Pseudoscalar decays to leptons in Weak effective theory

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The purpose of this note is to review the formulas for BR($P \to \ell^+\ell'^-$) decays in general weak effective theory (WET), keeping an eye on the subtleties such as signs and CP asymmetries, and carefully following the contemporary conventions as employed in, e.g., `flavio`.

The WET is defined by the Langrangian

$$\mathcal{L}_{\mathrm{WET(5)}} = \mathcal{L}_{QCD+QED}(\nu_{L\ell}, e, d, u, s, \mu, c, \tau, b) + \mathcal{L}_{\mathrm{eff}}\,, \tag{1}$$

the non-renormalizable part of which reads

$$\mathcal{L}_{\mathrm{eff}} = -\mathcal{H}_{\mathrm{eff}} = \sum_{A}^{\mathcal{O}_A=\mathcal{O}_A^\dagger} C_A\mathcal{O}_A + \sum_{B}^{\mathcal{O}_B\neq\mathcal{O}_B^\dagger}\left(C_B\mathcal{O}_B + C_B^*\mathcal{O}_B^\dagger\right). \tag{2}$$

We assume new physics relevant for $P^0 \to \ell^+\ell'^-$ to reside in the operators which are usually cast in the following form:

$$\mathcal{O}_{9\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_L}\gamma_\mu q_L\right)\left(\overline{\ell'}\gamma^\mu\ell\right) \qquad \mathcal{O}'_{9\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_R}\gamma_\mu q_R\right)\left(\overline{\ell'}\gamma^\mu\ell\right) \tag{3a}$$

$$\mathcal{O}_{10\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_L}\gamma_\mu q_L\right)\left(\overline{\ell'}\gamma^\mu\gamma_5\ell\right) \qquad \mathcal{O}'_{10\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_R}\gamma_\mu q_R\right)\left(\overline{\ell'}\gamma^\mu\gamma_5\ell\right) \tag{3b}$$

$$\mathcal{O}_{S\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_L}q_R\right)\left(\overline{\ell'}\ell\right) \qquad \mathcal{O}'_{S\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_R}q_L\right)\left(\overline{\ell'}\ell\right) \tag{3c}$$

$$\mathcal{O}_{P\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_L}q_R\right)\left(\overline{\ell'}\gamma_5\ell\right) \qquad \mathcal{O}'_{P\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_R}q_L\right)\left(\overline{\ell'}\gamma_5\ell\right) \tag{3d}$$

The normalization coefficients $\mathcal{N}$ and $\mathcal{N}\zeta$ differ among various papers. In general, they can be complex, which will turn out to be a bit cumbersome for the $K^0_{L,S}$ decays. We assume $\zeta \in \mathbb{R}$.

As the hadronic elements for the pseudoscalar mesons $P = \overline{q'}q$ read

$$\langle 0|\,\overline{q'}\gamma^\mu q\,|P(k)\rangle = 0, \qquad \langle 0|\,\overline{q'}\gamma^\mu\gamma^5 q\,|P(k)\rangle = if_P k^\mu, \tag{4a}$$

$$\langle 0|\,\overline{q'}q\,|P(k)\rangle = 0, \qquad \langle 0|\,\overline{q'}\gamma^5 q\,|P(k)\rangle = -if_P\frac{m_P^2}{m_q+m_{q'}} \equiv -if_P\overline{m}_P, \tag{4b}$$

only the combinations

$$C_{\Delta X\_qq'\ell\ell'} = C_{X\_qq'\ell\ell'} - C'_{X\_qq'\ell\ell'} \tag{5}$$

matter. As $\overline{\psi}_1(\gamma_\mu)\gamma_5\psi_2 \xrightarrow{C} +\overline{\psi_2}(\gamma_\mu)\gamma_5\psi_1$ and pseudoscalar mesons are C-even, there are no extra phases or signs for the antiparticles in the relations (4).

