## NUMERICAL LINEAR ALGEBRA

Exercises from previous exams on linear equation solving, the least square problem and the singular value decomposition.

**1.** Let  $\varepsilon$  be a small positive real number and consider the matrix and the vector

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$$
 and  $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

- (1) Compute the condition number with respect to the  $\infty$ -norm of A and of the matrices in its LU factorization.
- (2) Suppose that  $\varepsilon$  is smaller than the machine precision, so that  $1 \oplus \varepsilon = 1$ . Show that Gaussian elimination without pivoting is a numerically unstable algorithm for solving the linear equation Ax = b.
- (3) Show that solving Ax = b using Gaussian elimination with partial pivoting is numerically stable.

**2.** Given a symmetric and positive definite matrix  $A \in \mathbb{R}^{n \times n}$ , Cholesky's algorithm computes a factorization

$$A = L \cdot L^T$$

where L is a lower triangular matrix with positive diagonal entries.

- (1) Write down Cholesky's algorithm in pseudocode notation and describe how it works.
- (2) Using this pseudocode, derive a bound for the complexity of this algorithm in terms of floating point operations (flops).
- (3) Describe how you proceed to solve Ax = b, once computed the factorization  $A = L \cdot L^T$ .
- (4) Compute the Cholesky factorization of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

- **3.** Let  $A \in \mathbb{R}^{m \times n}$  be a matrix of rank n and  $b \in \mathbb{R}^m$ .
  - (1) Explain how to use the QR factorization of A to solve the least square problem (LSP) that asks to find the vector  $x_{\min} \in \mathbb{R}^n$  minimizing the quantity  $||Ax b||_2$  for  $x \in \mathbb{R}^n$ , and give the expression for the residual error  $||Ax_{\min} b||_2$ .
  - (2) Find the affine function  $\ell(x) = \alpha x + \beta$  whose graph fits better the points (-1,1), (0,0) and (1,1), in the sense that the Euclidean norm of the vector

$$(\ell(-1) - 1, \ell(0) - 0, \ell(1) - 1) \in \mathbb{R}^3$$

is minimal among all possible choices of  $\alpha, \beta \in \mathbb{R}$ .

4. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$$

- (1) Compute its QR factorization using Householder reflexions.
- (2) Compute the same factorization, but this time using Givens rotations instead of reflexions.
- **5.** Consider the singular value decomposition (SVD)

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \cdot \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \cdot \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- (1) Compute the condition number of the matrix A with respect to the 2-norm.
- (2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations.

**6.** Let  $A \in \mathbb{R}^{2 \times 2}$  such that the eigenvalues of  $A \cdot A^T$  are 9 and  $\frac{1}{4}$  with respective eigenvectors

$$\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix} \quad \text{ and } \quad \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix},$$

and the eigenvalues of  $A^T \cdot A$  are 9 and  $\frac{1}{4}$  with respective eigenvectors

$$\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \approx \begin{pmatrix} 0.50 \\ -0.87 \end{pmatrix} \quad \text{ and } \quad \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.50 \end{pmatrix}.$$

- (1) Compute a singular value decomposition (SVD) of A.
- (2) Using this SVD, compute condition number of A with respect to the 2-norm.
- (3) Determine the image of the unit disk of  $\mathbb{R}^2$  under the linear map defined by A.