LIMITI DI FUNZIONI

$f: X \subseteq \mathbb{R} \to \mathbb{R}$

X intervallo o unione finita di intervalli

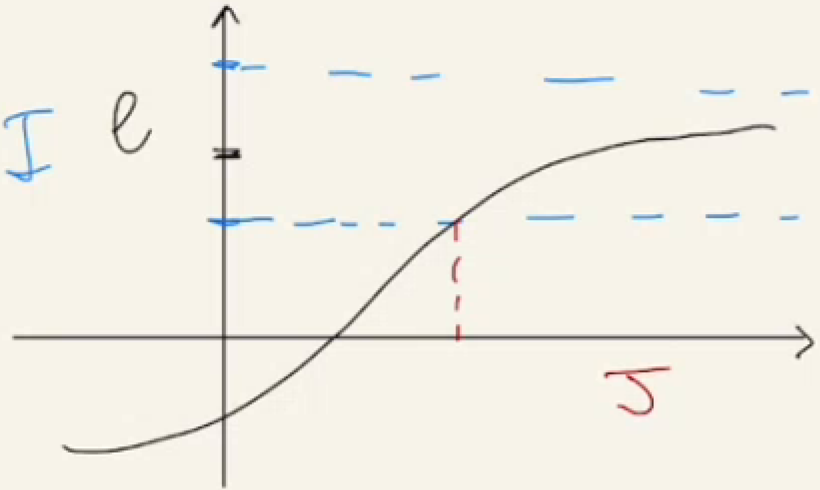
$x_0 \in X$ oppure x_0 è un estremo di uno degli intervalli che compongono X

1) $x_0 \in \mathbb{R}$, 2) $x_0 = +\infty$, 3) $x_0 = -\infty$

2) $x_0 = +\infty$

$$\lim_{x \to +\infty} f(x) = l \in \mathbb{R}$$

$\Leftrightarrow \forall I \in \mathcal{V}(\ell) \quad \exists J \in \mathcal{V}(+\infty) \,/\, x \in X \cap J \quad f(x) \in I$

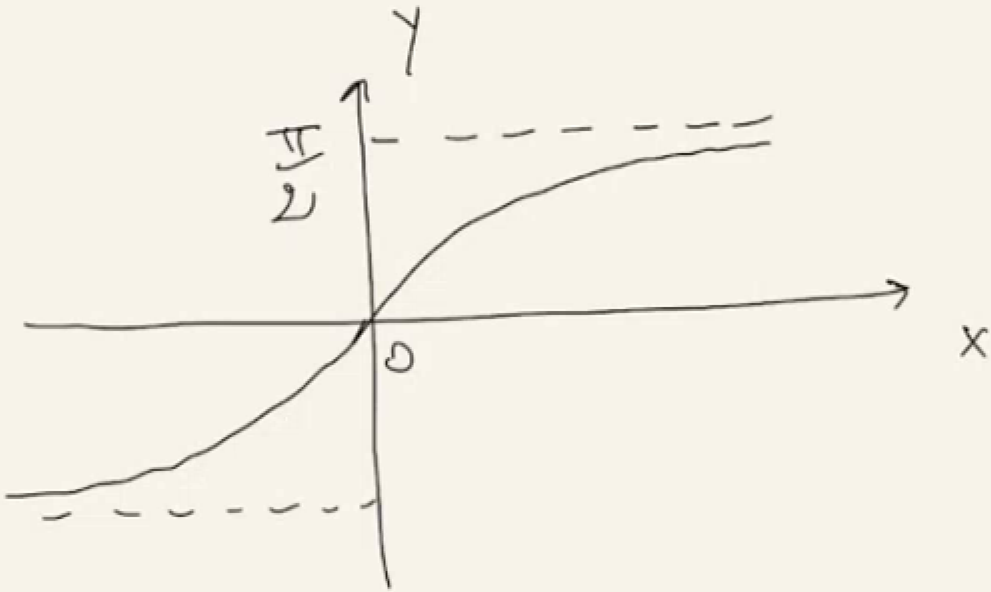


Equivalentemente

$\forall \varepsilon > 0 \quad \exists \delta > 0 \;/\; x \in X,\ x > \delta \qquad |f(x) - \ell| < \varepsilon$

es:

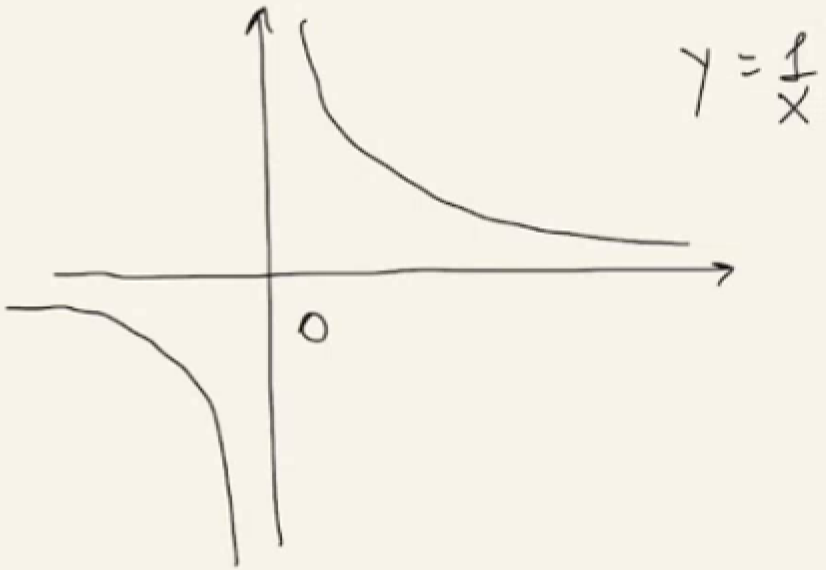
$$\lim_{x \to +\infty} \operatorname{arctg} x = \frac{\pi}{2}$$



$y = \operatorname{arctg} x$

es:

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$



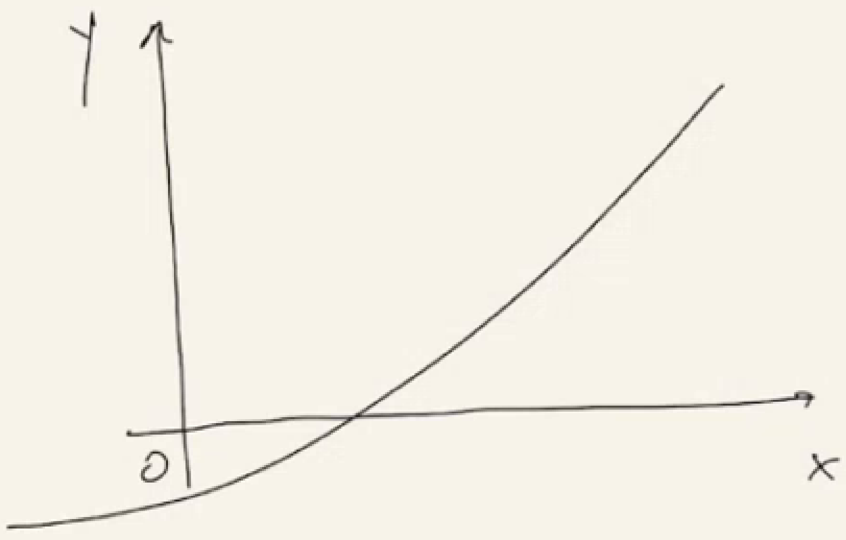
La retta $y = l$ è detta ASINTOTO ORIZZONTALE a $+\infty$.