The operators in Eqns. (3) are all non-hermitean (provided $q \neq q'$). Since

$$\left(\overline{\psi}\gamma_\mu\chi\right)^\dagger = +\overline{\chi}\gamma_\mu\psi, \qquad \left(\overline{\psi}\gamma_\mu\gamma_5\chi\right)^\dagger = +\overline{\chi}\gamma_\mu\gamma_5\psi, \tag{6}$$

$$\left(\overline{\psi}\chi\right)^\dagger = +\overline{\chi}\psi, \qquad \left(\overline{\psi}\gamma_5\chi\right)^\dagger = -\overline{\chi}\gamma_5\psi, \tag{7}$$

the operator $\mathcal{O}_{\Delta S\_qq'\ell\ell'} = \mathcal{O}_{S\_qq'\ell\ell'} - \mathcal{O}'_{S\_qq'\ell\ell'}$ spits an extra minus sign under hermitean conjugation while the other relevant operators, $\mathcal{O}_{\Delta 9,\Delta 10,\Delta P}$ in obvious notation, do not. In turn, hermiticity of the effective Hamiltonian implies the following relations among the relevant Wilson coefficients:

$$\mathcal{N}^*C_{\Delta 9\_qq'\ell\ell'}{}^* = +\mathcal{N}C_{\Delta 9\_q'q\ell'\ell}\,, \qquad \mathcal{N}^*C_{\Delta 10\_qq'\ell\ell'}{}^* = +\mathcal{N}C_{\Delta 10\_q'q\ell'\ell}\,, \tag{8a}$$

$$\mathcal{N}^*C_{\Delta S\_qq'\ell\ell'}{}^* = -\mathcal{N}C_{\Delta S\_q'q\ell'\ell}\,, \qquad \mathcal{N}^*C_{\Delta P\_qq'\ell\ell'}{}^* = +\mathcal{N}C_{\Delta P\_q'q\ell'\ell}\,. \tag{8b}$$

In flavio, the Wilson coefficients of the $\_sd\ell\ell'$, $\_bs\ell\ell'$ and $\_bd\ell\ell'$ types are defined directly while those of the $\_ds\ell\ell'$, $\_sb\ell\ell'$ and $\_db\ell\ell'$ kinds can be obtained by means of the above relations. For example, the part of effective Hamiltonian relevant for leptonic decays of B_s and $\bar{B}_s$ reads

$$\begin{aligned} -\mathcal{H}_{\text{eff}}^{B_s,\bar{B}_s\to\ell\ell'} = \sum_{\ell,\ell'} \Big[&- \frac{\mathcal{N}}{2} C_{\Delta 9\_bs\ell\ell'} \left(\bar{s}\gamma_\mu\gamma_5 b\right) \left(\overline{\ell'}\gamma^\mu \ell\right) - \frac{\mathcal{N}^*}{2} C^*_{\Delta 9\_bs\ell\ell'} \left(\bar{b}\gamma_\mu\gamma_5 s\right) \left(\bar{\ell}\gamma^\mu \ell'\right) \\ &- \frac{\mathcal{N}}{2} C_{\Delta 10\_bs\ell\ell'} \left(\bar{s}\gamma_\mu\gamma_5 b\right) \left(\overline{\ell'}\gamma^\mu\gamma_5 \ell\right) - \frac{\mathcal{N}^*}{2} C^*_{\Delta 10\_bs\ell\ell'} \left(\bar{b}\gamma_\mu\gamma_5 s\right) \left(\bar{\ell}\gamma^\mu\gamma_5 \ell'\right) \\ &+ \frac{\zeta\mathcal{N}}{2} C_{\Delta S\_bs\ell\ell'} \left(\bar{s}\gamma_5 b\right) \left(\overline{\ell'}\ell\right) - \frac{\zeta\mathcal{N}^*}{2} C^*_{\Delta S\_bs\ell\ell'} \left(\bar{b}\gamma_5 s\right) \left(\bar{\ell}\ell'\right) \\ &+ \frac{\zeta\mathcal{N}}{2} C_{\Delta P\_bs\ell\ell'} \left(\bar{s}\gamma_5 b\right) \left(\overline{\ell'}\gamma_5 \ell\right) + \frac{\zeta\mathcal{N}^*}{2} C^*_{\Delta P\_bs\ell\ell'} \left(\bar{s}\gamma_5 s\right) \left(\bar{\ell}\gamma_5 \ell'\right) \Big]. \end{aligned} \tag{9}$$

Of course, the effective interaction (9) is CP-invariant provided the coefficients are real.

