Advanced Mathematics for Scientific Challenges
Optimization

Universitat de Barcelona

Professor: Àngel Jorba

Flàvia Ferrús Marimón

December 19, 2022

Contents

Chapter 1

Introduction

We may focus on solving minimization problems, given as

$$\begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \leq 0,\ i = 1, \ldots, m,\ x \in S \subset \mathbb{R}^n \end{cases}$$

In other words, we aim to find $x \in S$ that minimizes the *objective function*, $f(x)$, subject to the *constraints*, $g_i(x)$, $x \in S$.
Note that if we have $h(x) \iff h(x) \leq 0$, and $-h(x) \leq 0$.

If a point x satisfies the constraints it is called a *solution*. We refer to a solution minimizing the objective function $f(x)$ as *optimal solution* (we don't need unity).

Definition 1.0.1. *x^0 is a local optimum $\iff$ there exists a neighbourhood $V(x^0)$ such that x^0 is a global optimum of*

$$\begin{cases} \textit{Minimize} & f(x) \\ \textit{subject to} & x \in S \cap V(x^0) \end{cases}$$

Chapter 2

Elements of complex analysis

2.1 General ideas

We may consider $x \in \mathbb{R}^n$ if not specified, and $||x||$ to be the Euclidean norm.
A set $S \subset \mathbb{R}^n$ is said to be *closed* if it contains all its limiting point, i.e. points whose any neighborhood contains infinitely many points belonging to S.
A point $x \in S$ is called *boundary point* of the set S if its any neighbourhood contains points both belonging to the set S and not belonging to it. The set $Cl(S)$ consisting of all boundary points of the set S is called the *boundary* of the set S.
A *compact set* is a non empty, closed and bounded set in $\mathbb{R}^n$.

Let's remember some basic results from topology theory.
We know that every sequence on a compact space has a subsequence that converges.

Theorem 2.1.1 (Weierstrass)**.** *Given f a real continuous function on a compact set $K \subset \mathbb{R}^n$, then the optimization problem*

$$\begin{cases} \textit{Minimize} & f(x), \\ & x \in K \end{cases}$$

has an optimal solution $x^* \in K$.

Proof. We may prove that there exists $x^* \in K$ such that $f(x) \geq f(x^*)$, $\forall x \in K$. Suppose K is a non empty space, otherwise, as established, $x^* = -\infty \notin K$.
Then, given it is a compact set, by definition it is closed and bounded in $\mathbb{R}^n$. Therefore, since it is bounded, there exists a lower bound $\alpha = Inf\{f(x) : x \in K\}$. Le'ts consider the sequence $K_j = \{x \in K : \alpha \leq f(x) \leq \alpha + \varepsilon^j\}$, $\forall j \in \mathbb{N}$ and for a fixed $\varepsilon \in (0, 1)$. Note that, by definition, $K_j \neq \emptyset$ (i.e. it is not empty), for all j.
Let's consider now a sequence given by $y^j \in K_j$, $\{y^j\}$. Since K is a compact space, we know that every sequence has a subsequence that converges, that we may denote as $\{x_j\}$. Therefore, we have a subsequence such that

$$\alpha \leq f(x_j) \leq \alpha + \varepsilon^j \tag{2.1}$$

that converges to x.
Since K is closed an x is a limit point, then $x \in K$, verifying $x_j \to x$ when $j \to \infty$. Given

the continuity of f, we also have $f(x_j) \to f(x)$, when $j \to \infty$. Then, taking limits into the expression (2.1) we get $\alpha \leq lim_{j\to\infty} f(x_j) = f(x) \leq \alpha \iff \alpha = f(x) \in K$.
Therefore, we have $\alpha = f(x) = x^* \in K$ as the minimum value that the function f reaches in the compact set K, as we wanted to prove.

$\square$

Corollary 2.1.2. *If f is a real continuous function defined on all $\mathbb{R}^n$ such that $f(x) \to \infty$ when $||x|| \to \infty$, then the problem*

$$\begin{cases} \textit{Minimize} & f(x), \\ & x \in \mathbb{R}^n \end{cases}$$

has an optimal solution.

Proof. Let's consider $y \in \mathbb{R}^n$. Note, that by hypothesis, there exists $M > 0$ such that $||x|| \geq M \Rightarrow f(x) \geq f(y)$. Then, considering the closed ball $B = \{x \in \mathbb{R}^n : ||x|| \leq M\}$ we get a compact set in which we can apply the previous theorem to see that there exists $x^* \in \mathbb{R}^n$ solving the minimizing problem, as we wanted to prove. $\square$

If $K \subset \mathbb{R}^n$ is an empty space we are not interested on the optimization problem. Therefore, when we find the optimization problem

$$\begin{cases} \text{Minimize} & f(x), \\ & x \in S = \emptyset \end{cases}$$

then the solution is $-\infty$. If we find a minimization problem without minimum boundary, the solution is also $-\infty$.

2.2 Convexity

Convexity gives us unity.

Definition 2.2.1. *A set $S \subset \mathbb{R}^n$ is convex if and only if $\forall x \in S$, $\forall y \in S$, $\forall \lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in S$.*

Definition 2.2.2. *Given p points in $\mathbb{R}^n$, $x^1, \ldots, x^p$ we say that $x \in \mathbb{R}^n$ is a convex combination of these p points if there exist coefficients verifying*

1. $\mu_1, \ldots, \mu_p \geq 0$,
2. $\mu_1 + \cdots + \mu_p = 1$,
3. $x = \sum_{i=1}^{p} \mu_i x^i$

Proposition 2.2.3. *$S \subset \mathbb{R}^n$ is convex if and only if every convex combination of points of S belongs to S.*

Definition 2.2.4. *Given $S \subset \mathbb{R}^n$ we denote by conv(S) the convex hull which is the set of points which are convex combinations of points of S.*

Observation 2.2.5. S convex $\iff$ S=conv(S)

Proposition 2.2.6. *The intersection of a finite number of convex sets is convex.*

Definition 2.2.7. *If $C \subset \mathbb{R}^n$ is a convex set, the relative interior, int(C) is the interior of C relative to the smallest affine variety that contains C.*

Theorem 2.2.8. *Every convex non-empty subset of $\mathbb{R}^n$ has a non-empty relative interior.*

Definition 2.2.9. *$f : \mathbb{R}^n \to \mathbb{R}$, $dom(f) \subset \mathbb{R}^n$, is a convex function if*

1. *dom(f) is convex*

2. *$\forall x \in dom(f)$, $\forall y \in dom(f)$, $\forall \lambda \in [0,1]$, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. If the inequality is strict, i.e. $<$, then f is strictly convex.*

Definition 2.2.10. *The epigraph of a function f is the set*

$$epi(f) = \{(\mu, x) : f(x) \leq \mu, x \in \mathbb{R}^n, \mu \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

Proposition 2.2.11. *f is convex $\iff$ epi(f) is convex.*

Proof. $\boxed{\Leftarrow}$ Assume $epi(f)$ is a convex set, and suppose that f is not convex. Then, there would exist $x, y \in dom(f)$, $\lambda \in [0,1]$ such that $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$. Then, picking $(x, f(x)), (y, f(y)) \in epi(f)$, we would have $(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \notin epi(f)$ contradicting the fact that $epi(f)$ is convex.

$\boxed{\Rightarrow}$ The other implication is direct with a similar argument.

□

Theorem 2.2.12. *A linear combination with positive coefficients of convex functions is a convex function.*

Proof. Let $f_1, \ldots, f_n$ be convex functions. Suppose we have a general linear combination $g = \lambda_1 f_1 + \cdots + \lambda_n f_n$, with $\lambda_1, \ldots, \lambda_n > 0$. Let's assume $dom\ f_i = dom\ f$, $\forall i = 1, \ldots, n$. Then, considering general $x, y \in dom\ g, \alpha \in (0,1)$ we have

$$\begin{aligned} g(\alpha x + (1-\alpha)y) &= \lambda_1 f_1(\alpha x + (1-\alpha)y) + \cdots + \lambda_n f_n(\alpha x + (1-\alpha)y)) \\ &\leq \lambda_1(\alpha f_1(x) + (1-\alpha)f_1(y)) + \cdots + \lambda_n(\alpha f_n(x) + (1-\alpha)f_n(y)) \\ &= \alpha(\lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)) + (1-\alpha)(\lambda_1 f_1(y) + \cdots + \lambda_n f_n(y)) \\ &= \alpha g(x) + (1-\alpha)g(y) \end{aligned}$$

□

Theorem 2.2.13. *Assume $f \in \mathcal{C}^1$, dom(f) convex. Then f is convex $\iff$*

$$f(y) \geq f(x) + [\nabla f(x)]^T(y-x), \ \forall x, y \in dom(f)$$

Proof. The geometric idea is that the tangent plane remains on one side of the function's image. We may, firstly prove it in one dimension, so then we can use the idea of the tangent line for greater dimensions.

$\boxed{n = 1}$ We may prove f convex $\iff$ $f(y) \geq f(x) + f'(x)(y-x), \ \forall x, y \in dom(f)$.

$\boxed{\Rightarrow}$ Assume f is convex, then $\forall x, y \in dom(f)$ we have

$$f(x+\lambda(y-x)) \leq (1-\lambda)f(x) + \lambda f(y),\ \ \lambda \in (0,1)$$

Since $\lambda > 0$, we can rearrange the inequality as follows:

$$f(y) \geq f(x) + \frac{f(x+\lambda(y-x)) - f(x)}{\lambda}.$$

Note that for $\lambda \to 0$ it accomplishes that

$$f(y) \geq f(x) + f'(x)(y-x).$$

This fact comes from the definition of a formal derivative, i.e. $\frac{f(x+h)-f(h)}{h} \overset{h\to 0}{\longrightarrow} f'(x)$, and therefore $\frac{f(x+h(y-x))-f(x)}{h}\frac{y-x}{y-x} \overset{h\to 0}{\longrightarrow} f'(x)(y-x)$.

$\boxed{\Leftarrow}$ We assume that $f(y) \geq f(x) + f'(x)(y-x)$, $\forall x, y \in dom(f)$. We choose $x \neq y$, $\lambda \in [0,1]$. Let's denote $z = \lambda x + (1-\lambda)y \in dom(f)$, since we have assumed that $dom(f)$ is convex. Then we have,

$$\begin{aligned} f(x) &\geq f(z) + f'(z)(x-z), \\ f(y) &\geq f(z) + f'(z)(y-z). \end{aligned}$$

Combining the two inequalities by multiplying the first by λ and the second by $(1-\lambda)$ and adding them we get:

$$\lambda f(x) + (1-\lambda)f(y) \geq f(z) + f'(z)(\underbrace{\lambda x + (1-\lambda)y - z}_{0})$$

$\boxed{n > 1}$ Let's consider now the general case, $f : \mathbb{R}^n \to \mathbb{R}$, consider any two general points $x, y \in dom(f) \subset \mathbb{R}^n$. Then we can define a new function $g(\lambda) = f(\lambda y + (1-\lambda)x)$, with $dom(g) = [0,1]$, which represents the restriction of f to the line across any two points from the domain of f. Since $f \in \mathcal{C}^1$, then $g'(\lambda) = \nabla f(\lambda y + (1-\lambda)x)^T(y-x)$.

$\boxed{\Rightarrow}$ Assume now that f is convex, then it is direct to see that g is convex, since it is a restriction of the function, and therefore may remain above any tangent line. Then, we have

$$g(\lambda_2) \geq g(\lambda_1) + g'(\lambda_1)(\lambda_2 - \lambda_1),$$

by fixing $\lambda_2 = 1, \lambda_1 = 0$ we get:

$$g(1) \geq g(0) + g'(0) \iff f(y) \geq f(x) + \nabla f(x)^T(y-x).$$

$\boxed{\Leftarrow}$ Assume now that $f(y) \geq f(x) + \nabla f(x)^T(y-x)$, $\forall x, y \in dom(f)$. Then we can consider $z = \lambda y + (1-\lambda)x$, $w = \mu y + (1-\mu)x$. Then they verify

$$f(z) \geq f(w) + \nabla f(w)^T(z-w).$$

Note that $z - w = (\lambda - \mu)(y-x)$, then

$$f(z) \geq f(w) + \underbrace{\nabla f(w)^T(y-x)}_{g'(\mu)}(\lambda - \mu).$$

Thus, equivalently,

$$g(\lambda) \geq g(\mu) + g'(\mu)(\lambda - \mu) \stackrel{n=1}{\Longrightarrow} g \text{ is convex.}$$

Finally, since g is f restricted to a segment between any two points of the domain of f, then is accomplishes for every segment. Therefore, the graphics of g remains bellow the segment line between any two points from $dom(f)$, and then we can affirm that f is convex.

$\square$

Remark. Consider now $f : (a, b) \to \mathbb{R}, \mathcal{C}^2$ convex function. Then we expect that $f''(x) \geq 0$, since it is convex, and then the tangent line keeps getting greater. Note that we could also argument this fact by taking any point $c \in (a, b)$ and considering x close to c, so we can compute Taylor's expansion as follows

$$f(x) \approx f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2.$$

Note that $f(c) + f'(c)(x - c)$ represents the tangent line of f at point c. By convexity, we know that the tangent line will remain bellow the function, therefore we have $f(x) - f(c) + f'(c)(x - c) \geq 0 \iff \frac{f''(x)}{2}(x - c)^2 \geq 0 \iff f''(c) \geq 0$.

Considering now the general dimension case, we have

$$f(x) = f(c) + \nabla f(c)^T(x - c) + \frac{1}{2}(x - c)^T H_f(c)(x - c) + \mathcal{O}(x^2).$$

where $H_f(x)$ is the Hessian of the function f:

$$Q(x) = H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right). \tag{2.2}$$

Using similar arguments we have that f is convex $\iff$ the Hessian is positive semi-definite, i.e. $y^T H_f(x) y \geq 0,\ \forall x, y \in dom(f)$

Theorem 2.2.14. *$f \in \mathcal{C}^2, f : C \subset \mathbb{R}^n \to \mathbb{R}$, C convex, then f is convex on C $\iff$ the Hessian matrix*

$$Q(x) = H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right), \tag{2.3}$$

is positive semi-definite for all $x \in C$

Proof. $\boxed{\Rightarrow}$ Assume f is a convex function. By hypothesis, $f \in C^2$, in particular it is C^1, therefore, by Theorem 2.2.13 we have

$$f(y) \geq f(x) + \nabla f(x)^T(y - x),\ \forall x, y \in dom\ f.$$

Let's consider Taylor's expansion of the function f around a general point $c \in C$ of degree 2:

$$f(x) = f(c) + \nabla f(c)^T(x - c) + \frac{1}{2}(x - c)^T H_f(c)(x - c) + \mathcal{O}(x^2).$$

By convexity, we have

$$0 \leq f(x) - f(c) + \nabla f(c)^T(x-c) \approx \frac{1}{2}(x-c)^T H_f(c)(x-c),$$

for x close enough to c. Since this is true for any $c \in C$, $x \in V(c)$, and we are on a convex open set we have $x - c \in C$. Then, we have that $(x-c)^T H_f(c)(x-c) \geq 0$, thus by definition of positive semi-definite we have that the Hessian is definite semi-positive for any point of C (since they all are convex combinations of points from C).

$\boxed{\Leftarrow}$ Similarly, assume that the Hessian matrix is positive semi-definite, $y^T H_f(x)y \geq 0,\ \forall x, y \in C$. In particular, it verifies for $x - c \in C$, then computing the Taylor's expansion as aforementioned,

$$0 \leq \frac{1}{2}(x-c)^T H_f(c)(x-c) \approx f(x) - f(c) + \nabla f(c)^T(x-c)$$

which in virtue of the Theorem 2.2.13, implies that the function f is convex, as we wanted to show.

□

Note that this proof could also been demonstrated using the fact that the convexity of f on S is equivalent to the convexity of f restricted to each segment of S, and using then arguments similar to the proof of theorem 2.2.13 and previous observations.

Corollary 2.2.15. *A positive semi-definite quadratic form is a convex function.*

Remember that a quadratic form is given by $Q(x) = x^T Ax + b^T x + c$. Note that clearly the Hessian of Q is given by the matrix A.

