider the circuit in Figure 21. This circuit is a source free RL circuit seeing as there is an inductor in series with a resistor (RL circuit) and no voltage/current sources (source free).

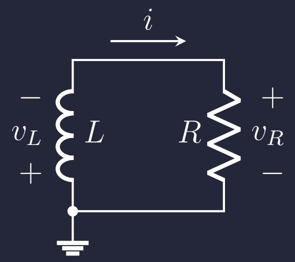


Figure 21: Source Free RL Circuit

Applying KVL on the circuit, the following equation can be derived:

$$v_L + v_R = 0$$

By definition:

$$v_L = L\frac{di}{dt} \quad \text{and} \quad v_R = iR$$

Thus, the KVL equation can be rewritten as:

$$\left[L\frac{di}{dt} + iR = 0\right] \to \left[\frac{di}{dt} + \frac{iR}{L} = 0\right]$$

Just like the previous RC Circuit, the basic RL Circuit is seen to be a first order circuit characterized by its first order differential equation.

$$\left[\frac{di}{dt} + \frac{iR}{L} = 0\right] \to \left[\frac{di}{dt} = -\frac{iR}{L}\right] \to \left[\frac{1}{i}di = -\frac{R}{L}dt\right]$$

Then, integrating both sides:

$$\left[\frac{1}{i}di = -\frac{R}{L}dt\right] \to \int \to \left[\ln|i| = -\frac{R}{L}t + C\right]$$

Again, setting $C = \ln|A|$:

$$\left[\ln|i| = -\frac{R}{L}t + C\right] \to \left[\ln|i| = -\frac{R}{L}t + \ln|A|\right] \to \left[\ln|i| - \ln|A| = -\frac{R}{L}t\right] \to \left[\ln\left|\frac{i}{A}\right| = -\frac{R}{L}t\right]$$

Then, exponentiating both sides:

$$\left[\ln\left|\frac{i}{A}\right| = -\frac{R}{L}t\right] \to e^x \to \left[e^{\ln\left|\frac{i}{A}\right|} = e^{-\frac{R}{L}t}\right] \to \left[\frac{i}{A} = e^{-\frac{R}{L}t}\right] \to \left[i = Ae^{-\frac{R}{L}t}\right]$$

Again, A represents the initial conditions of the circuit. Previously $A = V_i$ to account for the initial charge built up in the capacitor. Now with the inductor, $A = I_i$ for the initial current in the inductor.

$$\left[i = Ae^{-\frac{R}{L}t}\right] \rightarrow \left[i = I_i e^{-\frac{R}{L}t}\right]$$

This now gives the characteristic equation for a source free RL circuit.

Characteristic Equation of a RL Circuit $$i = I_i e^{-\frac{R}{L}t} = I_i e^{-\frac{t}{\tau}}$$	**2.4**

This shows that the RL Circuit still experiences exponential decay after starting with some initial condition ($i = I_i$). However, now the time constant is different, being:

$$\tau = \frac{L}{R}$$

2.2.2 Power and Energy

What would the voltage drop across the resistor be? By applying Ohm's Law:

$$v_R = iR = I_i e^{-\frac{t}{\tau}} \cdot R$$

Knowing the $p = vi$, the power dissipated by the resistor is:

$$p = vi_R = I_i R e^{-\frac{t}{\tau}} \cdot I_i e^{-\frac{t}{\tau}} = I_i^2 R e^{-\frac{2t}{\tau}}$$

then integrating to find the total energy absorbed by the resistor:

$$w = I_i^2 R \int_0^t e^{-\frac{2\lambda}{\tau}}\, d\lambda = -\frac{\tau}{2} I_i^2 R \left[e^{-\frac{2\lambda}{\tau}}\right]_0^t = -\frac{\tau}{2} I_i^2 R \left(e^{-\frac{2(0)}{\tau}} - e^{-\frac{2(t)}{\tau}}\right) = -\frac{\tau}{2} I_i^2 R \left(1 - e^{-\frac{2(t)}{\tau}}\right)$$

Notice that as $t \rightarrow \infty$, $w_R \rightarrow \frac{1}{2} L I_i^2$ which is the same as the initial energy stored in the inductor. Again, this shows the behavior of the resistor dissipating over time the energy initially stored in the inductor.

