1 Exercises 3: Distributions

Exercise 5.

(a) *Let* $f \in \mathcal{C}^1(\mathbb{R} \setminus \{a\})$ *be such that* $f(a^+) - f(a^-) = s$. *Prove that* $T_f' = T_{f'} + s\delta_a$.

Proof. Recall that, by definition the corresponding distributions induced by the functions f, f' are

$$T_f : \mathcal{C}^1(\mathbb{R}) \to \mathbb{C}, \quad \text{defined as} \quad T_f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbb{R}} \varphi \cdot f$$

Therefore, when considering the derivative of the function f, f', we have the distribution:

$$T_{f'} : \mathcal{C}^1(\mathbb{R}) \to \mathbb{C}, \quad \text{defined as} \quad T_{f'}(\varphi) = \langle f', \varphi \rangle = \int_{\mathbb{R}} \varphi \cdot f'.$$

On the other hand, by definition of the differentiation of $T \in D'(\mathbb{R})$, we have $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$.

Consider the equation $T_f' = T_{f'} + s\delta_a$ which we want to prove. Let's start by evaluating both sides, following the recalls aforementioned, on a general test function $\varphi \in \mathcal{C}_C^\infty(\mathbb{R})$:

$$T_f'(\varphi) = -\langle f, \varphi' \rangle$$
$$T_{f'}(\varphi) + s\delta_a(\varphi) = \langle f', \varphi \rangle + s\langle \delta_a, \varphi \rangle$$

Integrating by parts on the left side of the equation we obtain:

$$\begin{aligned}
\langle f, \varphi' \rangle &= \int_{\mathbb{R}} f(x)\varphi'(x)dx \\
&= [f(x)\varphi(x)]_{\mathbb{R}} - \int_{\mathbb{R}} f'(x)\varphi(x)dx \\
&= \big(\lim_{x\to\infty} f(x)\varphi(x) - f(a^+)\varphi(a)) + (f(a^-)\varphi(a) - \lim_{x\to-\infty} f(x)\varphi(x)\big) - \langle f', \varphi \rangle
\end{aligned}$$

Notice that the limit to infinity terms vanish because φ is a test function, so it has compact support. Thus, we have:

$$\begin{aligned}
\langle f, \varphi' \rangle &= -(f(a^+) - f(a^-))\varphi(a) - \langle f', \varphi \rangle \\
&= -s\langle \delta_a, \varphi \rangle - \langle f', \varphi \rangle
\end{aligned}$$

Substituting this back into our original equation, we obtain:

$$\begin{aligned}
T_f'(\varphi) &= -\langle f, \varphi' \rangle \\
&= s\langle \delta_a, \varphi \rangle + \langle f', \varphi \rangle \\
&= s\delta_a(\varphi) + T_{f'}(\varphi).
\end{aligned}$$

Therefore, we have shown that $T_f' = T_{f'} + s\delta_a$ as a formal manipulation of distributions.

□

(b) *More generally, if* $f \in \mathcal{C}^1(\mathbb{R} \setminus \{a_n\}_n)$ *with* $\lim_n |a_n| = \infty$, *and each* a_n *is a jump discontinuity of size* s_n, *then* $T_f' = T_{f'} + \sum_n s_n\delta_{a_n}$

Proof. Consider a similar approach to the previous one, therefore, once we get to the integration by parts, here we can partition the evaluation by the several discontinuities we have, and taking into account again that the test function has compact support, this is

$$\begin{aligned}
\langle f, \varphi' \rangle &= \int_{\mathbb{R}} f(x)\varphi'(x)dx = \Big[f(x)\varphi(x) \Big]_{\mathbb{R}} - \int_{\mathbb{R}} f'(x)\varphi(x)dx \\
&= \sum_{n\in\mathbb{Z}} \big(f(a_n^-) - f(a_n^+) \big)\varphi(a_n) - \langle f', \varphi \rangle \\
&= -\sum_{n\in\mathbb{Z}} s_n \langle \delta_{a_n}, \varphi \rangle - \langle f', \varphi \rangle.
\end{aligned}$$

Therefore, we have

$$T_f'(\varphi) = -\langle f, \varphi' \rangle = \langle f', \varphi \rangle + \sum_{n\in\mathbb{Z}} s_n \langle \delta_{a_n}, \varphi \rangle, \tag{1}$$

as desired. □

(c) *Let f be the T-periodic function with value $f(x) = x/T$ in $[0,T)$. Prove that, in the sense of distributions $f' = 1/T - \sum_{n\in\mathbb{Z}} \delta_{nT}$.*