2.3 Optimization

Definition 2.3.1. *An optimization problem (a mathematical programming problem) is convex if it consists of minimizing a convex function on a convex domain, i.e.*

$$\begin{cases} Min & f(x) \\ subject\ to & g_i(x) \leq 0,\ i = 1, \ldots, n,\ x \in S \subset \mathbb{R}^n \end{cases} \ \textit{is convex} \iff \begin{cases} f\ \textit{convex} \\ S\ \textit{convex} \\ g_i\ \textit{convex} \end{cases}$$

Theorem 2.3.2. *For a convex optimization problem, every local minimum is a global minimum.*

Proof. Let's consider the optimization problem

$$\begin{cases} Min & f(x) \\ \text{subject to} & x \in S \end{cases}, \quad f \text{ convex }, S \text{ convex }.$$

Let's assume x is a local minimum. Then for any $y \in S$ we want to prove that $f(x) \leq f(y)$. We may prove it by reduction to the absurd. Therefore, let's assume it is false, i.e. there exists a $y \in S$ such that $f(y) < f(x)$. Let's choose $\lambda \in (0, 1)$ and consider the segment $x + \lambda(y - x)$, then by convexity we have

$$f(x + \lambda(y-x)) \leq (1-\lambda)f(x) + \lambda f(y) < f(x),$$

which contradicts the fact that x is a local minimum, and therefore for small λ we are inside the neighbourhood of x. Then, we have that x is a global minimum. □

Definition 2.3.3. *We say that* $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ *is an extended convex function if and only if* $\forall x, y$ *such that* $f(x) \neq +\infty$, $\forall \lambda \in [0, 1]$, *then* $f(x + \lambda(y - x)) \leq (1 - \lambda)f(x) + \lambda f(y)$.

Note that we don't include the case in which $f(x) = -\infty$, because if we have it in our set, then it already is the minimum we are looking for, and therefore the problem is solved. Then we consider this definition of extension valid for

$$\overline{f}(x) = \begin{cases} f(x), & \text{if} \quad x \in dom(f) \\ +\infty, & \text{if} \quad x \notin dom(f) \end{cases}$$

2.4 Subgradient

Definition 2.4.1. *We call subgradient of* f *at a point* x_0 *to every vector* $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, *that satisfies*

$$f(x) \geq f(x_0) + \gamma^T(x - x_0), \ \forall x \in dom(f)$$

Note that if $f \in \mathcal{C}^1$, then in virtue of the Theorem 2.2.13 the gradient of the function is a subgradient for each point of the domain.

Definition 2.4.2. *We call subdifferential of* f *at the point* x_0 *to the set of all subgradients of* f *at* x_0. *We denote this set* $\partial f(x_0)$.

Example 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the absolute value $f(x) = |x|$. The let's consider the subgradient set at the point $x_0 = 0$. Note that each line $y = \mu x$ may leave the function's graphic above it for values of $\mu \in [-1, 1]$. Then, $\partial f(0) = [-1, 1]$. □

Remark. Note that when $f \in \mathcal{C}^1$ then $\nabla f = \partial f$ (then it is not a set). Therefore, we aim to define convexity for functions $f \notin \mathcal{C}^1$, throughout ∂f.

Theorem 2.4.3. *Let* f *be a convex function, then* x_0 *minimizes* $f \iff 0 \in \partial f(x_0)$.

Note that we don't ask for $f \in \mathcal{C}^1$. In particular, if $f \in \mathcal{C}^1$ then x_0 minimizes f if the gradient vanishes at that point, i.e. $\nabla f(x_0) = 0$.

Proof. $0 \in \partial f(x_0) \iff f(x) \geq f(x_0) + 0(x - x_0), \ \forall x \in dom(f) \iff f(x) \geq f(x_0), \ \forall x \in dom(f)$, and therefore x_0 is indeed a minimum. □

2.5 Exercises on elements of convex analysis

Exercise 1:

Theorem 2.5.1 (Weierstrass)**.** *Given f a real continuous function on a compact set $K \subset \mathbb{R}^n$, then the optimization problem*

$$\begin{cases} \textit{Minimize} & f(x), \\ & x \in K \end{cases}$$

has an optimal solution $x^* \in K$.

Proof. We may prove that there exists $x^* \in K$ such that $f(x) \geq f(x^*)$, $\forall x \in K$. Suppose K is a non empty space, otherwise, as established, $x^* = -\infty$ and the problem is not considered.
Then, given it is a compact set, by definition it is closed and bounded in $\mathbb{R}^n$. Therefore, since it is bounded, there exists a lower bound $\alpha = Inf\{f(x) : x \in K\}$, i.e. for all $x \in K, \alpha \leq f(x)$.
Let's consider the sequence $K_j = \{x \in K : \alpha \leq f(x) \leq \alpha + \varepsilon^j\}$, $\forall j \in \mathbb{N}$ and for a fixed $\varepsilon \in (0, 1)$.
Note that, by definition, $K_j \neq \emptyset$ for all j.
Let's consider now a sequence given by $y^j \in K_j$, $\{y^j\}$. Since K is a compact space, we know that every sequence has a subsequence that converges, that we may denote as $\{x_j\}$. Therefore, we have a subsequence such that

$$\alpha \leq f(x_j) \leq \alpha + \varepsilon^j \tag{2.4}$$

that converges to $\hat{x}$.
Since K is closed and $\hat{x}$ is a limit point, then $\hat{x} \in K$, verifying $x_j \to \hat{x}$ when $j \to \infty$. Given the continuity of f, we also have $f(x_j) \to f(\hat{x})$, when $j \to \infty$. Then, taking limits into the expression (2.4) we get $\alpha \leq lim_{j\to\infty} f(x_j) = f(\hat{x}) \leq \alpha \iff \alpha = f(\hat{x}) \leq f(x)$ for all $x \in K$, and therefore $\hat{x} \in K$ is an optimal solution of the initial minimization problem.

□

Exercise 2:

Corollary 2.5.2. *If f is a real continuous function defined on all $\mathbb{R}^n$ such that $f(x) \to \infty$ when $||x|| \to \infty$, then the problem*

$$\begin{cases} \textit{Minimize} & f(x), \\ & x \in \mathbb{R}^n \end{cases}$$

has an optimal solution.

Proof. Let's consider $y \in \mathbb{R}^n$. Note, that by hypothesis, there exists $M > 0$ such that $||x|| \geq M \Rightarrow f(x) \geq f(y)$. Then, considering the closed ball $B = \{x \in \mathbb{R}^n : ||x|| \leq M\}$ we get a compact set in which we can apply the previous theorem to see that there exists $x^* \in \mathbb{R}^n$ solving the minimizing problem, as we wanted to prove. □

Exercise 3: *Let S be a convex subset of $\mathbb{R}^n$, and let λ_1, λ_2 be positive scalars.*

(a) *Show that $(\lambda_1 + \lambda_2)S = \lambda_1 S + \lambda_2 S$.*

Proof. We may prove it by double inclusion.

$\boxed{\subseteq}$ Given $x \in (\lambda_1 + \lambda_2)S$, by definition we have $x = (\lambda_1 + \lambda_2)y = \lambda_1 y + \lambda_2 y \in \lambda_1 S + \lambda_2 S$, with $y \in S$. Therefore, $(\lambda_1 + \lambda_2)S \subseteq \lambda_1 S + \lambda_2 S$.

$\boxed{\supseteq}$ Given now $x \in \lambda_1 S + \lambda_2 S$, we have $x = \lambda_1 y + \lambda_2 z$, for $y, z \in S$. Since $\lambda_1, \lambda_2 > 0$, we can consider the equivalent equality

$$\frac{1}{\lambda_1 + \lambda_2} x = \frac{\lambda_1}{\lambda_1 + \lambda_2} y + \frac{\lambda_2}{\lambda_1 + \lambda_2} z.$$

Note that $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$, and $\frac{\lambda_1}{\lambda_1+\lambda_2}, \frac{\lambda_2}{\lambda_1+\lambda_2} \in (0,1)$. Therefore, by convexity we get

$$\frac{1}{\lambda_1 + \lambda_2} x \in S \iff x \in (\lambda_1 + \lambda_2) S,$$

and, thus $\lambda_1 S + \lambda_2 S \subseteq (\lambda_1 + \lambda_2)S$, as we wanted to prove[1].

□

(b) *Give an example that shows that this does not need to be true when S is not convex.*

Consider the noncovex subset of $\mathbb{R}^2$ given by the circumference of radius 1 and centered at the origin, i.e. $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Note that if we consider $(x,y) \in (\lambda_1 + \lambda_2)S$, we get $(x,y) = (\lambda_1 + \lambda_2)(u,v)$, with $(u,v) \in S$. Therefore, it is direct to see that (x,y) verifies $x^2 + y^2 = (\lambda_1 + \lambda_2)^2(u^2 + v^2) = (\lambda_1 + \lambda_2)^2$. Thus, $(x,y) \in (\lambda_1 + \lambda_2)S$ is contained in the circumference of radius $(\lambda_1 + \lambda_2)$. In fact, we have $(\lambda_1 + \lambda_2)S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = (\lambda_1 + \lambda_2)^2\}$.

Following the equivalent calculations for the case $(x,y) \in \lambda_1 S + \lambda_2 S \iff (x,y) = \lambda_1(u,v) + \lambda_2(w,z)$, with $(u,v),(w,z) \in S$, we get that (x,y) verifies the condition

$$x^2 + y^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2(uv + wz) \notin (\lambda_1 + \lambda_2)S.$$

Exercise 4: *Let S be a nonempty closed convex set in $\mathbb{R}^n$, not containing the origin. Show that there exists a hyperplane that strictly separates S and the origin.*

Proof. We want to prove that there exists a hyperplane that strictly separates S and the origin, $\mathbf{0}$. More specifically, we want to show that there exists $a \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $b \in \mathbb{R}$ such that $a^T x > b$ for all $x \in S$, and $a^T x < b$ for $x = \mathbf{0}$. We may denote this hyperplane by $H(a,b)$.

Note that there exists a point $y \in S$ that is the closest to $\mathbf{0}$, i. e. $d(y,\mathbf{0}) \le d(z,\mathbf{0})\ \forall z \in S$. This comes from the fact that S is closed. More explicitly, consider a general point $z \in S$ and let $r = d(z,\mathbf{0})$, then since $z \neq \mathbf{0}$ we have that $r > 0$. Thus, we can define the closed ball of radius r centered on the origin as $\overline{B} = cl(B(\mathbf{0}, r))$. Note that, since it is closed, $z \in \overline{B}$. In addition $\overline{B}$ is compact, and therefore $\overline{B} \cap S$ is nonempty (since $z \in \overline{B}$) and compact (because S is closed). Note now that the function $d(\cdot,\mathbf{0})$ is continuous, and therefore, from *Exercise 1 (Weierstrass theorem)* it attains its minimum on the compact set $\overline{B} \cap S$. Let's denote this minimum point $y \in \overline{B} \cap S$. Then, suppose there is a point closer to $\mathbf{0}$, $y' \in S$, then we would have $d(y',\mathbf{0}) < d(y,\mathbf{0}) \le r$, so we have $y' \in \overline{B}$ and therefore $y' \in \overline{B} \cap S$ with $d(y',\mathbf{0}) < d(y,\mathbf{0})$, contradicting the fact that y minimizes $d(x,\mathbf{0})$ on $\overline{B} \cap S$.

Then, let's denote this minimum by $y \in S$, that we already have proven that exists. Let's define $b = y^T y = ||y||^2$, and then consider the hyperplane $H(y,b)$. Note that clearly, $y^T \cdot \mathbf{0} = 0 < b$,

[1] Note that, by definition, it is direct to see that $\frac{1}{\lambda_1+\lambda_2} x \in S \iff \frac{1}{\lambda_1+\lambda_2} x = y \in S \iff x = (\lambda_1 + \lambda_2) y \in (\lambda_1 + \lambda_2) S$

since we assume $y \neq \mathbf{0} \notin S$.
Consider now any general point $x \in S$, we have to see that $y^t x \geq b = y^T y$. Let's assume, by way of contradiction, that $y^T x < y^T y$. From convexity, we have that for all $\lambda \in (0,1)$, $z_\lambda = \lambda x + (1-\lambda)y \in S$. We may see that for small values of λ, we have $d(z_\lambda, \mathbf{0}) < d(y, \mathbf{0})$, contradicting the fact that y minimizes $d(x, \mathbf{0})$ on S. Let's consider then the corresponding distance:

$$\begin{aligned} d(z_\lambda, \mathbf{0}) &= z_\lambda^T z = \lambda^2 x^T x + (1-\lambda)^2 y^T y + 2\lambda(1-\lambda)x^T y \\ &= y^T y + \lambda[\lambda(x^T x + y^T y - 2x^T y) + 2(x^T y - y^T y)] \\ &= d(y, \mathbf{0}) + \lambda\underbrace{[\lambda||x-y||^2 + 2(x^T y - y^T y)]}_{g(\lambda)} \end{aligned}$$

Note that by assumption $y^T x < y^T y \Rightarrow lim_{\lambda \to 0} g(\lambda) < 0$. Then for a λ small enough, i.e. inside a neighborhood of y it exists a point over the segment $\lambda x + (1-\lambda)y$ with $d(z_\lambda, \mathbf{0}) < d(y, \mathbf{0})$, that contradicts the fact that y minimizes this function in S. Thus, we have $y^t x \geq b = y^T y$, and consequently the hyperplane $H(y, y^T y)$ strictly separates S and the origin. ☐

Exercise 5: *Show that a convex function $f : (a,b) \to \mathbb{R}$ is continuous.*

Proof. Assume that f is a convex function, and we want to prove that it is continuous, i.e. that given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$, for all $x, y \in (a,b)$.

Firstly, note that for any $x \in (a,b)$ there exist $u, v \in (a,b)$ such that $u < x < v$. Consider $\frac{x-u}{v-u} \in (0,1)$, so we can express x in terms of u, v as follows:

$$x = \left(\frac{x-u}{v-u}\right) v + \left[1 - \left(\frac{x-u}{v-u}\right)\right] u.$$

Since f is convex, we have

$$f(x) \leq \left(\frac{x-u}{v-u}\right) f(v) + \left[1 - \left(\frac{x-u}{v-u}\right)\right] f(u). \tag{2.5}$$

Rearranging the terms from the inequality we get

$$f(x) - f(u) \leq \frac{x-u}{v-u}[f(v) - f(u)] \iff \frac{f(x) - f(u)}{x-u} \leq \frac{f(v) - f(u)}{v-u}. \tag{2.6}$$

Rewriting in a different way the inequality (2.5), we can also see that it verifies that

$$(f(v) - f(u))\left[1 - \left(\frac{x-u}{v-u}\right)\right] \leq f(v) - f(x) \iff \frac{f(v) - f(u)}{v-u} \leq \frac{f(v) - f(x)}{v-x} \tag{2.7}$$

Therefore, for any $x, y, \in [u, v]$, assuming without loss of generality that $x < y$ and since (a,b) is open, there exist $u_0, v_1 \in (a,b)$ such that $u_0 < u < y < x < v < v_1$. Thus, combining (2.6) applied to $x \leq y < v_1$ and (2.7) applied to $x < v < v_1$ we get:

$$\frac{f(y) - f(x)}{y-x} \leq \frac{f(v_1) - f(x)}{v_1 - x} \leq \frac{f(v_1) - f(v)}{v_1 - v}$$

Similarly, applying now (2.6) to the set $u_0 < u < y$ and then (2.7) applied to $u_0 < x < y$ we have:

$$\frac{f(u) - f(u_0)}{u - u_0} \leq \frac{f(y) - f(u)}{y - u} \leq \frac{f(y) - f(x)}{y - x} \iff$$
$$\frac{f(x) - f(y)}{y - x} \leq \frac{f(u) - f(y)}{y - u} \leq \frac{f(u_0) - f(u)}{u - u_0}$$

Note that this is equivalent to affirm that:

$$\frac{|f(y) - f(x)|}{y - x} \leq C = max\left\{\frac{|f(u_0) - f(u)|}{u - u_0}, \frac{|f(v_1) - f(v)|}{v_1 - v}\right\}.$$