Energy in a Inductor $$w_L = \frac{1}{2} L I_i^2$$	**2.5**

2.2.3 RL Circuit with External Voltage Source

Consider the circuit in Figure 22. A process similar to the process detailed in Subsubsection 2.1.3 can be done, starting with KVL applied to the circuit. However, rather than doing that, consider the equation:

$$i = i_n + i_f$$

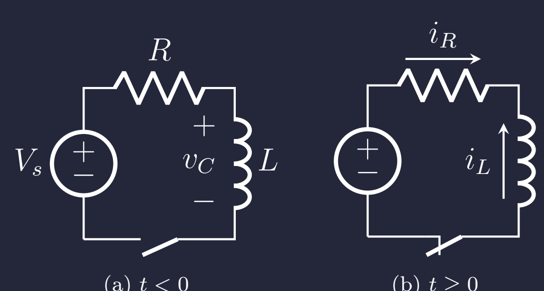


Figure 22: RL Circuit with a Step Input

as determined in Subsubsection 2.2.1, the natural response (i_n) of an RL circuit is generally a decaying exponential:

$$i_n = Ce^{-\frac{t}{\tau}};\ \tau = \frac{L}{R}$$

and the final behavior of the circuit, when the inductor becomes a short circuit, will be:

$$i_f = \frac{V_s}{R}$$

substituting back into the first equation:

$$i = Ce^{-\frac{t}{\tau}} + \frac{V_s}{R}$$

What is the constant C? At $t = 0$, the current equation for i becomes:

$$\left[i = Ce^{-\frac{t}{\tau}} + \frac{V_s}{R}\right] \to \left[I_0 = Ce^{-\frac{0}{\tau}} + \frac{V_s}{R}\right] \to \left[I_0 = C + \frac{V_s}{R}\right] \to \left[C = I_0 - \frac{V_s}{R}\right]$$

Then, substituting back into the equation:

$$\left[i = Ce^{-\frac{t}{\tau}} + \frac{V_s}{R}\right] \rightarrow \left[i = \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}} + \frac{V_s}{R}\right]$$

Which now gives the **complete response** of the RL Circuit:

Complete Response of an RL Circuit

$$\begin{aligned} i(t) &= i(\infty) + \big[i(0) - i(\infty)\big]e^{-\frac{t}{\tau}} \\ &= I_i e^{-\frac{t}{\tau}} + I_s\left(1 - e^{-\frac{t}{\tau}}\right) \\ &= \begin{cases} I_0, & t < 0 \\ I_s + (I_0 - I_s)e^{-\frac{t}{\tau}}, & 0 \le t \end{cases} \end{aligned} \tag{2.6}$$

Which again shows that solving such a circuit is as simple as finding:

- $i(0)$: the initial current across the inductor
- $i(\infty)$: the final current across the inductor
- $\tau = \frac{L}{R}$: the time constant of the circuit

2.3 Summary of First Order Circuits

A first order circuit is one that contains a single inductor or capacitor, and is thus characterized by a first order differential equation. All the circuits in Figure 23 are first order circuits.

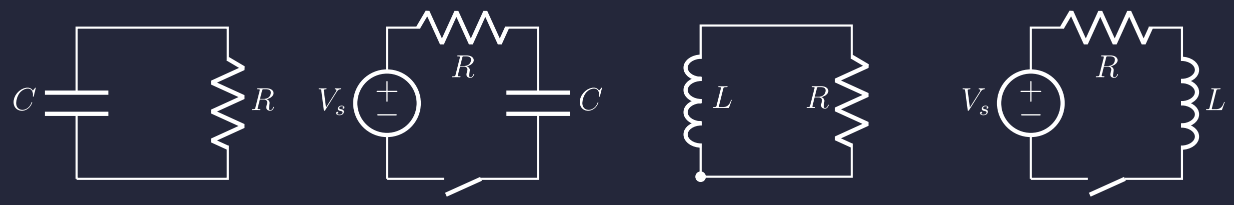


Figure 23: First Order Circuits

2.3.1 RC Circuits

The time constant for an RC Circuit is:

$$\tau = RC$$

The natural response of an RC Circuit is:

$$v(t) = V_i e^{-\frac{t}{\tau}}$$

and the complete response of the RC Circuit is:

$$\begin{aligned} v(t) &= v(\infty) + \big[v(0) - v(\infty)\big]e^{-\frac{t}{\tau}} \\ &= V_i e^{-\frac{t}{\tau}} + V_s\left(1 - e^{-\frac{t}{\tau}}\right) \\ &= \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & 0 \le t \end{cases} \end{aligned}$$

