1 Exercises 4: Short-time Fourier transform and Wavelets

Exercise 3. *Let $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ be the normalised Gaussian (so that $\|\varphi\|_2 = 1$) and let $f \in L^2(\mathbb{R})$.*

(a) *Letting $z = t + i\xi$, prove that*

$$V_\varphi f(t, -\xi) = e^{\pi i t\xi} Bf(z) e^{-\frac{\pi}{2}|z|^2},$$

where

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2}z^2} dt$$

is the so-called Bargmann transform of f.

Proof. Recall the definition of the Short-Time Fourier Transform (STFT) for general functions $f, g \in L^2(\mathbb{R})$:

$$V_g f(t, \xi) = \int_{\mathbb{R}} f(s)\overline{g(s-t)} e^{-2\pi i\xi s} ds$$

Thus, let us consider the corresponding STFT for the given function

$$\begin{aligned}
V_\varphi f(t, -\xi) &= \int_{\mathbb{R}} f(s)\overline{\varphi(s-t)} e^{2\pi i\xi s} ds \\
&= 2^{1/4} \int_{\mathbb{R}} f(s) e^{-\pi(s-t)^2} e^{2\pi i\xi s} ds \\
&= 2^{1/4} \int_{\mathbb{R}} f(s) \exp\left\{-\pi s^2 - \pi t^2 + 2\pi st + 2\pi i\xi s\right\} ds \\
&= 2^{1/4} \int_{\mathbb{R}} f(s) \exp\left\{2\pi s(t + i\xi) - \pi s^2 - \pi t^2\right\} ds \\
&= 2^{1/4} \int_{\mathbb{R}} f(s) \exp\left\{2\pi s(t + i\xi) - \pi s^2 - \frac{\pi}{2}(t^2 - \xi^2 + 2it\xi) - \frac{\pi}{2}(t^2 + \xi^2) + \pi it\xi\right\} ds \\
&= 2^{1/4} e^{\pi it\xi} e^{-\frac{\pi}{2}(t^2+\xi^2)} \int_{\mathbb{R}} f(s) e^{2\pi s(t+i\xi) - \pi s^2 - \frac{\pi}{2}(t+i\xi)^2} ds
\end{aligned}$$

Consider now the variable $z = t + i\xi$, so that the previous expression can be written as follows:

$$V_\varphi f(t, -\xi) = e^{\pi it\xi} e^{-\frac{\pi}{2}|z|^2} 2^{1/4} \int_{\mathbb{R}} f(s) e^{2\pi sz - \pi s^2 - \frac{\pi}{2}z^2} ds = e^{\pi it\xi} e^{-\frac{\pi}{2}|z|^2} Bf(z),$$

as wanted to prove. □

(b) *Check that Bf is an entire function belonging to the Bargmann-Fock space*

$$\mathcal{F}_\pi = \{F \in H(\mathbb{C}) : \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < +\infty\}$$

and that

$$\|f\|^2 = \int_{\mathbb{C}} |Bf(z)|^2 e^{-\pi|z|^2} dm(z).$$

Therefore B is, up to a constant, an isometry from $L^2(\mathbb{R})$ to $\mathcal{F}$.

Proof. In order to see check that Bf is an entire function belonging to the Bargmann-Fock space we need to see that it is well defined and holomorphic.

Notice that from the previous section, we have found a way to express the Bargmann transform of f in terms of the STFT $V_\varphi f(t, -\xi) \in L^2(\mathbb{R}^2)$, which is a well-known function with satisfactory properties. Thus, since Bf corresponds to a modulation, and a modification on the modulus of this function, we can suspect that it may have some equivalently good properties. Let us consider the equivalent expression of the Bargmann transform as deduced previously

$$Bf(z) = e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} V_\varphi f(t, -\xi)$$

where $z = t + i\xi$ is any complex number given. Thus, observe it is well defined within the Bargmann-Fock space

$$\begin{aligned}\int_{\mathbb{C}} |Bf(z)|^2 e^{-\pi|z|^2} dm(z) &= \int_{\mathbb{C}} \left|e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} V_\varphi f(t, -\xi)\right|^2 e^{-\pi|z|^2} dm(z) \\ &= \int_{\mathbb{R}^2} \left|V_\varphi f(t, -\xi)\right|^2 dt d\xi < \infty\end{aligned}$$

In order to check that Bf is holomorphic, we can apply the Morera's theorem that states that it is enough to check that the closed integral over any closed simple piecewise $\mathcal{C}^1$ curve is zero in order to prove holomorphy. This is, let γ be any closed simple piecewise $\mathcal{C}^1$ curve, then

$$\begin{aligned}\int_\gamma Bf(z)dz &= \int_\gamma e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} V_\varphi f(t, -\xi) dz \\ &= \int_\gamma e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} \left(2^{1/4} \int_{\mathbb{R}} f(s) e^{\pi(s-t)^2} e^{2\pi i \xi s} ds\right) dz\end{aligned}$$

