Problem 4.2

a) $T(n) = 36\,T(n/6) + 2n$

- Using Master Method

$a = 36$

$b = 6$ $\qquad \log_b a = \log_6 36$

$f(n) = 2n$ $\qquad n^{\log_6 36} = n^2$

We see that n^2 is greater than n for larger n.
This is the first case of Master Theorem
So the solution to this recurrence is $\Theta(n^{\log_b a})$

$$\Downarrow$$

$$T(n) = \Theta(n^2)$$

b) $T(n) = 5T(n/3) + 17n^{1.2}$

- Using Master Method

$a = 5$ $\qquad \log_b a = \log_3 5$

$b = 3$ $\qquad n^{\log_3 5} \approx n^{1.464}$

$f(n) = 17n^{1.2}$

We see that $n^{1.464}$ is polynomially greater than $17n^{1.2}$
This is the first case of Master Theorem
So the solution to this recurrence is $\Theta(n^{\log_b a})$

$$\Downarrow$$

$$T(n) = \Theta(n^{\log_3 5}) \approx \Theta(n^{1.464})$$

c) $T(n) = 12T(n/2) + n^2 \lg n$

- Using Master Method

$a = 12$

$b = 2$

$f(n) = n^2 \lg n$

$\log_b a = \log_2 12$

$n^{\log_2 12} \approx n^{3.58}$

We see that $n^{3.58}$ is greater than $n^2 \lg n$ as:

• $n^{3.58} > n^2$ • $\Theta(n) >> \Theta(\log n)$

This is the first case of Master Theorem

So the solution to this recurrence is $\Theta(n^{\log_b a})$

$\Downarrow$

$$T(n) = \Theta(n^{\log_2 12})$$

d) $T(n) = 3T(n/5) + T(n/2) + 2^n$

The lower bound: $T(n) \geq 4T(n/5) + 2^n$

Let's solve it $T(n) = 4T(n/5) + 2^n$

- Using Master Method

$a = 4$

$b = 5$

$f(n) = 2^n$

$\log_b a = \log_5 4$

$n^{\log_5 4} = n^{0.86}$

We see that $n^{0.86}$ is polynomially smaller than 2^n

This is the third case of Master Theorem

So the solution to this recurrence is $\Theta(f(n))$

$\Downarrow$

$$T(n) = \Theta(2^n)$$

Let's check the Regularity Condition

$af(n/b) \leq cf(n)$ where $c < 1$

$4 \cdot 2^{n/5} \leq c \cdot 2^n \Rightarrow 2^{-4n/5} \leq c/4$

- True as n grows $2^{-\frac{4}{5}} \to 0$ (in the end)

The upper bound: $T(n) \leq 4T(n/2) + 2^n$

Let's solve it $T(n) = 4T(n/2) + 2^n$

- Using Master Method

$a = 4$

$b = 2$ $\log_b a = \log_2 4$

$f(n) = 2^n$ $n^{\log_2 4} = n^2$

We see that n^2 is polynomially smaller than 2^n

This is the third case of Master Theorem.

So the solution to this recurrence is $\Theta(f(n))$

$$\Downarrow$$

$$T(n) = \Theta(2^n)$$

Let's check the Regularity Condition

$af(n/b) \leq cf(n)$ where $c < 1$

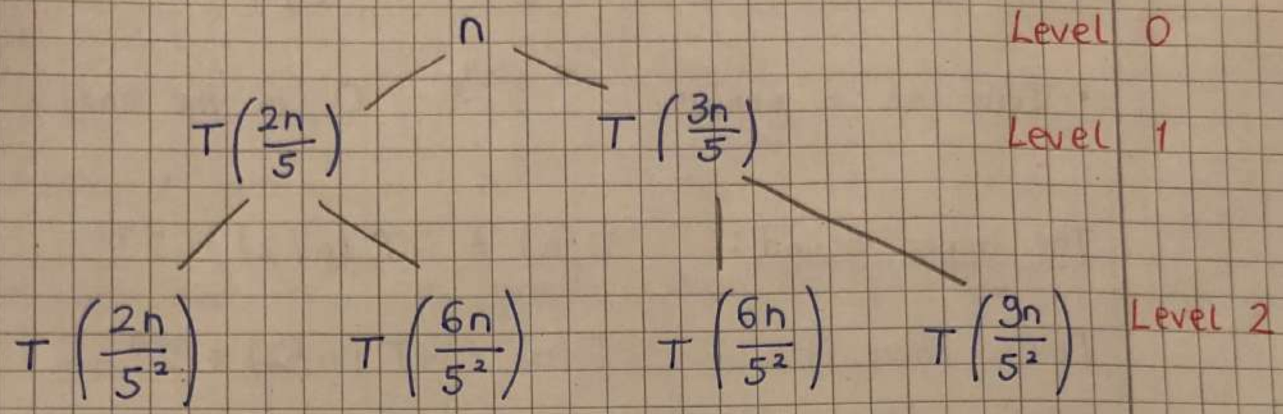
$4 \cdot 2^{n/2} \leq c \cdot 2^n \Rightarrow 2^{-n/2} \leq c/4$

- True as n grows $2^{-\frac{n}{2}} \to 0$ (in the end)

We see that the lower and upper bounds for $T(n)$ are equal. Therefore, the final result is $T(n) = \Theta(2^n)$

e) $T(n) = T(2n/5) + T(3n/5) + \Theta(n)$

- Using Recursion Tree



- Cost of level 0 $= n$
- Cost of level 1 $= \frac{2n}{5} + \frac{3n}{5} + n$
- Cost of level 2 $= \frac{2n}{5^2} + \frac{6n}{5^2} + \frac{6n}{5^2} + \frac{9n}{5^2} = n$

The total number of levels in the recursion tree

- Consider the rightmost sub tree as it goes to the deepest level

- Size of sub-problem at level 0 $\rightarrow (3/5)^0 n$
- Size of sub-problem at level 1 $\rightarrow (3/5)^1 n$
- Size of sub-problem at level 2 $\rightarrow \left(\frac{3}{5}\right)^2 n$
- Size of sub-problem at level i $\rightarrow \left(\frac{3}{5}\right)^i n$
- At last level (level x), size of sub-problem is 1

So:
$$\left(\frac{3}{5}\right)^x n = 1$$
$$\left(\frac{3}{5}\right)^x = \frac{1}{n}$$

$$x \log\left(\frac{3}{5}\right) = \log\left(\frac{1}{n}\right)$$

$$x = \log_{5/3} n$$

We understand that the total number of levels in the recursion tree is $\log_{5/3} n + 1$

Number of nodes in the last level

- Level 0 has 2^0 nodes $=$ 1 node
- Level 1 has 2^1 nodes $=$ 2 nodes
- Level 2 has 2^2 nodes $=$ 4 nodes

- Level $\log_{5/3} n$ has $2^{\log_{5/3} n}$ nodes

Cost of last level $\rightarrow$ $2^{\log_{5/3} n} \cdot T(1) = \Theta\left(2^{\log_{5/3} n}\right)$

$$= \Theta\left(n^{\log_{5/3} 2}\right)$$

The costs of all the levels of the recursion tree :

$$T(n) = \{\underbrace{n + n + n + \dots}_{\text{for } \log_{5/3} n \text{ levels}}\} + \Theta\left(n^{\log_{5/3} 2}\right)$$

$$= n \log_{5/3} n + \Theta\left(n^{\log_{5/3} 2}\right)$$

$$= \Theta\left(n \log_{5/3} n\right)$$ $\rightarrow$ Asymptotic notation