# 1 Exercises on one-dimensional optimization

**Exercise 1:** *Let us consider the function $g(x) = -e^{-x^2}$, that has a unique minimum at $x = 0$. Note that $g'(x) < 0$ if $x < 0$ and $g'(x) > 0$ if $x > 0$, which implies that any reasonable descent method should be able to find the minimum, no matter the starting point. Instead, let us use a Newton's method on the function $g'$ (i.e. to solve $g'(x) = 0$).*

(a) *Let $\{x_n\}_n$ be the sequence of points produced by the Newton's method starting at the seed $x_0 = 1$. Prove that $\lim_{n\to\infty} x_n = \infty$.*

*Proof.* Let us consider the succession generated by the Newton's method over the function $h(x) = g'(x)$ starting at point $x_0 = 1$:

$$x_{n+1} = x_n - \frac{h(x)}{h'(x)} = x_n - \frac{2x_n e^{-x_n^2}}{2e^{-x_n^2}(1-2x_n^2)} = x_n\Big(1 - \frac{1}{1-2x_n^2}\Big).$$

We want to see that this succession is divergent, i.e. $lim_{n\to\infty} x_n = \infty$. Let's suppose the succession does converge, and we may reach contradiction.

Consider the generating function of the $n$ iterative point $f(x) = x\Big(1 - \frac{1}{1-2x^2}\Big)$. Note that the recursive succession will converge if it has any fixed point, i.e. $f(x) = x$, which only verifies for $x = 0$ (this can be trivially deduced by solving the corresponding equation $f(x) = x$ and obtaining imaginary solutions other than $x = 0$). Thus, if the succession converges, it will converge to the limit point $x = 0$. Note however that it starts at the initial point $x_0 = 1$, therefore we trivially have $x_1 = f(x_0) = 2$. Observe that this generating function is monotonously increasing in this interval since:

$$f'(x) = 1 - \frac{1}{1-2x^2} - \frac{4x^2}{(1-2x^2)^2} = \frac{2x^2(2x^2-3)}{(1-2x^2)^2},$$

therefore $f'(x) > 0 \iff 2x^2 - 3 > 0 \iff x > \sqrt{\frac{3}{2}}$. Note that clearly the function increases and therefore, as $n$ increases for each new iteration it may not get closer to the fixed point, $x = 0$, as could be seen on Figure 1.

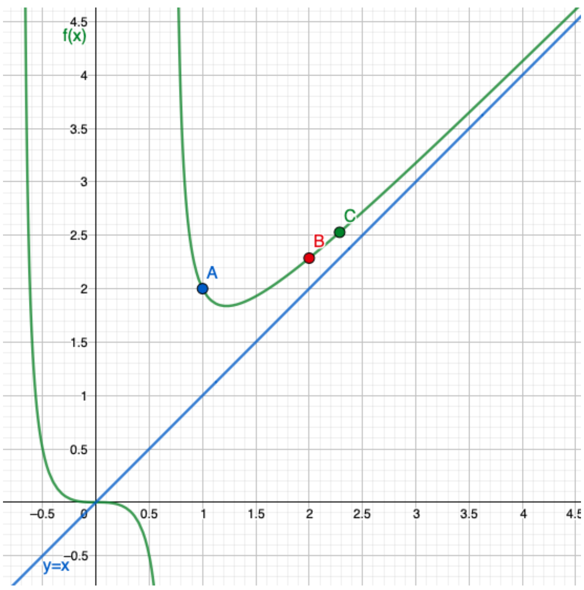


Figure 1: Illustration of the first points of the succession $\{x_n\}$ given by the Newton's method starting at point $x_0 = 1$.

Therefore, we will get further and further from the assumed limit point as $n$ increases, contradicting the initial hypothesis. Then, the succession diverges, and thus $lim_{n\to\infty}x_n = \infty$ as we wanted to see. $\square$

(b) *Find a value* $\alpha > 0$ *such that if* $x_0 \in [0, \alpha)$ *the Newton's method converges to 0, and if* $x_0 > \alpha$ *the Newton's method diverges.*
Following the previous arguments, note that for $x_0 = 0.5$, the succession does not have limit since, $f(0.5) = -0.5$ and iterativelly $f(-0.5) = 0.5$. Therefore we can find two subsuccessions that converge to different limit points, implying directly that the succession does not have a limit.
However, note that for any $x_0 \in [0, 0.5)$ then the Newton's method converges. An illustration of the first computed points for the case $x_0 = 4$ is shown in Figure 2.

