1 Exercises on Topological Data Analysis

1.1 Delivery 3. Persistence modules

Exercise 1. *Prove that a morphism f of persistence modules is an isomorphism if and only if f_t is an isomorphism of vector spaces for all t.*

Proof. $\boxed{\Rightarrow}$ Assume we have $f : (V, \pi) \to (V', \pi')$ isomorphism of persistence modules. Then, by hypothesis there exists a morphism g of persistence modules such that $(V, \pi) \xrightarrow{f} (V', \pi') \xrightarrow{g} (V, \pi)$ with $f \circ g = id_{V'}$. Note that by definition the composite $f \circ g$ is a morphism of persistence modules, defined as $(f \circ g)_t = f_t \circ g_t$, for all t. But we also know by the fact that $f \circ = id_{V'}$ that $id_{V'_t} = (f \circ g)_t = f_t \circ g_t$, for all t and hence we have that for all t there exists a morphism of vector spaces such that $f_t \circ g_t = id_{V'_t}$.
Similarly, we can see that for every t, $g_t \circ f_t = id_{V_t}$ for all t and consequently, we have that f_t is an isomorphism $\forall t$.

$\boxed{\Leftarrow}$ Similarly, assume that f_t is an isomorphism of vector spaces for all t, then there exists g_t for every t such that $g_t \circ f_t = id_{V_t}$ and $f_t \circ g_t = id_{V'_t}$. Thus, consider the morphism of persistence modules defined as the collection of the $\mathbb{F}$-linear maps g_t. Hence, consider the composite $g \circ f$ defined by $(g \circ f)_t = g_t \circ f_t = id_{V_t}$. Following the same arguments, we have that the composite $f \circ g$ satisfies $(f \circ g)_t = f_t \circ g_t = id_{V'_t}$, directly implying that f is an isomorphism of persistence modules, as wanted to prove.

□

Exercise 2. *Prove that two isomorphic persistence modules of finite type have the same spectrum.*

Proof. Consider two isomorphic persistence modules (V, π) and (V', π'), this is, there exists $f : (V, \pi) \to (V', \pi')$ isomorphism, i.e. there exists $g : (V', \pi') \to (V, \pi)$ such that $f \circ g = id_{V'}$ and $g \circ f = id_V$.

Recall now the definition of spectrum of a persistence module of finite type (V, π), this is a finite set $A = \{a_0, \dots, a_n\} \subseteq \mathbb{R}$ such that

i) For all $x \in \mathbb{R} \setminus A$ there exists $\delta > 0$ such that $\pi_{s,t}$ is an isomorphism for $x - \delta < s \leq x + \delta$.

ii) For all $a \in A$ there exists $\varepsilon > 0$ such that $\pi_{a,t}$ is an isomorphism if $a < t < a + \varepsilon$ and $\pi_{s,a}$ is not an isomorphism if $a - \varepsilon < s < a$.

iii) $V_t = \{0\}$ if $t < a_0$, given that $a_0 < a_1 < \dots < a_n$

Given therefore both spectrums, A, A' we want to see that $A = A'$. We may see it by double inclusion. However, given the symmetry over the definition of the two isomorphic persistence modules, we can see only one inclusion, noting that the other inclusion could be argued following the exact same procedure with a change on notation. Thus, it is enough to prove that $A \subseteq A'$.
Regarding now the first two conditions, note that it is enough to see that, for every $s, t \in \mathbb{R}$, if $\pi_{s,t}$ is an isomorphism then $\pi'_{s,t}$ is an isomorphism. Then, the intervals delimited by $\{a_i\}$ may be the same as the ones described by $\{a'_i\}$.
Thus, assume that for fixed $s, t \in \mathbb{R}$, we have that $\pi_{s,t}$ is an isomorphism, and thus there exists the corresponding inverse morphism of persistence modules, $\xi_{s,t}$, with $\xi_{s,t} \circ \pi_{s,t} = id_{V_s}$ for each s, t.
Considering the notation previously introduced note that we have the following commutative diagram, in which we seek to find the existence of $\xi'_{s,t}$ in order to prove that $\pi'_{s,t}$ is an isomorphism.

$$\begin{tikzcd}
V_s \arrow[r, "f_s"] \arrow[d, "\pi_{s,t}"] & V'_s \arrow[l, dotted, bend left, "g_s"] \arrow[d, "\pi'_{s,t}"] \\
V_t \arrow[u, dotted, bend left, "\xi_{s,t}"] \arrow[r, "f_t"] & V'_t \arrow[l, dotted, bend left, "g_t"] \arrow[u, dotted, bend left, "\xi'_{s,t}"]
\end{tikzcd}$$