The covariant S-matrix element for the decay of the weak eigenstate $\bar{P} \to \ell_1^+\ell_2^-$, where $\bar{P}(q\bar{q}') = \bar{K}^0(s\bar{d}), \bar{B}^0(b\bar{d})$ or $\bar{B}_s(b\bar{s})$, takes the form

$$\mathcal{M}_{\bar{P}\to\ell_1^+\ell_2^-} = -\frac{\mathcal{N}}{2} f_P\, \overline{u}(p_2) \left[m_P \mathcal{S}_{\_qq'\ell_1\ell_2} + m_P \mathcal{P}_{\_qq'\ell_1\ell_2}\gamma_5\right] v(p_1) \tag{10}$$

with

$$m_P\, \mathcal{S}_{\_qq'\ell_1\ell_2} = (m_2 - m_1)\, C_{\Delta 9\_qq'\ell_1\ell_2} + \overline{m}_P \zeta C_{\Delta S\_qq'\ell_1\ell_2}\,, \tag{11}$$

$$m_P \mathcal{P}_{\_qq'\ell_1\ell_2} = (m_2 + m_1)\, C_{\Delta 10\_qq'\ell_1\ell_2} + \overline{m}_P \zeta C_{\Delta P\_qq'\ell_1\ell_2}. \tag{12}$$

The prefactor m_P multiplying $\mathcal{S}$ and $\mathcal{P}$ is a mere convention (used in flavio or Refs. [1, 2] but not in [3]). For $P(q'\bar{q}) = K^0(d\bar{s}), B^0(d\bar{b})$ or $B_s(s\bar{b})$, the matrix elements read

$$\mathcal{M}_{P\to\ell_1^+\ell_2^-} = -\frac{\mathcal{N}^*}{2} f_P\, m_P\, \overline{u}(p_2)\Big[- \mathcal{S}^*_{\_qq'\ell_2\ell_1} + \mathcal{P}^*_{\_qq'\ell_2\ell_1}\gamma_5\Big] v(p_1) \tag{13}$$

Notice that the lepton index swap in $\mathcal{S}$ applies also to the lepton mass difference in the first term of Eq. (11).

1 Instantaneous B-meson decays

The B^0 and B_s mesons usually decay before the first oscillation. To some approximation, the oscillations can therefore be neglected and one can effectively consider the decay of the weak eigenstates. A more precise treatment will be discussed in Section 3.

From Eq. (10) or (13), one can derive, using the standard trace techniques,

$$\begin{aligned} &\text{BR}(B_s \to \ell_1^-\ell_2^+) = \text{BR}(\bar{B}_s \to \ell_1^+\ell_2^-) = \\ &= \tau_{B_s} \frac{|\mathcal{N}|^2}{32\pi} \frac{\sqrt{\lambda(m_{B_s}^2, m_1^2, m_2^2)}}{m_{B_s}} f_{B_s}^2 \left[\left(m_{B_s}^2 - (m_1+m_2)^2\right) |\mathcal{S}_{\_bs\ell_1\ell_2}|^2 + \left(m_{B_s}^2 - (m_1-m_2)^2\right) |\mathcal{P}_{\_bs\ell_1\ell_2}|^2\right] \end{aligned} \tag{14}$$

with $\lambda(a^2, b^2, c^2) = [a^2 - (b-c)^2][a^2 - (b+c)^2]$. The prediction is CP-symmetric regardless of possible complex phases of the Wilson coefficients.

Comparison with literature: The relevant equations in Ref. [3], cited in the flavio source code, is flawed by two subtle mistakes. It uses

$$\mathcal{N} = \frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \frac{e^2}{(4\pi)^2} = \frac{G_F\alpha}{\sqrt{2}\pi} V_{tb}V_{ts}^* = \frac{G_F\alpha}{\sqrt{2}\pi}\lambda_{bs}^t \tag{15}$$

