Pseudoscalar decays to leptons in Weak effective theory

Matěj Hudec

February 11, 2021

The purpose of this note is to review the formulas for $\mathrm{BR}(P \to \ell^+\ell'^-)$ decays in general weak effective theory (WET), keeping an eye on the subtleties such as signs and CP asymmetries, and carefully following the contemporary conventions as employed in, e.g., `flavio`.

The WET is defined by the Langrangian

$$\mathcal{L}_{\mathrm{WET}(5)} = \mathcal{L}_{QCD+QED}(\nu_{L\ell}, e, d, u, s, \mu, c, \tau, b) + \mathcal{L}_{\mathrm{eff}}\,, \tag{1}$$

the non-renormalizable part of which reads

$$\mathcal{L}_{\mathrm{eff}} = -\mathcal{H}_{\mathrm{eff}} = \sum_{A}^{\mathcal{O}_A=\mathcal{O}_A^\dagger} C_A \mathcal{O}_A + \sum_{B}^{\mathcal{O}_B\neq\mathcal{O}_B^\dagger} \left(C_B \mathcal{O}_B + C_B^* \mathcal{O}_B^\dagger\right). \tag{2}$$

The effective operators encode both SM weak interactions and new physics (NP). We assume that NP relevant for $P^0 \to \ell^+\ell'^-$ resides in the semileptonic operators which are usually cast in the following form:

$$\mathcal{O}_{9\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_L}\gamma_\mu q_L\right)\left(\overline{\ell'}\gamma^\mu\ell\right) \qquad \mathcal{O}'_{9\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_R}\gamma_\mu q_R\right)\left(\overline{\ell'}\gamma^\mu\ell\right) \tag{3a}$$

$$\mathcal{O}_{10\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_L}\gamma_\mu q_L\right)\left(\overline{\ell'}\gamma^\mu\gamma_5\ell\right) \qquad \mathcal{O}'_{10\_qq'\ell\ell'} = \mathcal{N}\left(\overline{q'_R}\gamma_\mu q_R\right)\left(\overline{\ell'}\gamma^\mu\gamma_5\ell\right) \tag{3b}$$

$$\mathcal{O}_{S\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_L}q_R\right)\left(\overline{\ell'}\ell\right) \qquad \mathcal{O}'_{S\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_R}q_L\right)\left(\overline{\ell'}\ell\right) \tag{3c}$$

$$\mathcal{O}_{P\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_L}q_R\right)\left(\overline{\ell'}\gamma_5\ell\right) \qquad \mathcal{O}'_{P\_qq'\ell\ell'} = \mathcal{N}\zeta\left(\overline{q'_R}q_L\right)\left(\overline{\ell'}\gamma_5\ell\right) \tag{3d}$$

The normalization coefficients $\mathcal{N}$ and $\mathcal{N}\zeta$ differ among various papers. In general, they can be {operatorBa
complex, which will turn out to be a bit cumbersome for the $K^0_{L,S}$ decays. We assume $\zeta \in \mathbb{R}$.

As the hadronic elements for the pseudoscalar mesons $P = \overline{q'}q$ read {hadronic˙el

$$\langle 0|\,\overline{q'}\gamma^\mu q\,|P(k)\rangle = 0, \qquad \langle 0|\,\overline{q'}\gamma^\mu\gamma^5 q\,|P(k)\rangle = i f_P k^\mu, \tag{4a}$$

$$\langle 0|\,\overline{q'}q\,|P(k)\rangle = 0, \qquad \langle 0|\,\overline{q'}\gamma^5 q\,|P(k)\rangle = -i f_P \frac{m_P^2}{m_q + m_{q'}} \equiv -i f_P \overline{m}_P, \tag{4b}$$

only the combinations

$$C_{\Delta X\_qq'\ell\ell'} = C_{X\_qq'\ell\ell'} - C'_{X\_qq'\ell\ell'} \tag{5}$$

{–defC-C'"}

matter. As $\overline{\psi}_1(\gamma_\mu)\gamma_5\psi_2 \xrightarrow{C} +\overline{\psi_2}(\gamma_\mu)\gamma_5\psi_1$ and pseudoscalar mesons are C-even, there are no extra phases or signs for the antiparticles in the relations (4).

