1 Exercises 1: Fourier Series

Exercise 8. *Let γ be a simple closed curve in $\mathbb{R}^2$ of length l, and let A denote the area of the region enclosed by this curve. Then*

$$A \leq \frac{l^2}{4\pi}$$

with equality if and only if γ is a circle.
Notice that when γ is a circle of length l then the radius is $l/(2\pi)$ and therefore the area is $l^2/(4\pi)$.

(a) *See that it is enough to consider the case $l = 2\pi$ and that γ is parametrised by the arc-length, so that the inequality to be proved has the form $A \leq \pi$*

Proof. By definition, the area enclosed by the simple closed curve γ is given by

$$A = \iint_D 1 dxdy$$

where D is the domain of the area defined by A. Since the curve is simple and closed, then it has a periodic regular parametrization with period T such that $\gamma|_{[t_0,t_1]}$ is injective. Consider $\gamma(t) = (x(t), y(t)),\ t \in [t_0, t_1]$. Then, the length of the curve is defined as

$$l = \int_\gamma \|\gamma'(t)\| dt.$$

Notice that we can define an alternative curve in terms of the initial one as follows: $\varphi(t) = \frac{2\pi\gamma(t)}{l}$, which has length:

$$l_\varphi = \int_0^{t_1} \|\varphi'(t)\| dt = \frac{2\pi}{l} \int_\gamma \|\gamma'(t)\| dt = 2\pi$$

Recall now the Green's Theorem:

Theorem 1.1. *Let C be a positively oriented, piecewise smooth, simple, closed curve, and let D be the region enclosed by the curve. Then if $f_1(x, y), f_2(x, y)$ have continuous first order partial derivatives on D, then*

$$\int_C (f_1(x, y)dx + f_2(x, y)dy) = \iint_D \left(\frac{\partial f_2(x, y)}{\partial x} - \frac{\partial f_1(x, y)}{\partial y} \right) dxdy$$

Then, by considering the function $f(x, y) = (f_1(x, y), f_2(x, y)) = (0, x)$ we have $\frac{\partial f_2(x,y)}{\partial x} - \frac{\partial f_1(x,y)}{\partial y} = 1$, and therefore the area enclosed by the curve γ can be expressed as follows:

$$A = \iint_D 1 dxdy = \oint_\gamma x dy \tag{1}$$

Observe now that, by the definition considered of the curve φ, we have the parametrization $\varphi(t) = (x_\varphi(t), y_\varphi(t)) = \frac{2\pi}{l}(x(t), y(t))$. Hence, the corresponding differential with respect to t is computed as follows $dy_\varphi = y'_\varphi(t)dt = \frac{2\pi}{l} y'(t)dt = \frac{2\pi}{l} dy$. Consequently, we can express the area given by the expression (1) as follows:

$$A = \frac{l^2}{4\pi^2} \oint_C x_\varphi(t) dy_\varphi = \frac{l^2}{4\pi^2} A_\varphi \tag{2}$$

Thus, observe that if we assume that the curve φ with length $l_\varphi = 2\pi$ verifies the isoperimetric inequality, i.e. $A_\varphi \leq \frac{l_\varphi^2}{4\pi} = \pi$ then, from (2) we directly have that $A \leq \frac{l^2}{4\pi^2}\pi = \frac{l^2}{4\pi}$. This is, if we consider the closed curve φ, to be parametrised by the arc-length, then the isoperimetric inequality $A \leq \frac{l^2}{4\pi}$ is equivalent to $A_\varphi \leq \pi$. Consequently, we may study the equivalent condition given as $A \leq \pi$ for a curve with length 1 in the subsequent steps. ☐

(b) *Prove that*

$$A = \frac{1}{2}\int_\gamma (xdy - ydx) = \frac{1}{2}\left|\int_\gamma (xdy - ydx)\right|$$

