FTML practical session 12 solution

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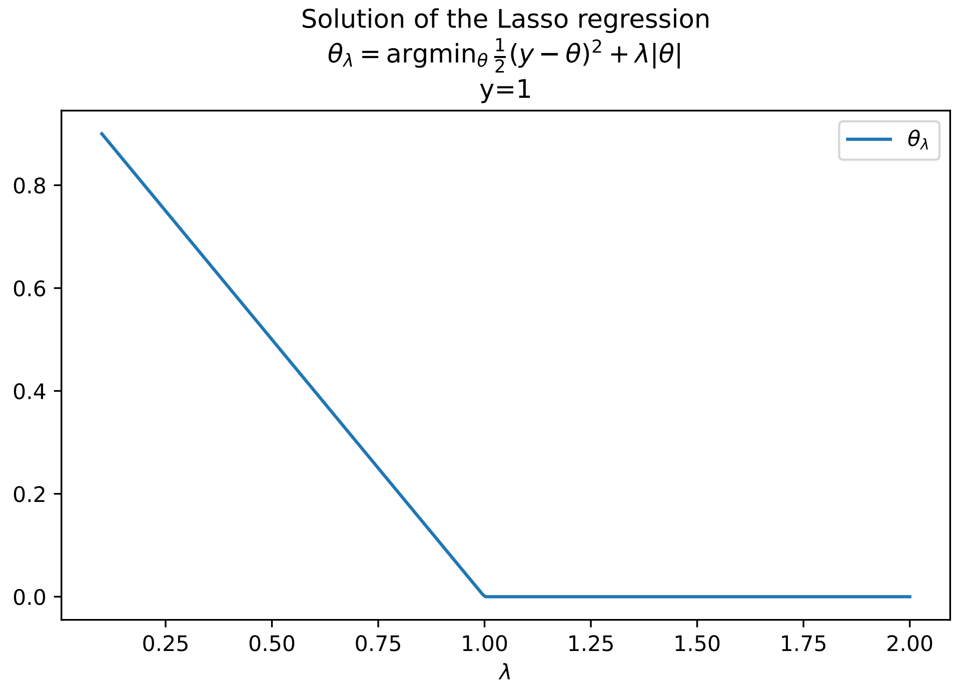

FIGURE 1 – Solution of the one-dimensional Lasso estimator

1 SOLUTION

Depending on the sign of θ, and the values of λ and y, the situation will be different. However, we need to keep in mind that we always have that $\lambda \geqslant 0$, as it is a regularization parameter.

1.0.1 $\theta \geqslant 0$

If $\theta \geqslant 0$, we have that

$$F_\lambda(\theta) = \frac{1}{2}(y-\theta)^2 + \lambda\theta \quad (1)$$

We can differentiate with respect to θ, and

$$F'_\lambda(\theta) = -(y-\theta) + \lambda = \theta - (y-\lambda) \quad (2)$$

In this case,

- If $\lambda \geqslant y$, then $y-\lambda \leqslant 0$, and we always have that $F'_\lambda(\theta) \geqslant 0$, the minimum on $\mathbb{R}_+$ is attained for $\theta = 0$, with value $\frac{1}{2}y^2$.
- If $\lambda \leqslant y$, then $y-\lambda \geqslant 0$, and we have that $F'_\lambda(\theta) \leqslant 0$ if $\theta \leqslant y-\lambda$ and $F'_\lambda(\theta) \geqslant 0$ if $\theta \geqslant y-\lambda$, the minimum on $\mathbb{R}_+$ is attained for $\theta = y-\lambda$, with value $\frac{1}{2}\lambda^2 + \lambda(y-\lambda) = \lambda(y-\frac{1}{2}\lambda)$

1.0.2 $\theta \leqslant 0$

If $\theta \leqslant 0$, we have that

$$F_\lambda(\theta) = \frac{1}{2}(y-\theta)^2 - \lambda\theta \tag{3}$$

We can differentiate with respect to θ, and

$$F'_\lambda(\theta) = -(y-\theta) - \lambda = \theta - (y+\lambda) \tag{4}$$

In this case,

- If $\lambda \geqslant -y$, then $y+\lambda \geqslant 0$, and we always have that $F'_\lambda(\theta) \leqslant 0$, the minimum on $\mathbb{R}_-$ is attained for $\theta = 0$, with value $\frac{1}{2}y^2$.
- If $\lambda \leqslant -y$, then $y+\lambda \leqslant 0$, and we have that $F'_\lambda(\theta) \leqslant 0$ if $\theta \leqslant y+\lambda$ and $F'_\lambda(\theta) \geqslant 0$ if $\theta \geqslant y+\lambda$, the minimum on $\mathbb{R}_-$ is attained for $\theta = y+\lambda$, with value $\frac{1}{2}\lambda^2 - \lambda(y+\lambda) = \lambda(-y-\frac{1}{2}\lambda)$

1.1 Applications

Let us consider some values of y and plot the θ_λ as a function of λ.

1.1.1 $y = 1$

In this case :

- If $\lambda \leqslant 1$,
  - the optimal $\theta \in \mathbb{R}_+$ is $\theta_+ = 1-\lambda$ with value $\lambda(1-\frac{1}{2}\lambda)$.
  - the optimal $\theta \in \mathbb{R}_-$ is $\theta_- = 0$ with value $\frac{1}{2}$.

  Let us show that is $0 \leqslant \lambda \leqslant 1$, then $g(\lambda) = \lambda(1-\frac{1}{2}\lambda) \leqslant \frac{1}{2}$. We have that $g(\lambda) = \lambda - \frac{1}{2}\lambda^2$, hence $g'(\lambda) = 1-\lambda \geqslant 0$. The maximum value of $g(\lambda)$ is thus attained in $\lambda = 1$, and is equal to $\frac{1}{2}$.
  Overall, we have

  $$\theta_\lambda = 1-\lambda \tag{5}$$

  and

  $$F_\lambda(\theta_\lambda) = \lambda(1-\frac{1}{2}\lambda) \tag{6}$$

- If $\lambda \geqslant 1$,
  - the optimal $\theta \in \mathbb{R}_+$ is $\theta_+ = 0$ with value $\frac{1}{2}$.
  - the optimal $\theta \in \mathbb{R}_-$ is $\theta_- = 0$ with value $\frac{1}{2}$.

  Overall

  $$\theta_\lambda = 0 \tag{7}$$

  and

  $$F_\lambda(\theta_\lambda) = \frac{1}{2} \tag{8}$$