The operators in Eqns. (3) are all non-hermitean (provided $q \neq q'$). Since {hc-of-biline

$$\left(\overline{\psi}\gamma_\mu\chi\right)^\dagger = +\,\overline{\chi}\gamma_\mu\psi, \qquad \left(\overline{\psi}\gamma_\mu\gamma_5\chi\right)^\dagger = +\,\overline{\chi}\gamma_\mu\gamma_5\psi, \tag{6a}$$

$$\left(\overline{\psi}\chi\right)^\dagger = +\,\overline{\chi}\psi, \qquad \left(\overline{\psi}\gamma_5\chi\right)^\dagger = -\,\overline{\chi}\gamma_5\psi, \tag{6b}$$

the operator $\mathcal{O}_{\Delta S\_qq'\ell\ell'} = \mathcal{O}_{S\_qq'\ell\ell'} - \mathcal{O}'_{S\_qq'\ell\ell'}$ spits an extra minus sign under hermitean conjugation while the other relevant operators, $\mathcal{O}_{\Delta 9,\Delta 10,\Delta P}$ in obvious notation, do not. In turn, hermiticity of the effective Hamiltonian implies the following relations among the relevant Wilson coefficients: {C-hc}

$$\mathcal{N}^* C_{\Delta 9\_qq'\ell\ell'}{}^* = +\mathcal{N} C_{\Delta 9\_q'q\ell'\ell}\,, \qquad \mathcal{N}^* C_{\Delta 10\_qq'\ell\ell'}{}^* = +\mathcal{N} C_{\Delta 10\_q'q\ell'\ell}\,, \tag{7a}$$

$$\mathcal{N}^* C_{\Delta S\_qq'\ell\ell'}{}^* = -\mathcal{N} C_{\Delta S\_q'q\ell'\ell}\,, \qquad \mathcal{N}^* C_{\Delta P\_qq'\ell\ell'}{}^* = +\mathcal{N} C_{\Delta P\_q'q\ell'\ell}\,. \tag{7b}$$

In `flavio`, the Wilson coefficients of the $\_sd\ell\ell'$, $\_bs\ell\ell'$ and $\_bd\ell\ell'$ types are defined directly while those of the $\_ds\ell\ell'$, $\_sb\ell\ell'$ and $\_db\ell\ell'$ kinds can be obtained by means of the above relations. For example, the part of effective Hamiltonian relevant for leptonic decays of B_s and $\bar{B}_s$ reads

$$\begin{aligned} -\mathcal{H}_{\text{eff}}^{B_s,\bar{B}_s\to\ell\ell'} = \sum_{\ell,\ell'} \Big[&-\frac{\mathcal{N}}{2} C_{\Delta 9\_bs\ell\ell'} \left(\bar{s}\gamma_\mu\gamma_5 b\right)\left(\overline{\ell'}\gamma^\mu \ell\right) - \frac{\mathcal{N}^*}{2} C^*_{\Delta 9\_bs\ell\ell'} \left(\bar{b}\gamma_\mu\gamma_5 s\right)\left(\bar{\ell}\gamma^\mu \ell'\right) \\ &- \frac{\mathcal{N}}{2} C_{\Delta 10\_bs\ell\ell'} \left(\bar{s}\gamma_\mu\gamma_5 b\right)\left(\overline{\ell'}\gamma^\mu\gamma_5 \ell\right) - \frac{\mathcal{N}^*}{2} C^*_{\Delta 10\_bs\ell\ell'} \left(\bar{b}\gamma_\mu\gamma_5 s\right)\left(\bar{\ell}\gamma^\mu\gamma_5 \ell'\right) \\ &+ \frac{\zeta\mathcal{N}}{2} C_{\Delta S\_bs\ell\ell'} \left(\bar{s}\gamma_5 b\right)\left(\overline{\ell'}\ell\right) - \frac{\zeta\mathcal{N}^*}{2} C^*_{\Delta S\_bs\ell\ell'} \left(\bar{b}\gamma_5 s\right)\left(\bar{\ell}\ell'\right) \\ &+ \frac{\zeta\mathcal{N}}{2} C_{\Delta P\_bs\ell\ell'} \left(\bar{s}\gamma_5 b\right)\left(\overline{\ell'}\gamma_5\ell\right) + \frac{\zeta\mathcal{N}^*}{2} C^*_{\Delta P\_bs\ell\ell'} \left(\bar{s}\gamma_5 s\right)\left(\bar{\ell}\gamma_5\ell'\right) \Big]. \end{aligned} \tag{8}$$