Proof. From the previous section and taking into account the Green's Theorem we deduce from (1) that:

$$A = \int_\gamma xdy = \int_{t_0}^{t_1} x(t)y'(t)dt$$

Now, since $\gamma(t)$ is a closed curve, which we assumed to be parametrised by the arc-length, as a consequence of the previous argumenrs, then we have $\gamma(t_0) = \gamma(t_1)$. Hence, in virtue of the Fundamental Theorem of Integration and Differentiation we have

$$\begin{aligned} A = \int_{t_0}^{t_1} x(t)y'(t)dt &= \int_{t-0}^{t_1} x'(t)y'(t)dt - \int_{t_0}^{t_1} x'(t)y(t)dt \\ &= [x(t_1)y(t_1) - x(0)y(0)] - \int_{t_0}^{t_1} x'(t)y(t)dt = -\int_{t_0}^{t_1} x'(t)y(t)dt \end{aligned}$$

Then, by following similar arguments, we have

$$\begin{aligned} A &= \int_{t_0}^{t_1} x(t)y'(t)dt = \frac{1}{2}\int_{t_0}^{t_1} x(t)y'(t)dt + \frac{1}{2}\int_{t_0}^{t_1} x(t)y'(t)dt \\ &= \frac{1}{2}\int_{t_0}^{t_1} x(t)y'(t)dt - \frac{1}{2}\int_{t_0}^{t_1} x'(t)y(t)dt = \frac{1}{2}\int_{t_0}^{t_1} (x(t)y'(t)dt - x'(t)y(t)dt) \\ &= \frac{1}{2}\int_\gamma (xdy - ydx) \end{aligned}$$

In order to prove the second equality it is enough to see that the integral is positive.
Observe that when considering the field $f(x, y) = (0, x)$ when applying the Green's theorem, we are setting the positive orientation of the closed curve. Hence, when considering this vector field, the positive orientation is given by the counterclockwise orientation. This can be seen directly from the graphical representation of the vectorfield and different locations of the closed curve, or either by considering the positive orientation given when considering the flow enclosed in the area in the direction $\vec{k} = \vec{i} \times \vec{j}$ (when studying the system from the physical point of view).
Therefore, since the orientation taken is positive and we are considering the according area differential then the area computed is positive, and hence the second equality holds. □

(c) *Let* $\gamma(t) = (x(t), y(t)),\ t \in [0, 2\pi]$ *be the parametrisation given by the arc-length and let* $x(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}, y(t) = \sum_{n\in\mathbb{Z}} b_n e^{int}$ *be the corresponding Fourier series. Prove that*

$$\sum_{n\in\mathbb{Z}} |n|^2(|a_n|^2 + |b_n|^2) = 1$$

Proof. Firstly, observe that if we take the arc-lenght parametrisation we have

$$\|\gamma'(t)\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |\gamma'(t)|^2 dt = 1.$$

On the other hand, we have $\|\gamma'(t)\|_2^2 = \|x'(t)\|_2^2 + \|y'(t)\|_2^2$. Now, notice that since we are considering the curve as simple closed, then $\gamma \in L^2[0, 2\pi]$, and hence $x, y \in L[0, 2\pi]$. Therefore the Fourier series of each component converge to x, y, respectively. Thus, the corresponding derivative of the series is computed by differentiating each component, this is

$$x'(t) = \sum_{n\in\mathbb{Z}} a_n(in)e^{int}$$

Now, the norm of each component is computed as follows

$$\begin{aligned}\|x'(t)\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} |x'(t)|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} x'(t)\overline{x'(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m\in\mathbb{Z}} a_n\overline{a_m}(nm)e^{i(n-m)t}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}} |a_n|^2 n^2 dt = \sum_{n\in\mathbb{Z}} |a_n|^2 n^2\end{aligned}$$

where the last two equalities come from the fact that $e_n(t) = e^{int}$ forms an orthonormal basis over $L^2[0, 2\pi]$.
Similarly, we have $\|y'(t)\|_2^2 = \sum_{n\in\mathbb{Z}} |b_n|^2 n^2$, so that we have:

$$\|\gamma'(t)\|_2^2 = \sum_{n\in\mathbb{Z}} |a_n|^2 n^2 + \sum_{n\in\mathbb{Z}} |b_n|^2 n^2 = \sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) = 1.$$

