1 Exercises on Topological Data Analysis

1.1 Delivery 2

Exercise. *Find the homology groups with coefficients in $\mathbb{Z}$ of the abstract simplicial complex whose maximal faces are:*

$$(12)\ (13)\ (14)\ (23)\ (25)\ (36)\ (456)$$

Proof. Consider the abstract simplicial complex as given by K, represented on figure 1.

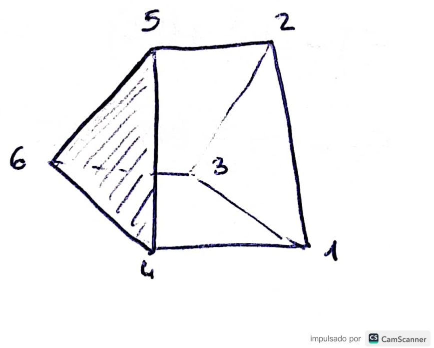

Figure 1: Qualitative representation of K.

Consider the corresponding free $\mathbb{Z}$-modules defined by the n-faces of K as follows[1]:

$$
\begin{aligned}
C_0(K) &= \mathbb{Z}(1) \oplus \mathbb{Z}(2) \oplus \mathbb{Z}(3) \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(5) \oplus \mathbb{Z}(6) \\
C_1(K) &= \mathbb{Z}(12) \oplus \mathbb{Z}(13) \oplus \mathbb{Z}(14) \oplus \mathbb{Z}(23) \oplus \mathbb{Z}(25) \oplus \mathbb{Z}(36) \oplus \mathbb{Z}(45) \oplus \mathbb{Z}(46) \oplus \mathbb{Z}(56) \\
C_2(K) &= \mathbb{Z}(456)
\end{aligned}
$$

Recalling the definition of the nth boundary group of homomorphism, ∂_n, we have corresponding $\mathbb{Z}$-modules' chain:

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Where the n-th boundary groups are defined as follows over K:

$$\partial_0(i_0) = 0,\ \forall i_0 = 1, \dots 6, \Rightarrow Im(\partial_0) = 0.$$

Hence, clearly $ker(\partial_0) = C_0 \cong \mathbb{Z}^6 = \langle (1), (2), (3), (4), (5), (6) \rangle$.
Similarly we have

$$
\begin{aligned}
\partial_1(12) &= (2) - (1) \\
\partial_1(13) &= (3) - (1) \\
\partial_1(14) &= (4) - (1) \\
\partial_1(23) &= (3) - (2) = \partial_1(13) - \partial_1(12) \\
\partial_1(25) &= (5) - (2) = \partial_1(45) + \partial_1(14) - \partial_1(12) \\
\partial_1(36) &= (6) - (3) = \partial_1(46) + \partial_1(14) - \partial_1(13) \\
\partial_1(45) &= (5) - (4) \\
\partial_1(46) &= (6) - (4) \\
\partial_1(56) &= (6) - (5) = \partial_1(46) - \partial_1(45)
\end{aligned}
$$

[1]By the sake of simplicity we may denote $C_n(K, \mathbb{Z}) = C_n(K)$.

which has been directly computed regarding the cycles observed on K. This could be similarly computed by defining the matrix of ∂_1 and by column-reduction. Hence, we get

$$
\begin{aligned}
&Im(\partial_1) = \langle \partial_1(12), \partial_1(13), \partial_1(14), \partial_1(45), \partial_1(46) \rangle \\
&Ker(\partial_1) = \langle (23) - (13) + (12), (56) - (46) + (45), (36) - (46) - (14) + (13), (25) - (45) - (14) + (12) \rangle
\end{aligned}
$$

and therefore $rank(\partial_1) = 5$.
Following the same procedure we get:

$$\partial_2(456) = (45) - (46) + (56) \Rightarrow Im(\partial_2) = \langle \partial_2(456) \rangle, \text{ and thus } rank(\partial_2) = 1,\ ker(\partial_2) = \{0\}$$

Then, the corresponding homology groups are given, as described by definition as:

$$
\begin{aligned}
H_0(K) &= {}^{ker(\partial_0)}\big/_{Im(\partial_1)} = {}^{C_0}\big/_{\langle \partial_1(12), \partial_1(13), \partial_1(14), \partial_1(45), \partial_1(46) \rangle} \cong \mathbb{Z} \\
H_1(K) &= {}^{ker(\partial_1)}\big/_{Im(\partial_2)} \cong \langle (23) - (13) + (12), (36) - (46) - (14) + (13), (25) - (45) - (14) + (12) \rangle \\
&\cong \mathbb{Z}^3 \\
H_2(K) &= {}^{ker(\partial_2)}\big/_{Im(\partial_3)} = \{0\}
\end{aligned}
$$

since clearly $Im(\partial_3) = \{0\}$, and we have seen that $ker(\partial_2) = \{0\}$. □