1 Exercises on linear programming

Exercise 1:
Let us consider the convex set (polyhedron),

$$C = \{(x, y) \in \mathbb{R}^2 \text{ such that } 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Write this set in the form $\{z \in \mathbb{R}^n$ such that $Az = b, z \geq 0\}$, compute the basic feasible solutions and, from them, the vertices.

Consider the slack variables $t_1, t_2 \geq 0$, so we can express the constraints

$$\begin{cases} x \leq 1, \\ y \leq 1, \\ x, y \geq 0 \end{cases}$$

as follows

$$\begin{cases} x + t_1 = 1, \\ y + t_2 = 1, \\ x, y, t_1, t_2 \geq 0 \end{cases}$$

More explicitly, we have $C = \{z \in \mathbb{R}^n \text{ such that } Az = b, z \geq 0\}$, for

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Consider now the basic feasible solutions of the polyhedron, which correspond to the ones obtained when setting the remaining variables to zero, i.e. $\mathbf{x} = [x_B, x_N]$, with $x_N = 0$ and such that $x_B \geq 0$. Note that these solutions are given by

$$\begin{aligned} (x, y, t_1, t_2)_1 &= (1, 1, 0, 0) \\ (x, y, t_1, t_2)_2 &= (1, 0, 0, 1) \\ (x, y, t_1, t_2)_3 &= (0, 1, 1, 0) \\ (x, y, t_1, t_2)_4 &= (0, 0, 1, 1). \end{aligned}$$

Recalling now a theorem previously states, we know that the set of vertices of a polytope corresponds to the set of basic feasible solutions. Therefore, the vertices of C, are given by the set of basic feasible solutions considered previously:

$$V_C = \{(1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 0, 1, 1)\}$$

Exercise 2:
Assume that $a_1, \ldots, a_m$ are given vectors in $\mathbb{R}^3$ (all different from 0). Let $b_1, \ldots, b_m$ be strictly positive numbers and let us define the set

$$M = \{x \in \mathbb{R}^3 \text{ such that } a_i^t x \leq b_i \text{ for } i = 1, \ldots, m\}.$$

(a) *Show that the interior of this set is not empty.*

(b) *We want to determine the centre and the radius of the biggest sphere contained in M. Write this problem as a linear problem.*

(a) Note that the origin is satisfies the condition since $b_i > 0$ strictly. Then we have that $0 \in M$. However, it could be in the closure of M, therefore we have see that there is a neighbourhood of 0, V_0 contained in M, so we would have $0 \in \text{Int}(M)$. Consider b_1 to be the minimum number i.e. $b_1 \leq b_i$, $\forall i = 2, \ldots, m$, without loss of generality. Since the vectors a_i are fixed, we can consider the vector with higher norm, let's suppose it is a_k, i.e. $||a_k||^2 \geq ||a_i||^2, \forall i = 1, \ldots, m$. Therefore, consider now the neighbourhood of 0 to be the ball centered on the origin and with radius $\varepsilon = \frac{b_1}{||a_k||^2}$, $V_0 = B(0, \varepsilon)$. Note that for any $v \in V_0$, we have $v \in M$, since for any general $i = 1, \ldots, m$ we have:

$$a_i^T v = \sum_{j=1}^{3} a_{ij} v_j \leq \sum_{j=1}^{3} a_{ij}\varepsilon \leq \varepsilon \underbrace{\sum_{j=1}^{3} a_{ij}^2}_{||a_i||^2} < \frac{b_1}{||a_k||^2} ||a_i||^2 \leq b_i \underbrace{\frac{||a_i||^2}{||a_k||^2}}_{\leq 1} \leq b_i.$$

Therefore, we have that $0 \in V_0 \subseteq M$, and consequently the origin belongs to the interior of M, i.e. $Int(M) \neq \emptyset$.

