1 Exercises on Topological Data Analysis

1.1 Delivery 1

Exercise 1: *Let K and L be the abstract simplicial complexes whose maximal faces are, respectively*

$$K: \quad (124)\ (125)\ (135)\ (136)\ (146)\ (234)\ (236)\ (256)\ (345)\ (456); \tag{1}$$
$$L: \quad (014)\ (015)\ (023)\ (027)\ (035)\ (047)\ (126)\ (128)\ (148) \tag{2}$$
$$(156)\ (236)\ (278)\ (346)\ (348)\ (358)\ (467)\ (567)\ (578). \tag{3}$$

Prove that the geometric realizations $|K|$ and $|L|$ are compact surfaces and find out which surfaces they are.

Proof. Consider the abstract simplicial complex, K, defined by the maximal faces as aforementioned on the expression (1). Consider the geometric realization of K which is the geometric simplicial complex X_K with, in this case, a $3-face$ $\Delta(e_{i_0},\dots,e_{i_k})$ in $\mathbb{R}^6$, in this particular problem, for each $k-face(i_0,\dots,i_k)$ of K.
Then the geometric simplicial complex X_K is defined by the 3-simplices $\sigma\subset\mathbb{R}^6$ explicitly specified in (1) in terms of the corresponding vertices. Note firstly that all of them are homeomorphic, since they represent the same set a exception of the specified vertices[1]. Aditionally, by definition, each of the faces are homeomorphic to the standard 3-simplex

$$\Delta^3 := \Delta(e_{i_0},e_{i_1},e_{i_2},e_{i_3}) = \{\lambda_0e_0+\lambda_1e_1+\lambda_2e_2+\lambda_3e_3 \mid \lambda_1+\lambda_2+\lambda_3=1, \lambda_i\geq 0,\ \forall i\},$$

e_i being the i-th vector of the canonical basis of $\mathbb{R}^6$ in this case. Note that this can be argued considering the same homeomorphisms as aforementioned. Intuitivelly, the 3-simpleces are geometrically visualized as shown on the figure 1.

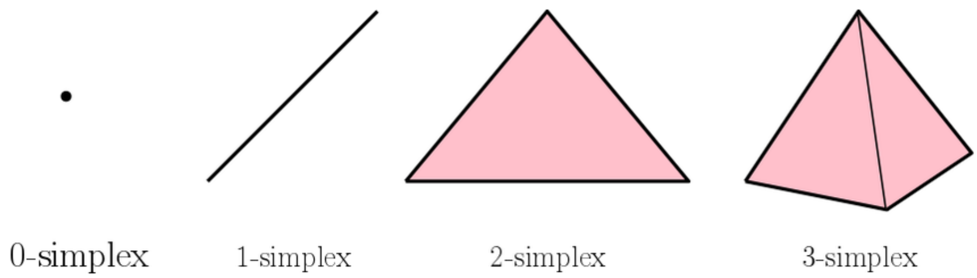

Figure 1: Geometrical intuitive representation of k-simpleces for low k.

Consider now the underlying topological space

$$|X_K| = \bigcup_{\sigma\in X_K}\sigma$$

endowed with the Euclidean topology.
Observe that under the Euclidean topology $\Delta^3\subset\mathbb{R}^3$ is a compact set since it is bounded and closed. In order to see that it is bounded note that for every $\epsilon\geq 1$, the ball $B(0,\varepsilon)$ verifies that $\Delta^3\subseteq B(0,\varepsilon)$. On the other hand, let us recall that the closure of a set S, noted as $\overline{S}=cl(S)$ is composed by all the points, a, for which there exists a neighbourhood V of a containing a point of the set S. Observe that for any point $y\notin\Delta^3$ we can find an $\epsilon>0$ such that the ball $B(y,\epsilon)$ does not intersect Δ^3. Thus $\Delta^3=cl(\Delta^3)$, and by definition we have that Δ^3 is closed.

[1]Note that this can be trivially proven by considering the homeomorphism for each 3-simplex that brings the corresponding vertices x_i of each 3-simplex to the canonical basis and acts like the identity over the remaining set.

Thus, since the finite union of compact sets is compact we have, by definition, that $|X_K|$ is compact.

