Complex Networks - Homework 1

Group 1: Luna Giesselbach, Iwan Pasveer,
Flavia Leotta, Noah Frinkings, Dylan Gavron

18^{th} September 2024

Homework 1.1

The entries b_{ij} and b_{ji} of the undirected adjacency matrix are always 1 when either of the entries a_{ij} or a_{ji} of the directed adjacency matrix is 1. The entries of the directed adjacency matrix are not always 1 if the entry of the undirected adjacency matrix is 1.

$$b_{ij} = b_{ji} = \begin{cases} 0 & if\ a_{ij} = 0\ and\ a_{ji} = 0 \\ 1 & else \end{cases} \tag{1}$$

or, mathematically:

$$b_{ij} = b_{ji} = max\{a_{ij}, a_{ji}\} \tag{2}$$

For fig. 1.1, the adjacency matrix is:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{3}$$

Using the expression (2) formulated above, we can compose the following adjacency matrix for fig. 1.2:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{4}$$

This is in line with fig. 1.2, as all links are described by this matrix.
Fig. 1.3 has exactly the same adjacency matrix as fig. 1.2, as all links are reciprocated. For all links we have $a_{ij} = a_{ji} = b_{ij} = b_{ji}$ and thus the expression in (2) is correct as well.

Homework 1.2

We start with the empirical degree distribution equation

$$P(k) \propto k^{-\gamma} \tag{5}$$

as given by (1.6) in the lecture notes. Computing the empirical cumulative distribution we use equations (1.8) from the lecture notes:

$$P_{>}(k) = \sum_{k' \geq k} P(k') \tag{6}$$

when approximating k with a continuous variable and, combining it with (5), we yield:

$$P_{>}(k) = \int_{k}^{\infty} P(k)dk = \int_{k}^{\infty} k^{-\gamma}dk = \left[\frac{k^{-\gamma+1}}{1-\gamma}\right]_{k}^{\infty} = \tag{7}$$

$$\frac{1}{-\gamma+1} \cdot 0 - \frac{1}{-\gamma+1} \cdot k^{-\gamma+1} = \frac{1}{-\gamma+1} \cdot k^{-\gamma+1}. \tag{8}$$

So, we prove that:

$$P_{>}(k) \propto k^{-\gamma+1}. \tag{9}$$

Now, consider $P(k) \propto e^{-ak}$, once again we approximate k with a continuous variable. In this case the empirical cumulative distribution will be as follows:

$$P_{>}(k) \propto \int_{k}^{\infty} P(k)dk = \int_{k}^{\infty} e^{-ak}dk = \tag{10}$$

$$P_{>}(k) \propto \left[\frac{e^{-ak}}{-a}\right]_{k}^{\infty} = \frac{1}{-a} \cdot 0 - \frac{1}{-a}e^{-ak} \tag{11}$$

So in this case we get:

$$P_{>}(k) \propto e^{-ak}. \tag{12}$$

For $P_{>}^{in}$ and $P_{>}^{out}$. We use expression (1.7) of the lecture notes:

$$P^{in}(k^{in}) \propto (k^{in})^{-\gamma_{in}} \tag{13}$$

$$P^{out}(k^{out}) \propto (k^{out})^{-\gamma_{out}} \tag{14}$$

When we approximate k with a continuous variable, we recognise the same exact form as in (7), so using (1.7) we obtain:

$$P_{>}^{in}(k^{in}) = \int_{1}^{\infty} P^{in}(k^{in}) d(k^{in}) = \int_{1}^{\infty} (k^{in})^{-\gamma_{in}} d(k^{in}) = \frac{(k^{in})^{-\gamma_{in}+1}}{-\gamma_{in}+1} \tag{15}$$

$$P_{>}^{out}(k^{out}) = \int_{1}^{\infty} P^{out}(k^{out}) d(k^{out}) = \int_{1}^{\infty} (k^{out})^{-\gamma_{out}} d(k^{out}) = \frac{(k^{out})^{-\gamma_{out}+1}}{-\gamma_{out}+1} \tag{16}$$

So we conclude that:

$$P_{>}^{in}(k^{in}) \propto (k^{in})^{-\gamma_{in}+1} \tag{17}$$

$$P_{>}^{out}(k^{out}) \propto (k^{out})^{-\gamma_{out}+1} \tag{18}$$

Homework 1.3

Once again let the empirical undirected degree distribution be:

$$P(k) \propto k^{-\gamma}, \quad 2 < \gamma \leq 3; \tag{19}$$

We approximate $P(k)$ with a continuous probability density defined for $k \in (1, \infty)$ and observe the following:

$$\bar{k} = \int_{1}^{\infty} k \cdot k^{-\gamma} dk = \left[\frac{1}{-\gamma+2} \cdot k^{-\gamma+2} \right]_{1}^{\infty}, \tag{20}$$

Since $\gamma > 2$ and $k^{-\gamma+2}$, k will always be elevated for a negative power, which means that $\lim_{k\to\infty} k^{-\gamma+2} = 0$. Further simplification now yields:

$$\bar{k} = lim_{k\to\infty} \frac{1}{-\gamma+2} \cdot k^{-\gamma+2} - \frac{1}{-\gamma+2} \cdot 1^{-\gamma+2} = \frac{1}{-\gamma+2}. \tag{21}$$

To compute the variance of the empirical undirected degree distribution as $\bar{k^2} - \bar{k}^2$ we first proceed to express $\bar{k^2}$. We consider $2 < \gamma < 3$:

$$\bar{k^2} = \int_{1}^{\infty} k^2 \cdot k^{-\gamma} dk = \left[\frac{1}{-\gamma+3} \cdot k^{-\gamma+3} \right]_{1}^{\infty} = lim_{k\to\infty} \frac{1}{-\gamma+3} \cdot k^{-\gamma+3} - \frac{1}{-\gamma+3}, \tag{22}$$

We now observe that $k^{-\gamma+3}$ is a strictly positive power for $\gamma < 3$, so $lim_{k\to\infty} k^{-\gamma+3} \to \infty$. Plugging this observation back into our integral, we yield:

$$\bar{k^2} = lim_{k\to\infty} \frac{1}{-\gamma+3} \cdot k^{-\gamma+3} - \frac{1}{-\gamma+3} \to \infty. \tag{23}$$

Now, we take $\gamma = 3$ and we get the following:

$$\bar{k^2} = \int_1^\infty k^2 \cdot k^{-3} dk = \int_1^\infty \frac{1}{k} dk = [log|k|]_1^\infty = lim_{k\to\infty} log|k| - 0 \to \infty, \tag{24}$$

Since $log(x)$ is monotonically increasing.

To sum everything up, when we examine the empirical undirected degree distribution of the form $P(k) \propto k^{-\gamma}$, for $2 < \gamma \leq 3$, we find that the mean value of the degree is defined as $\bar{k}$, which is equal to $\frac{1}{-\gamma+2}$; the mean, thus, is finite.
When we examine the variance of the degree, instead, which is defined as $\bar{k^2} - \bar{k}$, we find that it's infinite; since $\bar{k^2} \to \infty$.