and $\zeta^{[3]} = 1$ in the calculations but claims $\zeta = (4\pi)^2/g^2$ in their eq. $(3)_{[3]}$. Furthermore, the WCs defined in their eq. $(3)_{[3]}$, which can be related to the flavio conventions as $C_X^{[3](3)} = C_{X\_bs\ell_2\ell_1}$, are important for the $\bar{B}_s \to \ell_1^-\ell_2^+$ decays and for the conjugated mode $B_s \to \ell_1^+\ell_2^-$, but eq. $(5)_{[3]}$ uses them for $B_s \to \ell_1^- \ell_2^+$. In other words, the RHS of $(5)_{[3]}$ in fact applies to $B_s \to \ell_1^+ \ell_2^-$ but the LHS reads $\mathrm{BR}(B_s \to \ell_1^- \ell_2^+)$. The discussion under $(6)_{[3]}$, which mentions solely the $C_{\Delta 9}$-proportional term, indicates that the authors had not fully understood this subtlety. Apart from these issues, Eq. (14) agrees with $(5)_{[3]}$.

The `flavio` implementation properly takes into account different (pseudo)scalar normalization coefficients ($\zeta^{\texttt{flavio}} = m_b$, $\zeta^{[3]} = 1$, ignores the typo in $(3)_{[3]}$). Unfortunately, `flavio` adopts the incorrect formula in $(5)_{[3]}$ and employs $\mathcal{S}, \mathcal{P}_{\_bs\ell_1\ell_2}$ for predicting $B_s \to \ell_1^+ \ell_2^-$. This error probably is not very important at the moment as the standard `flavio` methods only yield predictions for the sum of both final states, $\mathrm{BR}(B_s \to \ell_1^\pm \ell_2^\mp)$.

Considering the special case $\ell_1 = \ell_2$, Eq. (14) simplifies to the form which can be found, e.g., in Ref. [1], where the normalization convention differs only by a overall minus sign in $\mathcal{N}$ from Eq. (15).

2 Decays of neutral kaons

Since the kaon particles are far from the weak basis. Neglecting the indirect CP violation, the relations between weak and mass eigenstates read

$$|K_L^0\rangle = \frac{|K^0\rangle + |\bar{K}^0\rangle}{\sqrt{2}}, \qquad |K_S^0\rangle = \frac{|K^0\rangle - |\bar{K}^0\rangle}{\sqrt{2}}. \tag{16}$$

Accordingly, one obtains

$$\mathcal{M}_{K_{L,S}^0 \to \ell_1^+ \ell_2^-} = -\frac{1}{2} f_{K^0}\, \overline{u}(p_1) \left[\mathcal{S}_{\ell_1\ell_2}^{K_{L,S}^0} + \mathcal{P}_{\ell_1\ell_2}^{K_{L,S}^0} \gamma_5 \right] v(p_2), \tag{17}$$

where

$$\mathcal{S}_{\ell_1\ell_2}^{K_{L,S}^0} = \frac{-\mathcal{N}^* \mathcal{S}^*_{\_sd\ell_2\ell_1} \pm \mathcal{N} \mathcal{S}_{\_sd\ell_1\ell_2}}{\sqrt{2}}, \qquad \mathcal{P}_{\ell_1\ell_2}^{K_{L,S}^0} = \frac{\mathcal{N}^* \mathcal{P}^*_{\_sd\ell_2\ell_1} \pm \mathcal{N} \mathcal{P}_{\_sd\ell_1\ell_2}}{\sqrt{2}}. \tag{18}$$

Furthermore, the SM brings a long-distance contribution to the lepton flavour conserving amplitudes, consisting in a $\gamma\gamma$ intermediate state, which is not encoded in the effective operators in Eqns. (3). Indlucing these long distance contributions, the amplitude coefficients for $\ell_1 = \ell_2 = \ell$ read

$$\mathcal{S}_{\ell\ell}^{K_L^0} = i\sqrt{2}\mathrm{Im}\left(\mathcal{N}\mathcal{S}_{\_sd\ell\ell}\right), \qquad \mathcal{P}_{\ell\ell}^{K_L^0} = \sqrt{2}\mathrm{Re}\left(\mathcal{N}\mathcal{P}_{\_sd\ell\ell}\right) - m_\ell\, \widetilde{\mathcal{A}}^L_{\gamma\gamma\_\ell}, \tag{19a}$$

$$\mathcal{S}_{\ell\ell}^{K_S^0} = -\sqrt{2}\mathrm{Re}\left(\mathcal{N}\mathcal{S}_{\_sd\ell\ell}\right) + m_\ell \widetilde{\mathcal{B}}^S_{\gamma\gamma\_\ell}, \qquad \mathcal{P}_{\ell\ell}^{K_S^0} = -i\sqrt{2}\mathrm{Im}\left(\mathcal{N}\mathcal{P}_{\_sd\ell\ell}\right). \tag{19b}$$