In virtue of the Fubini's theorem, we can consider the change in order of integration as follows

$$\int_\gamma Bf(z)dz = 2^{1/4} \int_{\mathbb{R}} f(s) \left(\int_\gamma e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} e^{\pi(s-t)^2} e^{2\pi i \xi s} dz\right) ds$$

Consider the corresponding integral over the closed curve:

$$I = \int_\gamma e^{-\pi i t\xi} e^{\frac{\pi}{2}|z|^2} e^{\pi(s-t)^2} e^{2\pi i \xi s} dz = \int_\gamma g(z) dz = \int_{w_0}^{w} g(\gamma(u))\gamma'(u) du$$

Alternatively, we can write the complex integral as follows, taking into account the complex variable $z = t + i\xi$ and considering the parametrization of the closed curve $\gamma(t) = x(t) + iy(t)$ for $a \leq t \leq b$, where $x(t)$ and $y(t)$ are real-valued functions, we can rewrite the integral as:

$$I = \int_\gamma g(z)dz = \int_\gamma (g_t + i g_\xi)(dt + i d\xi)$$

where $g_t = Re(g(z))$ and $g_\xi = Im(g(z))$.
Let's evaluate the integral I by substituting $z = t + i\xi$ and $dz = dt + i d\xi$. We also express $g(z)$ in terms of its real and imaginary parts:

$$\begin{aligned}I &= \int_\gamma (g_t + i g_\xi)(dt + i d\xi) \\ &= \int_\gamma \left(g_t + i g_\xi\right)\left(dt + i d\xi\right) \\ &= \int_\gamma (g_t dt - g_\xi d\xi) + i \int_\gamma (g_\xi dt + g_t d\xi) \\ &= \int_\gamma g_t dt - \int_\gamma g_\xi d\xi + i \int_\gamma g_\xi dt + i \int_\gamma g_t d\xi.\end{aligned}$$

Now, let's focus on each term separately, observe firstly that we have

$$g(z) = e^{\pi i\xi(2s-t)} e^{\frac{\pi}{2}|z|^2} e^{\pi(s-t)^2}$$

And therefore, for $2s = t$ then $Im(g(z)) = g_\xi = 0$ and the remaining terms for g_t are given by

$$g_t = Re(g(z)) = g(z)\Big|_{s=t/2} = e^{\frac{\pi}{2}|z|^2} e^{\pi(-t/2)^2} = e^{\frac{\pi}{2}(t^2+\xi^2)} e^{\pi(t^2/4)}$$

In this case, since the exponential e^{z^2} is real continuous function then it is holomorphic and both integrals over the closed curve are zero, then

$$\begin{aligned} I &= \int_\gamma g_t dt + i \int_\gamma g_t d\xi \\ &= \int_\gamma e^{\pi \frac{3t^2}{4}} e^{\pi \frac{\xi^2}{2}} dt + i \int_\gamma e^{\pi \frac{3t^2}{4}} e^{\pi \frac{\xi^2}{2}} d\xi = 0. \end{aligned}$$

Now, if $2s \neq t$ we can follow a similar approach considering

$$\begin{aligned} Re(g(z)) = g_t(z) = cos(\pi\xi(2s-t)) e^{\pi(s-t)^2} e^{\frac{\pi}{2}(t^2+\xi^2)}, \\ Im(g(z)) = g_\xi(z) = sin(\pi\xi(2s-t)) e^{\pi(s-t)^2} e^{\frac{\pi}{2}(t^2+\xi^2)}, \end{aligned}$$

which are both composition of real valued continuous functions and therefore, holomorphic. Then, when computing each term of the integral I we get that since they are all holomorphic functions $I = 0$ over any closed curve γ and hence, Bf is an holomorphic function.

Finally, since $\|\varphi\| = 1$, then using the previous section we have

$$\|f\|^2 = \|V_\varphi f\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{C}} |Bf(z)|^2 e^{-\pi|z|^2} dm(z), \tag{1}$$

Thus, we can conclude that B is an isometry up to a constant, mapping $L^2(\mathbb{R})$ to $\mathcal{F}_\pi$.