{–Ham˙bsll"

The covariant S-matrix element for the decay of the weak eigenstate $\bar{P} \to \ell_1^+\ell_2^-$, where $\bar{P}(q\bar{q}') = \bar{K}^0(s\bar{d}), \bar{B}^0(b\bar{d})$ or $\bar{B}_s(b\bar{s})$, takes the form

$$\mathcal{M}_{\bar{P}\to\ell_1^+\ell_2^-} = -\frac{\mathcal{N}}{2} f_P\, \overline{u}(p_2) \left[m_P \mathcal{S}_{\_qq'\ell_1\ell_2} + m_P \mathcal{P}_{\_qq'\ell_1\ell_2}\gamma_5\right] v(p_1) \tag{9}$$

{–amplitude

with {SP-def}

$$m_P\, \mathcal{S}_{\_qq'\ell_1\ell_2} = (m_2 - m_1)\, C_{\Delta 9\_qq'\ell_1\ell_2} + \overline{m}_P \zeta C_{\Delta S\_qq'\ell_1\ell_2}\,, \tag{10a}$$

{–S"}

$$m_P \mathcal{P}_{\_qq'\ell_1\ell_2} = (m_2 + m_1)\, C_{\Delta 10\_qq'\ell_1\ell_2} + \overline{m}_P \zeta C_{\Delta P\_qq'\ell_1\ell_2}. \tag{10b}$$

{–P"}

The prefactor m_P multiplying $\mathcal{S}$ and $\mathcal{P}$ is a mere convention (used in `flavio` or Refs. [1, 2] but not in [3]). For $P(q'\bar{q}) = K^0(d\bar{s}), B^0(d\bar{b})$ or $B_s(s\bar{b})$, the matrix elements read

$$\mathcal{M}_{P\to\ell_1^+\ell_2^-} = -\frac{\mathcal{N}^*}{2} f_P\, m_P\, \overline{u}(p_2)\Big[-\mathcal{S}^*_{\_qq'\ell_2\ell_1} + \mathcal{P}^*_{\_qq'\ell_2\ell_1}\gamma_5\Big] v(p_1)\,. \tag{11}$$

{–amplitude

Notice that the lepton index swap in $\mathcal{S}$ applies also to the lepton mass difference in the first term of Eq. (10a).

1 Instantaneous B-meson decays

The B^0 and B_s mesons usually decay before the first oscillation. To some approximation, the oscillations can therefore be neglected and one can effectively consider the decay of the weak eigenstates. A more precise treatment will be discussed in Section 3.

From Eq. (9) or (11), one can derive, using the standard trace techniques,

$$\begin{aligned} &\text{BR}(B_s \to \ell_1^-\ell_2^+) = \text{BR}(\bar{B}_s \to \ell_1^+\ell_2^-) = \\ &= \tau_{B_s} \frac{|\mathcal{N}|^2}{32\pi} \frac{\sqrt{\lambda(m_{B_s}^2, m_1^2, m_2^2)}}{m_{B_s}} f_{B_s}^2 \left[\left(m_{B_s}^2 - (m_1+m_2)^2\right) |\mathcal{S}_{\_bs\ell_1\ell_2}|^2 + \left(m_{B_s}^2 - (m_1-m_2)^2\right) |\mathcal{P}_{\_bs\ell_1\ell_2}|^2\right] \end{aligned} \tag{12}$$

{–BR(Bs-¿l1

with $\lambda(a^2,b^2,c^2) = [a^2-(b-c)^2][a^2-(b+c)^2]$. The prediction is CP-symmetric regardless of possible complex phases of the Wilson coefficients.

