Complex Networks - Homework 2

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Homework 2.1

The number of edges in $ER_n(p)$ is $\sum_e Y_e$ where $Y_e = 1_{\{e\ is\ retained\}}$ are i.i.d. random variables taking the value 1 with probability p. This sum runs over all $\binom{n}{2}$ edges of the complete graph with n vertices. We can see then that $P(X = k)$, the probability of having exactly k edges, is equal to the probability of having exactly k successes in $\binom{n}{2}$ Bernoulli trials with rate p. This is equal to the binomial distribution $Bin(\binom{n}{2}, p)$. So

$$P(X = k) = \binom{\binom{n}{2}}{k} p^k (1-p)^{\binom{n}{2}-k} \tag{1}$$

because it is a binomial distribution with parameters $(\binom{n}{2}, p)$ with

$$\mathbb{E}(X) = \binom{n}{2} p = \frac{n!}{2(n-2)!} p = \frac{n(n-1)}{2} p \tag{2}$$

$$Var(X) = \binom{n}{2} p(1-p) = \frac{n(n-1)}{2} p(1-p) \tag{3}$$

Furthermore the binomial distribution satisfies both the Central Limit Theorem (CLT) and the Law of Large Numbers (LLN) so it follows that our distribution satisfies them too. The Normal distribution has mean and variance of the same form as the Binomial distribution.

Homework 2.2

We know that the probability that a vertex v_i has degree k equals

$$f_{n-1,p}(k) = \mathbb{P}(N_1 = k) \tag{4}$$

and from Exercise 2.3, we found out that

$$\mathbb{P}(N_1 = k) = \binom{n-1}{k} p^k (1-p)^{n-k-1} \tag{5}$$

We're interested in analysing the case of n increasing towards infinite, so we compute the limit of this distribution.

$$\lim_{n\to+\infty} f_{n-1,\frac{\lambda}{n}}(k) = \lim_{n\to+\infty} \binom{n-1}{k} p^k (1-p)^{n-k-1} \tag{6}$$

We rewrite the factorial to ease our calculations.

$$\binom{n-1}{k} = \frac{(n-1)!}{k!(n-k-1)!} = \frac{(n-1)(n-2)...(n-k)}{n!} = \frac{n^k(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k}{n})}{k!} = \frac{n^k}{k!}\prod_{i=1}^{k} 1 - \frac{i}{n} \tag{7}$$

Now we use $p = \frac{\lambda}{n}$.

$$\lim_{n\to+\infty} f_{n-1,\frac{\lambda}{n}}(k) = \lim_{n\to+\infty} \left[\frac{(n-1)^k}{k!} \left(\prod_{i=1}^{k} 1 - \frac{i}{n} \right) \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k-1} \right] \tag{8}$$

We can solve this equation by considering it as a product of three different limits and by tackling each of them independently.

$$\lim_{n\to+\infty} f_{n-1,\frac{\lambda}{n}}(k) = \left[\lim_{n\to+\infty} \frac{(n-1)^k}{k!} \frac{\lambda^k}{n^k} \right] \cdot \left[\lim_{n\to+\infty} \left(\prod_{i=1}^{k} \left(1 - \frac{i}{n}\right) \right) \right] \cdot \left[\lim_{n\to\infty} \left(1 - \lambda n^{-1}\right)^{n-k-1} \right] \tag{9}$$

- For the first limit, we use that for $n \to +\infty$, $(n-1)^k = \Theta(n^k)$, so it becomes

$$\lim_{n\to+\infty} \left[\frac{(n-1)^k}{k!} \frac{\lambda^k}{n^k} \right] = \lim_{n\to+\infty} \left[\frac{n^k}{k!} \frac{\lambda^k}{n^k} \right] = \lim_{n\to+\infty} \left[\frac{\lambda^k}{k!} \right] = \frac{\lambda^k}{k!} \tag{10}$$

- For the second limit, we can see that, as $n \to +\infty$, all elements following 1 in the product tend to 0, so the total of the product is equal to 1, and consequentially the limit tends to that too.

$$\lim_{n\to+\infty} \left(\prod_{i=1}^{k} \left(1 - \frac{i}{n}\right) \right) = 1 \tag{11}$$

