# 1 Exercises 2: Fourier Transform

**Exercise 6.**

(a) *Prove that the Fourier transform of the function* $f(t) = e^{-2\pi|t|}$ *is*

$$\hat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$$

*Proof.* Consider the corresponding Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi|t|}e^{-2\pi i\xi t}dt = 2\underbrace{\int_0^{\infty} e^{-2\pi t}cos(2\pi\xi t)dt}_{I(\xi)}\,. \tag{1}$$

Observe that the integral $I(\xi)$ can be solved by considering integration by parts twice as follows:

$$\begin{aligned} I(\xi) &= \frac{1}{2\pi\xi}\underbrace{e^{-2\pi t}sin(2\pi\xi t)\Big|_0^{\infty}}_{0} + \frac{1}{\xi}\int_0^{\infty} e^{-2\pi t}sin(2\pi\xi t)dt \\ &= \frac{1}{\xi}\Bigg[-\frac{1}{2\pi\xi}\underbrace{e^{-2\pi t}cos(2\pi\xi t)\Big|_0^{\infty}}_{1} - \frac{1}{\xi}\underbrace{\int_0^{\infty} e^{-2\pi t}cos(2\pi\xi t)dt}_{I(\xi)}\Bigg]. \end{aligned}$$

Hence, we have the following relation:

$$\xi I(\xi) = \frac{1}{2\pi\xi} - \frac{1}{\xi}I(\xi) \iff I(\xi) = \frac{1}{2\pi\xi}\frac{1}{\xi^2+1}.$$

Finally, from (1) we have $\hat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$, as wanted to prove. □

(b) *Let* $g \in \mathcal{C}(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R})$. *Find* $u \in \mathcal{C}^2(\mathbb{R})$ *such that* $u, u', u'' \in \mathcal{L}^1(\mathbb{R})$ *and solving the differential equation*

$$u'' - u = g.$$

*Prove also that* $u(\infty) = 0$.

*Proof.* Applying Fourier on both sides of the differential equation we have:

$$(u'' - u)\hat{}\,(\xi) = \hat{g}(\xi)$$

Considering then the properties of the Fourier transfomr, such as the linearity and the properties of the Fourier transform of the derivatives, i.e. $(f')\hat{}\,(\xi) = 2\pi i\xi\hat{f}(\xi)$, then we have

$$\begin{aligned} (u'')\hat{}\,(\xi) - \hat{u}(\xi) &= 2\pi i\xi(u')\hat{}\,(\xi) - \hat{u}(\xi) = (2\pi i\xi)^2\hat{u}(\xi) - \hat{u}(\xi) \\ &= -[(2\pi\xi)^2 + 1]\hat{u}(\xi) = \hat{g}(\xi). \end{aligned}$$

Thus, by rearrenging this equation we get

$$\hat{u}(\xi) = -\frac{1}{(2\pi\xi)^2+1}\hat{g}(\xi) = -\pi(D_{2\pi}f)^{\wedge}(\xi)\hat{g}(\xi),$$

using the definition of the ditalation and its properties when considering the Fourier transformation: $\hat{f}(2\pi\xi) = \frac{1}{\pi}\frac{1}{(2\pi\xi)^2+1} = (D_{2\pi}f)^\wedge(\xi)$. Using now the convolution property over the Fourier transformation we have:

$$\hat{u}(\xi) = -\pi(D_{2\pi}f * g)^\wedge(\xi).$$

Thus, in virtue of the inversion formula we obtain:

$$u(t) = \int_{\mathbb{R}} \hat{u}(\xi)e^{2\pi i\xi t}d\xi = -\pi\int_{\mathbb{R}}(D_{2\pi}f * g)^\wedge(\xi)e^{2\pi i\xi t}d\xi = -\pi(D_{2\pi}f * g)(t).$$

More explicitly, by using the definition of the dilatation: $D_{2\pi}f(t) = \frac{1}{2\pi}f\big(\frac{1}{2\pi}\big) = \frac{1}{2\pi}e^{-|t|}$, hence

$$u(t) = -\frac{1}{2}(e^{-|t|} * g(t)).$$

Now, consider the corresponding derivatives:

$$u'(t) = \frac{d}{dt}u(t) = -\pi\frac{d}{dt}\underbrace{(D_{2\pi}f * g)(t)}_{w(t)}.$$

By taking into account the properties of the derivative of the Fourier transform:

$$\begin{aligned}(w'(t))^\wedge &= (w')^\wedge(\xi) = 2\pi i\xi\hat{w}(\xi) = 2\pi i\xi(D_{2\pi}f * g)^\wedge(\xi)\\ &= \underbrace{2\pi i\xi(D_{2\pi}f)^\wedge(\xi)}_{\big(\frac{d}{dt}(D_{2\pi}f)\big)^\wedge(\xi)}\hat{g}(\xi) = \big(\frac{d}{dt}(D_{2\pi}f)\big)^\wedge(\xi)\hat{g}(\xi)\\ &= \big(\big[\frac{d}{dt}(D_{2\pi}f)\big] * g\big)^\wedge(\xi),\end{aligned}$$

where

$$\frac{d}{dt}(D_{2\pi}f)(t) = \frac{1}{2\pi}\frac{d}{dt}(-|t|)e^{-|t|} = \begin{cases} -\frac{1}{2\pi}e^t, & t<0\\ \frac{1}{2\pi}e^{-t}, & t>0\end{cases}$$