Following the same arguments, we can see that the abstract simplicial complex L has a compact geometric realization $|L|$, since it is again the finite union of 3-simplices, this tome over $\mathbb{R}^9$ $\square$

Recall the theorem of classification of compact surfaces that states that any connected closed surface is homeomorphic to some member of one of these three families: the sphere S^2, the convex sum of n tors, $\mathbb{T}$ or either a convex sum of n projective planes, $\mathbb{P}$.

Consider now the Euler characteristic, defined for a compact surface M as $\chi(M) = V - E + F$, where V, E and F are the number of vertices, edges and faces, respectively, of the surface. Additionally, the Euler's characteristic is independent of the triangulation of the surface considered, and therefore it determines unequivocally the surface from the previous theorem at which each compact surface is homeomorphic.

Consider then the cases of the geometric realizations studied. Observe that the graph representation of the triangulation given by $|K|$, as shown on the Figure 2.

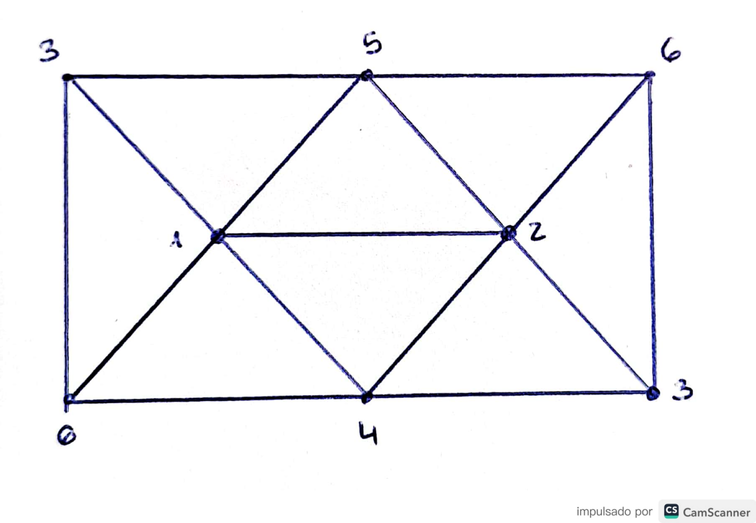

Figure 2: Triangulation given by $|K|$.

Thus, computing the Euler's characteristic we have $\chi(|K|) = V - E + F = 8 - 15 + 8 = 1 = \chi(\mathbb{P})$. Note that this corresponds to the Euler's characteristic of the projective plane, being therefore the surface homeomorphic to the geometric realization $|K|$. Consequently, we have that K is a triangulation of the topoogical space defined by a projective plane.

Consider now the triangulation given by $|L|$, present on Figure 3.

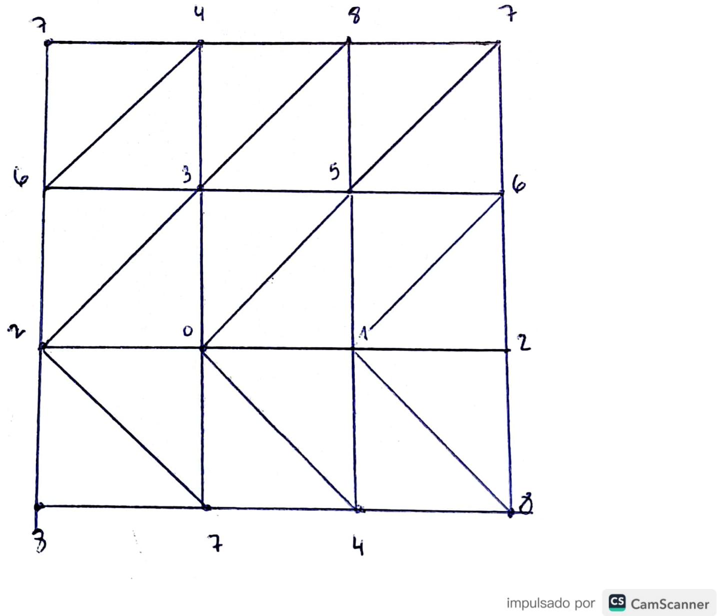

Figure 3: Triangulation given by $|L|$.

Considering similarly the Euler's characteristic of this surface we have $\chi(|L|) = 0$ and therefore we have that $|L| \cong \mathbb{T}$, since we can clearly distinguish a change of rotation by looking at the edges of the triangulation. Thus, we have that L is a triangulation of the Klein's bottle.

