PROPOSIZIONE: Sia $f: V \to W$ omomorfismo di spazi vettoriali e siano B_V e B_W due basi di V e W, rispettivamente. Sia inoltre M_f la matrice associata ad f rispetto a B_V e B_W. Se $\overline{B}_V$ e $\overline{B}_W$ sono due nuove basi di V e W e se $\overline{M}_f$ è la matrice associata a f rispetto a $\overline{B}_V$ e $\overline{B}_W$, allora:

$$\overline{M}_f = C^{-1} M_f A,$$

dove $C = M_{\overline{B}_W \to B_W}$ e $A = M_{\overline{B}_V \to B_V}$

ES. $f: M_2(\mathbb{R}) \to \mathbb{R}_2[x]$ t.c. $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + (b+c)x + dx^2$

1) f è omomorfismo di sp. vett.: $\forall A, B \in M_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} w & y \\ z & t \end{pmatrix}$

dobbiamo dimostrare che $f(A+B) = f(A) + f(B)$.

Ma $f(A+B) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & y \\ z & t \end{pmatrix}\right) = f\begin{pmatrix} a+w & b+y \\ c+z & d+t \end{pmatrix} =$

$= (a+w) + ((b+y) + (c+z))x + (d+t)x^2$.

$f(A) + f(B) = f\begin{pmatrix} a & b \\ c & d \end{pmatrix} + f\begin{pmatrix} w & y \\ z & t \end{pmatrix} = a + (b+c)x + dx^2 + w + (y+z)x + tx^2$

$= (a+w) + (b+c+y+z)x + (d+t)x^2 \Rightarrow f(A+B) = f(A) + f(B)$.

Inoltre, $\forall A \in M_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ e $\forall \alpha \in \mathbb{R}$ dobbiamo dimostrare che $f(\alpha A) = \alpha f(A)$.

Ma $f(\alpha A) = f\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = \alpha a + (\alpha b + \alpha c)x + \alpha d x^2 =$

$$= \alpha \big(\underbrace{a + (b+c)x + dx^2}_{= f(A)}\big) = \alpha f(A).$$

2) Costruire M_f rispetto alle basi canoniche di $M_2(\mathbb{R})$ ed $\mathbb{R}_2[x]$:

$$B_{M_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$B_{\mathbb{R}_2[x]} = \{ 1, \; x, \; x^2 \}$$

$$\bullet\ f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$\bullet\ f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$\bullet\ f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$\bullet\ f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2$$

$$M_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\operatorname{Ker} f = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a + (b+c)x + dx^2 = 0 \right\} = *$$

$$\begin{cases} a = 0 \\ b + c = 0 \\ d = 0 \end{cases}$$

$$\begin{cases} a = 0 \\ c = -b \\ d = 0 \end{cases}$$

$$* = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

$$\dim_{\mathbb{R}} \operatorname{Ker} f = 1 \quad e \quad \mathcal{B}_{\operatorname{Ker} f} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Per calcolare $\mathrm{Ker}\, f$ è sufficiente moltiplicare M_f per $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ e porre $= 0$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y+z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{cases} x = 0 \\ y + z = 0 \\ t = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ z = -y \\ t = 0 \end{cases} \qquad \mathrm{Ker}\, f = \left\{ \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} \mid y \in \mathbb{R} \right\}.$$

$$\dim_{\mathbb{R}} M_2(\mathbb{R}) = \dim_{\mathbb{R}} \mathrm{Ker}\, f + \dim_{\mathbb{R}} \mathrm{Im}\, f \Rightarrow 4 = 1 + \dim_{\mathbb{R}} \mathrm{Im}\, f$$

$$\Rightarrow \dim_{\mathbb{R}} \mathrm{Im}\, f = 3 = \dim_{\mathbb{R}} \mathbb{R}_2[x].$$

$\Rightarrow \dim_{\mathbb{R}} \operatorname{Im} f = 3 = \dim_{\mathbb{R}} \mathbb{R}_2[x] \Rightarrow \operatorname{Im} f = \mathbb{R}_2[x]$ ed f é surgettiva.

$$\dim_{\mathbb{R}} \operatorname{Im} f = r(M_f) = 3.$$

numero di colonne di M_f

$$\dim_{\mathbb{R}} \operatorname{Ker} f = 4 - r(M_f)$$

3) Scrivere $\overline{M}_f$ rispetto alle basi $\overline{B}_{M_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

$\overline{B}_{\mathbb{R}_2[x]} = \left\{ 1, 1 + x, 1 + x + x^2 \right\}.$

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = 1 \cdot 1 + 0 \cdot (1 + x) + 0 \cdot (1 + x + x^2)$$

$$f\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1 + x = 0 \cdot 1 + 1 \cdot (1 + x) + 0 \cdot (1 + x + x^2)$$

$$f\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = 1 + 2x = -1 \cdot 1 + 2 \cdot (1 + x) + 0 \cdot (1 + x + x^2)$$

$$f\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1 + 2x + x^2 = -1 \cdot 1 + 1 \cdot (1 + x) + 1 \cdot (1 + x + x^2)$$

$$\bar{M}_f = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad r(\bar{M}_f) = 3 = \dim_{\mathbb{R}} \operatorname{Im} f$$

$$\Rightarrow \dim_{\mathbb{R}} \operatorname{Ker} f = 1.$$

$$\bar{M}_f = C^{-1} M_f A, \quad \text{dove} \quad C = M_{\bar{B}_{\mathbb{R}_2[x]} \to B_{\mathbb{R}_2[x]}}$$

$$A = M_{\bar{B}_{M_2} \to B_{M_2}}$$

$$\bar{B}_{\mathbb{R}_2[x]} = \{1, \ 1+x, \ 1+x+x^2\} \qquad B_{\mathbb{R}_2[x]} = \{1, \ x, \ x^2\}$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$1 + x + x^2 = 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2$$

$$\bar{B}_{M_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \qquad B_{M_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Invertire C : 1) calcolare C^{-1} (regola di Gauss o complementi algebrici)