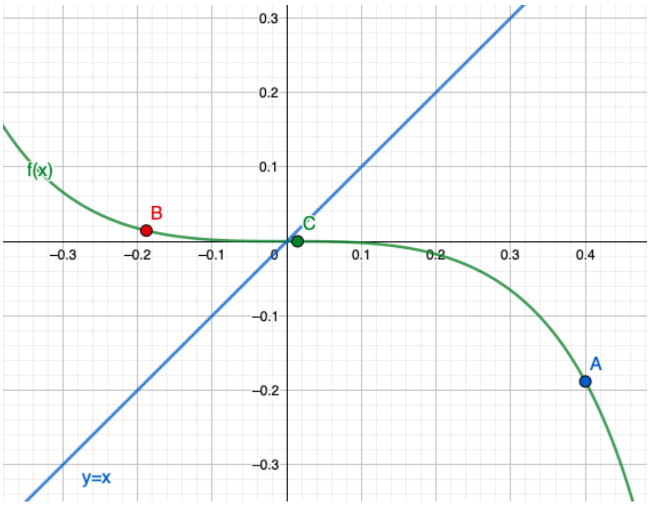


Figure 2: Illustration of the first points of the succession $\{x_n\}$ given by the Newton's method starting at point $x_0 = 1$.

Let's get a deepest sight into this problem. Note that we are seeking to apply the Newton method to the function $g'(x)$, i.e. we are solving $g'(x) = 2xe^{-x^2} = 0$, which is in fact a $\mathcal{C}^2$ function. Recall a known theorem that specifies:

**Theorem 1.1.** *Given* $f : I \to \mathbb{R}$ *a* $\mathcal{C}^2$ *function, and* $x^* \in I$ *a root of* $f$*, then it verifies:*

1. *Assume that there exist* $\alpha, A, B > 0$ *such that for any* $x \in (x_0-\delta, x_0+\delta)$ *we have* $|f'(x)| \geq A$, $|f''(x)| \leq B$*, then let* $x_0$ *be a initial value. Thus the succession given by the Newton method satisfies:*

$$|x_{n+1} - x^*| \leq \frac{B}{A}|x_n - x^*|^2$$

2. *Suppose that* $f'(x^*) \neq 0$*, then the existence of* $\alpha, A, B > 0$ *verifying the previous statement is assured.*

Observe that when computing the derivatives $h'(x) = g''(x)$ it does not vanish at point $x^* = 0$. Thus, we have the existence of the values as aforementioned. Note that for $\alpha = 1/2$ we have

$$\begin{aligned}
|h'(x)| = |g''(x)| = |4x^2 - 2|e^{-x^2} &\geq |2x|e^{-x^2} \geq g(\frac{1}{2}) \approx 0.7788, \ \forall x \in \Big(-\frac{1}{2}, \frac{1}{2}\Big) \\
|h''(x)| = |g^{(3)}(x)| = |8x^3 - 12x|e^{-x^2} &\leq |8x^3 - 12x| \\
&\leq |8x^3| + |12x| \leq 8\Big(\frac{1}{2}\Big) + 12\frac{1}{2} = 5, \ \forall x \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)
\end{aligned}$$

giving us the explicit values of $\alpha, A, B > 0$ which verify the statement *1.* of the previous theorem, ensuring quadratic convergence of the Newton's method, as wanted to prove.

