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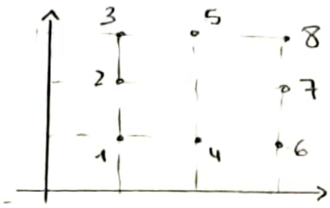
TDA· delivery 4

$\mathcal{X} = \{(1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \subseteq \mathbb{R}^2$



consider the distance matrix, which determines the Vietoris-Rips complexes:

	1	2	3	4	5	6	7	8
1	0	1	2	1	$\sqrt{5}$	2	$\sqrt{5}$	$2\sqrt{2}$
2	1	0	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{5}$	2	$\sqrt{5}$
3	2	1	0	$\sqrt{5}$	1	$2\sqrt{2}$	$\sqrt{5}$	2
4	1	$\sqrt{2}$	$\sqrt{5}$	0	2	1	$\sqrt{2}$	$\sqrt{5}$
5	$\sqrt{5}$	$\sqrt{2}$	1	2	0	$\sqrt{5}$	$\sqrt{2}$	1
6	2	$\sqrt{5}$	$2\sqrt{2}$	1	$\sqrt{5}$	0	1	2
7	$\sqrt{5}$	2	$\sqrt{5}$	$\sqrt{2}$	$\sqrt{2}$	1	0	1
8	$2\sqrt{2}$	$\sqrt{5}$	2	$\sqrt{5}$	1	2	1	0

Then note that the spectrum of the Vietoris-Rips persistence module is $A = \{0, 1, \sqrt{2}, 2, \sqrt{5}, 2\sqrt{2}\}$

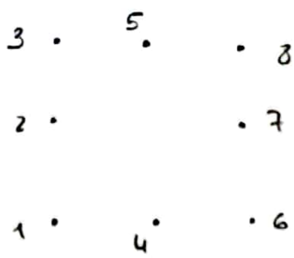
Consider the filtration $V_t = H_*(R_t(X))$ for all $t \in \mathbb{R}$ with $R_t(X) = 0$ for $t < 0$

Then let's consider the different complexes.

1) $0 \leq t < 1$. $R_t(x) = \{(1), (2), (3), (4), (5), (6), (7), (8)\}$

Then, $H_0(R_t(X)) = {}^{\ker \partial_0}/_{\operatorname{Im} \partial_1} = C_0 \cong \mathbb{F}^8$ since

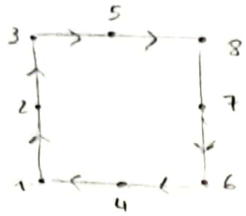
$C_0 = \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(8) \cong \mathbb{F}^8$, and we have $0 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$



And $H_n(R_t(X)) = 0 \quad \forall n \neq 0$

2) $1 \le t < \sqrt{2}$ · $R_t(X) = \{(12), (14), (23), (35), (46), (58), (67), (78)\}$

Here we have:



$$\begin{cases} \partial_1(12) = (2) - (1) \\ \partial_1(14) = (4) - (1) \\ \vdots \\ \partial_1(78) = (8) - (7) = -\partial(67) - \partial_1(46) - \partial_1(14) + \partial_1(12) + \partial_1(23) + \partial_1(35) + \partial_1(58) \end{cases}$$

Thus, $\operatorname{Im} \partial_1 = \langle \partial_1(12), \dots, \partial_1(67) \rangle$

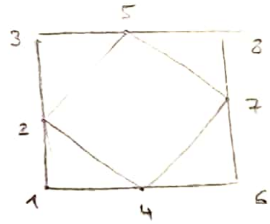
$\ker \partial_1 = \langle (78) + (67) + (46) + (14) - (12) - (23) - (35) - (58) \rangle = \langle \sigma_1 \rangle$

And we have: $H_0(R_t(X)) = C_0 / \operatorname{Im} \partial_1 \cong \mathbb{F}^8 / \mathbb{F}^7 \cong \mathbb{F}$ generated by any vertex x_i

$H_1(R_t(X)) = \ker \partial_1 / \operatorname{Im} \partial_2 \cong \ker \partial_1 \cong \mathbb{F}$ generated by the cycle σ_1

