SHEET 3

3.1) a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

$(A \times C) \cap (B \times D) \iff (x,y) \in (A \times C)$ and $(x,y) \in (B \times D)$

$\iff (x,y) \in A \times C$ and $(x,y) \in B \times D$

$\iff (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$

$\iff (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$

$\iff x \in A \cap B$ and $y \in B \cap D$

$\iff (x,y) \in (A \cap C) \times (B \cap D)$

$\Downarrow$

$(A \cap B) \times (C \cap B) = (A \times C) \cap (B \times D)$

b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

By counterexample

Let: $a \in A$, $b \in B$, $c \in C$, $d \in D$

- $(A \cup B) \times (C \cup D)$

$A \cup B = \{a, b\}$ $\quad C \cup D = \{c, d\}$

$(A \cup B) \times (C \cup D) = \{(a,c), (a,d), (b,c), (b,d)\}$

- $(A \times C) \cup (B \times D)$

$A \times C = \{(a,c)\}$ $\quad$ $B \times D = \{(b,d)\}$

$(A \times C) \cup (B \times D) = \{(a,c), (b,d)\}$

$\Downarrow$

So, we see that:

$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

3.2) a) $R = \{(a,b) \mid a, b \in \mathbb{Z} \wedge |a-b| \leq 3\}$

- Reflexive

$\forall a \in \mathbb{Z} \quad (a,a) \in \mathbb{Z}$

$|a-a| \leq 3$

$0 \leq 3$

Yes

- Symmetric

$\forall a, b \in \mathbb{Z} \quad (a,b) \in \mathbb{Z}$

$|a-b| \leq 3$

Yes

$-3 \leq a-b \leq 3$

$b-3 \leq a \leq b+3$

$a-3 \leq b \leq a+3 \Rightarrow |a-b| = |b-a|$

• Transitive

$\forall a, b \in Z \quad (a,b) \in Z, \ a \neq b$

$\forall b, c \in Z \quad (b,c) \in Z, \ b \neq c$

But maybe a is not equal to c

Example: $a = 5, \ b = 2, \ c = 1$

$|a-b| \leq 3 \qquad |b-c| \leq 3 \qquad |a-c| \leq 3$

$|5-2| \leq 3 \qquad |2-1| \leq 3 \qquad |5-1| \leq 3$

$3 \leq 3 \qquad 1 \leq 3 \qquad 4 \nleq 3$

Not transitive

b) $R = \{(a,b) \mid a, b \in Z \wedge (a \bmod 10) = (b \bmod 10)\}$

• Reflexive

$\forall a \in Z \quad (a,a) \in Z$

$a \bmod 10 = a \bmod 10$

Yes

• Symmetric

$a \bmod 10 = b \bmod 10 \qquad b \ \text{mo} \ 10 = a \bmod 10$

Example: $a = 33, \ b = 43$ Yes

- Transitive **Yes**

$\forall a, b \in Z \quad (a,b) \in Z, \; a \bmod 10 = b \bmod 10$

$\forall b, c \in Z \quad (b,c) \in Z, \; b \bmod 10 = c \bmod 10$

$$\Downarrow$$

$$a \bmod 10 = c \bmod 10$$

Example: $a = 56, \; b = 66, \; c = 76$

3.3) By induction

cnt x (con s t) == (cnt x s) + (cnt x t)

- Base case

s → empty list [ ]

cnt x (con [ ] t) == (cnt x [ ]) + (cnt x t)

cnt x t == 0 + cnt x t

cnt x t can be any number

So, if cnt x t = 1 :

$1 = 0 + 1$

$1 = 1$

• Assume $cnt\ x\ (con\ s\ t) == (cnt\ x\ s) + (cnt\ x\ t)$ is true

$(cnt\ x\ s) \to a$

$(cnt\ x\ t) \to b$ $\Rightarrow$ $a + b == a + b$

$cnt\ x\ (con\ s\ t) \to a + b$

We add another element (letter) to the list s. Now the list $s \in$

{SomeLettersBefore + NewLetter}

- If the added letter is a random letter other than x, it is still:

$$cnt\ x\ (con\ s\ t) == (cnt\ x\ s) + (cnt\ x\ t)$$

$$a + b == a + b$$

- If the added letter is x, then:

$$cnt\ x\ (con\ s\ t) == (cnt\ x\ s) + (cnt\ x\ t)$$

$$a + 1 + b == (a + 1) + b$$

This is true

So, by induction, we prove that:

$cnt\ x\ (con\ s\ t) == (cnt\ x\ s) + (cnt\ x\ t)$

is true