4.1) a) Let $\preceq \subset \Sigma^* \times \Sigma^*$ be a reflexive such that $p \preceq w$ for $p, w \in \Sigma^*$ if p is a prefix of w. Show that $\preceq$ is a partial order

If $\preceq$ is a partial order, it is reflexive, antisymmetric and transitive

- Reflexive

$\forall p, p \in \Sigma^* \quad (p, p) \in \Sigma^* \quad w = p$

A prefix p can be a prefix of itself

eg. p = jungle $\quad w$ = jungle

- Antisymmetric

$\forall p, w \in \Sigma^* \quad ((p, w) \in \Sigma^* \wedge (w, p) \in \Sigma^*)$

Assume q is an empty set, so $p = w$ and $w = p$

For example : p = jungle, w = jungle, q = ' '

• Transitive

Take w as a prefix of x

$\forall p, w \in \Sigma^*$ p is a prefix of w

$p \preceq w \quad [w = pq]$

$\rightarrow p = w$

$\forall w, x \in \Sigma^*$ p is a prefix of x

$w \preceq x \quad [x = wm]$

$\rightarrow x = m$

We get: $p \cdot q \cdot m = x \Rightarrow p \preceq x$ and $[p = x]$

b) Let $\preceq \subset \Sigma^* \times \Sigma^*$ be a relation such that for $p \prec w$, for $p, w \in \Sigma^*$ if p is a proper prefix of w. Show that $\prec$ is a strict partial order

If $\prec$ is a strict partial order on Σ^*, it is irreflexive, asymmetric and transitive

• Irreflexive

We have the condition $p \neq w$ in the question so the prefix p will always be different than w

For example: Let's take the word 'impossible'
im $\prec$ impossible (im is a prefix of impossible)
But impossible is not a prefix of im

• Asymmetric

By the definition in the question: $p \neq w$
Suppose $p \prec w$ is true
But $w \prec p$ cannot be true because w cannot be the prefix of p.
So the relation is asymmetric

• Transitive

$\forall\, p, w \in \Sigma^*$ p is a proper prefix of w

$p \prec w$ $pq = w$

$\rightarrow p \neq w$

$\forall w, x \in \Sigma^*$ w is a proper prefix of x

$w \prec x$ $x = w \cdot m$

$\rightarrow w \neq x$

We get : $p \cdot q \cdot m = x$

$p \prec x$ except when $p = w$ and $w = x$

For example :

$ef \prec efg$, $efg \prec efgh \Rightarrow ef \prec efgh$

c) The two order relations $\preceq$ and $\prec$ are not total

For example : $p =$ "cherry" $w =$ "kiwi"

$(p, w) \notin \prec$ and $(w, p) \notin \prec$

$(p, w) \notin \preceq$ and $(w, p) \notin \preceq$

4.2) a) If $g \circ f$ is bijective, then f is injective and g is surjective

By definition, if $g \circ f$ is bijective, it is also injective and surjective

- $g \circ f$ is injective, so we show that f is injective.

Let $x_1, x_2 \in A$ and suppose that $f(x_1) = f(x_2)$

$$(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$$

But since $g \circ f$ is injective, this implies that $x_1 = x_2$. Therefore f is injective.

- $g \circ f$ is surjective, so we show that g is surjective

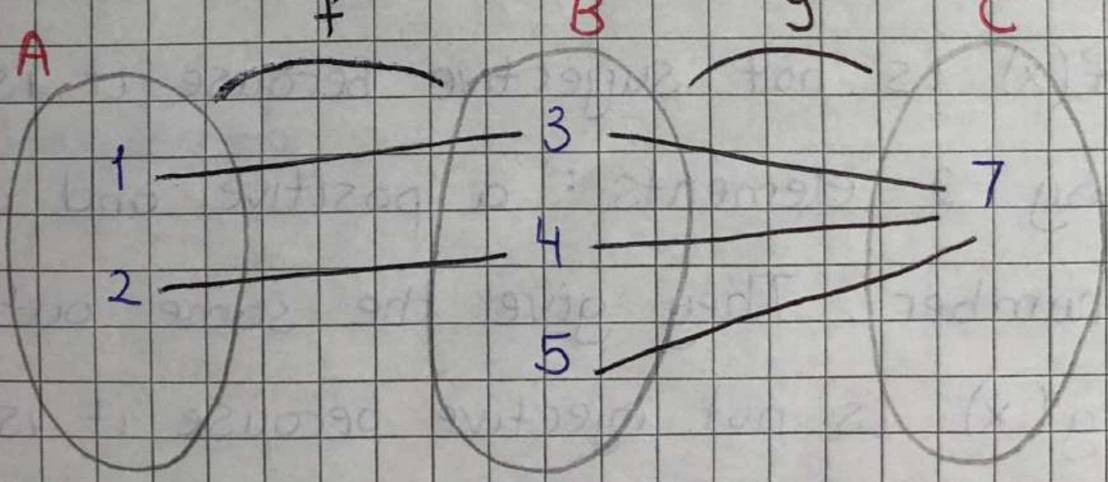
Let $z \in C$.

Then since $g \circ f$ is surjective, there exists $x \in A$ such that $(g \circ f)(x) = g(f(x)) = z$

So if we let $y = f(x) \in B$, then $g(y) = z$

Thus g is surjective

b) $g \circ f$ is not bijective, but f is injective and g is surjective



$f \rightarrow$ is injective because at most one element of domain A is mapped to every element of co-domain B.

$g \rightarrow$ is surjective because for every element in the codomain C of g, there is at least one element in the domain B of g

$g \circ f \rightarrow$ is not bijective because element 7 is mapped by more than one element

c) $g \circ f$ is bijective, but f is not surjective and g is not injective

$f : R \rightarrow R^+$ $\qquad$ $g : R^+ \rightarrow R$

$f(x) = x^2$ $\qquad$ $g(x) = \sqrt{x}$

- f(x) is not surjective because it is mapped by 2 elements : a positive and a negative number. They give the same output.
- g(x) is not injective because it is undefined for negative values.
- g(f(x)) is x, which is bijective since it is a one to one function