□

Exercise 4. *Let $\varphi \in L^2(\mathbb{R})$ be defined by $\hat{\varphi}(\xi) = \chi_{[-1/2,1/2)}(\xi)$.*

(a) *Prove that φ is the scaling function of a MRA. The MRA obtained in this way is called the Shannon MRA, and it can be viewed as the Fourier counterpart of the Haar MRA. (Hint: consider the Nyquist-Shannon formula for the various dyadic bandwidths.*

Proof. Let us recall the definition of a multi-resolution analysis (MRA):

Definition 1.1. *A MRA is an increasing sequence $\cdots V_n \subset V_{n+1} \subset \cdots$ of closed subspaces of $L^2(\mathbb{R})$ such that:*

1. *There exists $\varphi \in V_0$ such that the translates $\varphi_{0,k}(t) = \varphi(t-k), k \in \mathbb{Z}$ form an orthonormal base of V_0. This function is the scaling function of the MRA*

2. *$V_{n+1} = D_1(V_n)$ where $D_j f(t) = 2^{j/2} f(2^j t), t \in \mathbb{R}$. Equivalently, $f(t) \in V_n \iff f(2t) \in V_{n+1}$. This in fact implies that $V_n = D_n(V_0)$, and $\varphi_{n,k} = D_n(\varphi_{0,k}), k \in \mathbb{Z}$.*

3. *$\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$*

4. *$\bigcap_{n\in Z} V_n = \{0\}$*

Observe firstly, that in virtue of the *Theorem 13 (Chapter 6)* of the notes, we have that given the function $\varphi \in L^2(\mathbb{R})$, defined by $\hat{\varphi}(\xi) = \chi_{[-1/2,1/2)}(\xi)$ then $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system, since we have

$$\sum_{n\in Z} |\hat{\varphi}(\xi+n)|^2 = \sum_{n\in Z} |\chi_{[-1/2,1/2)}(\xi+n)|^2 = 1$$

where, it is direct to prove that this expression holds, since for any given $\xi \in \mathbb{R}$, then there is only one possible value of $n \in \mathbb{Z}$ such that the number $n+\xi$ lays in the interval $[-1/2, 1/2]$. Thus, let us consider the initial subspace $V_0 = \overline{\langle \varphi_{0,k} \rangle}_{k\in\mathbb{Z}}$
Therefore, in order to define the subspace, notice that it is direct to see that the scaling function corresponds to the cardinal sine, $\varphi(t) = sinc(t)$, since by computing the inverse Fourier transform we can easily find

$$\begin{aligned}\varphi(t) &= \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{2\pi i \xi t} d\xi = \int_{\mathbb{R}} \chi_{[-1/2,1/2)}(\xi) e^{2\pi i \xi t} d\xi = \int_{1/2}^{1/2} e^{2\pi i \xi t} d\xi \\ &= \left[\frac{e^{2\pi i t \xi}}{2\pi i t}\right]_{-1/2}^{1/2} = \frac{sin(\pi t)}{\pi t} = sinc(t)\end{aligned}$$

Hence, $V_0 = \overline{\langle \varphi_{0,k} \rangle}_{k\in\mathbb{Z}} = \overline{\langle \varphi(t-k) \rangle} = \overline{\langle sinc(t-k) \rangle} = \overline{\langle \frac{sin(\pi(t-s))}{\pi(t-s)} \rangle}$. Observe that this subspace contains any functions $f \in V_0$, such that $\text{supp}(\hat{f}) \subset [-1/2, 1/2]$ by Shannon's theorem (*The Kotelnikov-Shannon-Whittaker theorem.* on the notes). Using similar deductions, we can obtain the sequence of subspaces $V_n = D_n(V_0)$ that correspond to those such that $f \in V_n$ if $f(2t) \in V_{n+1}$ and this happens when $\text{supp}(\hat{f}) \subset [-2^{n-1}, 2^{n-1}]$.

Notice that clearly $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$. Thus, we have that the given function φ generates this MRA, called the Shannon MRA. Now we have to see that this MRA can be viewed as the Fourier counterpart of the Haar MRA.
Recall that the Haar MRA is the one defined by the scalling function $\phi(t) = \chi_{[0,1]}$ and formed by the subspaces

$$V_n = \{f \in L^2(\mathbb{R}) \ : \ f|_{I_{n,k}} = c_k \text{ constant for all } k \in \mathbb{Z}\}$$

where $I_{n,k} = [k/2^n, (k+1)/2^n), k \in \mathbb{Z}$.

The Nyquist-Shannon formula states that in order to avoid aliasing, a continuous-time signal must be sampled at a rate that is at least twice the maximum frequency present in the signal. This principle is fundamental in digital signal processing and sampling theory.
In the context of the Shannon MRA, the scaling function $\varphi(t)$ has a Fourier transform $\hat{\varphi}(\xi)$ that is nonzero within the interval $[-1/2, 1/2)$ and zero outside. This implies that the frequency content of $\varphi(t)$ is confined within the bandwidth of $[-1/2, 1/2)$ as stated on the beginning of this problem.
The Haar MRA, on the other hand, is based on the Haar wavelet, which has a square-shaped frequency response that spans the frequency range $[-1/2, 1/2)$. The Haar wavelet is known for its localization in both the time and frequency domains.

