Topological Data Analysis

2022–2023

Lecture 9

Stability Theorem

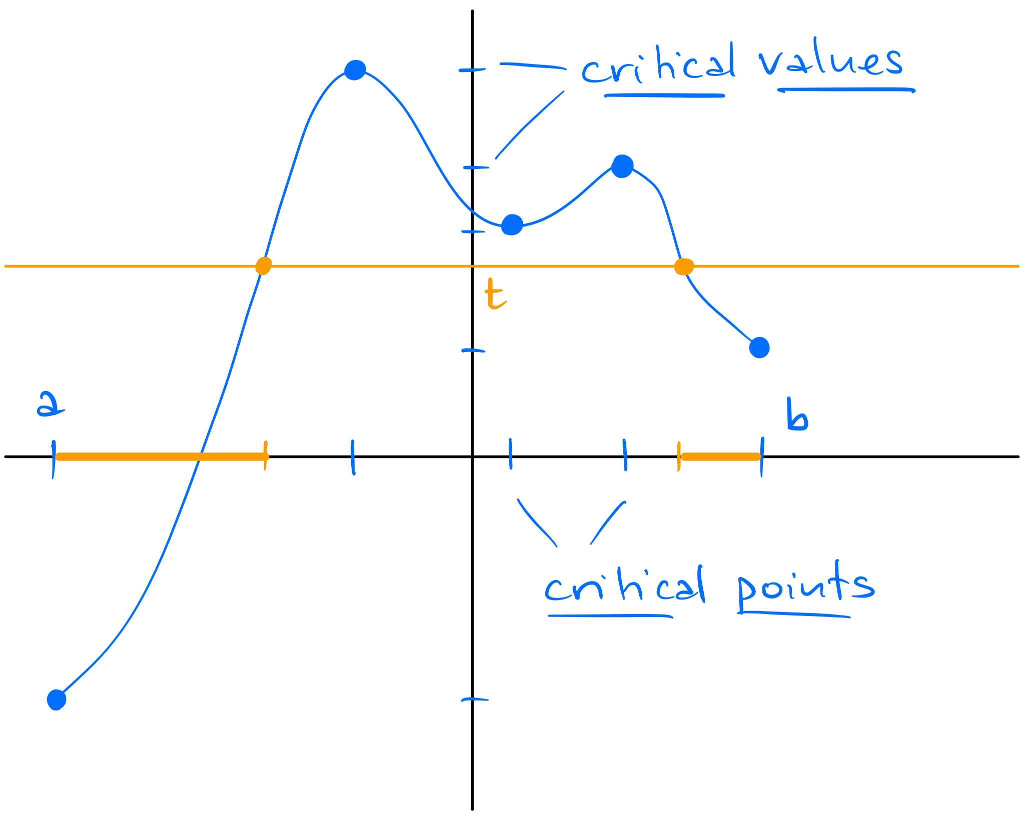
1 December 2022

Sublevel sets

For a continuous function $f:[a,b]\to\mathbb{R}$ denote, for each $t\in\mathbb{R}$,

$$L_t(f)=\{x\in[a,b]\mid f(x)\le t\},$$ called a <u>sublevel set</u>.

Note that if $s\le t$ then $L_s(f)\subseteq L_t(f)$.



$L_t(f)=\emptyset$ if $t<\inf(f)$

$L_t(f)=[a,b]$ if $t\ge\sup(f)$

We call $x_0\in[a,b]$ a <u>critical point</u> if it is a local maximum or a local minimum, including $x_0=a$ and $x_0=b$.

If x_0 is a critical point, then $f(x_0)$ is a <u>critical value</u>.

If f is differentiable and $x_0\in(a,b)$ is a critical point, then $f'(x_0)=0$.

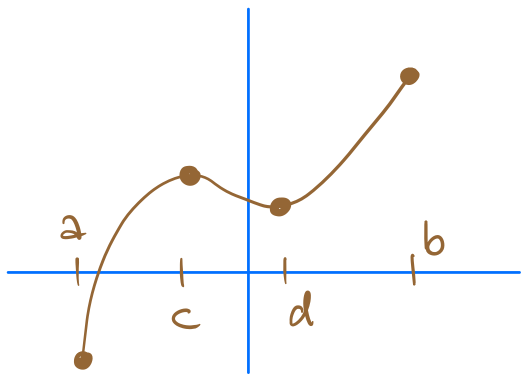
From now on we assume that f has finitely many critical points (hence each critical point is isolated).

Under this assumption, we associate to f a persistence module:

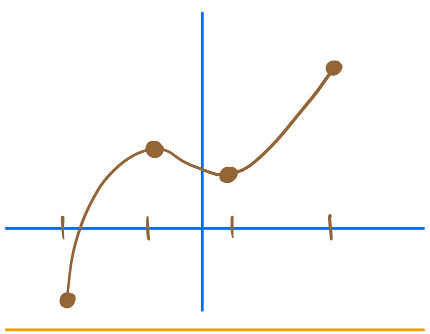
$$V_t(f) = H_0(L_t(f)),$$ where H_0 denotes zero-homology,

and we let $\pi_{s,t} : V_s(f) \to V_t(f)$ be induced by the inclusion $V_s(f) \hookrightarrow V_t(f)$ if $s \leq t$.

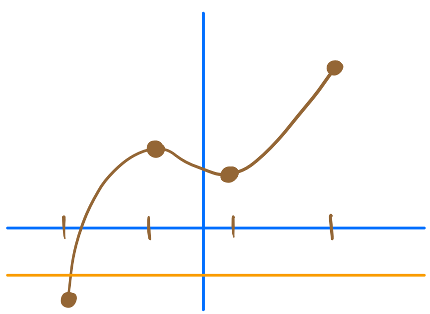
The spectrum of (V, π) is contained in the set of critical points of f.



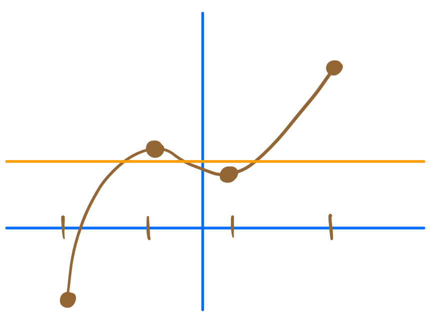
Here b is a critical point of f but the spectrum of $V(f)$ is $\{a, c, d\}$.



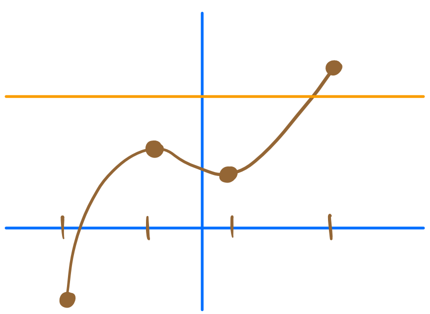
$\beta_0 = 0$



$\beta_0 = 1$



$\beta_0 = 2$



$\beta_0 = 1$

Stability Theorem:

$$\boxed{d_{int}(V(f), V(g)) \leq \|f-g\|_\infty}$$

Here $\|f-g\|_\infty = \sup\{ |f(x)-g(x)| : a \leq x \leq b \}$.