1 Exercises on unconstrained optimization with equalities

Exercise 2.
Consider the function $f(x, y) = \frac{x+y}{3+x^2+y^2+xy}$. We seek to find the necessary conditions for the extreme points. Note that the function is continuous in all the real 2D plane, since there is no real point where the denominator vanishes. Then, by computing the gradient we can also verify that it is continuous, following the same arguments, as well as the second derivative. Therefore, we have a $\mathcal{C}^2$ function. Thus, according to a theorem studied, a necessary condition for a point x^* to be an optimal point of f is that $\nabla f(x^*) = 0$, and that the Hessian evaluated at that point is definite semi positive, for x^* to be a minimum, or semi negative, for it to be a maximum. Then let's first consider the gradient,

$$\frac{\partial f}{\partial x}(x, y) = \frac{1}{3 + x^2 + y^2 + xy} - \frac{(x + y)(2x + y)}{3 + x^2 + y^2 + xy}$$
$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{3 + x^2 + y^2 + xy} - \frac{(x + y)(2y + x)}{3 + x^2 + y^2 + xy}$$

With some trivial calculations, we can see that the gradient vanishes if the following system of equation verifies:

$$\nabla f(x, y) = (0, 0) \iff \begin{cases} x^2 + 2xy - 3 = 0 \\ y^2 + 2xy - 3 = 0 \end{cases}$$

By solving this system we can see that the only real values that verify those equations are given by $(x_1, y_1) = (1, 1)$, and $(x_2, y_2) = (-1, -1)$. We have to verify now that they satisfy the second necessary condition. Computing the Hessian matrix and evaluating it in the corresponding points, we have that the eigenvalues of the Hessian evaluated on point $(1, 1)$ are $(-0.167, 0)$, and is therefore negative semi definite, and $(0.167, 0)$ on point $(-1, -1)$, verifying that the Hessian is positive semi definite on that point. Consequently, we have a minimum on point $(-1, -1)$ and a maximum on point $(1, 1)$.

Exercise 7.
Consider two companies, A and B that sell the same product on the market with demand function given by $Q(p) = 100 - p$ where p is the unit price and $Q = q_1 + q_2$ are the quantity of units sold by each company. We know that the cost function of each company are given by $C(q_1) = 2q_1$ and $C(q_2) = 3q_2$.

(a) Suppose that each company produces $q_1 = 10$, $q_2 = 20$. Then we have that $Q = 10 + 20 = 100 - p \iff p = 70$ price per unit. Therefore, the corresponding profits will be given by the difference between their income and their cost of production, i.e.

$$f(q_1) = pq_1 - C(q_1) = (70 - 2)q_1 = 680$$
$$g(q_2) = pq_2 - C(q_2) = (70 - 3)q_2 = 1340$$

(b) Consider now a situation in which company A knows in advance that company B is producing $q_2 = 20$ units. In order to determine the quantity of units that company A should produce we seek to maximize the possible profit that the company can take given that q_2 is fixed. Note that given this value we have $Q = q_1 + 20 = 100 - p \iff p = 100 - 20 - q_1 = 80 - q_1$. Therefore, this is traduced to an unconstrained maximization problem of the profit function:

$$f(q_1) = pq_1 - 2q_1 = (80 - q_1)q_1 - 2q_1 = 78q_1 - q_1^2.$$

By optimizing this $\mathcal{C}^2$ function over q_1 we find an extreme point which in fact is a maximum:

$$\frac{\partial f}{\partial q_1} = 78 - 2q_1 = 0 \iff q_1 = 39$$
$$\frac{\partial^2 f}{\partial q_1^2} = -2 < 0, \ \forall q_1.$$

Consequently, company A will produce $q_1 = 39$ units, earning a profit of $f(39) = (41-2)39 = 1521$. Similarly, the profit of company B will be $g(20) = (41-3)20 = 760$.

(c) We seek to find the Nash equilibrium, this is the pair (q_1^*, q_2^*) such that no company wants to deviate unilaterally. Therefore, we want to maximize the profit of both companies. Note that the best way to find Nash equilibrium is to consider the joint benefit, and split it equally by the two companies (so they can maximize the price, and get the optimal profit). Therefore, let's consider the equivalent problem given by the maximization of the joint profit along with the maximization of each company's profit:[1]:

$$\begin{aligned} f(x,y) &= (100-x-y)x - 2x = (98-x-y)x \\ g(x,y) &= (100-x-y)y - 3y = (97-x-y)y \\ h(x,y) &= f(x,y) + g(x,y) = 98x + 97y - 2xy - x^2 - y^2 \end{aligned}$$

By deriving the joint profit function it is direct to see that there is no point that verifies both conditions (the system is incompatible since we get two parallel plans). Therefore, we consider the point that maximizes jointly each profit:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 98 - 2x - y = 0 \iff y = 98 - 2x \\ \frac{\partial g}{\partial y} &= 97 - x - 2y = 0 \iff y = \frac{97-x}{2}, \end{aligned}$$

By solving this system we get $(x,y) = (33,32)$, which actually gives a joint profit of $h(x^*, y^*) = 3169$, which accounts for 1584.5 per company. Note that this joint profit is the maximum we could get and therefore, none of the companies may need to move from this equilibrium.