The long-distance contributions for the case $\ell = \mu$ are known [2]

$$m_\mu \widetilde{\mathcal{B}}^S_{\gamma\gamma\_\mu} = \sqrt{2}\, \frac{2G_F^2 m_W^2 m_\mu}{\pi^2 m_K} B^S_{\gamma\gamma\_\mu} = \sqrt{2}\ (-2.65 + 1.14i) \times 10^{-11}\mathrm{GeV}^{-2} \tag{20a}$$

$$m_\mu \widetilde{\mathcal{A}}^L_{\gamma\gamma\_\mu} = \sqrt{2}\, \frac{2G_F^2 m_W^2 m_\mu}{\pi^2 m_K} A^S_{\gamma\gamma\_\mu} = \pm\sqrt{2}\ (0.54 - 3.96i) \times 10^{-11}\mathrm{GeV}^{-2} \tag{20b}$$

up to an mere (but sometimes important) overall sign in one case. Our convention differs from [2] only in irrelevant overall phases of Eqns. (19) but the interference between short- and long-distance contribution matches well. Note that the current version of `flavio` uses the numerical values of $A^L_{\gamma\gamma\_\mu}, B^S_{\gamma\gamma\_\mu}$ as they follow from Eqs. (20) also for the electron channel; I don't know how important the differences are.

The BR can be be readily obtained in direct analogy with Eq. (14):

$$\begin{aligned} \mathrm{BR}(K_{L,S}^0 \to \ell_1^+ \ell_2^-) &= \frac{\tau_{K_{L,S}^0}}{32\pi} \frac{\sqrt{\lambda}}{m_{K_0}^2}\, m_{K^0}^3 f_{K_0}^2 \\ &\times \left[\left(1 - \frac{(m_1+m_2)^2}{m_{K^0}^2}\right) \left|\mathcal{S}_{\ell_1\ell_2}^{K_{L,S}^0}\right|^2 + \left(1 - \frac{(m_1-m_2)^2}{m_{K^0}^2}\right) \left|\mathcal{P}_{\ell_1\ell_2}^{K_{L,S}^0}\right|^2 \right] \end{aligned} \tag{21}$$

Taking $\mathcal{N} = 1$, $\zeta = m_s$ and $\ell_1 = \ell_2 = \mu$, the result is fully consistent with Eq. $(2.9)_{[2]}$. In `flavio`,

$$\zeta = m_s\,, \qquad \mathcal{N} = N \cdot \xi_t \equiv \left(\frac{G_F \alpha}{\sqrt{2}\pi}\right) \cdot (V_{ts} V_{td}^*) \tag{22}$$

is used, and the real prefactor N is withdrawn from the definition of $\mathcal{S}, \mathcal{P}$ in (19) and forms an extra factor of $|N|^2$ in (21).

Note that a general but awkwardly cast formula for $\mathrm{BR}(K_L^0 \to \ell^+ \ell'^-)$ can be found also in Ref. [4].

3 Time-dependent B decays

Nothing in there yet.

References

[1] I. Doršner, S. Fajfer, A. Greljo, J.F. Kamenik and N. Košnik, *Physics of leptoquarks in precision experiments and at particle colliders*, *Phys. Rept.* **641** (2016) 1 [1603.04993].

[2] V. Chobanova, G. D'Ambrosio, T. Kitahara, M. Lucio Martinez, D. Martinez Santos, I.S. Fernandez et al., *Probing SUSY effects in $K_S^0 \to \mu^+ \mu^-$*, *JHEP* **05** (2018) 024 [1711.11030].

[3] D. Bečirević, O. Sumensari and R. Zukanovich Funchal, *Lepton flavor violation in exclusive $b \to s$ decays*, *Eur. Phys. J. C* **76** (2016) 134 [1602.00881].

[4] O.U. Shanker, *Flavor Violation, Scalar Particles and Leptoquarks*, *Nucl. Phys. B* **206** (1982) 253.