- For the last limit, we can see that if it exists a limit for $log(f(x))$ then it also exists for $f(x)$. So we get that

$$log\left(\left(1 - \frac{\lambda}{n}\right)^{n-k-1} \right) = (n-k-1) \cdot \log\left(1 - \frac{\lambda}{n}\right) = (n-k-1) \left[-\sum_{m=1}^{\infty} \frac{(\frac{\lambda}{n})^m}{m} \right] = \tag{12}$$

$$= \left[n\left(1 - \frac{k+1}{n}\right) \right] \left[-\sum_{m=1}^{\infty} \frac{(\frac{\lambda}{n})^m}{m} \right] = \left[n\left(1 - \frac{k+1}{n}\right) \right] \left(-\frac{\lambda}{n}\right) + \left[-\sum_{m=2}^{\infty} \frac{(\frac{\lambda}{n})^m}{m} \right] \left[n\left(1 - \frac{k+1}{n}\right) \right] \tag{13}$$

For $lim_{n\to\infty}$ we have that $\frac{k+1}{n} = 0$ so

$$\left[n\left(1 - \frac{k+1}{n}\right) \right] \left(-\frac{\lambda}{n}\right) \to -\lambda \tag{14}$$

For the second part of the equation, we got that

$$\left[n\left(1 - \frac{k+1}{n}\right) \right] \left[-\sum_{m=2}^{\infty} \frac{(\frac{\lambda}{n})^m}{m} \right] = \left(1 - \frac{k+1}{n}\right) \left[-\sum_{m=2}^{\infty} \frac{1}{m} \left(\frac{\lambda^m}{n^{m-1}}\right) \right] \tag{15}$$

And it can be seen that as $n \to \infty$ this quantity is equal to 0. We're left with

$$\lim_{n\to\infty} (n-k-1) log\left(1 - \frac{\lambda}{n}\right) \to -\lambda \tag{16}$$

So, it is demonstrated that

$$\lim_{n\to\infty} (1 - \lambda n^{-1})^{n-k-1} = e^{-\lambda} \tag{17}$$

After all of the substitutions, we now have

$$\lim_{n\to+\infty} f_{n-1,\frac{\lambda}{n}}(k) = e^{-\lambda} \frac{\lambda^k}{k!} \tag{18}$$

which proves that as n increases, the probability that a vertex v_i has degree k follows a Poisson distribution with parameter λ.

Homework 2.3

Let the total number of wedges and triangles in the Erdös-Rényi random graph be denoted as $W_{ER_n(\lambda/n)}$ and $\Delta_{ER_n(\lambda/n)}$ respectively. The average number of wedges is then defined as:

$$\mathbb{E}(W_{ER_n(\lambda/n)}) = 3\binom{n}{3}p^2 \tag{19}$$

where $\binom{n}{3}$ is the total number of possible combinations of three vertices in the random graph and p is the probability that an edge exists between two vertices.

Similarly we define the average number of triangles as:

$$\mathbb{E}(\Delta_{ER_n(\lambda/n)}) = \binom{n}{3}p^3 \tag{20}$$

We then observe what happens as n increases by also considering that $p = \frac{\lambda}{n}$.

$$\lim_{n\to+\infty} n^{-1}\mathbb{E}(W_{ER_n(\lambda/n)}) = \lim_{n\to+\infty} \frac{3n(n-1)(n-2)}{6n}\frac{\lambda^2}{n^2} = \lim_{n\to+\infty} \frac{3n^3(1-\frac{3}{n}+\frac{2}{n^2})}{6n^3}\lambda^2 = \frac{1}{2}\lambda^2 \tag{21}$$

$$\lim_{n\to+\infty} \mathbb{E}(\Delta_{ER_n(\lambda/n)}) = \lim_{n\to+\infty} \frac{n(n-1)(n-2)}{6}\frac{\lambda^3}{n^3} = \lim_{n\to+\infty} \frac{n^3(1-\frac{3}{n}+\frac{2}{n^2})}{6n^3}\lambda^3 = \frac{1}{6}\lambda^3 \tag{22}$$

We use equation (2.11) to calculate the cluster coefficient C of the random graph.

$$C = \lim_{n\to+\infty} \frac{\mathbb{E}(\Delta_{ER_n(\lambda/n)})}{\mathbb{E}(W_{ER_n(\lambda/n)})} = \lim_{n\to+\infty} \frac{\binom{n}{3}p^3}{3\binom{n}{3}p^2} = \lim_{n\to+\infty} \frac{p}{3} = \lim_{n\to+\infty} \frac{\lambda}{3n} = 0 \tag{23}$$

This means that as n increases C decreases, meaning that the graph appears less and less clustered, having more of a tree shape.