Comparison with literature: The relevant equations in Ref. [3], cited in the `flavio` source code, is flawed by two subtle mistakes. It uses

$$\mathcal{N} = \frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \frac{e^2}{(4\pi)^2} = \frac{G_F\alpha}{\sqrt{2}\pi} V_{tb}V_{ts}^* = \frac{G_F\alpha}{\sqrt{2}\pi}\lambda_{bs}^t \tag{13}$$

{–Normaliza

and $\zeta^{[3]} = 1$ in the calculations but claims $\zeta = (4\pi)^2/g^2$ in their eq. $(3)_{[3]}$. Furthermore, the WCs defined in their eq. $(3)_{[3]}$, which can be related to the `flavio` conventions as $C_X^{[3](3)} = C_{X\_bs\ell_2\ell_1}$, are important for the $\bar{B}_s \to \ell_1^- \ell_2^+$ decays and for the conjugated mode $B_s \to \ell_1^+ \ell_2^-$, but eq. $(5)_{[3]}$ uses them for $B_s \to \ell_1^- \ell_2^+$. In other words, the RHS of $(5)_{[3]}$ in fact applies to $B_s \to \ell_1^+ \ell_2^-$ but the LHS reads $\mathrm{BR}(B_s \to \ell_1^- \ell_2^+)$. The discussion under $(6)_{[3]}$, which mentions solely the $C_{\Delta 9}$-proportional term, indicates that the authors had not fully understood this subtlety. Apart from these issues, Eq. (12) agrees with $(5)_{[3]}$.

The `flavio` implementation properly takes into account different (pseudo)scalar normalization coefficients ($\zeta^{\texttt{flavio}} = m_b$, $\zeta^{[3]} = 1$, ignores the typo in $(3)_{[3]}$). Unfortunately, `flavio` adopts the incorrect formula in $(5)_{[3]}$ and employs $\mathcal{S}, \mathcal{P}_{\_bs\ell_1\ell_2}$ for predicting $B_s \to \ell_1^+ \ell_2^-$. This error is unimportant at the moment as the `flavio` methods only yield predictions for the sum of both final states. Neglecting lighter masses, this sum is driven by

$$\mathrm{BR}(B_s \to \ell_1^\pm \ell_2^\mp) \propto \left|\mathcal{S}_{\_bs\ell_1\ell_2}\right|^2 + \left|\mathcal{S}_{\_bs\ell_2\ell_1}\right|^2 + \left|\mathcal{P}_{\_bs\ell_1\ell_2}\right|^2 + \left|\mathcal{P}_{\_bs\ell_2\ell_1}\right|^2 . \tag{14}$$

{–Bl1l2sum

Considering the special case $\ell_1 = \ell_2$, Eq. (12) simplifies to the form which can be found, e.g., in Ref. [1], where the normalization convention differs only by a overall minus sign in $\mathcal{N}$ from Eq. (13).

2 Decays of neutral kaons

A general but awkwardly cast formula for $\mathrm{BR}(K_L^0 \to \ell^+ \ell'^-)$ can be found also in Ref. [4]. Rather that adopting it, we modify our previous result to the case of kaons.

Experimentally studied neutral kaons are far from the weak basis. Neglecting the indirect CP violation, the relations between weak and mass eigenstates read

$$|K_L^0\rangle = \frac{|K^0\rangle + |\bar{K}^0\rangle}{\sqrt{2}}, \qquad |K_S^0\rangle = \frac{|K^0\rangle - |\bar{K}^0\rangle}{\sqrt{2}}. \tag{15}$$

Accordingly, one obtains

$$\mathcal{M}_{K^0_{L,S} \to \ell_1^+ \ell_2^-} = -\frac{1}{2} f_{K^0}\, \overline{u}(p_1) \left[\mathcal{S}^{L,S}_{\ell_1\ell_2} + \mathcal{P}^{L,S}_{\ell_1\ell_2}\gamma_5\right] v(p_2), \tag{16}$$

{–amp-KLS

where

$$\mathcal{S}^{L,S}_{\ell_1\ell_2} = \frac{-\mathcal{N}^*\mathcal{S}^*_{\_sd\ell_2\ell_1} \pm \mathcal{N}\mathcal{S}_{\_sd\ell_1\ell_2}}{\sqrt{2}}, \qquad \mathcal{P}^{L,S}_{\ell_1\ell_2} = \frac{\mathcal{N}^*\mathcal{P}^*_{\_sd\ell_2\ell_1} \pm \mathcal{N}\mathcal{P}_{\_sd\ell_1\ell_2}}{\sqrt{2}}. \tag{17}$$