Consequently, given $\epsilon > 0$, $x \in [u, v]$, let $\delta = min\{\frac{\epsilon}{C}, \frac{v-u}{2}\} > 0$, then for any $y \in (x - \delta, x + \delta) \subset [u, v]$, then $|f(y) - f(x)| \leq C|y - x| \leq C\frac{\epsilon}{C} = \epsilon$. Thus, we have f continuous, by definition in the interval (a, b). $\square$

Exercise extra:

Theorem 2.5.3. *A linear combination with positive coefficients of convex functions is a convex function.*

Proof. Let $f_1, \ldots, f_n$ be convex functions. Suppose we have a general linear combination $g = \lambda_1 f_1 + \cdots + \lambda_n f_n$, with $\lambda_1, \ldots, \lambda_n > 0$. Let's assume $dom\ f_i = dom\ g$, $\forall i = 1, \ldots, n$. Then, considering general $x, y \in dom\ g, \alpha \in (0, 1)$ we have

$$\begin{aligned}
g(\alpha x + (1 - \alpha)y) &= \lambda_1 f_1(\alpha x + (1 - \alpha)y) + \cdots + \lambda_n f_n(\alpha x + (1 - \alpha)y)) \\
&\leq \lambda_1(\alpha f_1(x) + (1 - \alpha)f_1(y)) + \cdots + \lambda_n(\alpha f_n(x) + (1 - \alpha)f_n(y)) \\
&= \alpha(\lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)) + (1 - \alpha)(\lambda_1 f_1(y) + \cdots + \lambda_n f_n(y)) \\
&= \alpha g(x) + (1 - \alpha)g(y)
\end{aligned}$$

$\square$

Exercise 6: *Consider a function $f : (a, b) \to \mathbb{R}$ of class C^2. Show that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.*

Proof. $\boxed{\Rightarrow}$ Consider $c \in (a, b)$ and x close to c, i.e. let $x \in V(c)$, where $V(c)$ denotes a neighbourhood of c. Then we can compute the Taylor expansion of order 2 around c as

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \mathcal{O}(|x - c|^3)$$

Note that $f(c) + f'(c)(x - c)$ represents the tangent line of f at point c. By convexity, we know that the tangent line will remain bellow the function, therefore we have $f(x) - f(c) + f'(c)(x - c) \geq 0$. Therefore, given that $\mathcal{O}(|x - c|^3)$ is given by the error added by terms of greater order on the expansion, with x close enough to c we can neglect those terms so we have $f(x) - f(c) + f'(c)(x - c) \approx \frac{f''(x)}{2}(x - c)^2 \geq 0 \iff f''(c) \geq 0$.

$\boxed{\Leftarrow}$ Similarly, let's consider any point of the interval, $c \in (a, b)$, then we can compute the Taylor expansion around this point as previously, so the fact that $f''(x) \geq 0$ for all $x \in (a, b)$ implies that

$$0 \leq \frac{f''(c)}{2}(x - c)^2 \approx f(x) - f(c) + f'(c)(x - c).$$

Therefore, for any point of the considered interval we have that the tangent line is bellow the function's image, so by definition, the function is convex.

□

Exercise 7: *Let f be a real valued function on an open convex set $S \subset \mathbb{R}^n$, of class C^2. Show that f is convex on S if and only if its Hessian matrix,*

$$Q(x) = H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right), \tag{2.8}$$

is positive semi-definite for all $x \in S$.

Proof. $\boxed{\Rightarrow}$ Assume f is a convex function. By hypothesis, $f \in C^2$, in particular it is C^1, therefore, by a previous theorem we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \ \forall x, y \in dom\ f.$$

Similarly to *Exercise 6*, let's consider Taylor's expansion of the function f around a general point $c \in S$ of degree 2:

$$f(x) = f(c) + \nabla f(c)^T (x - c) + \frac{1}{2}(x - c)^T H_f(c)(x - c) + \mathcal{O}(|x - c|^3).$$

By convexity, we have

$$0 \leq f(x) - f(c) + \nabla f(c)^T (x - c) \approx \frac{1}{2}(x - c)^T H_f(c)(x - c)$$

Since this is true for any $c \in S$, $x \in V(c)$, and we are on a convex open set we have $x - c \in S$. Then, we have that $(x - c)^T H_f(c)(x - c) \geq 0$, thus by definition of positive semi-definite, we have that the Hessian is positive semi-definite for any point of S (since they all are convex combinations of points of C) as we wanted to prove.

$\boxed{\Leftarrow}$ Similarly, assume that the Hessian matrix is positive semi-definite, $y^T H_f(x) y \geq 0, \ \forall x, y \in S$. Applying the Taylor's theorem, for a fixed point $c \in S$ and any $x \in S$ we have:

$$f(x) = f(c) + \nabla f(c)^T (x - c) + \frac{1}{2}(x - c)^T H_f(a)(x - c),$$

for some a in the segment between x and c, which belongs to S because of convexity. Then using that H_f is positive semi-definite we have

$$0 \leq \frac{1}{2}(x - c)^T H_f(a)(x - c) \approx f(x) - f(c) + \nabla f(c)^T (x - c) \iff f(x) \geq f(c) + \nabla f(c)^T (x - c),$$

which in virtue of a theorem previously proven, implies that the function f is convex, as we wanted to show.

□

Note that this proof could also been demonstrated using the fact that the convexity of f on S is equivalent to the convexity of f restricted to each segment of S, and using therefore the previous exercise.

Exercise 8: *Assume $S \subset \mathbb{R}^n$ is a convex set and that $g : S \to \mathbb{R}$. Show that the set $g(x) \leq 0$ is convex if g is convex. What about the opposite implication?*

Proof. Assume g is a convex function, therefore, as stated in a previous proposition, the epigraph set

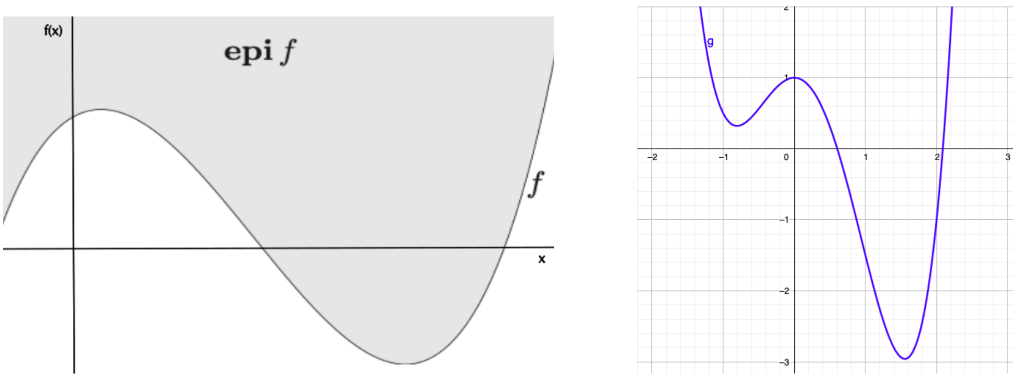
$$epi(g) = \{(\mu, x) : g(x) \leq \mu, x \in \mathbb{R}^n, \mu \in \mathbb{R}\}$$

is convex. Consider now the subset $L = \{z = (z_0, z_1, \ldots, z_n) \in \mathbb{R}^{n+1} : z_0 \leq 0\} \subset \mathbb{R}^{n+1}$, which is a convex set. Consider the finite intersection of the two convex sets, which is, in fact, also convex:

$$epi(g)|_{\mu \leq 0} = epi(g) \cap L = \{(\mu, x) : g(x) \leq \mu, x \in \mathbb{R}^n, \mu \in \mathbb{R}, \mu \leq 0\}$$

Note that, geometrically, $epi(g)|_{\mu \leq 0}$ represents the epigraph of g restricted to the values where the function has negative values, i.e. $g(x) \leq 0$. Then, it is direct to note that if the restricted epigraph is convex, then the function bellow, $\{g(x) \leq 0\}$ is convex.
This can be trivially deduced from the proof of the proposition that states that g is convex if and only if $epi(g)$ is convex. Explicitly, assume $epi(g)|_{\mu \leq 0}$ is a convex set, and suppose that $\{g(x) \leq 0\}$ is not convex. Then, there would exist $x, y \in \{z \in dom(g) : g(z) \leq 0\}$, $\lambda \in [0, 1]$ such that $g(\lambda x + (1 - \lambda)y) > \lambda g(x) + (1 - \lambda)g(y)$. Thus, picking $\hat{\mathbf{x}} = (x, g(x)), \hat{\mathbf{y}} = (y, g(y)) \in epi(g)|_{\mu \leq 0}$, we would have $\lambda \hat{\mathbf{x}} + (1 - \lambda)\hat{\mathbf{y}} = (\lambda x + (1 - \lambda)y, \lambda g(x) + (1 - \lambda)g(y)) \notin epi(g)|_{\mu \leq 0}$ contradicting the fact that $epi(g)|_{\mu \leq 0}$ is convex. $\square$

Let' take a closer look to the opposite implication. Assume that the set $g(x) \leq 0$ is convex and we want to know whether it can be affirmed that g is convex. Note that the hypothesis is a local affirmation, and we could have some regions of the whole epigraph of g on the positive hyperplane of $\mathbb{R}^{n+1}$ where the set is not globally convex. As a contra-example, consider the function $f : \mathbb{R} \to \mathbb{R}$ as shown on Figure 2.1a. More explicitly, we could consider the function $g(x) = x^4 - x^3 - \frac{5}{2}x + 1$, as shown on the Figure 2.1b.



(a) Illustrative contra-example of a function with $f(x) \leq 0$ convex but globally non convex.

(b) Graphical representation of $g(x) = x^4 - x^3 - \frac{5}{2}x + 1$.

Chapter 3

Linear optimization

3.1 Definitions

A linear optimization problem consists of minimizing a linear function subject to linear constraints:

$$\begin{cases} Min & f(x) \\ \text{subject to} & g_i(x) = 0,\ i \in I^0 \\ & g_i(x) \leq 0,\ i \in I^- \\ & g_i(x) \geq 0,\ i \in I^+ \\ & x = (x_1, \ldots, x_n) \geq 0 \end{cases}, \quad f \text{ linear }, g_i \text{ linear.}$$

Definition 3.1.1. *A linear optimization problem is in standard form is all constraints are equations (except for $x \geq 0$).*

EXAMPLE 2. Let's consider the linear general problem

$$\begin{cases} Min & 5x_1 - 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 2 \\ & 2x_1 + 3x_2 \leq 4 \\ & -x_1 + 6x_2 = 10 \\ & x = (x_1, x_2) \geq 0 \end{cases},$$

then in order to have this problem in standard form let's introduce new slack variables $x_3, x_4 \geq 0$, so we can rewrite the problem as:

$$\begin{cases} Min & 5x_1 - 3x_2 \\ \text{subject to} & x_1 - x_2 - x_3 = 2 \\ & 2x_1 + 3x_2 + x_4 = 4 \\ & -x_1 + 6x_2 = 10 \\ & x = (x_1, x_2, x_3, x_4) \geq 0 \end{cases},$$

□

In general, we may consider then the problem

$$\begin{cases} Min & cx \\ \text{subject to} & Ax = b \\ & x = (x_1, \ldots, x_n) \geq 0 \end{cases}, x \in \mathbb{R}^n, b \in \mathbb{R}^m, rk(A) = m < n$$

We denote the set of solutions as $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Note that X is a compact set (since it is a hyperplane). We are interested on X to be convex.

Definition 3.1.2 (Convex polytope)**.** *A convex polytope is a convex set of the form*

$$X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}.$$

A convex polyhedron is a convex polytope which is bounded.

Definition 3.1.3. *A vertex (extreme point) of a convex polytope X is any point, x, which cannot be expressed as a convex combination of other points $y \in X \setminus \{x\}$.*

We are interested on the boundaries of the polytopes, where the solutions are. In fact, the solution may be on the vertexs, which we have a finite number of. We may focus then on finding the vertexs.

Definition 3.1.4. *Every regular square $m \times m$ submatrix of A is called a basis.*

Let B be the basis, so permuting columns we can write $A = [B, N]$. Then we can divide the corresponding vectors as follows $x = [x_B, x_N]^T, c = [c_B, c_N]^T$. Then we can express the constraints equations as

$$Bx_B + Nx_N = b \tag{3.1}$$

Definition 3.1.5 (Basic solution)**.** *We call basic solution (associated with the basis B) the particular solution of (3.1) obtained by selecting $x_N = 0$. (Then, $x_B = B^{-1}b$).*

Definition 3.1.6 (Feasible)**.** *A solution is said to be feasible if $x_B \geq 0$.*

Definition 3.1.7 (Degenerate)**.** *A basic solution is said to be degenerate if x_B has some zero components.*

Theorem 3.1.8. *The set of vertices of a polytope X corresponds to the set of basic feasible solutions.*

Proof. Minoux □

Corollary 3.1.9. *The convex set X (polytope of polyhedron) has a finite number of vertices. In fact, num of vertices $\leq \binom{n}{m}$.*

Proposition 3.1.10. *Every point of a convex polyhedron X is a convex combination of the vertices of X.*

Proof. Minoux □

Let's study the case in which X is an unbounded polytope.

Definition 3.1.11. *We say that a vector $y \geq 0$ is an infinite array if $\forall x \in X,\ x + \lambda y \in X$, $\forall \lambda \geq 0$.*

Observation 3.1.12. $y \geq 0$ is an infinite ray $\iff$ y is a non-negative solution of $Ay = 0$.

Proof. $A(x + \lambda y) = \underbrace{Ax}_{b} + \lambda \underbrace{Ay}_{0}$ □

Observation 3.1.13. The set of infinite rays $Y = \{y \in \mathbb{R}^n : Ax = 0, y \geq 0\}$ is a cone.

Consider the hyperplane $\sum_{i=1}^{n} y_i = 1 := H$. Consider now the intersection $H \cap Y$ which is bounded (polytope). In fact, it is a convex polytope, since we are just adding a new constraint to the equation's system $Ax = 0$. More explicitely, we have

$$X' = H \cap Y = \{y \in \mathbb{R}^n : \begin{pmatrix} A \\ 1 \dots 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, y \geq 0\}$$

So we have the ellipsoid given by the intersection between the cone and this hyperplane. Then, we can apply corollary 3.1.9 to affirm that X' has a finite number of vertices $y_1, \dots, y_k$.

Since:

- Every point of Y comes (by scaling) from a point in $H \cap Y$,
- Every point of $H \cap Y$ is a convex combination of $y_1, \dots, y_k$,

it follows that every point of Y is a linear combination of $y_1, \dots, y_k$ with non-negative coefficients.

Definition 3.1.14. *The set Y is a convex polyhedral cone. The points $y_1, \dots, y_k$ are a generating set of Y and are called extreme rays of X.*

Observation 3.1.15. Extreme rays correspond to the basic feasible solutions of

$$\begin{cases} Ax = 0, \\ \sum_{i=1}^{n} y_i = 1, y \geq 0 \end{cases}$$

Proposition 3.1.16. *Every point of a convex polytope X is a convex combination of the vertices of X to which we may have to add a linear combination, with positive coefficients, of extreme rays.*

Theorem 3.1.17. *The optimum of a linear function $z = cx$ on a convex polyhedron X is obtained in at least one vertex. If it is obtained in more than one vertex it is obtained in every point which is a convex combination of these vertices.*

Proof. Let $x_1, \dots, x_k$ be the vertices of $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Consider the notation $z^* = min\{z(x_1), \dots, z(x_k)\}$, where $z(x_i) := cx_i$.
In virtue of the proposition 3.1.10, for any $x \in X$, we can express the point as a convex combination of the vertices of X, i.e. $x = \sum_{i=i}^{k} \lambda_i x_i$, with $\sum_{i=i}^{k} \lambda_i = 1$, $\lambda_i \geq 0$. Then, in particular applying linearity, we have

$$z(x) = \sum_{i=i}^{k} \lambda_i z(x_i) \geq \sum_{i=i}^{k} \lambda_i z^* = z^*$$

Then, for all $x \in X$, we have $z(x) \geq z^*$, which is attained on one of the vertices of X by definition.