(b) We want to find the biggest sphere $S = B(c, r)$ contained in M. Note that therefore, we seek to maximize the radius of this sphere, i.e. $r = ||x - c||$, since the volume of the 3-dimensional sphere is directly proportional to the radius, which is defined positive, $V = \frac{4}{3}\pi r^3$. Then the mos intuitive would be to consider the maximization of the radius subject to the fact that the whole sphere is contained in M, this is $x \in M$ and $c \in M$. This could be written as

$$\begin{cases} \text{Maximize} & ||x - c||^2 \\ \text{subject to} & x, c \in M \end{cases}$$

Note however that this is not a linear optimization problem. Let's reconsider this problem in terms of $r > 0$, this is maximizing r subject to the fact that $c \in M$, i.e. $a_i^t x \leq b_i$, and that any point inside the sphere, i.e. x such that $d(x, c) \leq r^2 \iff ||x - c|| \leq r$ then $x \in M$. This would imply that for every point in the sphere we would have $a_i^t x \leq b_i$. Note that if a point, y in the sphere was closer to any line $a_i^t x = b_i$ then the line would cross directly the sphere, leaving some points outside the set M. Then, given a point y inside the sphere, we need to have the distance between this point and any line $a_i^t x = b_i$ at least r. The distance between a point y and the line $a_i^t x = b_i$ is given by $d(y, a_i^t x - b_i) = \frac{||(y - b_i) \times a_i}{||a_i||}$. Considering the worst case, the possibly further point from the lines, the center of the sphere, we have the condition

$$d(c, a_i^t x - b_i) = \frac{||(c - b_i) \times a_i||}{||a_i||} = \frac{||a_i c - b_i||}{||a_i||} = \frac{b_i - a_i c}{||a_i||} \geq r$$

given the fact that we have $c \in M$. Then, the linear problem is given by:

$$\begin{cases} \text{Maximize} & r \\ \text{subject to} & a_i^t c \leq M \\ & r \leq \frac{b_i - a_i c}{||a_i||} \end{cases}$$

Exercise 3.

Use a software package to solve:

(a)

$$\begin{cases} \text{Minimize} & -8x_1 - 9x_2 - 5x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 \leq 2 \\ & 2x_1 + 3x_2 + 4x_3 \leq 3 \\ & 6x_1 + 6x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{cases}$$

Code used for a simple linear problem:

```
% Objective function
fa = [-8 -9 -5];
% Restrictions:
A = [1 1 2
    2 3 4
    6 6 2];
a = [2 3 8];
lb = zeros(3,1);
Aeq = [];
beq=[];
xa = linprog(fa, A, a,Aeq, beq, lb);
```

with solution $x_a = [1, 1/3, 0]$.

(b)

$$\begin{cases} \text{Minimize} & 5x_1 - 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 2 \\ & 2x_1 + 3x_2 \leq 4 \\ & -x_1 + 6x_2 = 10 \\ & x_1, x_2 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize} & 5x_1 - 3x_2 \\ \text{subject to} & -x_1 + x_2 \leq -2 \\ & 2x_1 + 3x_2 \leq 4 \\ & -x_1 + 6x_2 = 10 \\ & x_1, x_2 \geq 0 \end{cases}$$

Using a similar code:

```
% funció objectiu
fb = [5 -3];
% Restriccions:
B = [-1 1
    2 3
    ];
b = [-2 4];
Beq = [-1 6];
beq = 10;
la= zeros(2,1);
xb = linprog(fb, B, b, Beq, beq, la);
```

we obtaine that the problem has `no feasible solution`.

(c)

$$\begin{cases} \text{Maximize} & 3x_1 + 2x_2 - 5x_3 \\ \text{subject to} & 4x_1 - 2x_2 + 2x_3 \leq 4 \\ & -2x_1 + x_2 - x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize} & -3x_1 - 2x_2 + 5x_3 \\ \text{subject to} & -x_1 + x_2 \leq -2 \\ & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{cases}$$