{–PandS-KL

Recall that we define the relevant effective operators as $\sim (\bar{d}\,\Gamma s)(\bar{\ell}_2 \Gamma' \ell_1)$, while those with the $(\bar{s}d)$ qurak flavour are obtained by hermitean conjugation. The BR can be be readily obtained in direct analogy with Eq. (12):

$$\begin{aligned}\mathrm{BR}(K^0_{L,S} \to \ell_1^+ \ell_2^-) = {}& \frac{\tau_{K^0_{L,S}}}{32\pi} \frac{\sqrt{\lambda}}{m^2_{K_0}}\, m^3_{K^0} f^2_{K_0} \\ & \times \left[\left(1 - \frac{(m_1+m_2)^2}{m^2_{K^0}}\right)\left|\mathcal{S}^{L,S}_{\ell_1\ell_2}\right|^2 + \left(1 - \frac{(m_1-m_2)^2}{m^2_{K^0}}\right)\left|\mathcal{P}^{L,S}_{\ell_1\ell_2}\right|^2\right]\end{aligned} \tag{18}$$

{–BR(KLS-

LFV decays

Notice that, as far as we neglect the indirect CPV in K^0, we have $\left|\mathcal{S}^{L,S}_{e\mu}\right| = \left|\mathcal{S}^{L,S}_{\mu e}\right|$ and $\left|\mathcal{P}^{L,S}_{e\mu}\right| = \left|\mathcal{P}^{L,S}_{\mu e}\right|$, which leads to

$$\mathrm{BR}(K^0_{L,S} \to e^+\mu^-) = \mathrm{BR}(K^0_{L,S} \to e^-\mu^+). \tag{19}$$

Thus, the prediction for the sum of the two final states is, in the case of K_L^0, proportional to

$$\begin{aligned}\mathrm{BR}(K_L^0 \to e^{\pm}\mu^{\mp}) &\propto \frac{2}{|\mathcal{N}|^2}\left(\left|\frac{-\mathcal{N}^*\mathcal{S}^*_{\_sde\mu}+\mathcal{N}\mathcal{S}_{\_sd\mu e}}{\sqrt{2}}\right|^2+\left|\frac{\mathcal{N}^*\mathcal{P}^*_{\_sde\mu}+\mathcal{N}\mathcal{P}_{\_sd\mu e}}{\sqrt{2}}\right|^2\right)\\ &= |\mathcal{S}_{\_sde\mu}|^2+|\mathcal{S}_{\_sd\mu e}|^2+|\mathcal{P}_{\_sde\mu}|^2+|\mathcal{P}_{\_sd\mu e}|^2-2\mathrm{Re}\left[\frac{\mathcal{N}^2}{|\mathcal{N}|^2}\left(\mathcal{S}_{\_sd\mu e}\mathcal{S}_{\_sde\mu}+\mathcal{P}_{\_sd\mu e}\mathcal{P}_{\_sde\mu}\right)\right]. \quad (20)\end{aligned}$$

{–KL-¿emu

Notably enough, this structure differs from that in Eq. (14) by the last term. For K_S^0, this last term simply changes sign.

LF conserving decays and long distance contributions

For $\ell_1 = \ell_2 = \ell$, Eq. (18) simplifies to

$$\mathrm{BR}(K^0_{L,S} \to \ell^+\ell^-) = \frac{\tau_{K^0_{L,S}}\,\beta_\ell\, m_{K^0}^3 f_{K_0}^2}{32\pi}\left[\beta_\ell^2\left|\mathcal{S}^{L,S}_{\ell\ell}\right|^2+\left|\mathcal{P}^{L,S}_{\ell\ell}\right|^2\right] \quad (21)$$

with $\beta_\ell = \sqrt{\lambda(m^2_{K^0}, m^2_\ell, m^2_\ell)}/m^2_{K^0} = \sqrt{1-4m_\ell^2/m^2_{K^0}}$, and the expressions in (17) reduce to

{shortD-S,P

$$\mathcal{S}^L_{\ell\ell} = i\sqrt{2}\,\mathrm{Im}\,(\mathcal{N}\mathcal{S}_{\_sd\ell\ell})\,, \qquad \mathcal{P}^L_{\ell\ell} = \sqrt{2}\mathrm{Re}\,(\mathcal{N}\mathcal{P}_{\_sd\ell\ell})\,, \quad (22a)$$

{–shortD-S,I

$$\mathcal{S}^{K^0_S}_{\ell\ell} = -\sqrt{2}\,\mathrm{Re}\,(\mathcal{N}\mathcal{S}_{\_sd\ell\ell})\,, \qquad \mathcal{P}^{K^0_S}_{\ell\ell} = -i\sqrt{2}\,\mathrm{Im}\,(\mathcal{N}\mathcal{P}_{\_sd\ell\ell})\,. \quad (22b)$$

{–shortD-S,I

For LF conserving decays, the SM brings a long-distance (LD) contribution, consisting in a $\gamma\gamma$ intermediate state, which can not be encoded in the effective operators in Eqns. (3).