Now, if the minimum is attained in more than one vertex we could apply linearity to verify that it is indeed the minimum for every linear combination of these vertices. For instance, suppose it is attained on two of the vertices, $z^* = z(x_1) = z(x_2)$, then for $\lambda \in [0, 1]$,

$$z(\lambda x_1 + (1-\lambda)x_2) = \lambda z(x_1) + (1-\lambda)z(x_2) = \lambda z^* + (1-\lambda)z^* = z^*.$$

By induction, for any $1 < n \leq k$, suppose that the minimum is attained on the first n vertices, and consider $\sum_{i=i}^{n} \lambda_i = 1$, $\lambda_i \geq 0$, then

$$z\Big(\sum_{i=i}^{n} \lambda_i x_i\Big) = \sum_{i=i}^{n} \lambda_i z(x_i) = \sum_{i=i}^{n} \lambda_i z^* = z^*$$

□

Theorem 3.1.18. *Let us define $p = (p_1, \ldots, p_n)$ as $p = c_B B^{-1}$. A necessary and sufficient condition (we are assuming no degeneracy, i.e. $x_B = B^{-1}b > 0$) for B to be an optimal feasible basis is:*

$$\overline{c_N} = c_N - pN = c_N - c_B B^{-1} N \geq 0.$$

The vector p is called the vector of simplex multiplier. The components of $\overline{c_N}$ are the reduced sets of the non-basic variables.

Proposition 3.1.19. *Let B be any feasible basis, x^* be the corresponding basic solution and $\overline{c}_N = c_N - p_N$. If there exists a non basic variable x_S such that $\overline{c}_S < 0$ then, one of the next options holds:*

1. *The value of x_S can be indefinitely increased without leaving the set of feasible solutions. In this case, the optimum is unbounded, i.e. $-\infty$.*
2. *There exists another basis $\hat{B}$ and another feasible solution $\hat{x}$ such that $z(\hat{x}) < z(x^*)$.*

3.2 The (Primal) simplex method

Consider the system of constraints $Ax = b$, $A = [B, N]$. We have a feasible basis B^0, the we may consider an iterative algorithm as follows:

1. For $k = 0$, we have the initial feasible basis B^0.
2. We increase $k \leftarrow k + 1$.
3. Let B be the correct basis, B^k, then $x = [x_B, x_N]$ would be the corresponding basic solution. Then we compute

$$\overline{b} = B^{-1}b, \; p = c_B B^{-1}, \; \overline{c}_N = c_N - p_N$$

4. If $\overline{c}_N \geq 0$, STOP.
5. Else, if there exists s such that $\overline{c}_S < 0$, then let A_s be the s column of A and compute $\overline{A}_S = B^{-1}A_s$. If $\overline{a}_{is} \leq 0, \forall i = 1, \ldots, m$, then STOP, the problem is unbounded (solution $-\infty$), otherwise compute

$$\hat{x}_s = \frac{\overline{b}_s}{a_{rs}} = Min_{\{i:\overline{a}_{is}>0\}} \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \right\}$$

6. Let x_t be the variable which corresponds to the r^{th} column of the basis, that is, such that $B^{-1}A_t = e_r = (0, \ldots, 1, \ldots, 0)$. Then the variable s takes the value $\hat{x}_s > 0$ (*entries the basis*). The variable t becomes zero (*leaves the basis*) and therefore the new current solution x corresponds to the new feasible basis $\hat{B}$ obtained by replacing column A_t by column A_s.
   Go to step *2*.

Theorem 3.2.1. *The simplex method solves the problem in a finite number of steps.*

3.3 The concept of duality

We seek to convert the problem of minimizing to the problem of finding the maximum.

EXAMPLE 3. Consider the optimization problem:

$$\begin{cases} Min & 12x_1 - 3x_2 + 4x_3 \\ \text{subject to} & 4x_1 + 2x_2 + 3x_3 \geq 2 \\ & 8x_1 + x_2 + 2x_3 \geq 3 \\ & x = (x_1, x_2, x_3) \geq 0 \end{cases},$$

Since we restrict the solutions to the values with $x = (x_1, x_2, x_3) \geq 0$, we have that:

$$12x_1 + 3x_2 + 4x_3 \geq 4x_1 + 2x_2 + 3x_3 \geq 2$$

then the optimal solution z^* satisfies $z^* \geq 2$. Consider now the alternative variables $u_1, u_2 \geq 0$, then we have that it satisfies

$$u_1(4x_1 + 2x_2 + 3x_3) + u_2(8x_1 + x_2 + 2x_3) \geq 2u_1 + 3u_2 \iff$$
$$\underbrace{(4u_1 + 8u_2)}_{\leq 12} x_1 + \underbrace{(2u_1 + u_2)}_{\leq 3} x_2 + \underbrace{(3u_1 + 2u_2)}_{\leq 4} x_3 \geq 2u_1 + 3u_2.$$

Then, we have the equivalent maximizing problem:

$$\begin{cases} Max & 2u_1 + 3u_2 \\ \text{subject to} & 4u_1 + 8u_2 \leq 12 \\ & 2u_1 + u_2 \leq 3 \\ & 3u_1 + 2u_2 \leq 4 \\ & u = (u_1, u_2) \geq 0 \end{cases}$$

□

Definition 3.3.1 (Dual problem)**.** *Let us consider the general primal problem of minimization*

$$\begin{cases} Min & z = cx \\ \textit{subject to} & Ax \geq b \\ & x \geq 0, \end{cases}$$

then the dual problem is

$$\begin{cases} Max & w = ub \\ \textit{subject to} & uA \leq c \\ & u \geq 0, \end{cases}$$

Proposition 3.3.2 (Dual problem is in standard form)**.** *If the primal problem is*

$$\begin{cases} Min & z = cx \\ subject\ to & Ax = b \\ & x \geq 0, \end{cases}$$

then its dual problem is

$$\begin{cases} Max & w = ub \\ subject\ to & uA \leq c \end{cases}$$

Proof. Consider the primal problem

$$\begin{cases} Min & z = cx \\ \text{subject to} & Ax \geq b \\ & -Ax \geq b \\ & x \geq 0, \end{cases}$$

Then, by definition its dual problem is given by

$$\begin{cases} Max & w = \overline{u}\overline{b} \\ \text{subject to} & \overline{u}\overline{A} \leq c \\ & \overline{u} \geq 0, \end{cases}$$

with $\overline{A} = [A, -A], \overline{b} = [b, -b], \overline{u} = [u_1, u_2]$. Then, computing the corresponding operations we have

$$\begin{aligned} \overline{u}\overline{b} &= u_1 b - u_2 b = (u_1 - u_2)b \\ \overline{u}\overline{A} &= (u_1 - u_2)A. \end{aligned}$$

Thus, the resulting dual problem is given by

$$\begin{cases} Max & w = ub = (u_1 - u_2)b \\ \text{subject to} & (u_1 - u_2)A \leq c \\ & u_1, u_2 \geq 0, \end{cases}$$

□

Lemma 3.3.3. *If $\overline{x}$ and $\overline{u}$ are respectively solutions of the primal and dual, then*

$$z = c\overline{x} \geq \overline{w} = \overline{u}b$$

Proof. Let $\overline{x}$ be the solution of the primal problem, $A\overline{x} = b$. Then we have

$$\begin{cases} \overline{u}A\overline{x} = \overline{u}b \\ \overline{u}A \leq c \\ \overline{x} \geq 0 \end{cases} \Longrightarrow \overline{u}A\overline{x} \leq c\overline{x} \Rightarrow \overline{u}b \leq c\overline{x}$$

□

Corollary 3.3.4. *Let x^*, u^* be the solutions of the primal and dual problems, respectively, such that $cx^* = u^*b$. Then, x^* is an optimal solution of the primal and u^* is an optimal solution of the dual.*

Theorem 3.3.5 (Duality theorem)**.** *Assume we have a primal problem P, and its dual problem D. Then:*

1. *If (P) and (D) have solutions, then each of them has an optimal solution and*

$$z^* = Min(P) = Max(D) = u^*$$

2. *If one of them has an unbounded optimum, then the other has no solutions.*

3.4 Exercises on linear programming

Exercise 1:
Let us consider the convex set (polyhedron),

$$C = \{(x, y) \in \mathbb{R}^2 \text{ such that } 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Write this set in the form $\{z \in \mathbb{R}^n$ *such that* $Az = b, z \geq 0\}$*, compute the basic feasible solutions and, from them, the vertices.*

Consider the slack variables $t_1, t_2 \geq 0$, so we can express the constraints

$$\begin{cases} x \leq 1, \\ y \leq 1, \\ x, y \geq 0 \end{cases}$$

as follows

$$\begin{cases} x + t_1 = 1, \\ y + t_2 = 1, \\ x, y, t_1, t_2 \geq 0 \end{cases}$$

More explicitly, we have $C = \{z \in \mathbb{R}^n \text{ such that } Az = b, z \geq 0\}$, for

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Consider now the basic feasible solutions of the polyhedron, which correspond to the ones obtained when setting the remaining variables to zero, i.e. $\mathbf{x} = [x_B, x_N]$, with $x_N = 0$ and such that $x_B \geq 0$. Note that these solutions are given by

$$\begin{aligned} (x, y, t_1, t_2)_1 &= (1, 1, 0, 0) \\ (x, y, t_1, t_2)_2 &= (1, 0, 0, 1) \\ (x, y, t_1, t_2)_3 &= (0, 1, 1, 0) \\ (x, y, t_1, t_2)_4 &= (0, 0, 1, 1). \end{aligned}$$

Recalling now a theorem previously states, we know that the set of vertices of a polytope corresponds to the set of basic feasible solutions. Therefore, the vertices of C, are given by the set of basic feasible solutions considered previously:

$$V_C = \{(1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 0, 1, 1)\}$$

Exercise 2:
Assume that $a_1, \ldots, a_m$ *are given vectors in* $\mathbb{R}^3$ *(all different from* 0*). Let* $b_1, \ldots, b_m$ *be strictly positive numbers and let us define the set*

$$M = \{x \in \mathbb{R}^3 \text{ such that } a_i^t x \leq b_i \text{ for } i = 1, \ldots, m\}.$$

(a) *Show that the interior of this set is not empty.*

(b) *We want to determine the centre and the radius of the biggest sphere contained in M. Write this problem as a linear problem.*

(a) Note that the origin is satisfies the condition since $b_i > 0$ strictly. Then we have that $0 \in M$. However, it could be in the closure of M, therefore we have see that there is a neighbourhood of 0, V_0 contained in M, so we would have $0 \in \text{Int}(M)$. Consider b_1 to be the minimum number i.e. $b_1 \leq b_i$, $\forall i = 2, \dots, m$, without loss of generality. Since the vectors a_i are fixed, we can consider the vector with higher norm, let's suppose it is a_k, i.e. $||a_k||^2 \geq ||a_i||^2, \forall i = 1, \dots, m$. Therefore, consider now the neighbourhood of 0 to be the ball centered on the origin and with radius $\varepsilon = \frac{b_1}{||a_k||^2}$, $V_0 = B(0, \varepsilon)$. Note that for any $v \in V_0$, we have $v \in M$, since for any general $i = 1, \dots, m$ we have:

$$a_i^T v = \sum_{j=1}^{3} a_{ij} v_j \leq \sum_{j=1}^{3} a_{ij} \varepsilon \leq \varepsilon \underbrace{\sum_{j=1}^{3} a_{ij}^2}_{||a_i||^2} < \frac{b_1}{||a_k||^2} ||a_i||^2 \leq b_i \underbrace{\frac{||a_i||^2}{||a_k||^2}}_{\leq 1} \leq b_i.$$

Therefore, we have that $0 \in V_0 \subseteq M$, and consequently the origin belongs to the interior of M, i.e. $Int(M) \neq \emptyset$.

(b) We want to find the biggest sphere $S = B(c, r)$ contained in M. Note that therefore, we seek to maximize the radius of this sphere, i.e. $r = ||x - c||$, since the volume of the 3-dimensional sphere is directly proportional to the radius, which is defined positive, $V = \frac{4}{3}\pi r^3$. Then the mos intuitive would be to consider the maximization of the radius subject to the fact that the whole sphere is contained in M, this is $x \in M$ and $c \in M$. This could be written as

$$\begin{cases} \text{Maximize} & ||x - c||^2 \\ \text{subject to} & x, c \in M \end{cases}$$

Note however that this is not a linear optimization problem. Let's reconsider this problem in terms of $r > 0$, this is maximizing r subject to the fact that $c \in M$, i.e. $a_i^t x \leq b_i$, and that any point inside the sphere, i.e. x such that $d(x, c) \leq r^2 \iff ||x - c|| \leq r$ then $x \in M$. This would imply that for every point in the sphere we would have $a_i^t x \leq b_i$. Note that if a point, y in the sphere was closer to any line $a_i^t x = b_i$ then the line would cross directly the sphere, leaving some points outside the set M. Then, given a point y inside the sphere, we need to have the distance between this point and any line $a_i^t x = b_i$ at least r. The distance between a point y and the line $a_i^t x = b_i$ is given by $d(y, a_i^t x - b_i) = \frac{||(y - b_i) \times a_i}{||a_i||}$. Considering the worst case, the possibly further point from the lines, the center of the sphere, we have the condition

$$d(c, a_i^t x - b_i) = \frac{||(c - b_i) \times a_i||}{||a_i||} = \frac{||a_i c - b_i||}{||a_i||} = \frac{b_i - a_i c}{||a_i||} \geq r$$

given the fact that we have $c \in M$. Then, the linear problem is given by:

$$\begin{cases} \text{Maximize} & r \\ \text{subject to} & a_i^t c \leq M \\ & r \leq \frac{b_i - a_i c}{||a_i||} \end{cases}$$

Exercise 3.
Use a software package to solve:

(a)

$$\begin{cases} \text{Minimize} & -8x_1 - 9x_2 - 5x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 \leq 2 \\ & 2x_1 + 3x_2 + 4x_3 \leq 3 \\ & 6x_1 + 6x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{cases}$$

Code used for a simple linear problem:

```
% Objective function
fa = [-8 -9 -5];
% Restrictions:
A = [1 1 2
    2 3 4
    6 6 2];
a = [2 3 8];
lb = zeros(3,1);
Aeq = [];
beq=[];
xa = linprog(fa, A, a,Aeq, beq, lb);
```

with solution $x_a = [1, 1/3, 0]$.

(b)

$$\begin{cases} \text{Minimize} & 5x_1 - 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 2 \\ & 2x_1 + 3x_2 \leq 4 \\ & -x_1 + 6x_2 = 10 \\ & x_1, x_2 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize} & 5x_1 - 3x_2 \\ \text{subject to} & -x_1 + x_2 \leq -2 \\ & 2x_1 + 3x_2 \leq 4 \\ & -x_1 + 6x_2 = 10 \\ & x_1, x_2 \geq 0 \end{cases}$$

Using a similar code:

```
% funció objectiu
fb = [5 -3];
% Restriccions:
B = [-1 1
    2 3
    ];
b = [-2 4];
Beq = [-1 6];
beq = 10;
la= zeros(2,1);
xb = linprog(fb, B, b, Beq, beq, la);
```

we obtaine that the problem has `no feasible solution`.