Consequently, we have

$$u'(t) = -\pi\big(\big[\frac{d}{dt}(D_{2\pi}f)\big] * g\big)(t) = \begin{cases} \frac{1}{2}(e^t * g(t)), & t<0\\ -\frac{1}{2}(e^{-t} * g(t)), & t>0\end{cases}$$

Similarly,

$$u''(t) = -\pi\big(\big[\frac{d^2}{dt^2}(D_{2\pi}f)\big] * g\big)(t) = \frac{1}{2}(e^t * g(t)).$$

Finally, when taking the limit on both sides of the equality and by considering the Dominated Convergence Theorem, since $g \in L^1(\mathbb{R})$, and clearly $e^{-|t|} \in L^1(\mathbb{R})$ we have

$$\begin{aligned}u(\infty) &= \lim_{t\to\infty} u(t) = -\pi\lim_{t\to\infty}(D_{2\pi}f * g)(t) = -\pi\lim_{t\to\infty}\int_{\mathbb{R}} D_{2\pi}f(t-s)g(s)ds\\ &= -\pi\int_{\mathbb{R}}\underbrace{\lim_{t\to\infty}\frac{1}{2\pi}e^{-|t-s|}}_{0}g(s)ds = 0\end{aligned}$$

□

**Exercise 12.**

(a) *Show that the functions* $\varphi_n(x) = \frac{sin\ (x/2)}{\pi x} e^{inx}$, $n \in \mathbb{Z}$, *are pairwise orthogonal in* $L^2(\mathbb{R})$.

*Proof.* First of all consider the Fourier transform of the function $h(t) = (1-|t|)\chi_{(-1,1)}(t)$:

$$\hat{h}(\xi) = \int_{\mathbb{R}} h(t)e^{-2\pi it\xi}dt = \int_{-1}^{1}(1-|t|)e^{-2\pi it\xi}dt = \int_{-1}^{1} e^{-2\pi it\xi}dt - \int_{-1}^{1}|t|e^{-2\pi it\xi}dt,$$

where the computation of the first integral is:

$$\int_{-1}^{1} e^{-2\pi it\xi}dt = -\frac{1}{2\pi i\xi}(e^{-2\pi i\xi} - e^{2\pi i\xi}) = \frac{\sin\ (2\pi\xi)}{\pi\xi}$$

Regarding the second integral, integrating by parts we have

$$\begin{aligned}\int_{-1}^{1}|t|e^{-2\pi it\xi}dt &= 2\int_0^1 t\ \cos\ (2\pi\xi t)dt = 2\big(t\frac{1}{2\pi\xi}\ \sin\ (2\pi\xi t)\Big|_0^1 - \frac{1}{2\pi\xi}\int_0^1\ \sin\ (2\pi\xi t)dt\big)\\ &= 2\left(\frac{\sin\ (2\pi\xi)}{2\pi\xi} + \frac{\cos\ (2\pi\xi)}{(2\pi\xi t)^2}\Big|_0^1\right) = 2\left(\frac{\sin\ (2\pi\xi)}{2\pi\xi} + \frac{\cos\ (2\pi\xi)}{(2\pi\xi)^2} - \frac{1}{(2\pi\xi)^2}\right)\\ &= \frac{\sin\ (2\pi\xi)}{\pi\xi} - \frac{\sin\ ^2(\pi\xi)}{(2\pi\xi)^2}.\end{aligned}$$

Consequently,

$$\hat{h}(\xi) = \frac{\sin\ ^2(\pi\xi)}{(2\pi\xi)^2}.$$

Now, in order to see that the functions $\varphi_n(x)$ are pairwise orthogonal in $L^2(\mathbb{R})$ we need to check that $\langle\varphi_n, \varphi_m\rangle = 0,\ \forall n \neq m$. Considering the corresponding inner product we have:

$$\langle\varphi_n, \varphi_m\rangle = \int_{\mathbb{R}} \overline{\varphi_n(x)}\varphi_m(x)dx = \int_{\mathbb{R}}\left(\frac{\sin\ (x/2)}{\pi x}\right)^2 e^{-ix(n-m)}dx.$$

By considering the variable $x/2 = \pi\xi$, this is $dx = 2\pi\xi$, we have

$$\langle\varphi_n, \varphi_m\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin\ (\pi\xi)}{\pi\xi}\right)^2 e^{-2\pi i\xi(n-m)}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{h}(\xi)e^{-2\pi i\xi(n-m)}d\xi = 0, \quad \forall n \neq m,$$

which is given from the fact that $\{e_n\}_{n\in\mathbb{Z}} = \{e^{2\pi i\xi n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis over $L^2(\mathbb{R})$. □

(b) *Determine the constants* $c_n$ *such that*

$$I(c_n) = \int_{\mathbb{R}}\left|\frac{1}{1+x^2} - \sum_{n=-N}^{N} c_n\varphi_n(x)\right|^2 dx$$