We will adopt the LD amplitudes from Ref. [2]. Their EFT conventions are summarized here:

$$\begin{aligned}\mathcal{H}_{\rm eff} &= -C_AQ_A - \tilde{C}_A\tilde{Q}_A - C_SQ_S - \tilde{C}_S\tilde{Q}_S - C_PQ_P - \tilde{C}_P\tilde{Q}_P + \mathrm{H.c.},\\ Q_A &= (\bar{s}\gamma^\mu P_L d)(\bar{\ell}\gamma_\mu\gamma_5\ell), \quad \tilde{Q}_A = (\bar{s}\gamma^\mu P_R d)(\bar{\ell}\gamma_\mu\gamma_5\ell),\\ Q_S &= m_s(\bar{s}P_R d)(\bar{\ell}\ell), \qquad \tilde{Q}_S = m_s(\bar{s}P_L d)(\bar{\ell}\ell),\\ Q_P &= m_s(\bar{s}P_R d)(\bar{\ell}\gamma_5\ell), \quad \tilde{Q}_P = m_s(\bar{s}P_L d)(\bar{\ell}\gamma_5\ell),\end{aligned}$$

and will be denoted by the referencing super-/subscript whenever confusion might arise. For example, $\mathcal{N}^{[2]} = 1$, $\zeta^{[2]} = m_s$ and, denoting $C_X^{[2]} - \tilde{C}_X^{[2]} \equiv C^{[2]}_{\Delta X}$,

$$C^{[2]}_{\Delta A} = +\mathcal{N}^* C^*_{\Delta 10\_sd\ell\ell} \qquad C^{[2]}_{\Delta S} = -\mathcal{N}^* C^*_{\Delta S\_sd\ell\ell} \qquad C^{[2]}_{\Delta P} = +\mathcal{N}^* C^*_{\Delta P\_sd\ell\ell}\,. \quad (23)$$

Cf. Eq. (7) for the signs in the above equation. The LD contributions can be accounted for by the substituting

$$|\mathcal{P}^{L,S}_{\ell\ell}|^2 \longrightarrow 2|A^{[2]}_{L,S}|^2\,, \qquad |\mathcal{S}^{L,S}_{\ell\ell}|^2 \longrightarrow 2|B^{[2]}_{L,S}|^2 \quad (24)$$

where [2]

{AB-SL}

$$A_S^{[2]} = \mathrm{Im}\left(m_s\overline{m}_{K^0}\,C^{[2]}_{\Delta P} + \frac{2m_\mu}{m_{K^0}}C^{[2]}_{\Delta A}\right) = -\,\mathrm{Im}\,(\mathcal{N}\mathcal{P}_{\_sd\ell\ell}) \quad (25a)$$

$$B_S^{[2]} = \frac{2G_F^2 m_W^2 m_\ell}{\pi^2 m_K}B^\ell_{S\gamma\gamma} - \mathrm{Re}\left(m_s\overline{m}_{K^0}C^{[2]}_{\Delta S}\right) = \frac{2G_F^2 m_W^2 m_\ell}{\pi^2 m_K}B^\ell_{S\gamma\gamma} + \mathrm{Re}\,(\mathcal{N}\mathcal{S}_{\_sd\ell\ell}) \quad (25b)$$

$$A_L^{[2]} = \frac{2G_F^2 m_W^2 m_\ell}{\pi^2 m_K}A^\ell_{L\gamma\gamma} - \mathrm{Re}\left(m_s\overline{m}_{K^0}C^{[2]}_{\Delta P} + \frac{2m_\mu}{m_{K^0}}C^{[2]}_{\Delta A}\right) = \frac{2G_F^2 m_W^2 m_\ell}{\pi^2 m_K}A^\ell_{L\gamma\gamma} - \mathrm{Re}\,(\mathcal{N}\mathcal{P}_{\_sd\ell\ell}) \quad (25c)$$

$$B_L^{[2]} = \mathrm{Im}\left(m_s\overline{m}_{K^0}C^{[2]}_{\Delta S}\right) = +\,\mathrm{Im}\,\mathcal{S}_{\_sd\ell\ell} \quad (25d)$$

On each line, the first equality is adopted form [2] while the second step is a translation to our notation. The whole point of going to such a detailed comparison is to keep track of the relative sign between the short- and long-distance amplitudes.