(c)

$$\begin{cases} \text{Maximize} & 3x_1 + 2x_2 - 5x_3 \\ \text{subject to} & 4x_1 - 2x_2 + 2x_3 \leq 4 \\ & -2x_1 + x_2 - x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize} & -3x_1 - 2x_2 + 5x_3 \\ \text{subject to} & -x_1 + x_2 \leq -2 \\ & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{cases}$$

Trying directly from to define the maximization problem from Matlab and let it convert it to a minimization problem:

```
x = optimvar('x','LowerBound',0);
y = optimvar('y','LowerBound',0);
z = optimvar('z','LowerBound',0);
prob = optimproblem('Objective',3*x + 2*y - 5*z,'ObjectiveSense','max');
prob.Constraints.c1 = 4*x - 2*y + 2*z <= 4;
prob.Constraints.c2 = -2*x + y - z <= -1;
problem = prob2struct(prob);
[sol,fval,exitflag,output] = linprog(problem);
```

We obtain `Problem is unbounded`. Converting manually the problem to a minimization problem, we get the same result, following the computation:

```
fc = [-3 -2 5];
C = [4 -2 2
    -2 1 -1
];
c = [4 -1];
Aeq=[];
beq=[];
lb= zeros(3,1);
xc = linprog(fc, C, c, Aeq, beq, lb);
```

(d)

$$\begin{cases} \text{Maximize} & 4x_1 + 6x_2 + 3x_3 + x_4 \\ \text{subject to} & 1.5x_1 + 2x_2 + 4x_3 + 3x_4 \leq 550 \\ & 4x_1 + x_2 + 2x_3 + x_4 \leq 700 \\ & 2x_1 + 3x_2 + x_3 + 2x_4 \leq 200 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases} \iff \begin{cases} \text{Minimize } -4x_1 - 6x_2 - 3x_3 - x_4 \\ \text{subject to} \\ 1.5x_1 + 2x_2 + 4x_3 + 3x_4 \leq 550 \\ 4x_1 + x_2 + 2x_3 + x_4 \leq 700 \\ 2x_1 + 3x_2 + x_3 + 2x_4 \leq 200 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

By following the same procedure and creating the corresponding minimization problem

```
% funció objectiu
fc = [-4 -6 -3 -1];
% Restriccions:
C = [1.5 2 4 3
```

```
        4 1 2 1
        2 3 1 2
        ];
    c = [550 700 200];
    Aeq=[];
    beq=[];
    lb= zeros(4,1);
    xc = linprog(fc, C, c, Aeq, beq, lb);
```

the solution $(x, y, z, w) = (0, 25, 125, 0)$, is found. We could also have used a code similar to the previous one.

Exercise 4. *Consider the linear programme (P) in standard form and its dual programme (D),*

$$\left\{ \begin{array}{l} \text{Min } z = c \cdot x \\ A \cdot x = b, \; x \geq 0 \end{array} \right., \quad \left\{ \begin{array}{l} \text{Max } w = u \cdot b \\ u \cdot A \leq c \end{array} \right.$$

Let us denote by A_j the jth column of A. Prove that two solutions $(\overline{x}, \overline{u})$ of respectively, (P) and (D) are optimal if and only if

$$(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$$

Proof. $\boxed{\Rightarrow}$ Consider $(\overline{x}, \overline{u})$ to be two optimal solutions of (P) and (D), then by recalling a theorem stated in class we have $z^* = c \cdot \overline{x} = \overline{u} \cdot b = w^* \iff \overline{u} \cdot b - c \cdot x = 0$. But since $\overline{x}$ is a solution, it verifies the constraints of the primal linear programme, i.e. $A \cdot \overline{x} = b$ and therefore we have $\overline{u} \cdot A \cdot \overline{x} - c \cdot x = 0 \iff (\overline{u} \cdot A - c) \cdot \overline{x} = 0$, Since $\overline{x} \geq 0$ and $\overline{u} \cdot A - c \leq 0$, the zero scalar product corresponds to the set of equations given by $(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$.

$\boxed{\Leftarrow}$ Consider now $(\overline{x}, \overline{u})$ to be two solutions of (P) and (D), that verify $(\overline{u} \cdot A_j - c_j)\overline{x}_j = 0, \forall j = 1, \ldots, n$. Since $\overline{x} \geq 0$ and $\overline{u} \cdot A - c \leq 0$, this is equivalent to say that $(\overline{u} \cdot A - c) \cdot \overline{x} = 0 \iff \overline{u} \cdot A \cdot \overline{x} - c \cdot \overline{x}$, and again using that $\overline{x}$ verifies the constraints from (P) we have $\overline{u} \cdot b = c \cdot \overline{x}$, which in virtue of the same theorem afforementioned implies that $\overline{x}$ and $\overline{u}$ are optimal solutions of the linear programmes $(P), (D)$, as wanted to prove.

□

Chapter 4

Non linear optimization

4.1 One dimensional optimization

Let $f : \mathbb{R} \to \mathbb{R}$ be an unconstrained function. If the function is $\mathcal{C}^2$, in order to optimize this function we can approach by using the corresponding derivatives as follows:

1. Consider $g(x) = f'(x)$.
2. Find the points such that $g(x) = 0$, by, for instance the Newton-Raphson method:

$$\alpha_{n+1} = \alpha_n - \frac{g(\alpha_n)}{g'(\alpha_n)}. \tag{4.1}$$

   The convergence of this method is quadratic, but it is local convergence. Therefore, we have to start relatively close to the zeros we are looking for. In addition, sometimes it can be difficult to compute $g' = f''$. In order to improve this method, we could use the trick of considering

$$g'(\alpha_n) \approx \frac{g(\alpha_n) - g(\alpha_{n-1})}{\alpha_n) - \alpha_{n-1}},$$

   consequently, the equation (4.2) becomes

$$\alpha_{n+1} = \alpha_n - g(\alpha_n)\frac{\alpha_n - \alpha_{n-1}}{g(\alpha_n) - g(\alpha_{n-1}},$$

   providing the secant method.
   The error for the Newton method in the n-step is $\epsilon_n \approx \epsilon_{n-1}^2$. Meanwhile, the error for the secant method in the n-step is $\epsilon_n \approx \epsilon_{n-1}^{\gamma}$, for $\gamma \approx 1.618$.

Note now that sometimes the functions considered are not differentiable, and we may need to use methods without computing the derivatives.

Exercise: *Search the information of the Fibonacci method, Golden section.*

The Golden section search function is a one-dimensional optimization method without derivatives that can be used when the number N of computations wished to carry out is not known a priori. It is based on the elimination of subintervals for every newly computed point by using unimodality. This can be explained by considering four points in an interval, $a < b < c < d \in [A, B]$. Knowing this four values of the function, we have a key for where the minimum are. Note that we could actually remove an interval. Consider then the initial interval $[a_1, d_1]$ of length $d_1 - a_1 = \Delta_1$, and we consider we know the value of the function we seek to minimize, f, in two inner intermediate points $a_1 < b_1 < c_1 < d_1$ such that $c_1 - a_1 = d_1 - b_1 = \Delta_2$. Then, by unimodality we can discard one of the intervals considered. Without loss of generalization, suppose this intervals corresponds to $[c_1, d_1]$, then we are left with the interval $[a_1, c_1]$ of length Δ_2, then we have

$$\Delta_1 = d_1 - a_1 = c_1 - a_1 + d_1 - c_1 = c_1 - a_1 + b_1 - a_1 = \Delta_2 + \Delta_3$$

Then, by induction we have

$$\Delta_k = \Delta_{k+1} + \Delta_{k+2} \tag{4.2}$$

where Δ_k represents the length of the interval after $k-1$ iterations.
Assume now that the length of the intervals has a given fixed ration, i.e. $\frac{\Delta_1}{\Delta_2} = \frac{\Delta_2}{\Delta_3} = \cdots = \frac{\Delta_k}{\Delta_{k+1}} = \cdots = \gamma$. Combined with (4.2), then we have

$$\frac{\Delta_k}{\Delta_{k+1}} = 1 + \frac{\Delta_{k+2}}{\Delta_{k+1}} \iff \gamma = 1 + 1/\gamma \iff \gamma^2 - \gamma - 1 = 0 \iff \gamma = \frac{\sqrt{5}+1}{2} \approx 1.618$$

which corresponds to the golden section number. Therefore, this method has a linear rate of convergence of rate $1/\gamma = 0.618$.

Definition 4.1.1 (Unimodal)**.** *We say that a function is unimodal at the real interval $[A, B]$ if it has a minimum $\overline{\alpha} \in [A, B]$ and if $\alpha_1, \alpha_2 \in [A, B], \alpha_1 < \alpha_2$ we have:*

$$\alpha_2 \leq \overline{\alpha} \Rightarrow g(\alpha_1) > g(\alpha_2)$$
$$\alpha_1 \geq \overline{\alpha} \Rightarrow g(\alpha_1) < g(\alpha_2)$$

Observation 4.1.2. Consider $a < b < c < d \in [A, B]$. Knowing this four values of the function, do we have a key for where the minimum are. Note we could actually remove an interval.

4.2 Non linear unconstrained optimization

Our goal is to find x^* such that $\forall x \in \mathbb{R}^n, f(x^*) \leq f(x)$, i.e. we seek to find a global minimum. If $f(x^*) < f(x), \forall x \neq x^* \Rightarrow$ the minimum is unique.

Theorem 4.2.1. *A necessary condition for x^* to be a (local or global) $f \in \mathcal{C}^2$ minimum of is*

1. $\nabla f(x^*) = 0$.
2. *The Hessian* $\nabla^2(x^*) = \left(\frac{\partial f}{\partial x_i \partial x_j}(x^*)\right)$ *is a positive definite matrix.*

Then necessary $\iff$ semidefinite positve, sufficient $\iff$ definite.

Theorem 4.2.2. *If $f \in \mathcal{C}^1$ but convex, then a necessary and sufficient condition for x^* to be a global minimum is $\nabla f(x^*) = 0$.*

4.2.1 Numerical methods

Given a function $f \in \mathcal{C}^1$, $\nabla f(x) = 0$, we could consider the Newton method in n-variables.

On the other hand, the decent method always goes down:

$$x^k$$
$$x^{k+1} = x^k + \lambda_k d_k,$$

we have to choose the values λ_k which indicates the size of every step, and

$$d_k = \begin{cases} -\nabla f(x^k) \\ \nabla f(x^k) \cdot d_{k-1} < 0 \end{cases}$$

On the gradient methods we consider

$$d_k = \frac{-\nabla f(x^k)}{||\nabla f(x^k)||}, \quad x^{k+1} = x^k - \lambda_k d_k.$$

Steepest decent method: $g(\lambda) = f(x^k - \lambda \nabla f(x^k)), \lambda \geq 0$.

1. Choose $x^0, k = 0$.
2. Step k, $d_k = -\nabla f(x^k)$. Find λ_k such that $f(x^k + \lambda_k d_k) = Min_{\lambda \geq 0} f(x^k + \lambda d_k)$,
$$x^{k+1} = x^k + \lambda_k d_k.$$
3. Stepping test: if satisfied END. Otherwise, $k \leftarrow k+1$ and go to step 2.

Theorem 4.2.3. *If f is $\mathcal{C}^1$ and $f(x) \to \infty$ when $||x|| \to \infty$. Then, for any starting point X^0, the method of the steepest descents converges to a stationary point of f, (i.e. such that $\nabla f(x) = 0$).*

Accelerated steepest descent

At step k: $x^k \to y^k$ throughout p additional steps of Stochastic Descent Method (SDM) and then $d_k = y^k - x^k$.

Let x^* be a local minimum of f, i.e. $\nabla f(x^*) = 0$. Let us suppose that we have $c^r, r \geq 1$, then

$$f(x) = f(x^*) + \frac{1}{2}(x - x^*)^T \nabla^2 f(x^*)(x - x^*) + R_3$$

Let us skip the reminder R_3 (we suppose we are already so close). Then, note that the level curves are defined by ellipsoids:

$$\hat{f}(x) = f(x^*) + \frac{1}{2}(x - x^*)^T \nabla^2 f(x^*)(x - x^*)$$

Consider $q(x) = \frac{1}{2} x^T A x + b^T x + c, x \in \mathbb{R}^n, A = A^T$.

Definition 4.2.4 (Mutually conjugate)**.** *A set of directions, $d_0, \ldots, d_p$ are mutually conjugate with respect to the quadratic form q if $d_i^T A d_j = 0, \forall i \neq j$.*

Observation 4.2.5. The eigenvectors of A satisfy that, i.e. if $v_1, \ldots, v_p$ are eigenvectors of A, then they are mutually conjugate:

$$v_i^T A v_j = \lambda_j v_i^T v_j = 0$$

Note that a symmetric matrix diagonalize in an orthogonal basis.

In fact, given $x^0, x^{k+1} = x^k + \lambda_k d_k$, we have that $\{d_k\}_k$ are mutually conjugate directions. Using the scalar product $\langle u, v\rangle = u^T A v$, we can compute a basis, with gram-schmidt, to get an orthogonal basis with respect to $\langle\rangle$ that will be mutually conjugate (we have as much vectors as initial vectors we choose for the algorithm).

Remember that λ_k is such that $q(x^k + \lambda d_k)$. Note that

$$\begin{aligned}\frac{\partial q}{\partial x_i} &= \frac{1}{2} e_i^T A x + \frac{1}{2} x^T A e_i + b^T e_i \\ &= e_i^T A x + e_i^T b = e_i^T (Ax + b) \Rightarrow \nabla q = Ax + b\end{aligned}$$

Then, we have

$$0 = d_k^T \nabla q(x^{k+1}) = d_k^T (Ax^{k+1} + b) = d_k^T A(x^k + \lambda_k d_k) + d_k^T b$$

with $\lambda_k = -\frac{d_k^T(Ax^k+b)}{d_k^T A d_k}$.

Theorem 4.2.6. *For all k, $0 \le k \le n$, the point*

$$x^k = x^0 + \sum_{j=0}^{k-1} \lambda_j d_j$$

is the optimum of $q(x)$ restricted to the affine variety V^k generated by $\{d_0, \ldots d_{k-1}\}$ and passing through x^0.

In particular, $x^n = x^0 + \sum_{j=0}^{n-1} \lambda_j d_j$ is the optimum of q on $\mathbb{R}^n$.

Observation 4.2.7. Note that if we have $\nabla q(x^k) \perp V^k$ we are done. Given x^k, $0 \le i \le k-1$

$$\begin{aligned}d_i^T A x^k &= d_i^T A x^0 + \sum_{j=0}^{k-1} \lambda_j d_i^T A d_j = d_i^T A x^0 + \lambda_i d_i^T A d_i \\ &= d_i^T A x^0 - d_i^T (Ax^0 + b) \Rightarrow d_i^T (Ax^k + b) = 0\end{aligned}$$

The conjugate gradient method

Consider the quadratic form $q(x) = \frac{1}{2} x^T A x + b^T x + c$, then the algorithm is given as follows:

1. x^0 starting point, $g_0 = \nabla q(x^0) = Ax^0 + b$, then $d_0 = -g_0$
2. Step k:

$$\begin{aligned}x^{k+1} &= x^k + \lambda_k d^k, \lambda_k = -\frac{g_k^T d_k}{d_k^T A d_k} \\ g_{k+1} &= \nabla q(x^{k+1}) \\ d^{k+1} &= -g_{k+1} + \beta_k d_k \text{ such that } \beta_k = \frac{g_{k+1}^T A d_k}{d_k^T A d_k}\end{aligned}$$

3. $k \leftarrow k+1$, do to step 2

If we proof d^{k+1} is mutually conjugate to the previous ones, the theorem gives us that this algorithm works.

Theorem 4.2.8. *At each stage of the algorithm where the optimum has not been reached, we have:*

1. $\lambda_k = \frac{g_k^T g_k}{d_k^T A d_k} \neq 0$
2. $\beta_k = \frac{g_{k+1}^T [g_{k+1} - g_k]}{g_k^T g_k} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}$
3. *The direction $d_0, d_1, \ldots, d_k$ are mutually conjugate.*

Proof. 1. We may prove it by induction over $k > 1$. For $k = 0$ it is direct. Let's see it for k general. Assume $d_0, \ldots, d_k$ are mutually conjugate, then

$$\begin{aligned} d_k &= -g_k + \beta_{k-1} d_{k-1} \\ \lambda_k &= -\frac{g_k^T[-g_k + \beta_{k-1} d_{k-1}]}{d_k^T A d_k} = \frac{g_k^T g_k}{d_k^T A d_k} - \beta_{k-1} \frac{g_k^T d_{k-1}}{d_k^T A d_k} \end{aligned}$$

Note that $g_k^T d_{k-1} = 0$, since g_k is the gradient in the hyperplane. If this would not be zero, we would be improving in the hyperplane.
Note that we would find the minimum on the hyperplane (directions $d_0, \ldots, d_k$)) and then, the gradient is orthogonal to the hyperplane. And so on.