Exercise 2: *List the maximal faces fo the Cech complex $C_\epsilon(X)$ and the Vietoris-Rips complex $R_\epsilon(X)$, depending on ϵ, if X is the set of vertices of a regular hexagon of radius 1.*

Consider the hexagon, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, of radius 1 over the plane $\mathbb{R}$, as shown on the Figure 4

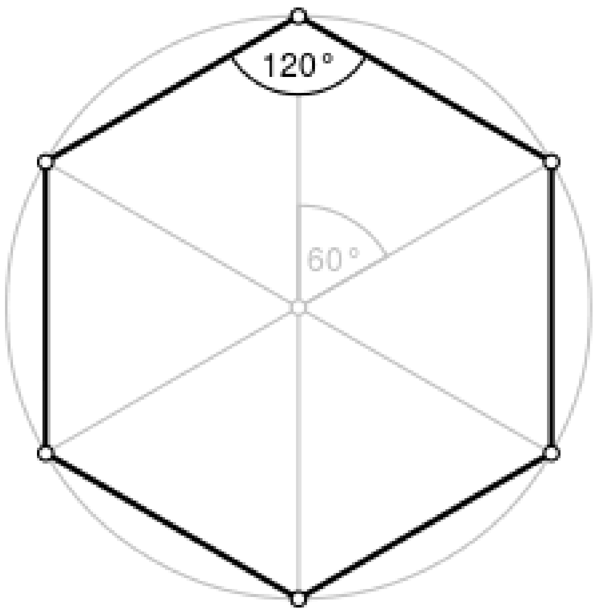

Figure 4: Qualitative representation of the hexagon centered on the origin.

By a trivial computation taking into account that the angle between two vertices is $\theta = \pi/3$ for every two edges of the hexagon we can compute the vertices:

$$x_1 = (1,0),\ x_2 = \Big(\frac{1}{2}, \frac{\sqrt{3}}{2}\Big),\ x_3 = \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big),$$
$$x_4 = (-1,0),\ x_5 = \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big),\ x_6 = \Big(\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big)$$

In order to compute the Cech complex $C_\varepsilon(X)$ and the Vietoris-Rips complex $R_\varepsilon(X)$ observe that each vertex is appart from the consecutive edges by a distance of $\varepsilon = 1$, by definition. Considering then the corresponding balls centred at each center and distance matrix of X:

$$D_X = \begin{pmatrix} 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\ 1 & 0 & 1 & \sqrt{3} & 2 & \sqrt{3} \\ \sqrt{3} & 1 & 0 & 1 & \sqrt{3} & 2 \\ 2 & \sqrt{3} & 1 & 0 & 1 & \sqrt{3} \\ \sqrt{3} & 2 & \sqrt{3} & 1 & 0 & 1 \\ 1 & \sqrt{3} & 2 & \sqrt{3} & 1 & 0 \end{pmatrix}$$

we can compute the corresponding complexes:

1. For $0 \leq \varepsilon < 1$, we have

$$C_\varepsilon(X) : (1)\ (2)\ (3)\ (4)\ (5)\ (6)$$

   And in fact, for this case, $C_\varepsilon(X) = R_\varepsilon(X)$.

2. For $1 \leq \varepsilon < \sqrt{3}$,

$$C_\varepsilon(X) :\ (12)\ (23)\ (34)\ (45)\ (56)\ (16)$$

   and again $R_\varepsilon(X) = C_\varepsilon(X)$.

3. For $\sqrt{3} \leq \varepsilon < 2$, we have

$$\begin{aligned} R_\varepsilon(X) &:\ (123)\ (126)\ (135)\ (156)\ (234)\ (246)\ (345)\ (456) \\ C_\varepsilon(X) &:\ (123)\ (126)\ (156)\ (234)\ (345)\ (456) \end{aligned}$$

   Since clearly all the consecutive balls intersect in a nonempty space, while the intersection between three consecutive balls does not intersect. Meanwhile, the distance between any two edges is less than2, except for the opposite ones.

4. For $\varepsilon \geq 2$:

$$\begin{aligned} R_\varepsilon(X) :\ & (123)\ (126)\ (135)\ (156)\ (124)\ (125)\ (134)\ (136)\ (145) \\ & (146)\ (234)\ (246)\ (235)\ (245)\ (256)\ (345)\ (346)\ (456) \end{aligned}$$