Topological Data Analysis

2022–2023

Lecture 3

Persistent Homology

14 November 2022

Functoriality of homology

A function $f: K \to L$ between abstract simplicial complexes is a <u>simplicial map</u> if it sends vertices of K to vertices of L and $f(v_{i_0}), \dots, f(v_{i_n})$ form a face in L whenever $\{v_{i_0}, \dots, v_{i_n}\}$ is a face of K.

Note that $f(v_{i_0}), \dots, f(v_{i_n})$ need not be distinct.

Every simplicial map $f: K \to L$ between finite ordered abstract simplicial complexes induces a group homomorphism

$$f_n : C_n(K) \longrightarrow C_n(L)$$

or an R-module homomorphism if coefficients in a ring R are used

for each $n \geq 0$ as follows. Let us denote $V_K = \{v_1, \dots, v_k\}$ and $V_L = \{w_1, \dots, w_\ell\}$. Given an n-face $\{v_{i_0}, \dots, v_{i_n}\}$ with $v_{i_0} < \dots < v_{i_n}$, write $f(v_{i_s}) = w_{j_s}$ for all s. Then we define

with the usual reordering convention

$$f_n(i_0 \dots i_n) = \begin{cases} (j_0 \dots j_n) & \text{if } w_{j_0}, \dots, w_{j_n} \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that $f_{n-1} \circ \partial_n^K = \partial_n^L \circ f_n$ for all n.

$$\begin{tikzcd} C_n(K) \arrow[r, "\partial_n^K"] \arrow[d, "f_n"'] & C_{n-1}(K) \arrow[d, "f_{n-1}"] \\ C_n(L) \arrow[r, "\partial_n^L"] & C_{n-1}(L) \end{tikzcd}$$

commutes:

$$f_{n-1}\left(\partial_n^K(i_0 \dots i_n)\right) = f_{n-1}\left(\sum_{r=1}^{n} (-1)^r (i_0 \dots \hat{i}_r \dots i_n)\right) = \sum_{r=1}^{n} (-1)^r f_{n-1}(i_0 \dots \hat{i}_r \dots i_n) =$$

$$= \sum_{r=1}^{n} (-1)^r (j_0 \dots \hat{j}_r \dots j_n) = \partial_n^L (j_0 \dots j_n) = \partial_n^L\left(f_n(i_0 \dots i_n)\right)$$

if $\omega_{j_0}, \dots, \omega_{j_n}$ are distinct, or otherwise if $\omega_{j_s} = \omega_{j_t}$ with $s<t$ then

$$f_{n-1}\left(\partial_n^K(i_0 \dots i_n)\right) = (-1)^s (j_0 \dots \hat{j}_s \dots j_n) + (-1)^t (j_0 \dots \hat{j}_t \dots j_n) =$$

$$= (-1)^s (j_0 \dots \hat{j}_s \dots j_n) + (-1)^t (-1)^{t-s-1} (j_0 \dots \hat{j}_s \dots j_n) = 0 .$$ ✓

Consequently, f induces homomorphisms

$$f_*: H_n(K) \longrightarrow H_n(L) \quad \text{for all } n \geq 0.$$

These are defined as $f_*([z]) = [f_n(z)]$ for each n-cycle $z \in Z_n(K)$.

Let us check that f_* is well defined:

1) $z \in Z_n(K) \Rightarrow f_n(z) \in Z_n(L)$, since $\partial_n^L(f_n(z)) = f_{n-1}(\partial_n^K z) = 0$.