2. Note now that since $g_k = Ax^k + b$ we have

$$g_{k+1} - g_k = A(x^{k+1} - x^k) = \lambda_k A d_k \iff$$

$$g_{k+1}^T A d_k = \frac{1}{\lambda_k} g_{k+1}^T [g_{k+1} - g_k],$$

then, keeping in mind that $\lambda_k = \frac{g_k^T g_k}{d_k^T A d_k}$ we have

$$\begin{aligned} \beta_k &= \frac{g_{k+1}^T A d_k}{d_k^T A d_k} = \frac{d_k^T A d_k}{g_k^T g_k} \frac{g_{k+1}^T [g_{k+1} - g_k]}{d_k^T A d_k} \qquad \text{first equality} \\ &= \frac{g_{k+1}^T g_{k+1} - g_{k+1}^T g_k}{g_k^T g_k} \qquad \text{second inequality} \end{aligned}$$

Note that $g_{k+1}^T g_k = 0$, $g_k = -d_k + \beta_{k-1} d_{k-1}$. Since g_{k+1}^T is orthogonal to d_k and d_{k-1} we are done.

3. Assume now that $d_0, \ldots, d_k$ are mutually conjugate, then

- $d_{k+1}^T A d_k = (-g_{k+1} + \beta_k d_k)^T A d_k = -g_{k+1}^T A d_k + \beta_k d_k^T A d_k$, and β_k is such that this is zero. Then, it is mutually conjugate to the previous one.
- $d_{k+1}^T A d_i, i = 0, \ldots, k-1$, is mutually conjugate to all of them if $d_{k+1}^T A d_i = g_{k+1}^T A d_i + \beta_k \underbrace{d_{k+1}^T A d_i}_{0} = 0$. Therefore, we have to see that $g_{k+1}^T A d_i = 0$:

$$x^{i+1} = x^i + \lambda_i d_i, \quad A d_i = \frac{1}{\lambda_i}(Ax^{i+1} - Ax^i) = \frac{1}{\lambda_1}(g_{i+1} - g_i),$$

with $g_{i+1} = -d_{i+1} + \beta_i d_i$ and $g_i = -d_i + \beta_{i-1} d_{i-1}$. Note that $A d_i$ is a linear combination of d_{i+1}, d_i, d_{i-1}. Since g_{k+1} is orthogonal to $d_{i+1}, d_i, d_{i-1} \Rightarrow g_{k+1}^T A d_i = 0$.

□

Note that $Ax = b$, with $A = A^T$ is positive definite. Then we would have to minimize $q(x) = \frac{1}{2}x^T Ax - b^T x$.

Method of Fletcher and Reeves

We have to consider this algorithm for functions non quadratic, $f : \mathbb{R}^n. \to \mathbb{R}$:

1. $k = 0, x^0$ starting point. Consider $d_0 = -\nabla f(x^0)$
2. Step k, find λ_k that minimizes $g(\lambda) = f(x^k + \lambda_k d_k)$, which is a 1D function that we can minimize like a general function. Then
$$x^{k+1} = x^k + \lambda_k d_k$$
$$d_{k+1} = -\nabla f(x^{k+1}) + \beta_k d_k, \beta_k = \frac{||\nabla f(x^{k+1})||^2}{||\nabla f(x^k)||^2}$$
3. Stopping criteria: if satisfied, STOP, otherwise $k \leftarrow k + 1$. Go to step *2*.

Note that it is the same idea as before, but now with a general function.

Variant: Polak-Riviere method

We consider $\beta_k = \frac{g_{k+1}^T(g_{k+1}-g_k)}{g_k^T g_k}$. For quadratic functions it is exactly the same method we have studied. For a general function, it is a different method, since the equality does not hold for a general function, only for quadratics.

Note that for a general function, near the minimum (Taylor) the function has quadratic behaviour (but only near the minimum). If the function is similar to a quadratic shape it is a good choice, but if it is not, then gradient descent.

Newton's method

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ to be at least $\mathcal{C}^2$. Consider the Taylor's expansion of order 2:
$$f(x) \approx g(x) = f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{1}{2}(x - x^k)^T \nabla^2 f(x^k)(x - x^k)$$
Then the Newton's method seeks to solve $\nabla f(x) = 0 \iff \nabla g(x) = 0 \to \nabla f(x^k) + \nabla^2 f(x^k)^T(x - x^k) \approx 0 \to \nabla^2 f(x^k)^T(x - x^k) \approx -\nabla f(x)$. Therefore, we consider the iteration steps given by
$$x^{k+1} = x^k - [\nabla^2 f(x^k)]^{-1}\nabla f(x^k)$$

Observation 4.2.9. Newton's method does not need to work when we are far from the minimum. If the step we move is very big, the Newton's method is not a good idea.

Therefore, we add λ_k to correct the problem. Note that if the step is not too big, we can take $\lambda_k = 1$ and we have Newton's method, otherwise we can choose $\lambda_k < 1$ to correct the big step. The idea is illustrated as follows:
$$g(\lambda) = f(x^k + \lambda d_k), \quad d_k = -[\nabla^2 f(x^k)]^{-1}\nabla f(x^k),$$
where λ_k is the value that minimizes $g(\lambda)$.
Another idea could be to use a condition statement $f(x^k) \geq f(x^k + d_k)$. If it is, then we are going on the right direction. If it is not, the value of λ_k that minimizes $g(\lambda) = f(x^k + \lambda d_k)$ is between 0 and 1.

Proof. $g'(0) = \nabla f(x^k)d_k = -\nabla f(x^k)[\nabla^2 f(x^k)]^{-1}\nabla f(x^k) \Rightarrow g'(0) < 0 \Rightarrow d_k$ is the descent direction, since the Hessian is positive definite. □

Consider the situation in which $\nabla^2 f(x^k)$ is not positive definite. Note that we would have to fins $M_k \approx \nabla^2 f(x^k)$ with $M_k = M_k^T$, M_k positive definite. Consider $M_k = \mu_k I + \nabla^2 f(x^k)$. If the matrix is not defined positive, i.e. the eigenvalues are not all positive, we have to move the eigenvalues in the right directions $\mu_k I$. If μ_k is very large, this matrix is too different to do Newton, then $M_k \approx \mu_k I$ and we can recover the steepest descent method.

Quasi-Newton method (variable metric method)

Consider

$$x^{k+1} = x^k - \lambda_k[\nabla^2 f(x^k)]^{-1}\nabla f(x^k)$$

Since we approach a minimum, $H_k = H_k^T$ is a positive definite matrix and therefore:

$$x^{k+1} = x^k - \lambda_k H_k \nabla f(x^k), \quad d_k = -H_k\nabla f(x^k)$$

Observation 4.2.10. d_k is a descent direction, then $-\nabla f(x^k)^T d_k = \nabla f(x^k)^T \underbrace{H_k}_{>0} \nabla f(x^k)$.

Consider then

$$q(x) = \frac{1}{2}x^T Ax + b^T x + c, A = A^T \text{ positive definite}$$

Given H_0, H_1, H_2 we can compute $A^{-1}[\nabla^2 f(x^k)]^{-1}$.
We impose $H_k[\nabla f(x^K) - \nabla f(x^{k-1})] = x^k - x^{k-1}$. Then we consider the iteration step over the hessian matrix as $H_{k+1} = H_k + \Delta_k$, where we can compute Δ_k using corrections of rank 1 o 1 as follows, using the lemma:

Lemma 4.2.11. *For any two vectors u, v we have*

$$uu^T = \frac{(uu^Tv)(uu^Tv)^T}{(u^Tv)(u^Tv)}$$

rank 1: $\Delta_k = \alpha_k u_k u_k^T, \alpha_k \in \mathbb{R}, u_k \in \mathbb{R}^n$, and $\delta_k = x^{k+1} - x^k, \gamma_k = \nabla f(x^{k+1}) - \nabla f(x^k)$. Then

$$\delta_k = H_{k+1}\gamma_k = [H_k + \alpha_k u_k u_k^T]\gamma_k \iff$$
$$\delta_k - H_k\gamma_k = \alpha_k u_k u_k^T \gamma_k \iff \alpha_k(u_k\gamma_k^T)^2 = \gamma_k^T(\delta_k - H_k\gamma_k)$$

Using now the afforementioned lemma along with the previous computations we can see that:

$$\alpha_k(u_k u_k^T) = \frac{(\alpha_k u_k u_k^T\gamma_k)(\alpha_k u_k u_k^T\gamma_k)^T}{\alpha_k(u_k\gamma_k^T)^2} = \frac{(\delta_k - H_k\gamma_k)(\delta_k - H_k\gamma_k)^T}{\gamma_k^t(\delta_k - H_k\gamma_k)} = \Delta_k$$

In this method we choose H_k and then use Newton's method. We consider a quadratic function and we modified H_k in every step (we have seen that we have a descent direction if the matrix is positive definite).

Theorem 4.2.12. *Let f be a quadratic function $A = \nabla^2 f$ positive definite. Consider an iterative method which, starting from a point x_0 generates successively the points $x^1 = x^0 + \delta_1, \dots, x^n = x^{n-1} + \delta_n$ by displacing along n independent directions $\delta_1, \dots, \delta_n$. Then the sequence of matrices H_k is given by*

$$\begin{cases} H_0 \text{ any symmetric matrix} \\ H_{k+1} = H_k + \frac{(\delta_k - H_k\gamma_k)(\delta_k - H_k\gamma_k)^T}{\gamma_k^t(\delta_k - H_k\gamma_k)} \end{cases}$$

converge in at most n steps to A^{-1} (like the newton's method!).

In practice, if H_{k+1} is not positive definite then we choose H_k.

Algorithm of Daviden-Fletcher-Pavell (DFP)

In this algorithm the step is given by

$$H_{k+1} = H_k + \underbrace{\frac{\delta_k\delta_k^T}{\delta_k^T\gamma_k} - \frac{H_k\gamma_k\gamma_k^T H_k}{\gamma_k^T H_k \gamma_k}}_{\text{rank 2}}$$

Theorem 4.2.13. *Assume H_k is positive definite. Then if $\delta_k^T\gamma_k > 0$ then H_{k+1} is positive definite.*

Theorem 4.2.14. *If f is a quadratic function, $A = \nabla^2 f$ is positive definite, then the algorithm DFP generates the directions $\delta_0, \dots, \delta_k$ which satisfy*

$$\delta_i^T A \delta_j = 0, \quad 0 \le i \le j \le k$$
$$(H_{k+1} \text{ is the } A^{-1} \Leftarrow H_{k+1}A\delta_i = \delta_i, \quad 0 \le i \le k$$

Note that the directions are mutually conjugate! Which is perfect for going down.

Algorithm of Brayden-Fletcher-Godforb-Shanon (BFGS)

$$H_{k+1}\gamma_k = \delta_k \to H_k \Rightarrow A^{-1}$$
$$G_{k+1}\delta_k = \gamma_k \to G_k \Rightarrow A$$

The idea is to arbitrary symmetric positive definite:

$$G_{k+1} = G_k + \frac{\gamma_k\gamma_k^T}{\gamma^T\delta_k} - \frac{G_k\delta_k\delta_k^T G_k}{\delta_k^T G_k \delta_k}$$

Some experiences show that this method works better.

4.3 Nonlinear constrained optimization

Consider now the optimization problem

$$\begin{cases} Min & f(x) \\ s.t. & g_i(x) \le 0, \ i \in I = \{1, \dots, m\}, x \in \mathbb{R}^n \end{cases}$$

with $f, g_i \in \mathcal{C}^1$ for all $i \in I$. Then the set of solutions is given by $X = \{x \in \mathbb{R}^n | g_i(x) \le 0 \forall i \in I\} \neq \emptyset$. Consider $x^0 \in X$ to be a local optimum (minimum), and Γ the arc through x^0.

Definition 4.3.1 (Admissible arc)**.** *We say that*

$$\begin{aligned} \varphi :&\mathbb{R}^+ \to \mathbb{R}^n, \ \mathcal{C}^1 \\ &\theta \to \varphi(\theta) = (\varphi_1(\theta), \dots, \varphi_n(\theta)) \end{aligned}$$

is an admissible arc if it verifies:

(a) $\varphi(0) = x^0$

(b) For $\theta > 0$ *small enough,* $\varphi(\theta) \in X$*.*

Definition 4.3.2 (Admissible direction)**.** *An admissible direction*[1] *at* x^0 *is a vector of the form*

$$g = \frac{d\varphi}{\theta}(\theta) = \left[\frac{d\varphi_1}{\theta}(\theta), \dots, \frac{d\varphi_n}{\theta}(\theta)\right]^T$$

for any admissible arc at x^0*. We call* C_{ad} *the core of admissible directions at* x^0*.*

Consider $I^0 \subset I$ the set of indices of the constraints which are restricted at x^0, $I^0 = \{i \in I | g_i(x^0) = 0\}$. Then we can define $G = \{y \in \mathbb{R}^n | \nabla g_i(x^0)^T y \leq 0, \ \forall i \in I^0\}$.

Lemma 4.3.3. $C_{ad} \subset G$*.*

Proof. Let φ be an admissible arc at x_0, $y = \frac{d\varphi}{d\theta}(0)$ admissible direction. Then for $i \in I^0$, $g_i(\varphi(\theta)) \leq 0$ for $\theta > 0$ small enough. Expanding Taylor's expansion around $\theta = 0$:

$$\begin{gathered} \underbrace{g_i(x^0)}_{0} + \theta \nabla g_i(x^0) \frac{d\varphi}{d\theta}(0) + \mathcal{O}(\theta) \leq 0 \iff \\ \nabla g_i(x^0)^T y + \mathcal{O}(\theta) \leq 0 \iff \nabla g_i(x^0)^T y \leq 0 \end{gathered}$$

□

Example 4. Consider the next functions defined over the first quadrant

$$\begin{aligned} g_1(x) &= -x_1 \leq 0 \\ g_2(x) &= -x_2 \leq 0 \\ g_3(x) &= (1 - x_1)^3 + x_2 \leq 0 \end{aligned}$$

Note $G \not\subset C_{ad}$. Consider $x^0 = (1, 0)$, therefore, clearly $g_2(x^0) = 0 = g_3(x^0)$, and by definition, $I^0 = \{2, 3\}$. Then, let us compute the gradients

$$\nabla g_2(x^0) = (0, -1)^T, \quad \nabla g_3(x^0) \qquad = (0, 1)^T$$

Now we. can compute G,

$$\begin{aligned} (0, -1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= -y_2 \leq 0 \\ (0, 1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= y_2 \leq 0 \end{aligned}$$

which can only occur on the first quadrant when $y_2 = 0$. Then, we have $G = \langle (1, 0) \rangle$, which is not an admissible direction, and thus $G \not\subset C_{ad}$. □

[1] set of linear combination of admissible arcs

Definition 4.3.4. *A domain* $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}$ *satisfies at* $x^0 \in X$ *the constraints qualification assumption (CG) if* $\overline{C_{ad}} = G$.

Lemma 4.3.5. *For (CG) to hold at every point* $x \in X$*, it is sufficient that one of the following conditions holds:*

(a) All functions g_i *are linear*

(b) All functions g_i *are convex and* $X \neq \emptyset$.

Moreover, for (CG) *to hold at a point* $x^0 \in X$ *it is sufficient that:*

(c) The gradients $\nabla g_i(x^0), (i \in I)$ *are linearly independent.*

4.3.1 The Karush-Kuhu-Tucker conditions

Lemma 4.3.6 (Farkas and Mikoski). *Let* A *be a* $p \times q$ *matrix,* $b \in \mathbb{R}^p$. *In order to exist* $x \geq 0$ *such that* $Ax = b$ *it is necessary and sufficient that* $u^T b \geq 0$, $\forall u \in \mathbb{R}^p$ *such that* $u^T A \geq 0$.

