SHEET 6

6.1) We need to prove that the two elementary boolean functions $\rightarrow$ and $\neg$ are universal. To do this, it's necessary to show that $\wedge$, $\vee$ and $\neg$ are produced with $\rightarrow$ and $\neg$

Let's keep in mind that $\uparrow$ (nand) is a universal function since it can be used to derive all elementary Boolean functions

- $X \wedge Y = (X \uparrow Y) \uparrow (X \uparrow Y)$

$X \rightarrow \neg Y$ is equivalent to $\uparrow$ so we can write $\wedge$ using $\rightarrow$ and $\neg$

$X \wedge Y = (X \rightarrow \neg Y) \rightarrow \neg (X \rightarrow \neg Y)$

Proof:

x	y	$\neg y$	$x \to \neg y$	$\neg(x \to \neg y)$	$(x \to \neg y) \to \neg (x \to \neg y)$	$x \wedge y$
0	0	1	1	0	0	0
0	1	0	1	0	0	0
1	0	1	1	0	0	0
1	1	0	0	1	1	1

- $x \vee y = (x \uparrow x) \uparrow (y \uparrow y)$

$x \to \neg y$ is equivalent to $\uparrow$, so we can write $\vee$ using $\to$ and $\neg$

$x \vee y = (x \to \neg x) \to \neg (y \to \neg y)$

Proof:

x	y	$\neg x$	$\neg y$	$x \to \neg x$	$y \to \neg y$	$\neg(y \to \neg y)$	$(x \to \neg x) \to \neg(y \to \neg y)$	$x \vee y$
0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	1
1	0	0	1	0	1	0	1	1
1	1	0	0	0	0	1	0	1

- $\neg X = X \uparrow X$

$X \to \neg X$ is equivalent to $\uparrow$, so we can write $\neg$ using $\to$ and $\neg$

$\neg X = X \to \neg X$

Proof:

X	$\neg X$	$X \to \neg X$
0	1	1
1	0	0

6.2)

a) $\varphi(A,B) = (\neg A \vee \neg B) \wedge (\neg A \vee B) \wedge (A \vee \neg B)$

$= (\neg A \vee (B \wedge \neg B)) \wedge (A \vee \neg B)$ ← distributivity

$= (\neg A \vee 0) \wedge (A \vee \neg B)$ ← complementation

$= \neg A \wedge (A \vee \neg B)$ ← identity

$= (\neg A \wedge A) \vee (\neg A \wedge \neg B)$ ← distributivity

$= 0 \vee (\neg A \wedge \neg B)$ ← complementation

$= \neg A \wedge \neg B$ ← identity

$= \neg (A \vee B)$ ← de Morgan

b) $\varphi(A,B,C) = (A \wedge \neg B) \vee (A \wedge \neg B \wedge C)$

$= A \wedge (\neg B \vee (\neg B \wedge C))$ distributivity

$= A \wedge \neg B$ absorption

c) $\varphi(A,B,C,D) = (A \vee \neg(B \wedge A)) \wedge (C \vee (D \vee C))$

de Morgan $= (A \vee \neg B \vee \neg A) \wedge (C \vee (C \vee D))$ commutativity

commutativity $= (A \vee \neg A \vee \neg B) \wedge ((C \vee C) \vee D)$ associativity

complementation $= (1 \vee \neg B) \wedge (C \vee D)$ idempotency

$= 1 \wedge (C \vee D)$ domination

$= C \vee D$ identity

d) $\varphi(A,B,C) = (\neg(A \wedge B) \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$

de Morgan $= (\neg A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$

commutativity $= (\neg A \vee \neg C \vee \neg B) \wedge (\neg A \vee \neg C \vee B)$

distributivity $= (\neg A \vee \neg C) \vee (\neg B \wedge B)$

complementation $= (\neg A \vee \neg C) \vee 0$

identity $= \neg A \vee \neg C$

de Morgan $= \neg(A \wedge C)$

e) $\varphi(A,B) = (A \vee B) \wedge (\neg A \vee B) \wedge (A \vee \neg B) \wedge (\neg A \vee \neg B)$

$= (B \vee (A \wedge \neg A)) \wedge (\neg B \vee (A \wedge \neg A))$ distributivity

$= (A \wedge \neg A) \vee (B \wedge \neg B)$ distributivity

$= 0 \vee 0$ complementation

$= 0$

6.3)

P	Q	R	S	$\neg P$	$\neg P \vee Q$	$\neg Q$	$\neg Q \vee R$	$\neg R$	$\neg R \vee S$	$\neg S$	$\neg S \vee P$	$(\neg P \vee Q) \wedge (\neg Q \vee R)$	$(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S)$	$(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$
0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1	1	0	0	1	1	0
0	0	1	0	1	1	1	1	0	0	1	1	1	0	0
0	1	0	0	1	1	0	0	1	1	1	1	0	0	0
1	0	0	0	0	0	1	1	1	1	1	1	0	0	0
1	1	0	0	0	1	0	0	1	1	1	1	0	0	0
1	0	1	0	0	0	1	1	0	0	1	1	0	0	0
1	0	0	1	0	0	1	1	1	1	0	1	0	0	0
0	1	1	0	1	1	0	1	0	0	1	1	1	0	0
0	1	0	1	1	1	0	0	1	1	0	0	0	0	0
0	0	1	1	1	1	1	1	0	0	0	0	1	0	0
1	1	1	0	0	1	0	1	0	0	1	1	1	0	0
1	1	0	1	0	1	0	0	1	1	0	1	0	0	0
1	0	1	1	0	0	1	1	0	1	0	1	0	0	0
0	1	1	1	1	1	0	1	0	1	0	0	1	1	0
1	1	1	1	0	1	0	1	0	1	0	1	1	1	1

a) 2 interpretations of the variables P, Q, R, S satisfy φ, as we can see from the truth table that only 2 rows result in 1

b) Row 1 and row 16 satisfy φ, so their result is 1. To write the formula for φ in DNF for each interpretation, firstly we negate the variables that are 0 and combine them with $\wedge$. Then we put $\vee$ between the two parentheses.

DNF: $(\neg P \wedge \neg Q \wedge \neg R \wedge \neg S) \vee (P \wedge Q \wedge R \wedge S)$