2) If $[z] = [z']$ then $z' = z + \partial_{n+1}^K \omega$ for some $\omega \in C_{n+1}(K)$. Then

$f_n(z') = f_n(z) + f_n(\partial_{n+1}^K \omega) = f_n(z) + \partial_{n+1}^L f_{n+1}(\omega)$, and hence

$[f_n(z')] = [f_n(z)]$. ✓

Induced homomorphisms satisfy the <u>functoriality</u> relations:

$$\begin{cases} (g \circ f)_* = g_* \circ f_* \\ id_* = id \end{cases}$$

since $(g \circ f)_*([z]) = [g_n(f_n(z))] = g_*([f_n(z)]) =$

$= g_*(f_*([z]))$. ✓

since $id_*([z]) = [id_n(z)] = [z]$. ✓

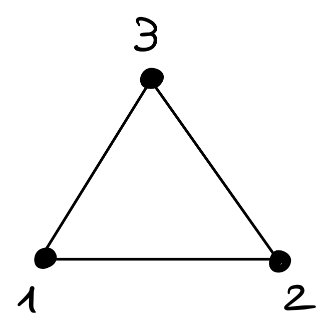
If $i: K \hookrightarrow L$ is the inclusion of a subcomplex, then $i_n(z) = z$ for all $n \geq 0$ and all $z \in Z_n(K)$. Hence

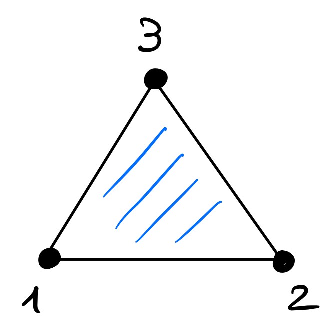
$$i_* : H_n(K) \longrightarrow H_n(L)$$

sends $i_*([z]) = [z]$. Yet, it is possible that $[z] = 0$ in $H_n(L)$ but not in $H_n(K)$. Hence i_* need <u>not</u> be a monomorphism.

<u>Example:</u> K: $(1\,2)(1\,3)(2\,3)$

L: $(1\,2\,3)$





$H_1(K) \cong \mathbb{Z}$ but $H_1(L) = 0$.

Hence $i_* : H_1(K) \longrightarrow H_1(L)$ is not injective.

$z = (1\,2) - (1\,3) + (2\,3) \in Z_1(K)$,

$z \notin B_1(K)$ since K has no 2-faces; hence $[z] \neq 0$ in $H_1(K)$.

$z = \partial_2^L (1\,2\,3) \in B_1(L)$, so $[z] = 0$ in $H_1(L)$.

Filtrations

A finite filtration of an abstract simplicial complex K is a nested family of subcomplexes

$$K_0 \subseteq K_1 \subseteq \dots \subseteq K_{m-1} \subseteq K_m = K.$$

Examples:

1. The family of skeleta of a finite complex is a filtration:

$$K^{(0)} \subseteq K^{(1)} \subseteq \dots \subseteq K^{(d)} = K.$$

2. The sequences of distinct Čech complexes or Vietoris–Rips complexes of a point cloud X are finite filtrations of the complex of all nonempty subsets of X:

$$X = C_0 \subset C_1 \subset \dots \subset C_m = \mathcal{P}(X) \setminus \{\emptyset\}$$

$$X = R_0 \subset R_1 \subset \dots \subset R_n = \mathcal{P}(X) \setminus \{\emptyset\}$$

Fix any coefficient field $\mathbb{F}$, which will be omitted from the notation.
Suppose given a finite filtration of a finite ordered abstract simplicial complex K:

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{m-1} \subseteq K_m = K.$$

For all $i \leq j$ and every $n \geq 0$, consider the homomorphism

$$\varphi_n^{i,j} : H_n(K_i) \longrightarrow H_n(K_j) \qquad \text{$\mathbb{F}$-linear map}$$

induced by the inclusion $K_i \hookrightarrow K_j$.

- A nonzero homology class $\alpha \in H_n(K_j)$ is <u>born</u> at K_j if $\alpha \notin \operatorname{Im} \varphi_n^{i,j}$ for any $i<j$.
- A nonzero homology class $\alpha \in H_n(K_i)$ <u>dies</u> or <u>vanishes</u> at K_j for $j>i$ if $\varphi_n^{i,j}(\alpha)=0$ but $\varphi_n^{i,j-1}(\alpha) \neq 0$.
- If α is born at K_i and dies at K_j with $j>i$, then $j-i$ is called the <u>life</u> or <u>persistence</u> of α.
- If α survives until $K_m = K$, then α is called <u>essential</u> or <u>permanent</u>.