Theorem 4.3.7. *Suppose that the functions* $f, g_i \in \mathcal{C}^1$ *and that the* (CG) *condition holds at* $x^0 \in X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}$. *Then, a necessary condition for* x^0 *to be a local minimum of*

$$\begin{cases} Min & f(x) \\ s.t. & g_i(x) \leq 0,\ i \in I \end{cases}$$

is the existence of numbers $\lambda_i \geq 0, i \in I$ *called KKT multipliers such that*

$$\begin{cases} \nabla f(x^0) + \sum_{i \in I} \lambda_i \nabla g_i(x^0) = 0 \\ \lambda_i g_i(x^0) = 0,\ \forall i \in I \end{cases}$$

Proof. For x^0 to be a local minimum it is necessary that

$$f(\varphi(\theta)) \geq f(\varphi(0)) = f(x^0),\ \forall \theta \in B(0, \varepsilon)$$

For $\theta > 0$ small enough we have

$$f(\varphi(\theta)) = f(x^0) + \theta \nabla f(x^0)^T y + \theta \mathcal{O}(\theta)$$

where $y = \frac{d\varphi}{d\theta}(0) \in C_{ad}$ and $\mathcal{O}(\theta) \to 0$ when $\theta \to 0$. If $\nabla f(x^0)^T y \leq 0$ we could choose θ small such that $\theta \nabla f(x^0)^T y + \theta \mathcal{O}(\theta) < 0$ and then we would have $f(\varphi(\theta)) \leq f(\varphi(0))$ contradicting the hypothesis. Therefore, $\nabla f(x^0)^T y \geq 0$, for all $y \in C_{ad}$ and thus $\nabla f(x^0)^T y \geq 0, \forall y \in \overline{C_{ad}} = G$. Consequently, we have seen that a necessary condition for x^0 to be a local minimum is $\nabla f(x^0)^T y \geq 0$ for any y such that $\nabla g_i(x^0)^T y \leq 0,\ \forall i \in I$.

Consider now $A = [-\nabla g_i(x^0)^T], i \in I^0$, $b = \nabla f(x^0)$, $u = y, x = \lambda = (\lambda_i)_{i \in I^0}$. Let us check the conditions of the Lemma 4.3.6:

$$u^T b = \nabla f(x^0)^T y \geq 0, \text{ and } u^T A = \nabla g_i(x)^T y \geq 0$$

then it exists $x \geq 0$ such that $Ax = b$ implies that there exists $(\lambda_i)_{i \in I^0}$ such that $\sum_{i \in I^0} \lambda_i(-\nabla g_i(x^0)) = \nabla f(x^0)$. Let us consider $\lambda_i = 0, \forall i \in I \setminus I^0$, then

$$\nabla f(x^0) + \sum_{i \in I^0} \lambda_i(\nabla g_i(x^0)) = 0$$
$$\lambda_i g_i(x^0) = 0, \forall i \in I$$

□

Consider now the optimization problem

$$\begin{cases} Min & f(x) \\ s.t. & g_i(x) \leq 0, \ i \in I = \{1, \dots, m\} \\ & h_j(x) = 0, \ i \in L = \{1, \dots, p\} \\ & x \in \mathbb{R}^n \end{cases}$$

Then we have $G = \{y \in \mathbb{R}^n \mid \nabla g_i(x^0)^T y \leq 0, i \in I^0 \text{ and } \nabla h_j(x^0)^T y = 0, j \in L\}$. Note that this gives us the result that $(CG) \Rightarrow \overline{C_{ad}} = G_1$.

Theorem 4.3.8 (KKT). *A necessary condition for x^0 to be a local minimum is that there exist number $\lambda_i \geq 0, (i \in I)$ and $\mu_j > 0, (j \in L)$ such that*

$$\nabla f(x^0) + \sum_{i \in I} \lambda_i \nabla g_i(x^0) + \sum_{j \in L} \mu_j h_j(x^0) = 0$$
$$\lambda_i g_i(x^0) = 0, \ \forall i \in I$$

Example 5. Consider the optimization problem given by

$$\begin{cases} Min & x_1^2 + x_2^2 \\ s.t. & 2x_1 + x_2 \leq -4 \end{cases}$$

In order to keep consistency with the notation let's consider $f(x) = x_1^2 + x_2^2$ and $g(x) = 2x_1 + x_2 + 4$, then

$$\begin{cases} 2x_1 + \lambda 2 = 0 \\ 2x_2 + \lambda = 0 \\ \lambda(2x_1 + x_2 + 4) = 0 \end{cases}$$

Note that if $\lambda = 0 \Rightarrow (x_1, x_2) = (0, 0)$, but $(0, 0)$ is not satisfying the restrictions. Similarly if $2x_1 + x_2 + 4 = 0 \Rightarrow \lambda = \frac{2}{5}, x_1 = -\frac{8}{5}, x_2 = -\frac{4}{5} \Rightarrow g\Big(-\frac{8}{5}, -\frac{4}{5}\Big) = 0 \leq 0$. If we have a minimum, it is in $\Big(-\frac{8}{5}, -\frac{4}{5}\Big)$. □

Consider the optimization problem given as

$$\begin{cases} Min & f(x) \\ s.t. & h_l(x) = 0, \ l \in L = \{1, \dots, m\}, x \in \mathbb{R}^m \end{cases}$$

We already know, due to Theorem 4.3.8, that the optimal solution is given by

$$\nabla f(x) + \sum_{l \in L} \mu_l \nabla h_l(x) = 0$$

Sufficient optimality conditions:

$$(P) = \begin{cases} Min & f(x) \\ s.t. & g_i(x) \leq 0, \ i \in I, x \in S \subset \mathbb{R}^m \end{cases}$$

Definition 4.3.9. *To each constrain, we associate a real number $\lambda_i \geq 0$, so called Lagrange multiplier. The Lagrange function associated with problem (P) is*

$$L(x, y) = f(x) + \sum_{i \in I} \lambda_i g_i(x)$$

Definition 4.3.10. *Let $\overline{x} \in S$ and $\overline{\lambda} \geq 0$ we say that $(\overline{x}, \overline{\lambda})$ is a saddle point of L if*

- $L(\overline{x}, \overline{\lambda}) \leq L(x, \overline{\lambda}), \forall x \in S$
- $L(\overline{x}, \overline{\lambda}) \geq L(\overline{x}, \lambda), \forall \lambda \geq 0$

Theorem 4.3.11. *Let $\overline{x} \in S$ and $\overline{\lambda} \geq 0$, then $(\overline{x}, \overline{\lambda})$ is a saddle point for L iff*

1. $L(\overline{x}, \overline{\lambda}) = min_{x \in S} L(x, \overline{\lambda})$
2. $g_i(\overline{x}) \leq 0, \ \forall i \in I$
3. $\overline{\lambda}_i g_i(\overline{x}) = 0, \forall i \in I.$

Theorem 4.3.12. *If $(\overline{x}, \overline{\lambda})$ is a saddle point of L, them $\overline{x}$ is a global optimum of (P).*

Proof. Using the previous Theorem, we can see that condition *1.* implies that $f(\overline{x}) + \sum_{i \in I} \overline{\lambda}_i g_i(\overline{x}) \leq f(x) + \sum_{i \in I} \overline{\lambda}_i g_i(x), \forall x \in S$. Note now that statement *3.* implies that $\overline{\lambda}_i g_i(\overline{x}) = 0, \ \forall i \in I$. Thus, combining these statements we have that $f(\overline{x}) \leq f(x) + \sum_{i \in I} \overline{\lambda}_i \underbrace{g_i(x)}_{\leq 0} \leq f(x)$ and consequently, we have that $\overline{x}$ is in fact a global minimum of (P), as wanted to prove. ☐

Consider now the convex differentiable case:

$$(P) + S = \mathbb{R}^n := (P')$$

with all functions being $\mathcal{C}^1$.

Theorem 4.3.13. *For $\overline{x}$ to be a local minimum of (P') it is necessary and sufficient that the KKT conditions hold at $\overline{x}$. That is, that there exists $\overline{\lambda} \geq 0$ such that*

$$\begin{cases} \nabla x L(\overline{x}, \overline{\lambda}) = 0 \\ \overline{\lambda}_i g_i(\overline{x}) = 0, \ \forall i \in I \end{cases}$$

Consider now the non-convex case, i.e. the problem (P'), with the functions $f_i, g_i \in \mathcal{C}^1$. Thus, sufficient conditions for $\overline{x}$ to be a local minimum of (P) are

1. There exists a neighbourhood $V(\overline{x})$ of $\overline{x}$ in which the functions f_i, g_i are convex.
2. There exists $\overline{\lambda}$ such that
   - $L(\overline{x}, \overline{\lambda}) \leq L(x, \overline{\lambda}), \forall x \in S \cap V(\overline{x})$
   - $L(\overline{x}, \lambda) \leq L(\overline{x}, \overline{\lambda}), \ \forall \lambda \geq 0$

Let's consider the primal methods:

- Restriction of the form $a \leq x \leq b$. We convert this into a cylinder, of the form $x = a + (b-a) sin^2 y$
- Restrictions of the type $h(x) = 0$. We reduce the dimension in 1 and remove the constraints.

Method of feasible directions

Note that feasible means:

- A small skip in that directions does not leave the set of solutions.
- f increases strictly

Consider the optimization problem

$$\begin{cases} Min & f(x) \\ s.t. & g_i(x) \leq 0 \end{cases}$$

Let X be the set of solutions. Consider $x_0 \in X$, and let's define $I^0 = \{i \in I : g_i(x_0) = 0\}$. Then we say that the direction y is *feasible* when

$$\frac{d}{d\theta} g_i(x_0 + \theta y)\Big|_{\theta=0} = \nabla g_i(x_0)^T y \leq 0, \ i \in I^0$$
$$\frac{d}{d\theta} f(x_0 + \theta y)\Big|_{\theta=0} = \nabla f(x_0)^T y < 0$$

Then we can consider the optimization problem as follows:

$$\begin{cases} Min & \nabla f(x_0)^T y \\ s.t. & \nabla g_i(x_0)^T y \leq 0, \ i \in I^0 \\ & \sum_{i=1}^m |y_i| = 1 \end{cases}$$

Equivalently,

$$\begin{cases} Min \ \xi & \\ s.t. & \nabla f(x_0)^T y - \xi \leq 0 \\ & g_i(x_0) + \nabla g_i(x_0)^T y - u_i \xi \leq 0, \ i \in I^0 \end{cases}$$

where we unknown y_i, ξ and u_i are positive given parameters.

Theorem 4.3.14. *If at $x_0 \in X$ the optimal value satisfies $\xi = 0$ and if x_0 is a neighbour point of X, then the KKT condition holds at x_0.*

Observation 4.3.15. If f is nonlinear and g_i are linear, then the gradient project method works well for the linear conditions. We may just need to find x_0 and compute $\nabla f(x_0)$ and check $I_0 = \{i \in I : g_i(x_0) = 0\}$.

Consider the problem

$$\begin{cases} Min & f(x) \\ s.t. & Ax \leq b \end{cases}$$

Note that this is a good problem for the gradient projection method.
Consider now the optimization problem

$$\begin{cases} Min & f(x) \\ s.t. & g(x) = (g_1(x), \ldots, g_m(x)9^T = 0 \end{cases}$$

In order to solve this problem a good idea could be to use the implicit function theorem, in order to make the problem become unconstrained.

For example, consider we have $f(x, y)$ subject to $g(x, y) = x + y + 1 = 0 \iff y = -1 - x$, which is equivalent to the unconstrained problem $f(x, -1 - x)$.
More generally, consider the general constrained function $g(x, y) = 0$, note that when $g : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{C}^1$, and given $(x_0, y_0) \in \mathbb{R}^2$ such that $g(x_0, y_0) = 0$ and $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$ then the implicit function theorem assures us that there exists $h : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ such that $\frac{\partial}{\partial x} g(x, h(x)) = 0, \ \forall x \in (x_0 - \delta, x_0 + \delta)$. Thus we know:

$$\frac{\partial}{\partial x} g(x_0, y_0) + \frac{\partial}{\partial y} g(x_0, y_0) h'(x_0) = 0$$

In $\mathbb{R}^n$, consider $g(x_0) = 0$, using the aforementioned notation we have $g(x_B, x_N) = 0$, with $D_{x_B} g(x_0)$ regular. Then the implicit function theorem gives us that $x_B = x_B(x_N) \in \mathbb{R}^m$ in a neighbourhood of x_0. Then given an objective function $f(x)$ with $x = (x_B, x_N), x_B \in \mathbb{R}^m, x_N \in \mathbb{R}^{n-m}$ with x_0 such that $D_{x_B} g(x_0)$ is regular, subject to the function $g(x) = 0$. Then the implicit function theorem gives us $f(x_B(x_N), x_N)$ such that at x_0

$$D_{x_N} f = \underbrace{D_{x_B} f}_{\text{known}} D_{x_N} x_B + \underbrace{D_{x_N} f}_{\text{known}}$$

$$g(x_B(x_N), x_N) = 0 \Rightarrow \underbrace{D_{x_B} g}_{\text{known}} D_{x_N} x_B + \underbrace{D_{x_N} g}_{\text{known}} = 0$$

Note that $x_N^0 \to x_N^0 + d_N^0 = x'_N$, but we also need $x_B^1 = x'_B(x'_N)$.

Imagine the optimization problem:

$$\begin{cases} Min & f(x, y) \\ s.t. & g(x, y) = 0 \iff y(x) = h(x) \end{cases}$$

with initial conditions $(x_0, y_0), g(x_0, y_0) = 0$. Then we consider $f(x, h(x))$ so we have $x_0 \to x_1 \to y' = (h(x_1))$. Then we solve $g(x_1, y) = 0$ using the Newton method, with initial point:

$$y' \approx h(x_0) + D_x h(x_0)(x_1 - x_0) + \mathcal{O}_2$$

where we have already seen that $D_x h(x_0) = D_{x_N}(x_B)$.

Penalty function methods

Consider the optimization problem

$$\begin{cases} Min & f(x, y) \\ s.t. & g_i(x) \leq 0, \ i \in 1, \dots m, \ x \in \mathbb{R}^n \end{cases}$$

The idea surrounding this method is to consider $h : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$h(y) = \begin{cases} 0, \text{ if } & y \leq 0 \\ \infty \text{ if } & y > 0 \end{cases}$$

Then, recalling the implicit function theorem gives us the equivalent unconstrained optimization problem:

$$\begin{cases} Min & \varphi(x) = f(x) + H(x), \ x \in \mathbb{R}^n \\ & H(x) = \sum_{i=1}^m h(g_i(x)) \end{cases}$$

However, note that we can not do anything practical here, ∞ is too much penalization.

Method of exterior penalties

Consider the equivalent penalty function:

$$h(y) = \begin{cases} 0, \text{ if } & y \leq 0 \\ y^2 \text{ if } & y > 0 \end{cases}$$

Then let us consider the optimizatio problem:

$$\begin{cases} Min & \varphi(x,r) = f(x) + rH(x),\ x \in \mathbb{R}^n,\ r > 0 \\ & H(x) = \sum_{i=1}^m h(g_i(x)) \end{cases}$$

where r is the penalty coefficient, and we can note that the function φ is $\mathcal{C}^1$, not $\mathcal{C}^2$.
Now we can do descent methods. We can find the minimum on the other side, if $r = 1$, or increase the penalty.

Interior penalty methods

Consider now the function $B(x) = -\sum_{i=1}^m \frac{1}{g_i(x)}$, and the set $X = \{x \in \mathbb{R}^n : g(x) \leq 0\}$. Note that

- $B(x) \geq 0$ on X
- $B(x) \to \infty$ if $x \to$ boundary of X

Then the optimization problem considered is:

$$\begin{cases} Min & \varphi(x,t) = f(x) + tB(x),\ x \in X^0,\ r > 0 \\ & B(x) = \sum_{i=1}^m \frac{1}{g_i(x)} \end{cases}$$

Note that now φ is as differentiable as f and g_i. Additionally, note that when we approach the boundary, we find a singularity, thus, if this function has a minimum it has to be in the inside of X. Observe that when considering t small, φ, f are similar in X^0, and therefore the minimum is close to the boundary.