2.3.2 RL Circuits

The time constant for an RL Circuit is:

$$\tau = \frac{L}{R}$$

The natural response of an RL Circuit is:

$$i(t) = I_i e^{-\frac{t}{\tau}}$$

and the complete response of the RL Circuit is:

$$\begin{aligned} i(t) &= i(\infty) + \left[i(0) - i(\infty)\right] e^{-\frac{t}{\tau}} \\ &= I_i e^{-\frac{t}{\tau}} + I_s \left(1 - e^{-\frac{t}{\tau}}\right) \\ &= \begin{cases} I_0, & t < 0 \\ I_s + (I_0 - I_s) e^{-\frac{t}{\tau}}, & 0 \leq t \end{cases} \end{aligned}$$

3 Analysis Through the Laplace Transform

As has been seen in previous sections, circuits with capacitors and inductors can be modeled by differential equations. Through solving these differential equations, the behavior of the circuit can be explained. However, this process is sometimes quite cumbersome and tedious.

The method of using the **Laplace transform** can turn these differential equations into algebraic equations to solve, reducing the procedural bottleneck that the differential equations create.

3.1 Defining the Laplace Transform

3.1.1 Laplace Transform

Laplace Transform

Given some function $f(t)$, the Laplace transform is defined as:

$$\mathcal{L}\big\{f(t)\big\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \tag{3.1}$$

where s is a complex variable given by:

$$s = \sigma + j\omega$$

Notice that, after its transformation, the function $f(t)$ becomes a new function $F(s)$, with a different independent variable. Whereas $f(t)$ functioned in the time domain (t-domain), the transformed function functions in the complex domain (s-domain). Thus, the Laplace transform can be understood as follows:

Laplace Transform **3.1**

The Laplace transform is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

A function $f(t)$ may not always *have* a Laplace transform. In order for $f(t)$ to have a Laplace transform, the integral in the Laplace transform:

$$\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

must converge to some finite value. Luckily, all functions used in circuit analysis properly satisfy this criteria, so it can be assumed that all functions used throughout this course have a Laplace transform.

3.1.2 Inverse Laplace Transform

There also exists an inverse Laplace transform. Whereas the Laplace transform took a function in the time domain and returned a function in the complex domain, the inverse Laplace transform does the opposite. A function $F(s)$ can be inverse Laplace transformed into some other function $f(t)$.

The inverse Laplace transform is even trickier to compute manually. For this reason, rather than relying on the definition given below, common Laplace and inverse Laplace transforms will be generalized into a table for reference.

Inverse Laplace Transform

Given some function $F(S)$, the inverse Laplace transform is defined as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

3.2

3.2 Properties of the Laplace Transform

3.2.1 Linearity

For some real numbers α and β, consider the following Laplace Transform:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\}$$

By the definition of the Laplace Transform given in Subsection 3.1:

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty \left[\alpha f(t) + \beta g(t)\right]e^{-st}\,dt \\
&= \int_0^\infty \alpha f(t)e^{-st} + \beta g(t)e^{-st}\,dt \\
&= \int_0^\infty \alpha f(t)e^{-st}\,dt + \int_0^\infty \beta g(t)e^{-st}\,dt \\
&= \alpha\int_0^\infty f(t)e^{-st}\,dt + \beta\int_0^\infty g(t)e^{-st}\,dt \\
&= \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}
\end{aligned}$$

Thus, it can be seen that, through the linearity of integrals, so too is the Laplace transform linear.

Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

3.3

3.2.2 Scaling

For some real number α where $\alpha > 0$, consider:

$$\mathcal{L}\{f(\alpha t)\} = \int_{0^-}^\infty f(at)e^{-st}\,dt$$

If $x = \alpha t$, and consequently $dx = a\ dt$, then:

$$\mathcal{L}\{f(\alpha t)\} = \int_{0^-}^\infty f(x)e^{-s\frac{x}{\alpha}}\,\frac{dx}{\alpha} = \frac{1}{\alpha}\int_{0^-}^\infty f(x)e^{-x\frac{s}{a}}\,dx$$

By comparing this current equation with the definition of the Laplace transform, it can be seen that:

- s must be replaced by $\frac{s}{\alpha}$: $s \mapsto \frac{s}{\alpha}$
- t is replaced by x: $t \mapsto x$

Scaling Property

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \tag{3.4}$$

3.2.3 Frequency Shift

Consider the Laplace Transform of:

$$\mathcal{L}\{e^{at} f(t)\}$$

This can be rewritten as:

$$\int_0^\infty e^{at} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s-a)t}\, dt = F(s-a)$$