- $\lim_{x \to +\infty} f(x) = +\infty$

$\iff \forall I \in \mathcal{I}(+\infty) \ \exists J \in \mathcal{I}(+\infty) \ / \ x \in X \cap J \quad f(x) \in I$

Equivalentemente

$\forall k > 0 \quad \exists \delta > 0 \ / \ x \in X, \ x > \delta \quad f(x) > k$



$$\lim_{x \to +\infty} x^2 = +\infty, \quad \lim_{x \to +\infty} a^x = +\infty \text{ se } a > 1$$

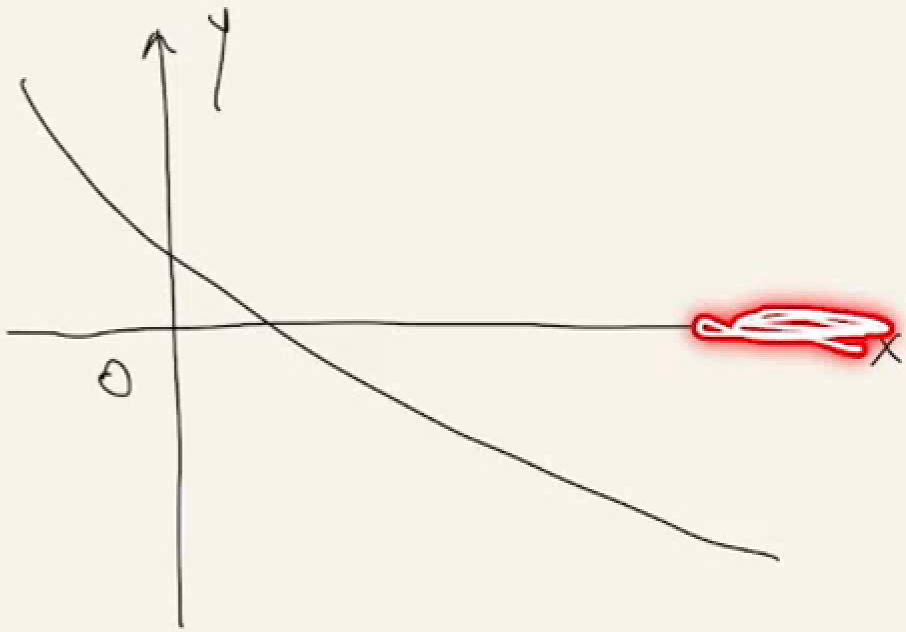
$$\lim_{x \to +\infty} \log_a x = +\infty \text{ se } a > 1$$

- $\lim\limits_{x \to +\infty} f(x) = -\infty$

$\Leftrightarrow \forall I \in \mathcal{I}(-\infty) \ \exists J \in \mathcal{I}(+\infty) \ / \ x \in X \cap J \quad f(x) \in I$

Equivalentemente

$\forall k > 0 \quad \exists \delta > 0 \ / \ x \in X, \ x > \delta \quad f(x) < -k$.



<u>es</u>:

$$\lim_{x \to +\infty} \log_a x = -\infty \quad \text{se } 0 < a < 1$$

<u>Oss</u>: c'è un'analogia tra le definizioni appena viste e quelle di successioni convergenti, divergenti pos. e neg. Per esempio

$$\lim_{n} a_n = \ell \iff \forall \varepsilon > 0 \ \exists \nu \in \mathbb{N} \mid n > \nu \quad |a_n - \ell| < \varepsilon$$

$$\lim_{x \to +\infty} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 \mid x > \delta \quad |f(x) - \ell| < \varepsilon$$