3) $\sqrt{2} \le t < 2$, $R_t(X) = \{(124), (235), (467), (578)\}$

with $\partial_2(124) = (24) - (14) + (12)$ for each 3-face. Note they are linearly independent from each other and thus:



$\operatorname{Im} \partial_2 = \langle \partial_1(124), \partial_1(235), \partial_1(467), \partial_1(578) \rangle$

However, we have

$\partial_1(24) = (4) - (2) = \partial_1(12) - \partial_1(14)$

$\partial_1(25) = (5) - (2) = \partial_1(23) + \partial_1(35)$

$\partial_1(57) = (7) - (5) = \partial_1(58) - \boxed{\partial_1(78)} =$ Here $\partial_1(78)$ in terms of the other faces as aforementioned

$\partial_1(47) = (7) - (4) = \partial_1(46) + \partial_1(67)$

Then $\operatorname{Im} \partial_1 \cong \mathbb{F}^7$ like before but

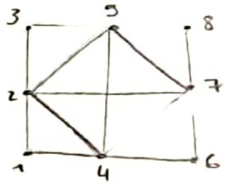
$\ker \partial_1 \cong \mathbb{F}^5$ adding there 4 new cycles.

Hence, $H_0 = C_0 / \operatorname{Im} \partial_1 \cong \mathbb{F}$

$H_1 = \ker \partial_1 / \operatorname{Im} \partial_2 \cong \mathbb{F}$

$H_2 = \ker \partial_2 / \operatorname{Im} \partial_2 \cong 0$

4) $2 \leq t < \sqrt{5}$: $R_t(X) = \{ (257), (247), (245), (457),$
$(124), (235), (467), (578) \rangle$



Here $\partial_2(257) = (57) - (27) + (25)$

and the new faces have images through ∂_2 also independent, i.e.

$\text{Im}\, \partial_2 \cong \mathbb{F}^8$

Meanwhile, we have new cycles:

$\partial_1(13) = (3) - (1) = \partial_1(12) + \partial_1(23)$

$\vdots$

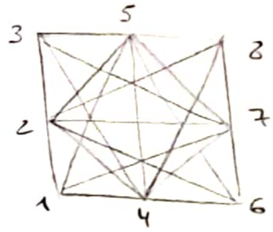
in terms of the others

And $\partial_1(27) = (7) - (1) = \partial_1(23) + \partial_1(35) + \partial_1(58) - \partial_1(78)$

$\partial_1(45) = (5) - (4) = \partial_1(46) + \partial_1(67) + \partial_1(78) - \partial_1(58)$

So we have two new elements of $\ker \partial_1$

$\ker \partial_1 = \langle \sigma_1, (12) - (14), (23) + (35), (58) + (78) - \sigma_1, (46) + (67),$
$2[(23) + (35) + (58)] - (67) - (46) - (14) + (12),$
$-(14) + (12) + (23) + (35) \rangle$



Thus we have: $H_0 \cong \mathbb{F}$

$H_1 = \ker \partial_1 / \text{Im}\, \partial_2 \cong \mathbb{F}^{11} / \mathbb{F}^8 \cong \mathbb{F}^3$

$H_2 = \ker \partial_2 / \text{Im}\, \partial_3 \cong 0$

5) $\sqrt{5} \leq t < 2$: $R_t(X) = \{ (126), (167), (238), (378), (468), (134)$
$(135), (568),$
$(257), (247), (245), (457),$
$(124), (235), (467), (578) \}$

We added the 2-faces:

$\partial_1(17) = (7) - (1) = \partial(14) + \partial_1(46) + \partial_1(67)$

$\partial_1(15) = \partial_1(12) + \partial_1(23) + \partial_1(35)$

$\left.\begin{array}{l} \partial_1(34) \\ \partial_1(37) \\ \partial_1(56) \\ \partial_1(28) \\ \partial_1(26) \end{array}\right\}$ All LD of the initial ones while the new 3-faces are LI with the previous ones, i.e. it is clear we cannot express them as a linear combination of the previous 3-faces. Then:

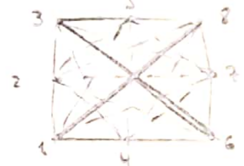
$$H_0 \cong \mathbb{F}$$

$$H_1 = \ker \partial_1 / \mathrm{Im}\, \partial_2 \cong \mathbb{F}^{18} / \mathbb{F}^{16} \cong \mathbb{F}^2$$

$$H_2 = \ker \partial_2 / \mathrm{Im}\, \partial_3 = 0$$

6) $2\sqrt{2} \leq t$ $\quad R_t\{X\} = \{(12345678)\}$

We've added the edges (18) and (36) also LD with the space of $\mathrm{Im}\,\partial_1$. Therefore:



$$H_0 \cong \mathbb{F}$$

$$H_1 = \ker \partial_1 / \mathrm{Im}\, \partial_2 \cong \mathbb{F}^{20} / \mathbb{F}^{19} \cong \mathbb{F}$$

$$H_2 = \ker \partial_2 / \mathrm{Im}\, \partial_3 \cong 0$$

New 3-faces are not dependent to the previous ones

we have therefore:

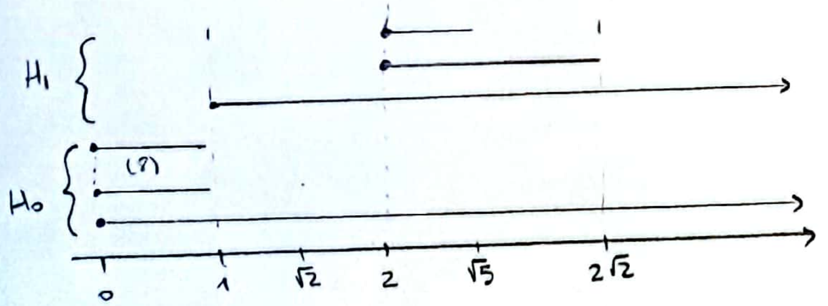
$$\begin{array}{cccccc} H_0(K_0) \longrightarrow & H_0(K_1) \longrightarrow & H_0(K_2) \longrightarrow & H_0(K_3) \longrightarrow & H_0(K_4) \longrightarrow & H_0(K_5) \\ \mathbb{F}^8 & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \end{array}$$

$\Rightarrow$ There is one permanent 0-cycle (vertex)

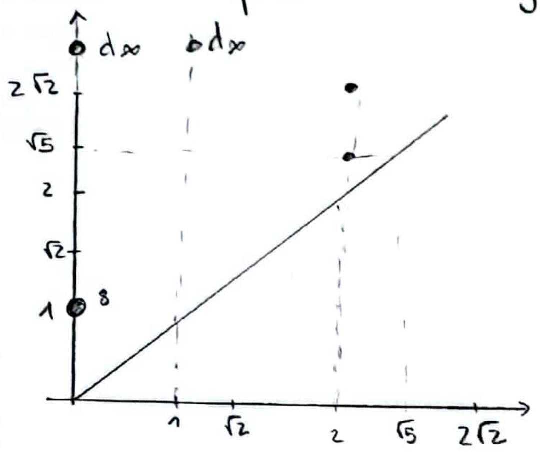
$$\begin{array}{cccccc} H_1(K_0) \longrightarrow & H_1(K_1) \longrightarrow & H_1(K_2) \longrightarrow & H_1(K_3) \longrightarrow & H_1(K_4) \longrightarrow & H_1(K_5) \\ 0 & \mathbb{F} & \mathbb{F} & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} \end{array}$$

So the barcode is the following:

$$V \cong \mathbb{F}[0,\infty) \oplus \mathbb{F}[0,1)^7 \oplus \mathbb{F}[1,\infty) \oplus \mathbb{F}[2,\sqrt{5}) \oplus \mathbb{F}[2,2\sqrt{2})$$



And the persistent diagram is:



PD: Sorry, didn't have time to deliver it on Latex this time.

Flàvia.