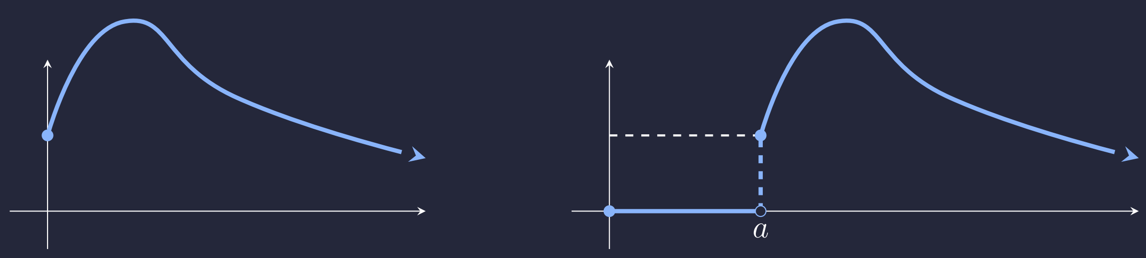
What this shows is that, when a function $f(t)$ is multiplied by an exponential function of e^{at}, the resulting Laplace Transform will be whatever the Laplace Transform of $f(t)$ is, with $s \mapsto s - a$.

Frequency Shift

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \text{or} \quad \mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s \mapsto s-a} \tag{3.5}$$

3.2.4 Time Shift

Considering some function shifted by some value a with the unit step function:



(a) $y = f(t)$ (b) $y = f(t-a)\mathcal{U}(t-a)$

The Laplace Transform of the function is:

$$\int_0^\infty f(t-a)\mathcal{U}(t-a) e^{-st}\, dt = \int_a^\infty f(t-a)\mathcal{U}(t-a) e^{-st}\, dt$$

this equivalency is valid since, up until $t = a$, the value of the entire function is 0 since the unit step function has yet to be "activated". Then, mapping $(t-a) \mapsto T$:

$$\int_0^\infty f(T)\mathcal{U}(T) e^{-s(T+a)}\, dt = \int_0^\infty f(T)\mathcal{U}(T) e^{-sT-sa}\, dT = e^{-sa} \int_0^\infty f(T) \cdot 1 \cdot e^{-sT}\, dT = e^{-as} F(s)$$

Thus:

$$\mathcal{L}\left\{f(t-a)\mathcal{U}(t-a)\right\} = e^{-as} F(s) \quad \text{or} \quad \mathcal{L}\left\{f(t)\mathcal{U}(t-a)\right\} = e^{-as} \mathcal{L}\left\{f(t+a)\right\}$$

Time Shift

$$\mathcal{L}\left\{f(t-a)\mathcal{U}(t-a)\right\} = e^{-as} F(s) \tag{3.6}$$

3.2.5 Time Differentiation

If $f(t)$ is continuous on the interval $[0, \infty)$, of exponential order, and if $f'(t)$ is piecewise continuous on $[0, \infty)$, then:

$$\mathcal{L}\big\{f'(t)\big\} = s\mathcal{L}\big\{f(t)\big\} - f(0) = sF(s) - f(0)$$

To prove this, consider the following Laplace Transform:

$$\mathcal{L}\big\{f'(t)\big\} = \int_0^\infty f'(t)e^{-st}\,dt$$

Through integration by parts, this can be written as:

$$\begin{aligned}\int_0^\infty f'(t)e^{-st}\,dt &= \big[f(t)e^{-st}\big]_0^\infty - \int_0^\infty -s\cdot f(t)e^{-st}\,dt \\ &= \big[f(t)e^{-st}\big]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= \Big(\lim_{b\to\infty} f(b)e^{-sb} - f(0)e^0\Big) + sF(s)\end{aligned}$$

Under the assumption that $f(t)$ is of exponential order, for a sufficiently large value of s:

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

thus showing that:

$$\mathcal{L}\big\{f'(t)\big\} = \Big(\lim_{b\to\infty} f(b)e^{-sb} - f(0)e^0\Big) + sF(s) = sF(s) - f(0)$$

This process of proof can be used to prove:

$$\begin{aligned}\mathcal{L}\big\{f''(t)\big\} &= s^2F(s) - sf(0) - f'(0) \\ \mathcal{L}\big\{f'''(t)\big\} &= s^3F(s) - s^2f(0) - sf'(0) - f''(0) \\ &\vdots \\ \mathcal{L}\big\{f^{(n)}(t)\big\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)\end{aligned}$$

Provided that, for the Laplace Transform of some derivative $f^n(t)$, $f^n(t)$ is piecewise continuous and all previous functions $\big[f(t),\ f'(t), \ldots\big]$ are continuous over $[0, \infty)$ and of exponential order. The Laplace of the n^{th} derivative can also be expressed as:

Time Differentiation

$$\mathcal{L}\big\{f^{(n)}(t)\big\} = s^nF(s) - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0) \qquad \textbf{3.7}$$

3.2.6 Time Integration

Consider the Laplace transform of the integral of $f(t)$:

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \int_{0^-}^{\infty}\left[\int_0^t f(x)\,dx\right]e^{-st}\,dt$$

Integrating by parts:

$$u = \int_0^t f(x)\,dx, \quad du = f(t)\ dt$$
$$dv = e^{-st}\ dt, \quad v = -\frac{1}{s}e^{-st}$$

Then:

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \left[\left[\int_0^t f(x)\,dx\right]\left(-\frac{1}{s}e^{-st}\right)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty}\left(-\frac{1}{s}e^{-st}\right)f(t)\,dt$$

Evaluating the first term on the right side:

$$\left[\int_0^t f(x)\,dx\right]\left(-\frac{1}{s}e^{-st}\right)$$

at $t = \infty$ and $t = 0$ both give zero since

$$\left[e^{-st}\right]_{t\mapsto\infty} = e^{-s(\infty)} = \frac{1}{e^{\infty}} = 0$$

and

$$\left[\int_0^t f(x)\,dx\right]_{t\mapsto 0} = \int_0^0 f(x)\,dx = 0$$

thus, the first term is zero and the equation can be rewritten as:

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = 0 - \int_{0^-}^{\infty}\left(-\frac{1}{s}e^{-st}\right)f(t)\,dt = \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)$$

thus showing the time integration of the Laplace transform.

Time Integration

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s) \qquad \textbf{3.8}$$

3.2.7 Summary of the Properties of the Laplace Transform

Property	$f(t)$	$F(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$
Scaling	$f(\alpha t)$	$\frac{1}{\alpha}F\left(\frac{s}{\alpha}\right)$
Frequency Shift	$e^{at}f(t)$	$F(s-a)$
Time Shift	$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
Time Differentiation	$f^{(n)}(t)$	$s^nF(s) - \sum_{i=0}^{n-1}\left[s^{n-1-i}f^{(i)}(0)\right]$
Time Integration	$\int_0^t f(x)\,dx$	$\frac{1}{s}F(s)$

3.2.8 Common Laplace Transforms

Laplace	**Inverse Laplace**
$\mathcal{L}\big\{\delta(t)\big\} = 1$	$\mathcal{L}^{-1}\big\{1\big\} = \delta(t)$
$\mathcal{L}\big\{\mathcal{U}(t)\big\} = \frac{1}{s}$	$\mathcal{L}^{-1}\big\{\frac{1}{s}\big\} = \mathcal{U}(t)$
$\mathcal{L}\big\{t^n\big\} = \frac{n!}{s^{n+1}},\ s > 0$	$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$
$\mathcal{L}\big\{t^n e^{-\alpha t})\big\} = \frac{1}{(s+\alpha)^{n+1}}$	$\mathcal{L}^{-1}\big\{\frac{1}{(s+\alpha)^n}\big\} = \frac{1}{(n-1)!} t^{n-1} e^{\alpha t}$
$\mathcal{L}\big\{e^{\alpha t}\big\} = \frac{1}{s-\alpha},\ s > \alpha$	$\forall \alpha \in \mathbb{R},\ \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$
$\mathcal{L}\big\{\cos(\beta t)\big\} = \frac{s}{s^2+\beta^2},\ s > 0$	$\forall \beta \in \mathbb{R},\ \beta > 0 \to \mathcal{L}^{-1}\left\{\frac{s}{s^2+\beta^2}\right\} = \cos(\beta t)$
$\mathcal{L}\big\{\sin(\beta t)\big\} = \frac{\beta}{s^2+\beta^2},\ s > 0$	$\forall \beta \in \mathbb{R},\ \beta > 0 \to \mathcal{L}^{-1}\left\{\frac{1}{s^2+\beta^2}\right\} = \frac{1}{\beta}\sin(\beta t)$
$\mathcal{L}\big\{\cosh(kt)\big\} = \frac{s}{s^2-k^2},\ s > k$	$\forall k \in \mathbb{R},\ k > 0 \to \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$
$\mathcal{L}\big\{\sinh(kt)\big\} = \frac{k}{s^2-k^2},\ s > k$	$\forall k \in \mathbb{R},\ k > 0 \to \mathcal{L}^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{1}{k}\sinh(kt)$