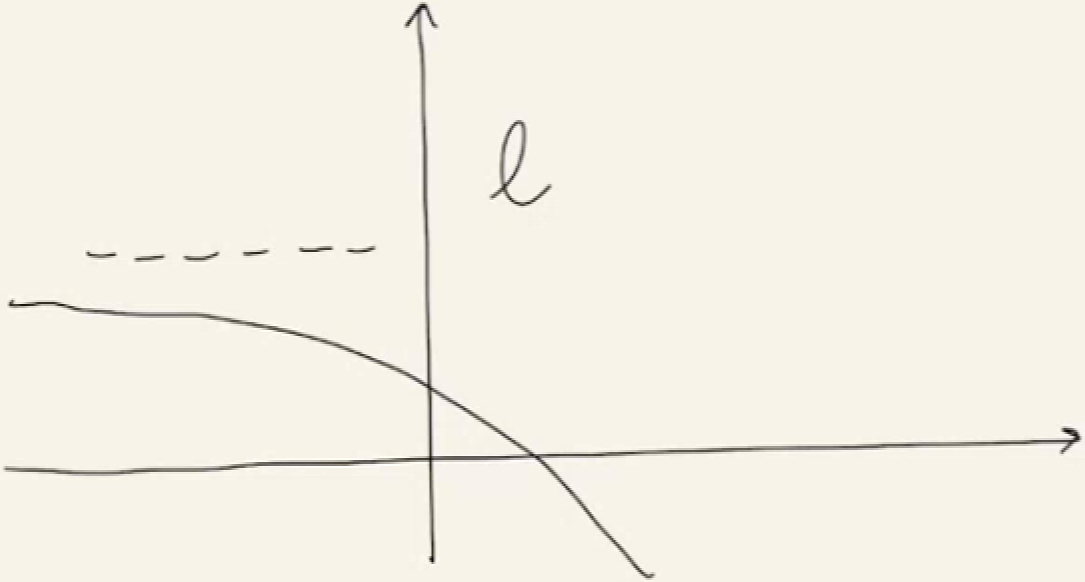
3) $x_0 = -\infty$

$$\lim_{x \to -\infty} f(x) = \ell \in \mathbb{R}$$

$$\iff \forall I \in \mathcal{I}(\ell) \ \exists J \in \mathcal{I}(-\infty) \mid x \in X \cap J \quad f(x) \in I$$

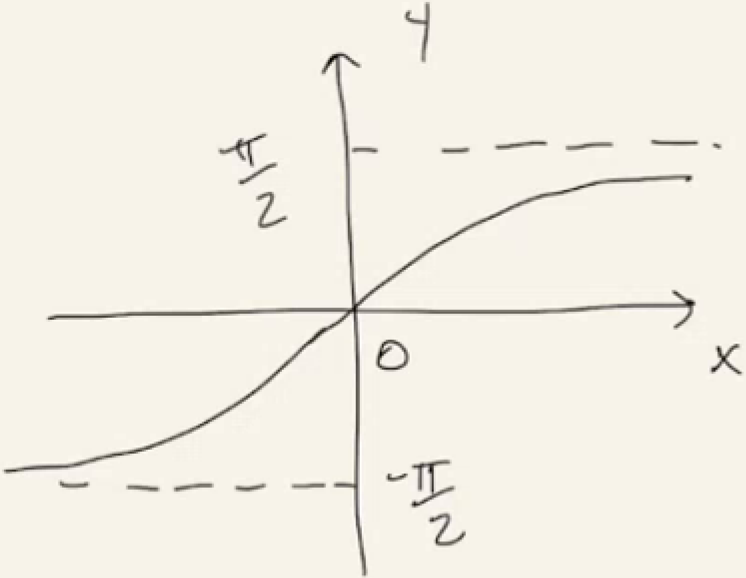
Equivalentemente

$$\forall \varepsilon > 0 \ \exists \delta > 0 \mid x \in X,\ x < -\delta \quad |f(x) - \ell| < \varepsilon$$