*is minimal.*

*Proof.* First, observe that considering the results obtained on **Exercise 6.a** we have that $f(t) = e^{-2\pi|t|}$ has Fourier transform $\hat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$, thus

$$I(c_n) = \int_{\mathbb{R}}\left|\pi\hat{f}(x) - \sum_{n=-N}^{N} c_n\varphi_n(x)\right|^2 dx.$$

Hence, minimizing $I(c_n)$ is equivalent to find the $c_n$ such that $\varphi = \sum_{n=-N}^{N} c_n\varphi_n(x)$ defines the projection of the map $\pi\hat{f}(x)$ over the space generated by the span of $\{\varphi_n\}_{|n|\leq N}$. Therefore, since we want to find the corresponding projection, the corresponding coefficients are given by the following inner product:

$$c_n = \langle \pi\hat{f}(x), \varphi_n(x)\rangle = \pi\int_{\mathbb{R}} \overline{\hat{f}(x)}\varphi_n(x)dx = \int_{\mathbb{R}} \frac{1}{1+x^2}\frac{\sin\,(x/2)}{\pi x}e^{inx}dx.$$

By considering again the variable change given by $x/2 = \pi\xi$ we have

$$\begin{aligned} c_n &= \int_{\mathbb{R}} \frac{1}{1+(2\pi\xi)^2}\frac{\sin\,(\pi\xi)}{\pi\xi}e^{2\pi in\xi}d\xi = \int_{\mathbb{R}} \pi\hat{f}(2\pi\xi)\hat{g}(\xi)e^{2\pi in\xi}d\xi \\ &= \pi\int_{\mathbb{R}} (D_{2\pi}f)^\wedge(\xi)\hat{g}(\xi)e^{2\pi in\xi} = \pi\int_{\mathbb{R}} (D_{2\pi}f * g)^\wedge(\xi)e^{2\pi in\xi}d\xi \\ &= \pi(D_{2\pi}f * g)(n), \end{aligned}$$

where the last equality is given in virtue of the inversion formula. Notice that here, we have defined $D_{2\pi}f(t) = \frac{1}{2\pi}f(t/(2\pi)) = \frac{1}{2\pi}e^{-|t|}$, and $g(t) = \chi_{[-1/2,1/2]}(t)$, which clearly has the cardinal sinus as its Fourier transform, as studied in theory class. □

(c) *Is the system $(\varphi_n)_{n\in\mathbb{Z}}$ complete?*

We say that a set of orthogonal functions is complete if the closure of the span $\overline{\langle\varphi_n(x)\rangle_{n\in\mathbb{Z}}}$ is the whole $L^2(\mathbb{R})$. That is, any function squared integrable $w \in L^2(\mathbb{R})$ is completely determined by its Fourier series in the basis given by $\{\varphi_n\}$.

Observe that the functions $\varphi_n(x)$ are modulations of the cardinal sinus:

$$\begin{aligned} \varphi_n(x) = \varphi_n(2\pi\xi) &= \frac{1}{\pi}\frac{\sin\,(\pi\xi)}{\pi\xi}e^{2\pi in\xi} = \frac{1}{\pi}\,\text{sinc}\,(\xi)e^{2\pi in\xi} \\ &= \frac{1}{\pi}M_n(\,\text{sinc}\,(\xi)) = \frac{1}{\pi}M_n(\hat{g}(\xi)) = \frac{1}{\pi}M_n(\hat{g}(x/(2\pi))), \end{aligned}$$

where we have used the notation of the previous section. Notice that if the system was complete, then we would be able to express any function squared integrable $w \in L^2(\mathbb{R})$ as a combination $(\varphi_n)_{n\in\mathbb{Z}}$ :

$$w(x) = \sum_{n\in\mathbb{Z}} c_n\varphi_n(x),$$

with $c_n = \langle w, \varphi_n\rangle$. Notice that it is not possible to express all the possible squared integrable functions as combinations of modulations of one fixed function. In order to illustrate this observe that given the linearity of the Fourier transform we have

$$\begin{aligned} \hat{w}(\xi) &= \sum_{n\in\mathbb{Z}} c_n\hat{\varphi}_n(\xi) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} c_n(M_n(\hat{g}))^\wedge\big(\frac{\xi}{2\pi}\big) \\ &= \frac{1}{\pi}\sum_{n\in\mathbb{Z}} c_n\tau_n\hat{g}\big(\frac{\xi}{2\pi}\big) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} c_n\hat{g}\big(\frac{\xi}{2\pi} - n\big) \\ &= \frac{1}{\pi}\sum_{n\in\mathbb{Z}} c_n\chi_{[-\frac{1}{2},\frac{1}{2}]}\big(\frac{\xi}{2\pi} - n\big) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} c_n\chi_{[n-\pi,n+\pi]}(\xi) \end{aligned}$$

This would mean that the Fourier transform of any $w \in L^2(\mathbb{R})$ is determined by the system defined by the characteristic function defined over the interval $[n-\pi, n+\pi]$, of length $2\pi$, which is clearly not true.