Therefore, we can draw a parallel between the Shannon MRA and the Haar MRA by considering their frequency responses. Both MRAs have basis functions (scaling function and wavelet) whose Fourier transforms are nonzero within the interval $[-1/2, 1/2)$.
The Shannon MRA can be viewed as the Fourier counterpart of the Haar MRA because it utilizes a scaling function whose Fourier transform resembles the frequency response of the Haar wavelet. In both cases, the bandwidth of interest is $[-1/2, 1/2)$, reflecting a dyadic scale

By using the Nyquist-Shannon formula, which relates sampling rate and bandwidth, the Shannon MRA captures the essential frequency content of signals within the bandwidth of $[-1/2, 1/2)$, just as the Haar MRA does in the time domain.

In summary, the Shannon MRA and the Haar MRA are connected through the Nyquist-Shannon formula and their similar frequency characteristics. The Shannon MRA can be seen as the Fourier domain counterpart of the Haar MRA, as both MRAs effectively capture and represent signals within their respective bandwidths. □

(b) *Determine the detail spaces $W_n = V_{n+1} \ominus V_n,\ n \in \mathbb{Z}$ and the associated wavelet ψ. (You can use the Note in Exercise 5, if you want)*

Proof. From the previous exercise, we have seen that:

- V_n are the signals with frequencies in the range $[-2^{n-1}, 2^{n-1}]$, and similarly,
- V_{n+1} are the signals with frequencies in the range $[-2^n, 2^n]$,

clearly satisfying $V_n \subset V_{n+1}$. Then, $W_n = V_{n+1} \ominus V_n,\ n \in \mathbb{Z}$ is the space of details we have to add to V_n to obtain V_{n+1}, that is, the space of frequencies in the range $[-2^n, -2^{n-1}] \cup [2^{n-1}, 2^n]$.

In order to find the expression of the wavelet ψ, let us consider the *Note in exercise 5:*
It can be proved that Mallat's wavelet ψ associated to the MRA can be defined through the Fourier identity $\hat{\psi}(\xi) = e^{i\pi\xi}\overline{H(\xi/2 + 1/2)}\hat{\varphi}(\xi/2)$ being φ the scaling function of the MRA $\{V_n\}_{n\in\mathbb{Z}}$.

In this context, from exercise *5* we have that for a MRA $\{V_n\}_n$ with scaling function φ there exists a 1-periodic function $H(\xi)$ such that

1. $\hat{\varphi}(\xi) = H(\xi/2)\hat{\varphi}(\xi/2)$
2. If $\hat{\varphi}$ is continuous at 0, then $\hat{\varphi}(\xi) = \hat{\varphi}(0)\prod_{k=1}^{\infty} H(\xi/2^k)$.

Observe therefore that the scaling function determines the H function, since considering the expression *1.* and the definition of the given scaling function we have

$$\begin{aligned} \hat{\varphi}(\xi) = H(\xi/2)\hat{\varphi}(\xi/2) \iff \\ \chi_{[-1/2,1/2)}(\xi) = H(\xi/2)\chi_{[-1,1)}(\xi) \iff \\ H(\xi/2) = \chi_B(\xi) \ \text{ such that } \ B \cap [-1, 1) = [-1/2, 1/2) \end{aligned}$$

This is, $B = [-1/2, 1/2)$, and therefore $H(\xi/2) = \chi_{[-1/2,1/2)}(\xi)$ and consequently

$$H(\xi) = \chi_{[-1/2,1/2)}(2\xi) = \chi_{[-1/4,1/4)}(\xi)$$

Thus, for the expression of the Fourier transform of ψ we have

$$\hat{\psi}(\xi) = e^{i\pi\xi}\overline{H(\xi/2 + 1/2)}\hat{\varphi}(\xi/2) = e^{i\pi\xi}\overline{\chi_{[-1/4,1/4)}(\xi/2 + 1/2)}\chi_{[-1,1)}(\xi)$$

Notice that $\chi_{[-1/4,1/4)}(\xi/2 + 1/2) \in \mathbb{R}$ and hence $\overline{\chi_{[-1/4,1/4)}(\xi/2 + 1/2)} = \chi_{[-1/4,1/4)}(\xi/2 + 1/2)$. On the other hand, let us compute this characteristic function by noticing that

$$\xi/2 + 1/2 \in [-1/4, 1/4) \iff \xi/2 \in [-3/4, -1/4) \iff \xi \in [-3/2, -1/2)$$

Hence:

$$\chi_{[-1/4,1/4)}(\xi/2 + 1/2) = \chi_{[-3/4,-1/4)}(\xi/2) = \chi_{[-3/2,-1/2)}(\xi)$$

Consequently, we have

$$\hat{\psi}(\xi) = e^{i\pi\xi}\chi_{[-3/2,-1/2)}(\xi)\chi_{[-1,1)}(\xi) = e^{i\pi\xi}\chi_{[-1,-1/2)}(\xi)$$

□