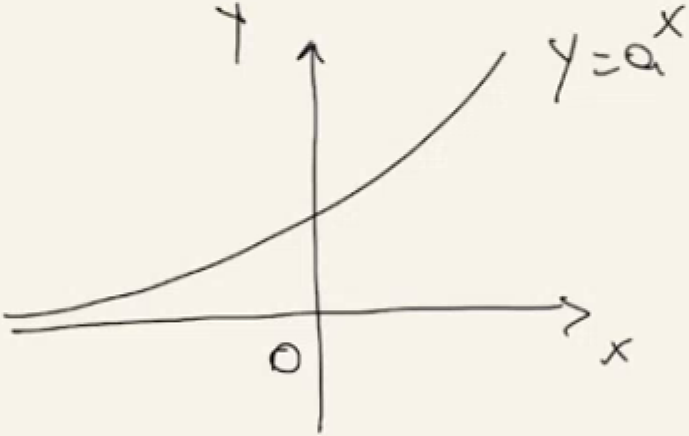


es:

$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$, $\lim_{x \to -\infty} a^x = 0$ se $a > 1$



$y = \operatorname{arctg} x$



$a > 1$

La retta $y = l$ è detta ASINTOTO ORIZZONTALE a $-\infty$.

- $\lim_{x\to -\infty} f(x) = +\infty$

$\Leftrightarrow \forall I \in \mathcal{I}(+\infty)\ \exists J \in \mathcal{I}(-\infty) \mid x \in X \cap J \quad f(x) \in I$

Equivalentemente

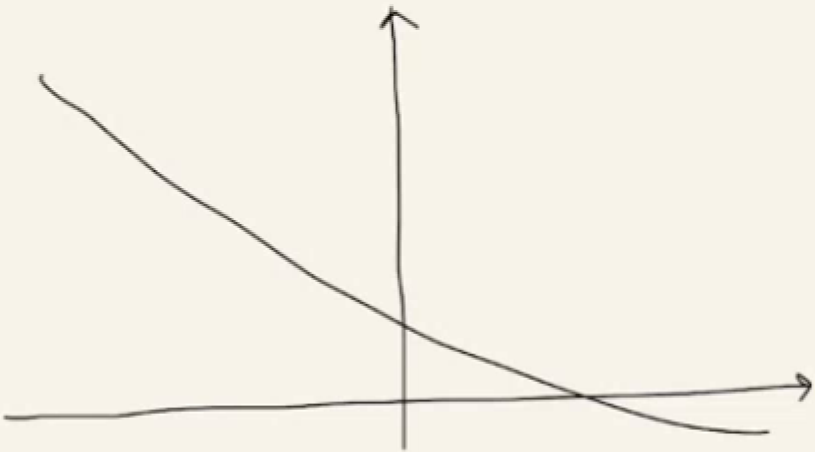
$\forall k > 0 \quad \exists \delta > 0 \mid x \in X,\ x < -\delta \quad f(x) > k$

- $\lim_{x\to -\infty} f(x) = -\infty$

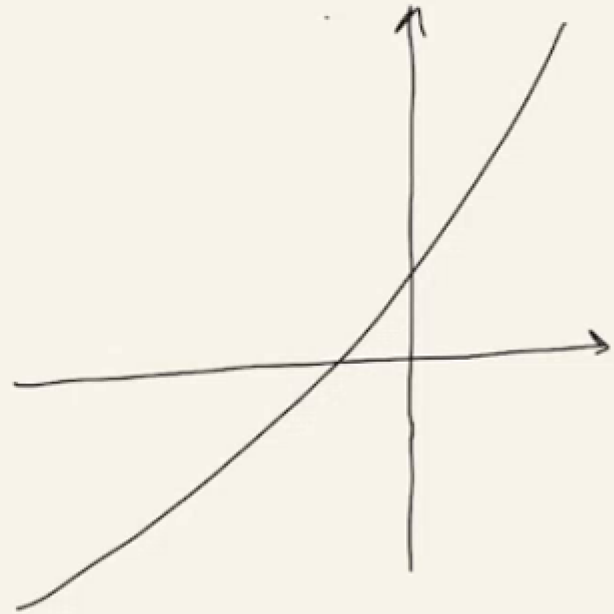
$\Leftrightarrow \forall I \in \mathcal{I}(-\infty)\ \exists J \in \mathcal{I}(-\infty) \mid x \in X \cap J \quad f(x) \in I$