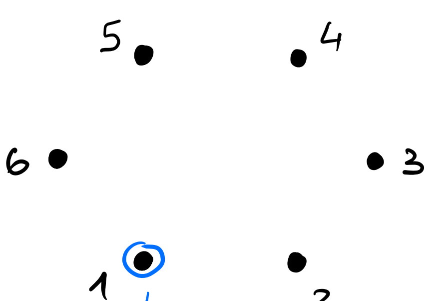
Notation: $H_n^{i,j}(K) = \operatorname{Im}\left(\varphi_n^{i,j} : H_n(K_i) \longrightarrow H_n(K_j)\right)$.

These are called persistent homology groups of K with respect to the filtration $\{K_i\}_{0 \le i \le m}$.

in fact, $\mathbb{F}$-vector spaces

We also denote
$$\beta_n^{i,j}(K) = \dim_{\mathbb{F}} H_n^{i,j}(K)$$
and call them persistent Betti numbers.

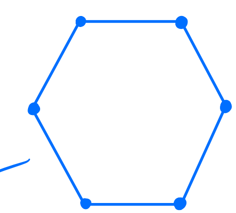
Example: Let $\{K_i\}$ be the Vietoris–Rips filtration of the point cloud X formed by the vertices of a regular hexagon of radius 1:



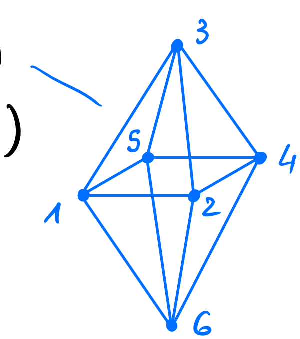
a permanent 0-cycle

$K_0 = R_0(X)$: (1) (2) (3) (4) (5) (6)

$K_1 = R_1(X)$: (12) (16) (23) (34) (45) (56)



$K_2 = R_{\sqrt{3}}(X)$: (123) (126) (135) (156)
(234) (246) (345) (456)



$K_3 = R_2(X)$: (123456)

ε

$$\begin{tikzcd}
H_0(K_0) \arrow[r, "\varphi_0^{0,1}"] & H_0(K_1) \arrow[r, "\varphi_0^{1,2}", "\mathrm{id}"'] & H_0(K_2) \arrow[r, "\varphi_0^{2,3}", "\mathrm{id}"'] & H_0(K_3) \\
\mathbb{F}^6 & \mathbb{F} & \mathbb{F} & \mathbb{F}
\end{tikzcd}$$

$$\begin{tikzcd}
H_1(K_0) \arrow[r, "\varphi_1^{0,1}"] & H_1(K_1) \arrow[r, "\varphi_1^{1,2}", "(*)"'] & H_1(K_2) \arrow[r, "\varphi_1^{2,3}"] & H_1(K_3) \\
0 & \mathbb{F} & 0 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
H_2(K_0) \arrow[r, "\varphi_1^{0,1}"] & H_2(K_1) \arrow[r, "\varphi_1^{1,2}"] & H_2(K_2) \arrow[r, "\varphi_1^{2,3}", "(**)"'] & H_2(K_3) \\
0 & 0 & \mathbb{F} & 0
\end{tikzcd}$$

$$\beta_0^{0,1} = \beta_0^{1,2} = \beta_0^{2,3} = 1$$

$$\beta_1^{0,1} = \beta_1^{1,2} = \beta_1^{2,3} = 0$$

$$\beta_2^{0,1} = \beta_2^{1,2} = \beta_2^{2,3} = 0$$

(*) $H_1(K_1)$ is generated by the homology class of the 1-cycle $z = (12) + (23) + (34) + (45) + (56) - (16)$. This 1-cycle becomes a boundary in K_2:

$$\partial\big((123) + (345) + (135) + (156)\big) = z.$$

(**) $H_2(K_2)$ is generated by the homology class of the 2-cycle $\omega = (123) - (126) + (135) + (156) - (234) - (246) + (345) - (456)$. This 2-cycle becomes a boundary in K_3:

$$\partial\big(-(1234) + (1345) - (1456) - (1246)\big) = \omega. \checkmark$$