2) $C^{-1} = M_{B_{\mathbb{R}_2[x]} \to \bar{B}_{\mathbb{R}_2[x]}}$

$$B_{\mathbb{R}_2[x]} = \{1, \ x, \ x^2\} \qquad \overline{B}_{\mathbb{R}_2[x]} = \{1, \ 1+x, \ 1+x+x^2\}$$

$$1 = 1 \cdot 1 + 0 \cdot (1+x) + 0 \cdot (1+x+x^2)$$

$$x = -1 \cdot 1 + 1(1+x) + 0 \cdot (1+x+x^2)$$

$$x^2 = 0 \cdot 1 + (-1) \cdot (1+x) + 1 \cdot (1+x+x^2)$$

$$C^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{M}_f = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M_f} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A}$$

DEF. Un omomorfismo f è un endomorfismo se dominio e codominio coincidono: $f: V \to V$.

OSS. $f \in End(V)$ allora $M_f \in M_n(F)$ dove $n = \dim_F V$.

$$\bar{M}_f = A^{-1} M_f A$$

ES. $f: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ $\quad f(a + bx + cx^2) = bx + c$

1) Det. M_f rispetto alla base canonica

2) Det. $\text{Ker} f$ e $\text{Im} f$

3) È vero che $\mathbb{R}_2[x] = \text{Ker} f \oplus \text{Im} f$?

4) Det. M_f rispetto a $\bar{B} = \{1,\ 1+x,\ x+x^2\}$

Sol. 1) $B = \{1,\ x,\ x^2\}$

$f(1) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$f(x) = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$ $\qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$f(x^2) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

2) $r(M_f) = 2 = \dim_{\mathbb{R}} \operatorname{Im} f$, $\dim_{\mathbb{R}} \operatorname{Ker} f = 1$. Inoltre $x^2 \in \operatorname{Ker} f \Rightarrow$

$\Rightarrow \operatorname{Ker} f = \langle x^2 \rangle = \{ c x^2 \mid c \in \mathbb{R} \}$ e $B_{\operatorname{Ker} f} = \{x^2\}$.

$\operatorname{Im} f = \langle f(1), f(x), f(x^2) \rangle = \langle 1, x, 0 \rangle = \langle 1, x \rangle =$

$= \{ a + bx \mid a, b \in \mathbb{R} \}$, $\quad B_{\operatorname{Im} f} = \{1, x\}$.

3) Ker f e Im f sono in somma diretta se $\ker f \cap \operatorname{Im} f = \{0\}$

Ma poiché $\ker f = \langle x^2 \rangle$, $\operatorname{Im} f = \langle 1, x \rangle \Rightarrow \ker f \cap \operatorname{Im} f = \{0\}$

quindi Ker f e Im f sono in somma diretta.

$$\dim_{\mathbb{R}} (\ker f \oplus \operatorname{Im} f) = \dim_{\mathbb{R}} \ker f + \dim_{\mathbb{R}} \operatorname{Im} f = 1 + 2 = 3$$

$$\ker f \oplus \operatorname{Im} f \subseteq \mathbb{R}_2[x] \qquad \Rightarrow \qquad \mathbb{R}_2[x] = \ker f \oplus \operatorname{Im} f$$

4) $\bar{B} = \{1,\ 1+x,\ x+x^2\}$

$f(1) = 1 = 1 \cdot 1 + 0 \cdot (1+x) + 0 \cdot (x+x^2)$

$f(1+x) = 1+x = 0 \cdot 1 + 1 \cdot (1+x) + 0 \cdot (x+x^2)$

$f(x+x^2) = x = -1 \cdot 1 + 1 \cdot (1+x) + 0 \cdot (x+x^2)$

$$\bar{M}_f = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$r(\bar{M}_f) = 2$

$\bar{M}_f = A^{-1} M_f A$, dove $A = M_{\bar{B} \to B}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

1
$1+x$
$x+x^2$

$$\det A = (-1)^{3+3} \cdot 1 \cdot (1 - 0) = 1$$

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & \end{pmatrix}$$

$$a'_{11} = \frac{1}{\det A} (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \qquad a'_{12} = \frac{1}{\det A} (-1)^{1+2} \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0$$

$$a'_{13} = \frac{(-1)^{1+3}}{\det A} \det \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \qquad a'_{21} = \frac{(-1)^{1+2}}{\det A} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$$a'_{22} = \frac{(-1)^{2+2}}{\det A} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad a'_{23} = \frac{(-1)^{2+3}}{\det A} \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$a'_{31} = \frac{(-1)^{3+1}}{\det A} \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1 \qquad a'_{32} = \frac{(-1)^{2+3}}{\det A} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$$a'_{33} = \frac{(-1)^{3+3}}{\det A} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

$$\bar{M}_f = \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M_f} \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A}$$

$$\| $$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$