Equivalentemente

$\forall K>0 \ \exists \delta>0 \mid x\in X,\ x<-\delta \quad f(x)<-K.$



$\lim_{x\to-\infty} f(x)=+\infty$

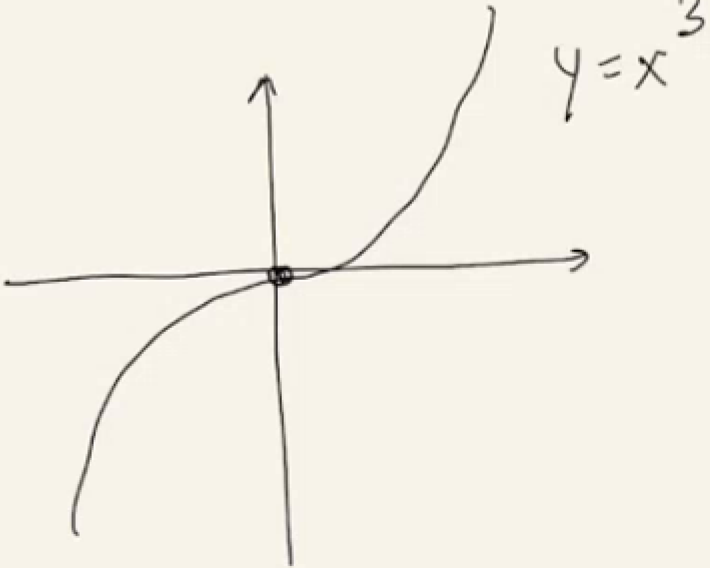


$\lim_{x\to-\infty} f(x)=-\infty$

es:

$$\lim_{x \to \infty} x^2 = +\infty, \quad \lim_{x \to -\infty} a^x = +\infty \quad \text{se } 0 < a < 1$$

$$\lim_{x \to -\infty} x^3 = -\infty$$



Oss: Scriviamo esplicitamente il legame che intercorre tra limiti di funzioni e limiti di successioni.

TEOR. PONTE

$f : X \subseteq \mathbb{R} \to \mathbb{R}$

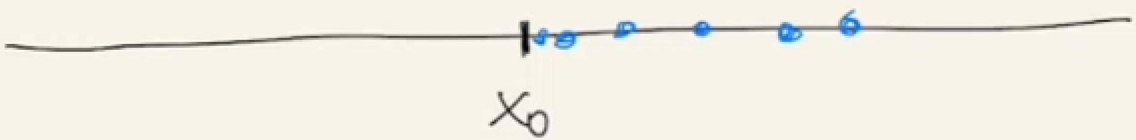
X intervallo o unione finita di intervalli

$x_0 \in X$ o x_0 estremo di uno degli intervalli che compongono X

$$\lim_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}} \iff \forall \{x_n\} \subseteq X, \ \lim_n x_n = x_0,$$

$$x_n \neq x_0 \quad \forall n \in \mathbb{N},$$

$$\lim_n f(x_n) = \ell .$$



Il teor. ponte è utile per dimostrare che un limite non esiste.

es.:

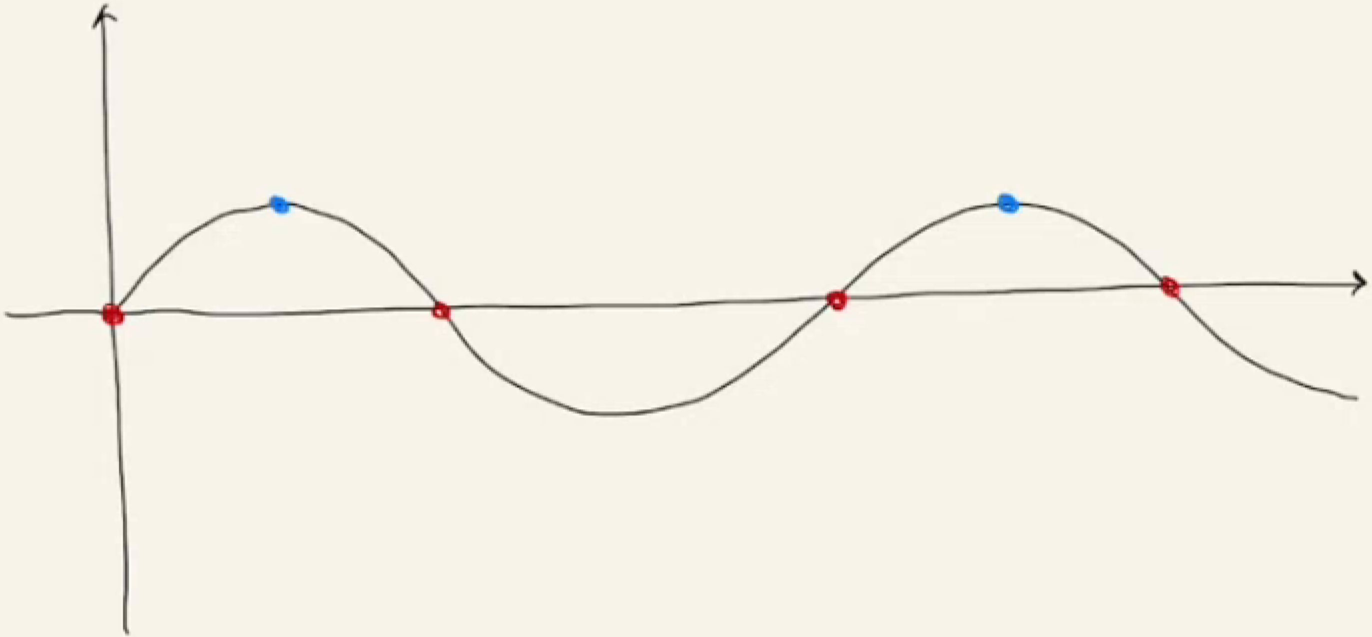
$$\nexists \lim_{x \to +\infty} \sin x .$$

Siano $x_n = 0 + n\pi$, $\lim_n (0 + n\pi) = +\infty$

$$\lim_n \sin(0 + n\pi) = 0$$

$y_n = \frac{\pi}{2} + 2n\pi$, $\lim_n \left(\frac{\pi}{2} + 2n\pi\right) = +\infty$

$$\lim_n \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 .$$



es. per caso

$\nexists \lim_{x \to +\infty} \cos x$, $\nexists \lim_{x \to -\infty} \cos x$, $\nexists \lim_{x \to -\infty} \sin x$.

Inoltre, il teor. ponte consente di ricondurre lo studio di gran parte delle proprietà dei limiti di funzioni a quello delle analoghe proprietà dei limiti di successioni.

Teor. di unicità di limite

Se f è regolare per $x \to x_0 \in \overline{\mathbb{R}}$, allora essa ammette un unico limite.

Teor. della permanenza del segno

f regolare per $x \to x_0 \in \overline{\mathbb{R}}$

- $\lim_{x \to x_0} f(x) = \ell > 0 \Rightarrow \exists I \in \mathcal{I}(x_0) \mid f(x) > 0, x \in I$

< 0 < 0

- • $f(x) > 0 \ \forall x \in X \Rightarrow \lim_{x \to x_0} f(x) \geq 0$

< 0 ≤ 0

eg:

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \quad e \quad \frac{1}{x} > 0 \quad \forall x > 0$$

$$\lim_{x\to-\infty} e^x = 0 \quad e \quad e^x > 0 \quad \forall x \in \mathbb{R}.$$

Teor. del confronto

f, g regolari per $x \to x_0 \in \overline{\mathbb{R}}$

- $\lim_{x\to x_0} f(x) > \lim_{x\to x_0} g(x) \Rightarrow \exists I \in \mathcal{I}(x_0) \left(f(x) > g(x), \ x \in I \right.$

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