Ref. [2] enumerates the LD contributions for the case $\ell = \mu$: {SP˙LD}

$$\frac{2G_F^2 m_W^2 m_\mu}{\pi^2 m_K} B^\mu_{S\gamma\gamma} = (-2.65 + 1.14i) \times 10^{-11}\mathrm{GeV}^{-2} \tag{26a}$$

$$\frac{2G_F^2 m_W^2 m_\mu}{\pi^2 m_K} A^\mu_{L\gamma\gamma} = \pm\,(0.54 - 3.96i) \times 10^{-11}\mathrm{GeV}^{-2} \tag{26b}$$

from which the values of $B_{S\gamma\gamma}, A_{L\gamma\gamma}$ alone can be easily derived, if desired. Notice that the sign of the latter is not known.

Flavio

The `flavio` basis adopts

$$\zeta = m_s = \zeta^{[2]}\,, \qquad\qquad \mathcal{N} = N \cdot \xi_t \equiv \left(\frac{G_F \alpha}{\sqrt{2}\pi}\right) \cdot (V_{ts} V_{td}^*)\,. \tag{27}$$

Within this convention, one can cast

$$B_S = N\left[\frac{2G_F^2 m_W^2 m_\ell}{N\pi^2 m_K} B^\ell_{S\gamma\gamma} + \mathrm{Re}\,(\xi_t\,\mathcal{S}_{\_sd\ell\ell})\right] = N\left[\frac{2m_\ell}{m_K \sin^2\theta_w} B^\ell_{S\gamma\gamma} + \mathrm{Re}\,(\xi_t\,\mathcal{S}_{\_sd\ell\ell})\right] \tag{28}$$ {–flavio-amp

and similarly for A_L. Concerning the implementation, the weak-eigenstate amplitudes $\xi_t(\mathcal{P},\mathcal{S})_{\_qq'\ell\ell'}$ from Eqs. (10a) and (10b) are calculated by `amplitudes_weak_eigst`, while the amplitudes for $K^0_{L,S}$, normalized as $\sqrt{2}\xi_t\,(\mathcal{P},\mathcal{S})^{L,S}_{\ell_1\ell_2}$, are obtained by `amplitudes`. The `amplitudes_LD` yield the first term within the bracket in (28) etc. The effective amplitudes $(A_{L,S}, B_{L,S})$ in (25), as adopted from [2], are rephased compared to (22), which is, of course, physically irrelevant. Nevertheless, to comply with those different conventions, `amplitudes_eff` *subtracts* the `amplitudes` from `amplitudes_LD` in the proposed implementation of `flavio`.

Note that the current version of `flavio` uses the numerical values of $A^\mu_{L\gamma\gamma}, B^\mu_{S\gamma\gamma}$ as they follow from Eqs. (26) also for the electron channel; I don't know how important the differences are.

3 Time-dependent LFV B decays

{sec:B-time-

Nothing in there yet.

References

[1] I. Doršner, S. Fajfer, A. Greljo, J.F. Kamenik and N. Košnik, *Physics of leptoquarks in precision experiments and at particle colliders*, *Phys. Rept.* **641** (2016) 1 [`1603.04993`].

[2] V. Chobanova, G. D'Ambrosio, T. Kitahara, M. Lucio Martinez, D. Martinez Santos, I.S. Fernandez et al., *Probing SUSY effects in $K^0_S \to \mu^+\mu^-$*, *JHEP* **05** (2018) 024 [`1711.11030`].

[3] D. Bečirević, O. Sumensari and R. Zukanovich Funchal, *Lepton flavor violation in exclusive $b \to s$ decays*, *Eur. Phys. J. C* **76** (2016) 134 [`1602.00881`].

[4] O.U. Shanker, *Flavor Violation, Scalar Particles and Leptoquarks*, *Nucl. Phys. B* **206** (1982) 253.