4.4 Exercises on one-dimensional optimization

Exercise 1: *Let us consider the function $g(x) = -e^{-x^2}$, that has a unique minimum at $x = 0$. Note that $g'(x) < 0$ if $x < 0$ and $g'(x) > 0$ if $x > 0$, which implies that any reasonable descent method should be able to find the minimum, no matter the starting point. Instead, let us use a Newton's method on the function g' (i.e. to solve $g'(x) = 0$).*

(a) *Let $\{x_n\}_n$ be the sequence of points produced by the Newton's method starting at the seed $x_0 = 1$. Prove that $\lim_{n\to\infty} x_n = \infty$.*

Proof. Let us consider the succession generated by the Newton's method over the function $h(x) = g'(x)$ starting at point $x_0 = 1$:

$$x_{n+1} = x_n - \frac{h(x)}{h'(x)} = x_n - \frac{2x_n e^{-x_n^2}}{2e^{-x_n^2}(1 - 2x_n^2)} = x_n\Big(1 - \frac{1}{1 - 2x_n^2}\Big).$$

We want to see that this succession is divergent, i.e. $lim_{n\to\infty} x_n = \infty$. Let's suppose the succession does converge, and we may reach contradiction.

Consider the generating function of the n iterative point $f(x) = x\Big(1 - \frac{1}{1-2x^2}\Big)$. Note that the recursive succession will converge if it has any fixed point, i.e. $f(x) = x$, which only verifies for $x = 0$ (this can be trivially deduced by solving the corresponding equation $f(x) = x$ and obtaining imaginary solutions other than $x = 0$). Thus, if the succession converges, it will converge to the limit point $x = 0$. Note however that it starts at the initial point $x_0 = 1$, therefore we trivially have $x_1 = f(x_0) = 2$. Observe that this generating function is monotonously increasing in this interval since:

$$f'(x) = 1 - \frac{1}{1 - 2x^2} - \frac{4x^2}{(1 - 2x^2)^2} = \frac{2x^2(2x^2 - 3)}{(1 - 2x^2)^2},$$

therefore $f'(x) > 0 \iff 2x^2 - 3 > 0 \iff x > \sqrt{\frac{3}{2}}$. Note that clearly the function increases and therefore, as n increases for each new iteration it may not get closer to the fixed point, $x = 0$, as could be seen on Figure 4.1.

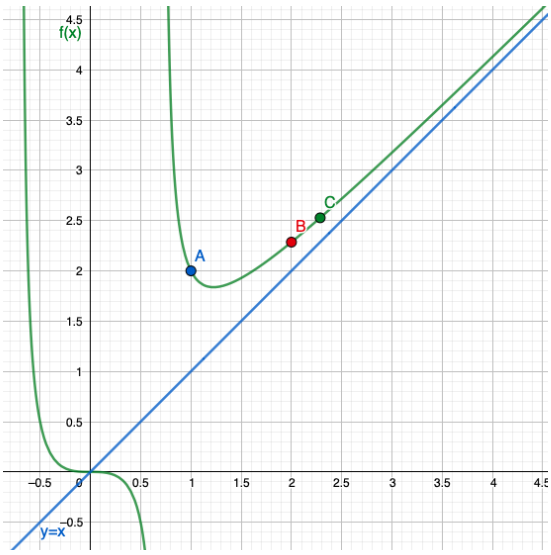


Figure 4.1: Illustration of the first points of the succession $\{x_n\}$ given by the Newton's method starting at point $x_0 = 1$.

Therefore, we will get further and further from the assumed limit point as n increases, contradicting the initial hypothesis. Then, the succession diverges, and thus $lim_{n\to\infty} x_n = \infty$ as we wanted to see. $\square$

(b) *Find a value $\alpha > 0$ such that if $x_0 \in [0, \alpha)$ the Newton's method converges to 0, and if $x_0 > \alpha$ the Newton's method diverges.*
Following the previous arguments, note that for $x_0 = 0.5$, the succession does not have limit since, $f(0.5) = -0.5$ and iteratively $f(-0.5) = 0.5$. Therefore we can find two sub-successions that converge to different limit points, implying directly that the succession does not have a limit.
However, note that for any $x_0 \in [0, 0.5)$ then the Newton's method converges. An illustration of the first computed points for the case $x_0 = 4$ is shown in Figure 4.2.

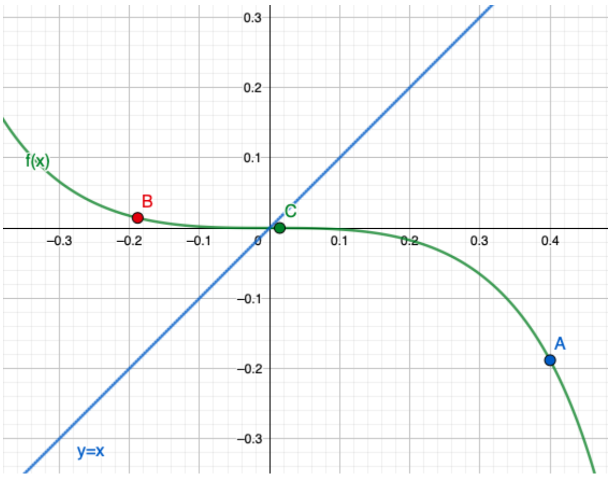


Figure 4.2: Illustration of the first points of the succession $\{x_n\}$ given by the Newton's method starting at point $x_0 = 1$.

Let's get a deepest sight into this problem. Note that we are seeking to apply the Newton method to the function $g'(x)$, i.e. we are solving $g'(x) = 2xe^{-x^2} = 0$, which is in fact a $\mathcal{C}^2$ function. Recall a known theorem that specifies:

Theorem 4.4.1. *Given $f : I \to \mathbb{R}$ a $\mathcal{C}^2$ function, and $x^* \in I$ a root of f, then it verifies:*

1. *Assume that there exist $\alpha, A, B > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$ we have $|f'(x)| \geq A$, $|f''(x)| \leq B$, then let x_0 be a initial value. Thus the succession given by the Newton method satisfies:*

$$|x_{n+1} - x^*| \leq \frac{B}{A}|x_n - x^*|^2$$

2. *Suppose that $f'(x^*) \neq 0$, then the existence of $\alpha, A, B > 0$ verifying the previous statement is assured.*

Observe that when computing the derivatives $h'(x) = g''(x)$ it does not vanish at point $x^* = 0$. Thus, we have the existence of the values as aforementioned. Note that for

$\alpha = 1/2$ we have

$$|h'(x)| = |g''(x)| = |4x^2 - 2|e^{-x^2} \geq |2x|e^{-x^2} \geq g(\frac{1}{2}) \approx 0.7788, \ \forall x \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)$$
$$|h''(x)| = |g^{(3)}(x)| = |8x^3 - 12x|e^{-x^2} \leq |8x^3 - 12x|$$
$$\leq |8x^3| + |12x| \leq 8\Big(\frac{1}{2}\Big) + 12\frac{1}{2} = 5, \ \forall x \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)$$

giving us the explicit values of $\alpha, A, B > 0$ which verify the statement *1.* of the previous theorem, ensuring quadratic convergence of the Newton's method, as wanted to prove.

Exercise 2. *Discuss if the following functions are unimodal:*

(a) $g(x) = x^3 - x$ *on* $x \in [-2, 0]$, *and on* $x \in [0, 2]$.
Let's find the optimal points of the function:

$$g'(x) = 2x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{2}}$$
$$g''(x) = 4x,$$

therefore, $x = -\frac{1}{\sqrt{2}} \in [-2, 0]$ is a maximum and $x = \frac{1}{\sqrt{2}} \in [0, 2]$ is a local minimum. Thus, in virtue of the Definition 4.1.1, we can affirm that the function is unimodal on the interval $[0, 2]$ but not on $[-2, 0]$.

(b) $g(x) = exp(-x)$ *on* $x \in [0, 1]$.
Let's consider the derivative of this $\mathcal{C}^1$ function:

$$g'(x) = -e^{-x} < 0, \ \forall x$$

Note that it is strictly decreasing for all x, and therefore, it has a local minimum at the extreme of the interval, $x = 1$. Thus this could be considered as an unimodal function, since it actually satisies the corresponding conditions.

(c) $g(x) = |x| + |x - 1|$ *on* $x \in [-2, 2]$.
Note that by definition of the absolute value we have

$$\frac{d}{dx}|x| = \begin{cases} 1, & x > 0 \\ -1, & x > 0 \end{cases}$$

Thus, taking this fact into consideration, the derivative of f is given by

$$g'(x) = \begin{cases} 1 + 1 = 2 & x \geq 1 \\ 1 - 1 = 0 & x \in (0, 1) \\ -1 - 1 = -2 & x < 0 \end{cases}$$

Therefore, recalling Definition 4.1.1 we can conclude that the function is not unimodal in the interval $[-2, 2]$, since the minimum value of the function is attained in an interval and thus, for any two values $\alpha_1, \alpha_2 \in (0, 1)$ the strict condition given in the definition is not satisfied.

4.5 Exercises on multi-dimensional optimization

Exercise 1. (Optional) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that -f is also a convex function. Prove that there exists $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $f(x) = a^T x + c$.*

Proof. Recall the definition of convexity of a function:

Definition 4.5.1. *$f : \mathbb{R}^n \to \mathbb{R}$, $dom(f) \subset \mathbb{R}^n$, is a convex function if*

1. *dom(f) is convex*
2. *$\forall x \in dom(f)$, $\forall y \in dom(f)$, $\forall \lambda \in [0,1]$, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. If the inequality is strict, i.e. $<$, then f is strictly convex.*

Thus, we have that for any $x, y \in \mathbb{R}^n$, $\forall \lambda \in [0,1]$ then, since f is convex

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

and given that $-f$ is convex

$$-f(\lambda x + (1-\lambda)y) \leq -\lambda f(x) - (1-\lambda)f(y) \iff f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

Consequently,

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$$

and we have that the function $f|_L$ is linear, this is that f restricted to the line between x and y, denoted as L is linear, and this holds for any two points from $\mathbb{R}^n$. Then we have that f is linear, and consequently, there exist $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $f(x) = a^T x + c$ as wanted to prove. □

Exercise 2. *Use the Kuhn-Tucker conditions to solve the following problems*

(a)

$$(P)\begin{cases} Min & f(x) = x_1 x_2 \\ s.t. & x_1 + x_2 \geq 2 \\ & x_2 \geq x_1 \end{cases} \iff \begin{cases} Min & f(x) = x_1 x_2 \\ s.t. & g_1(x) = 2 - x_1 - x_2 \leq 0 \\ & g_2(x) = x_1 - x_2 \leq 0 \end{cases}$$

According to the KKT theorem, since clearly the (CG) condition holds, then $x^0 \in X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}$ is a local minimum of (P) if there exist $\lambda_i \geq 0, i \in I$ called KKT multipliers such that

$$\begin{cases} \nabla f(x^0) + \sum_{i \in I} \lambda_i \nabla g_i(x^0) = 0 \\ \lambda_i g_i(x^0) = 0, \ \forall i \in I \end{cases}$$

Computing the corresponding gradients we have

$$\nabla f(x) = (x_2, x_1)$$
$$\nabla g_1(x) = (-1, -1)$$
$$\nabla g_2(x) = (1, -1)$$

thus, the KKT conditions are given as follows:

$$\begin{cases} x_2 - \lambda_1 + \lambda_2 = 0 \\ x_1 - \lambda_1 - \lambda_2 = 0 \\ \lambda_1(2 - x_1 - x_2) = 0 \\ \lambda_2(x_1 - x_2) = 0 \end{cases}$$

From the two first linear equations, we have $x_2 = \lambda_1 - \lambda_2$ and $x_1 = \lambda_1 + \lambda_2$. Now, regarding the last two equations, the nonlinear ones, we can see that they hold if

$$\lambda_1(2 - x_1 - x_2) = 0 \iff 2\lambda_1(1 - \lambda_1) = 0 \iff \begin{cases} \lambda_1 = 0 \\ \lambda_1 = 1 \end{cases}$$

$$\lambda_2(x_1 - x_2) = 0 \iff -\lambda_2^2 = 0 \iff \lambda_2 = 0$$

Thus, if $\lambda_2 = 0$, we have $x = (0, 0)$, but note that this point does not verify the restriction $g_1(x) \leq 0$. Therefore, consider the case $\lambda_1 = 1$, and we have $x = (1, 1)$, which in fact satisfies $g_1(x) = 2 - 1 - 1 = 0 \leq 0$. Then, if the optimization problem (P) has a minimum it is in $\hat{x} = (1, 1)$.

(b)

$$(P) \begin{cases} Min & f(x) = (x_1 - 1)^2 + x_2 - 2 \\ s.t. & x_2 - x_1 = 1 \\ & x_1 + x_2 - 2 \leq 0 \end{cases}$$

Recalling the KKT theorem for optimization problems that have equality constraints $h_i(x) = 0$, we have that a necessary condition for x^0 to be a local minimum of (P) is that there exist $\lambda_i \geq 0, (i \in I)$ and $\mu_j > 0, (j \in L)$ such that

$$\nabla f(x^0) + \sum_{i \in I} \lambda_i \nabla g_i(x^0) + \sum_{j \in L} \mu_j h_j(x^0) = 0$$
$$\lambda_i g_i(x^0) = 0, \ \forall i \in I$$

Computing the corresponding gradients as follows

$$\nabla f(x) = (2(x_1 - 1), 1)$$
$$\nabla g(x) = (1, 1)$$
$$\nabla h(x) = (-1, 1)$$

so the KKT conditions correspond to:

$$\begin{cases} 2(x_1 - 1) + \lambda - \mu = 0 \\ 1 + \lambda + \mu = 0 \\ \lambda(x_1 + x_2 - 2) = 0 \end{cases}$$

Following the same procedure as before, note that from the linear equations we get $\lambda = -1 - \mu$ and $x_1 = 1 + \frac{\mu - \lambda}{2} = \frac{3}{2} + \mu$. Thus, by considering the last equation we have:

$$\lambda(x_1 + x_2 - 2) = 0 \iff \begin{cases} \lambda = 0 \\ (x_1 + x_2 - 2) = 0 \end{cases}$$

Note that if $\lambda = 0$, then $\mu = -1$, $x_1 = \frac{1}{2}$. Therefore, from the constraint $h(x) = 0$ we get $x_2 = \frac{3}{2}$, which also verifies $g(x) = 0 \leq 0$. Then, $\hat{x} = (\frac{1}{2}, \frac{3}{2})$ is an optimal solution of (P).

Note that by considering the other condition we obtain the same result, since we get $(x_1 + x_2 - 2) = 0 \iff (x_1, x_2) = (\frac{3}{2} + \mu, \frac{1}{2} - \mu)$, and when restricting this solution to the remaining constraint we get $h(x^0) = 0 \iff \mu = -1$ obtaining again the point $\hat{x}$.

(c)

$$(P) \begin{cases} Min & f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\ s.t. & x_1 - x_2 - 2x_3 - 12 \leq 0 \\ & x_1 + 2x_2 - 3x_3 - 8 \leq 0 \end{cases}$$

Consider the corresponding gradients:

$$\begin{aligned} &\nabla f(x) = (2x_1, 4x_2, 6x_3) \\ &\nabla g_1(x) = (1, -1, -2) \\ &\nabla g_2(x) = (1, 2, -3) \end{aligned}$$

Then, the KKT conditions are given as follows:

$$\begin{cases} 2x_1 + \lambda_1 + \lambda_2 = 0 \\ 4x_2 - \lambda_1 + 2\lambda_2 = 0 \\ 6x_3 - 2\lambda_1 - 3\lambda_3 = 0 \\ \lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \\ \lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \end{cases}$$

Following again the same procedure as previously, from the linear equations we have

$$x_1 = -\frac{\lambda_1 + \lambda_2}{2}, \quad x_2 = \frac{\lambda_1 - 2\lambda_2}{4}, \quad x_3 = \frac{2\lambda_1 + 3\lambda_2}{3}$$

Consider now the non-linear part of the KKT system. Note that we have 4 possibilities:

$$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \iff \begin{cases} \lambda_1 = 0 \\ (x_1 - x_2 - 2x_3 - 12) = 0 \end{cases}$$

$$\lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \iff \begin{cases} \lambda_2 = 0 \\ (x_1 + 2x_2 - 3x_3 - 8) = 0 \end{cases}$$

Consider the case in which $\lambda_1 = 0 = \lambda_2$. Note that for this case we have $\hat{x} = (0, 0, 0)$ which actually verifies all the constraints $g_i(\hat{x}) \leq 0$, and in fact gives the minimum possible value for the optimization function, since $f(x) \geq 0$, $\forall x$, and is therefore the optimal solution of (P).