Note that by definition of morphism we have $f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$. Given now that g_t and $\xi_{s,t}$ are the inverse morphisms of the isomorphisms f and $\pi_{s,t}$, respectively, we have by the commutativity of the diagram that

$$\pi'_{s,t} \circ f_s \circ \xi_{s,t} = f_t \iff \pi'_{s,t} \circ f_s \circ \xi_{s,t} \circ g_t = id_{V'_t} \iff \pi'_{s,t} \circ \underbrace{(f_s \circ \xi_{s,t} \circ g_t)}_{\xi'_{s,t}} = id_{V'_t}$$

And thus we have that whenever $\pi_{s,t}$ is isomorphism then there exists $\xi'_{s,t} = f_s \circ \xi_{s,t} \circ g_t$ such that $\pi'_{s,t} \circ \xi'_{s,t} = 0$. Using similar arguments we have that $\xi'_{s,t} \circ \pi'_{s,t} = id_{V'_s}$. And thus, $\pi_{s,t}$ isomorphism implies that $\pi'_{s,t}$ is isomorphism.

Finally, note that if $V_t = \{0\}$, clearly since it exists an isomorphism between V_t and V'_t then $V'_t = \{0\}$. Thus, we have $A \subseteq A'$, and consequently, by the symmetry of the problem $A = A'$, as we wanted to prove. $\square$

Exercise 3. *Prove that there is a nonzero morphism $\mathbb{F}[a,b) \to \mathbb{F}[c,d)$ if and only if $c \leq a$ and $a < d \leq b$.*

Proof. $\boxed{\Leftarrow}$ Assume that $c \leq a$ and $a < d \leq b$. Then note that, in particular, we have $a \in [a,b)$ and $a \in [c,d)$, and therefore $\mathbb{F}[a,b)_a = \mathbb{F} = \mathbb{F}[c,d)_a$. Then, at least we can consider the nonzero morphism f containing $f_t = f_a = id : \mathbb{F} \to \mathbb{F}$.

$\boxed{\Rightarrow}$ Assume now that there exists a nonzero morphism $f : \mathbb{F}[a,b) \to \mathbb{F}[c,d)$, i.e. there exists at least one t such that $f_t : \mathbb{F}[a,b)_t \to \mathbb{F}[c,d)_t$ is a nonzero $\mathbb{F}$-linear map. By simplicity, we may denote the intervals $[a,b) = I_1, [c,d) = I_2$. Then, recalling the definition of interval modules we have that

$$\mathbb{F}(I)_t \begin{cases} \mathbb{F} \; if \; t \in I \\ 0 \; if \; t \notin I \end{cases}$$

We may consider the alternative positions of the intervals we can have and argue by reduction to the absurd. Consider first the case in which $I_1 \cap I_2 = \emptyset$, then for all t we have at least two $\{0\}$ spaces on the diagram

$$\begin{tikzcd}
V_s \arrow[r, "f_s"] \arrow[d, "\pi_{s,t}"] & V'_s \arrow[d, "\pi'_{s,t}"] \\
V_t \arrow[r, "f_t"] & V'_t
\end{tikzcd}$$

and it is direct that the only way that this diagram commutes under these situations is that all the morphisms are 0.

Consider the remaining case, in which we have $a \leq c$ and $c \leq b < d$. Note therefore that in the best of the possible cases, there exists at least some $t \in I_2 \setminus I_1$. In this situation assume that $s \in I_1$, because otherwise we already have the same situation as before and there is not any nonzero morphism verifying that the corresponding diagram commutes. Hence, assuming that $s \in I_1$, and $s \in I_2$ we have the following diagram

$$\begin{tikzcd} \mathbb{F} \arrow[r, "i"] \arrow[d, "0"] & \mathbb{F} \arrow[d, "\pi'_{s,t}=id"] \\ 0 \arrow[r, "i"] & \mathbb{F} \end{tikzcd}$$

Where i represents the inclusion morphism. Hence, we would have $0 = id \circ 0 = id \circ i \neq 0$, getting to contradiction.
Note now that if we would have $s, t \in I1 \cap I2$, then the diagram will commute, but eventually, increasing the value of t, we will reach the situation previously described and reaching contradiction again.

Hence we are left with the situation described by $c \leq a$ and $a < d \leq b$, in which case, as described on the other inclusion there is at least one nonzero morphism. Additionally, note that we may not enter in contradiction since in this situation we would have $t \in I_2 \setminus I_1 \Rightarrow s \notin I_1$, but we could have $t \in I_1 \setminus I_2$ in which case the situation we get is the following

$$\begin{tikzcd} 0 \arrow[r, "i"] \arrow[d, "0"] & \mathbb{F} \arrow[d, "\pi'_{s,t}=id"] \\ \mathbb{F} \arrow[r, "f_t"] & \mathbb{F} \end{tikzcd}$$

which commutes, verifying the definition of morphism without necessarily being zero.

$\square$