**Definition 1.1** (Unimodal)**.** *We say that a function is unimodal at the real interval* $[A, B]$ *if it has a minimum* $\overline{\alpha} \in [A, B]$ *and if* $\alpha_1, \alpha_2 \in [A, B], \alpha_1 < \alpha_2$ *we have:*

$$\alpha_2 \leq \overline{\alpha} \Rightarrow g(\alpha_1) > g(\alpha_2)$$
$$\alpha_1 \geq \overline{\alpha} \Rightarrow g(\alpha_1) < g(\alpha_2)$$

**Exercise 2.** *Discuss if the following functions are unimodal:*

(a) $g(x) = x^3 - x \quad on \quad x \in [-2, 0], \quad and\ on \quad x \in [0, 2].$
Let's find the optimal points of the function:

$$g'(x) = 2x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{2}}$$
$$g''(x) = 4x,$$

therefore, $x = -\frac{1}{\sqrt{2}} \in [-2, 0]$ is a maximum and $x = \frac{1}{\sqrt{2}} \in [0, 2]$ is a local minimum. Thus, in virtue of the Definition 1.1, we can affirm that the function is unimodal on the interval $[0, 2]$ but not on $[-2, 0]$.

(b) $g(x) = exp(-x) \quad on \quad x \in [0, 1].$
Let's consider the derivative of this $\mathcal{C}^1$ function:

$$g'(x) = -e^{-x} < 0, \ \forall x$$

Note that it is strictly decreasing for all $x$, and therefore, it has a local minimum at the extreme of the interval, $x = 1$. Thus this could be considered as an unimodal function, since it actually satisies the corresponding conditions.

(c) $g(x) = |x| + |x - 1| \quad on \quad x \in [-2, 2].$
Note that by definition of the absolute value we have

$$\frac{d}{dx}|x| = \begin{cases} 1, \ x > 0 \\ -1, \ x > 0 \end{cases}$$

Thus, taking this fact into consideration, the derivative of $f$ is given by

$$g'(x) = \begin{cases} 1 + 1 = 2 & x \geq 1 \\ 1 - 1 = 0 & x \in (0, 1) \\ -1 - 1 = -2 & x < 0 \end{cases}$$

Therefore, recalling Definition 1.1 we can conclude that the function is not unimodal in the interval $[-2, 2]$, since the minimum value of the function is attained in an interval and thus, for any two values $\alpha_1, \alpha_2 \in (0, 1)$ the strict condition given in the definition is not satisfied.

**Exercise 3.** *Look for the Golden section search method and explain it.*

The Golden section search function is a one-dimensional optimization method without derivatives that can be used when the number $N$ of computations wished to carry out is not known a priori. It is based on the elimination of subintervals for every newly computed point by using unimodality. This can be explained by considering four points in an interval, $a < b < c < d \in [A, B]$. Knowing this four values of the function, we have a key for where the minimum are. Note that we could actually remove an interval. Consider then the initial interval $[a_1, d_1]$ of length $d_1 - a_1 = \Delta_1$, and we consider we know the value of the function we seek to minimize, $f$, in two inner intermediate points $a_1 < b_1 < c_1 < d_1$ such that $c_1 - a_1 = d_1 - b_1 = \Delta_2$. Then, by unimodality we can discard one of the intervals considered. Without loss of generalization, suppose this intervals corresponds to $[c_1, d_1]$, then we are left with the interval $[a_1, c_1]$ of length $\Delta_2$, then we have

$$\Delta_1 = d_1 - a_1 = c_1 - a_1 + d_1 - c_1 = c_1 - a_1 + b_1 - a_1 = \Delta_2 + \Delta_3$$

Then, by induction we have

$$\Delta_k = \Delta_{k+1} + \Delta_{k+2} \tag{1}$$

where $\Delta_k$ represents the length of the interval after $k-1$ iterations.
Assume now that the length of the intervals has a given fixed ration, i.e. $\frac{\Delta_1}{\Delta_2} = \frac{\Delta_2}{\Delta_3} = \cdots = \frac{\Delta_k}{\Delta_{k+1}} = \cdots = \gamma$. Combined with (1), then we have

$$\frac{\Delta_k}{\Delta_{k+1}} = 1 + \frac{\Delta_{k+2}}{\Delta_{k+1}} \iff \gamma = 1 + 1/\gamma \iff \gamma^2 - \gamma - 1 = 0 \iff \gamma = \frac{\sqrt{5}+1}{2} \approx 1.618$$

which corresponds to the golden section number. Therefore, this method has a linear rate of convergence of rate $1/\gamma = 0.618$.