$\square$

(d) *Prove that*

$$A = \pi\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right|$$

and deduce that $A \leq \pi$

Proof. From section (b) and following a similar approach as in the previous section we have

$$\begin{aligned}A &= \frac{1}{2}\left|\int_\gamma (xdy - ydx)\right| = \frac{1}{2}\left|\int_0^{2\pi} (x(t)\overline{y'(t)}dt - y(t)\overline{x'(t)}dt)\right| \\ &= \frac{1}{2}\left|\int_0^{2\pi} \left(\sum_{n\in\mathbb{Z}} a_n e^{int}\sum_{n\in\mathbb{Z}}\overline{b_n}(-in)e^{-int} - \sum_{n\in\mathbb{Z}} b_n e^{int}\sum_{n\in\mathbb{Z}}\overline{a_n}(-in)e^{-int}\right)dt\right| \\ &= \frac{1}{2}\left|\int_0^{2\pi} \left(\sum_{n,m\in\mathbb{Z}} a_n\overline{b_m}(-im)e^{i(n-m)t} - \sum_{n,m\in\mathbb{Z}} b_n\overline{a_m}(-im)e^{i(n-m)t}\right)dt\right| \\ &= \frac{1}{2}\left|\int_0^{2\pi} (-i)\left(\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right)dt\right| = \pi\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right|.\end{aligned}$$

For the second part, we want to deduce that $A \leq \pi$, i.e.

$$\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right| \leq 1$$

Applying the triangular inequality:

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right| &\leq \sum_{n\in\mathbb{Z}} |n(a_n\overline{b_n} - b_n\overline{a_n})| \\ &\leq \sum_{n\in\mathbb{Z}} n^2|a_n\overline{b_n} - b_n\overline{a_n}|.\end{aligned}$$

Now, observe that

$$\|a_n - b_n\|^2 = (a_n - b_n)\overline{(a_n - b_n)} = |a_n|^2 + |b_n|^2 - a_n\overline{b_n} - b_n\overline{a_n} \geq 0$$

Therefore, $|a_n|^2 + |b_n|^2 \geq (a_n\overline{b_n} + b_n\overline{a_n})$. Hence,

$$\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right| \leq \sum_{n\in\mathbb{Z}} n^2(|a_n\overline{b_n}| + |b_n\overline{a_n}|) \leq \sum_{n\in\mathbb{Z}} |n|^2(|a_n|^2 + |b_n|^2) = 1 \tag{3}$$

□

(e) *Assume $A = \pi$. Prove that*

$$x(t) = a_{-1}e^{-it} + a_0 + a_1e^{it}, \quad y(t) = b_{-1}e^{-it} + b_0 + b_1e^{it},$$

with $a_{-1} = \overline{a}_{-1}, b_{-1} = \overline{b}_{-1}$, and deduce that γ is a circle.

Proof. In order to have the equality, we need $A = \pi$ which, by the previous section, is equivalent to considering

$$\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right| = 1.$$

Observe that this is then equivalent to consider all the inequalities of the expression (3) to be equalities, this is

$$\left|\sum_{n\in\mathbb{Z}} n(a_n\overline{b_n} - b_n\overline{a_n})\right| = \sum_{n\in\mathbb{Z}} n^2(|a_n\overline{b_n}| + |b_n\overline{a_n}|) \iff |n| = n^2 \iff n = 0, 1.$$

Therefore, we have $a_n = b_n = 0,\ \forall n \geq 2$. Hence, we get

$$x(t) = a_{-1}e^{-it} + a_0 + a_1e^{it}, \quad y(t) = b_{-1}e^{-it} + b_0 + b_1e^{it}.$$

Now, since we are considering the real curve γ, we have x, y real valued, and therefore, by definition of the Fourier coefficients:

$$a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-int}dt = \overline{a}_n.$$

In particular, $a_{-1} = \overline{a}_{-1}, b_{-1} = \overline{b}_{-1}$, as wanted to prove.
Finally, from the second inequality of (3) together with the aforementioned deductions, we have

$$|a_1\overline{b}_1| + |b_1\overline{a}_1| = |a_1|^2 + |b_1|^2 \iff |a_1| = |b_1|$$

and consequently $\gamma(t) = 2(Re(a_1), Re(b_1))cos(t) - 2(Im(a_1), Im(b_1))sin(t) + (a_0, b_0)$ is clearly a circle[1]. □

[1] Here, the notation used is: $a_1 = Re(a_1) + Im(a_1)i$, and the deduction of the expression comes directly from using this notation and the expression $e^{\pm it} = cos(t) \pm i\ sin(t)$ on the definitions of $x(t), y(t)$.