Exercise 13.

(a) Consider the function $f(x) := \sum_{j=1}^m w_j ||x - y_j||$, with $x \in \mathbb{R}^2$, $y_1, \dots, y_m \in \mathbb{R}^2$, $w_1, \dots, w_m > 0$. Note that given that the norms are continuous and the linear combination of continuous functions is continuous, we have that f is continuous. Additionally, since norms are positive and the weights are positive, we have that $f(x) \geq 0$ for all $x \in \mathbb{R}^2$. Observe now that since the vectors y_k are fixed, we have that when $||x|| \to \infty$, then $f(x) \to \infty$. Therefore, for any $z \in \mathbb{R}^2$, there exists $M > 0$ such that for any x that verifies $||x|| \geq M$ then $f(x) \geq f(z)$. Thus, the problem of minimizing f for $x \in \mathbb{R}^2$ reduces to the optimization problem over the closed ball $B(0,M) := \{x \in \mathbb{R}^2 : ||x|| \leq M\}$. Since this ball is clearly compact, in virtue of the Weierstrass theorem we know that there exists a minimum of this continuous function in $B(0,M)$.

Consider now a mechanical system of m particles situated at points y_i in the 2 dimensional space, each with a weight w_i (i.e. mass). From the mechanics point of view, this problem can be studied by considering their center of mass, which represents the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero. Note that by definition, and taking into account that $f(x) \geq 0, \forall x \in \mathbb{R}$ this is the minimum we are considering on the optimization problem. In fact, for any mechanical model, its center of mass is given by

$$x^* = \frac{1}{\sum_{j=1}^m w_j} \sum_{j=1}^m w_j y_j,$$

which clearly minimizes f. Note that by definition, $\sum_{j=1}^m w_j (x^* - y_j) = 0$, which is equivalent to the definition given of the point that minimizes f.

[1] for the sake of simplicity I may denote $(q_1, q_2) \to (x, y)$

(b) Note that clearly this function is convex, since a linear combination with positive coefficients of convex functions is a convex function, and by using the triangular inequality it is direct to see that every norm is convex[2]. Therefore for the sake of a known theorem we have that the minimum is unique. Moreover, by the fact of knowing that this minimum corresponds to the center of mass, we also have the unity.

(c) Consider now the mechanical system in which we have m masses hanging on cables of length L each of them and tied by a point, located at position x over a table. Note that if we let the system attain its equilibrium without any external force (except but gravitational) the system will attain a state of minimum energy, and remain at equilibrium with x being the minimal solution of f, i.e. $x = x^*$ as defined previously. In order to give a more technical explanation of this fact, let's consider more explicitly the computation of the potential energy of the system. Let L be long enough so this length does not modify the system's structure, i.e. let's suppose the length L is long enough so we cannot have a weight that hits the table to balance any other weight. Then, by fixing the system's reference at the table height we can compute the potential energy by fixing $h = 0$ at the table:

$$\Delta U_{grav} = -W_{grav} = \sum_{i=1}^{m} w_i g \Delta h_i = g \sum_{i=1}^{m} w_i h_i,$$

where h_i is defined as the vertical distance between the table and weight i[3]. Therefore, due to the system's geometry, $h_i = ||y_i - x|| - L$ and therefore

$$\Delta U_{grav} = g \sum_{i=1}^{m} w_i(||y_i - x|| - L) = g \sum_{i=1}^{m} w_i ||y_i - x|| - gL \underbrace{\sum_{i=1}^{m} w_i}_{const}$$

which is clearly minimum when $f(x) = \sum_{i=1}^{m} w_i ||y_i - x||$ is minimum, i.e. when considering the optimal solution of f, which is, as aforementioned the center of mass, x^*.

[2] $||\lambda v + (1-\lambda)w|| \leq \lambda ||v|| + (1-\lambda)||w||$.

[3] Here clearly W_{grav} stands for the work due to the gravitational force, i.e. $W_{ab} = -\int_a^b mgdy = -mg(b-a)$.