Proof. Firstly, observe that if we extend this T-periodic function to $\mathbb{R}$, we get a discontinuous function with a jump discontinuity at each multiple of T, of size 1, this is:

$$s_n = f(a_n^+) - f(a_n^-) = \lim_{x\to T+} f(x) - \lim_{x\to T-} f(x) = 0 - \frac{T}{T} = -1,$$

where $\{a_n\}_n = \{nT\}_n$. Therefore, taking into account the equality proven on the previous section, (1) we have

$$T_f'(\varphi) = \langle f', \varphi \rangle - \sum_n \langle \delta_{nT}, \varphi \rangle = \langle \frac{1}{T}, \varphi \rangle - \sum_n \langle \delta_{nT}, \varphi \rangle$$

Consequently, this is equivalent to have $f' = \frac{1}{T} - \sum_n \langle \delta_{nT} \rangle$ in the sense of distributions, as wanted to prove. □

(d) *Let $\{\alpha_n\}_n$ be such that $\lim_n |\alpha_n| = \infty$ slowly, meaning that $|\alpha_n| = O(|n|^k)$, for some $k \geq 1$. Let $a \in \mathbb{R}$. Prove that $\sum_n \alpha_n \delta_{na}$ is a tempered distribution and that its Fourier transform is $\sum_n \alpha_n e^{-2\pi i a n t}$.*

Proof. Recall the definition of a tempered distribution, which is a linear continuous map on the space of Schwartz functions $\mathcal{S}(\mathbb{R})$, $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$. By definition, a map is continuous if and only if for all compact $K \subset \mathbb{R}$ there exist $m = m(K) \geq 1, C = C(K) > 0$ such that $|T(\varphi)| \leq C\|\varphi\|_{K,m}$, for all $\varphi \in \mathcal{C}_C^\infty(\mathbb{R})$ with compact support (i.e. $\text{supp}(\varphi) \subset K$). Here, we have used the definition of the seminorms: $\|\varphi\|_{K,m} := \sup_{j\leq m} \sup_{x\in K} |\varphi^{(j)}(x)|$.

Therefore, consider the linear map T defined as $T(\varphi) = \sum_n \alpha_n \delta_{na}(\varphi)$ for $\varphi \in \mathcal{S}(\mathbb{R})$. Let us consider the absolute value of $T(\varphi)$:

$$\big|T(\varphi)\big| = \left| \sum_{n\in\mathbb{Z}} \alpha_n \varphi(an) \right| = \left| \sum_{n\in\mathbb{Z}} \alpha_n (1 + (ax)^{k+1}) \frac{\varphi(an)}{1 + (ax)^{k+1}} \right|,$$

where $k \geq 1$ is such that $|\alpha_n| = O(|n|^k)$. Therefore, observe that we can bound the sum as follows

$$\big|T(\varphi)\big| \leq \sum_{n\in\mathbb{Z}} |\alpha_n| \frac{|(1 + (ax)^{k+1})\varphi(an)|}{|1 + (ax)^{k+1}|} \leq \sup_{x\in\mathbb{R}} |(1 + (ax)^{k+1})\varphi(ax)| \sum_{n\in\mathbb{Z}} \frac{|\alpha_n|}{|1 + (ax)^{k+1}|}$$

Now by applying a change of variables $t = ax \in \mathbb{R}$ we have

$$\big|T(\varphi)\big| \leq \underbrace{\sup\ _{t\in\mathbb{R}}(1+|t|)^{k+1}(\varphi(t))|}_{P_{k+1,1}(\varphi)<\infty} \underbrace{\sum_{n\in\mathbb{Z}} \frac{|\alpha_n|}{|1+(ax)^{k+1}|}}_{C<\infty}$$

Notice that the first part of the expression corresponds to the partial seminorm in the Schwarz space, which is convergent since $\varphi \in \mathcal{C}_C^\infty(\mathbb{R})$ and the second part is convergent given the property of slow grouth of the succession $\{\alpha_n\}$.

Now that we know that the distribution is tempered, we can consider the corresponding Fourier transform. By definition, $\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle$. Therefore, recalling the linearity of the Fourier transform, we have

$$\begin{aligned}
\langle \hat{T}, \varphi\rangle &= \sum_{n\in\mathbb{Z}} \alpha_n \langle (\delta_{an})^\wedge, \varphi\rangle = \sum_{n\in\mathbb{Z}} \alpha_n \langle \delta_{an}, \hat{\varphi}\rangle = \sum_{n\in\mathbb{Z}} \alpha_n \hat{\varphi}(an) \\
&= \sum_{n\in\mathbb{Z}} \alpha_n \int_{\mathbb{R}} \varphi(t) e^{-2\pi itan} dt = \sum_{n\in\mathbb{Z}} \alpha_n \langle e^{-2\pi itan}, \varphi\rangle
\end{aligned}$$

Hence, in the sense of distributions, we have $\hat{T} = \sum_{n\in\mathbb{Z}} \alpha_n e^{-2\pi itan}$ as desired. $\square$