Trying directly from to define the maximization problem from Matlab and let it convert it to a minimization problem:

```
x = optimvar('x','LowerBound',0);
y = optimvar('y','LowerBound',0);
z = optimvar('z','LowerBound',0);
prob = optimproblem('Objective',3*x + 2*y - 5*z,'ObjectiveSense','max');
prob.Constraints.c1 = 4*x - 2*y + 2*z <= 4;
prob.Constraints.c2 = -2*x + y - z <= -1;
problem = prob2struct(prob);
[sol,fval,exitflag,output] = linprog(problem);
```

We obtain `Problem is unbounded`. Converting manually the problem to a minimization problem, we get the same result, following the computation:

```
fc = [-3 -2 5];
C = [4 -2 2
    -2 1 -1
];
c = [4 -1];
Aeq=[];
beq=[];
lb= zeros(3,1);
xc = linprog(fc, C, c, Aeq, beq, lb);
```

(d)

$$\begin{cases} \text{Maximize} & 4x_1 + 6x_2 + 3x_3 + x_4 \\ \text{subject to} & 1.5x_1 + 2x_2 + 4x_3 + 3x_4 \leq 550 \\ & 4x_1 + x_2 + 2x_3 + x_4 \leq 700 \\ & 2x_1 + 3x_2 + x_3 + 2x_4 \leq 200 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize } -4x_1 - 6x_2 - 3x_3 - x_4 \\ \text{subject to} \\ 1.5x_1 + 2x_2 + 4x_3 + 3x_4 \leq 550 \\ 4x_1 + x_2 + 2x_3 + x_4 \leq 700 \\ 2x_1 + 3x_2 + x_3 + 2x_4 \leq 200 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

By following the same procedure and creating the corresponding minimization problem

```
% funció objectiu
fc = [-4 -6 -3 -1];
% Restriccions:
C = [1.5 2 4 3
    4 1 2 1
    2 3 1 2
    ];
c = [550 700 200];
Aeq=[];
beq=[];
lb= zeros(4,1);
xc = linprog(fc, C, c, Aeq, beq, lb);
```

the solution $(x, y, z, w) = (0, 25, 125, 0)$, is found. We could also have used a code similar to the previous one.

Exercise 4. *Consider the linear programme (P) in standard form and its dual programme (D),*

$$\begin{cases} \text{Min } z = c \cdot x \\ A \cdot x = b, \; x \geq 0 \end{cases}, \quad \begin{cases} \text{Max } w = u \cdot b \\ u \cdot A \leq c \end{cases}$$

Let us denote by A_j the jth column of A. Prove that two solutions $(\overline{x}, \overline{u})$ of respectively, (P) and (D) are optimal if and only if

$$(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$$

Proof. $\boxed{\Rightarrow}$ Consider $(\overline{x}, \overline{u})$ to be two optimal solutions of (P) and (D), then by recalling a theorem stated in class we have $z^* = c \cdot \overline{x} = \overline{u} \cdot b = w^* \iff \overline{u} \cdot b - c \cdot x = 0$. But since $\overline{x}$ is a solution, it verifies the constraints of the primal linear programme, i.e. $A \cdot \overline{x} = b$ and therefore we have $\overline{u} \cdot A \cdot \overline{x} - c \cdot x = 0 \iff (\overline{u} \cdot A - c) \cdot \overline{x} = 0$, Since $\overline{x} \geq 0$ and $\overline{u} \cdot A - c \leq 0$, the zero scalar product corresponds to the set of equations given by $(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$.

$\boxed{\Leftarrow}$ Consider now $(\overline{x}, \overline{u})$ to be two solutions of (P) and (D), that verify $(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$. Since $\overline{x} \geq 0$ and $\overline{u} \cdot A - c \leq 0$, this is equivalent to say that $(\overline{u} \cdot A - c) \cdot \overline{x} = 0 \iff \overline{u} \cdot A \cdot \overline{x} - c \cdot \overline{x}$, and again using that $\overline{x}$ verifies the constraints from (P) we have $\overline{u} \cdot b = c \cdot \overline{x}$, which in virtue of the same theorem afforementioned implies that $\overline{x}$ and $\overline{u}$ are optimal solutions of the linear programmes $(P), (D)$, as wanted to prove.

□