ES. $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

$(+)$: $(a,b) + (c,d) = (a+b,\ c+d)$

$(\cdot)$: $\alpha \cdot (a,b) = (\alpha a,\ b)$

- $\mathbb{R}^2$ gruppo abeliano additivo:
  1) associativa
  2) elem. neutro
  3) opposti
  4) propr. commutativa

$$\big((a,b) + (c,d)\big) + (e,f) = (a+b,\ c+d) + (e,f) =$$

$$= \boxed{(a+b+c+d,\ e+f)}$$

$$(a,b) + \big((c,d) + (e,f)\big) = (a,b) + (c+d,\ e+f) =$$

$$= \boxed{(a+b,\ c+d+e+f)}$$

$+$ non è associativa $\Rightarrow \mathbb{R}$ non è uno sp. vett.

**ES.** $\mathbb{R}^2$

$$(+): \quad (a,b)+(c,d) = (a+c,\ b+d)$$

$$(\cdot): \quad \alpha(a,b) = (\alpha a,\ b)$$

additivo

$-\ (\mathbb{R}^2, +)$ è un gruppo abeliano.

$$\begin{cases} \forall\, \alpha, \beta \in \mathbb{R},\ \forall\, (a,b) \in \mathbb{R}^2, \quad (\alpha+\beta)(a,b) \overset{?}{=} \alpha(a,b)+\beta(a,b) \\ (\alpha+\beta)\cdot(a,b) = ((\alpha+\beta)a,\ b) = (\alpha a + \beta a,\ b) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \neq \\ \alpha(a,b)+\beta(a,b) = (\alpha a,\ b)+(\beta a,\ b) = (\alpha a+\beta a,\ 2b) \end{cases}$$

$\Rightarrow \mathbb{R}^2$ non è uno sp. vett. rispetto a queste operazioni.

ES. $V = M_2(F)$ $\quad S = \{ A \in M_2(F) \mid A^t = A \}$ sottosp. vett.

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in F \right\}$$

a) $T$ sottosp. vett. di $V$: 1) $\forall A, B \in T, \ A + B \in T$?

2) $\forall A \in T, \forall \alpha \in F, \ \alpha A \in T$?

1) $\forall A, B \in T, \quad A = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} \Rightarrow$

$$A + B = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix} \in T.$$

2) $\forall A \in T, \forall \alpha \in F, \quad A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow \alpha A = \alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} =$

$$= \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \in T.$$

$\dim_F T = 2 \quad \Rightarrow \quad B_T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in F \right\} \Rightarrow \dim_F S = 3$

$B_S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

b) Poiché $T \subseteq S \Rightarrow S \cap T = T$

c) Poiché $T \subseteq S \Rightarrow S + T = S \neq V$

b) (reprise): $\forall A \in S \cap T \Rightarrow A \in S \Rightarrow A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$A \in T \Rightarrow A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \Rightarrow \begin{cases} a = x \\ b = 0 \\ \cancel{b = 0} \\ c = y \end{cases} \Rightarrow b = 0$$

$$S \cap T = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \middle| \ a, c \in F \right\} = T$$

ES. $V = F_2[x]$

$$U = \{ a_1 x + a_2 x^2 \mid a_1, a_2 \in F \}$$

$$W = \{ a_0 + a_0 x + a_0 x^2 \mid a_0 \in F \}$$

a) Dim. che $U$ e $W$ sono sottosp. vett.

$\dim_F U = 2 \qquad B_U = \{ x, x^2 \}$

$\dim_F W = 1 \qquad B_W = \{ 1 + x + x^2 \}$

b) $\forall\, p(x) \in U \cap W$ $\qquad p(x) \in U \Rightarrow p(x) = a_1 x + a_2 x^2$

$$\parallel \qquad \Rightarrow$$

$\qquad p(x) \in W \Rightarrow p(x) = a_0 + a_0 x + a_0 x^2$

$$\begin{cases} 0 = a_0 \\ a_1 = a_0 \\ a_2 = a_0 \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases} \Rightarrow p(x) = 0 \Rightarrow U \cap W = \{0\} \qquad B_{W \cap U} = \emptyset$$

$\Rightarrow U$ e $W$ sono in somma diretta

$$\dim_F (U \oplus W) = \dim_F U + \dim_F W = 2 + 1 = 3 = \dim_F V \Rightarrow$$

$$\Rightarrow V = U \oplus W. \qquad B_{U \oplus W} = \{x, x^2, 1 + x + x^2\}.$$

**ES.** $V = F_3[x]$

$$U = \langle \overset{u_1}{1+x+x^2+x^3},\ \overset{u_2}{2+2x+x^2+x^3},\ \overset{u_3}{1+x+2x^2+2x^3} \rangle$$

$\dim_F U = ?$ $\qquad\qquad u_2 + u_3 = 3u_1$

$$\alpha(1+x+x^2+x^3) + \beta(2+2x+x^2+x^3) + \gamma(1+x+2x^2+2x^3) = 0$$

$$(\alpha+2\beta+\gamma) + (\alpha+2\beta+\gamma)x + (\alpha+\beta+2\gamma)x^2 + (\alpha+\beta+2\gamma)x^3 = 0$$

$$\begin{cases} \alpha+2\beta+\gamma = 0 \\ \cancel{\alpha+2\beta+\gamma = 0} \\ \alpha+\beta+2\gamma = 0 \\ \cancel{\alpha+\beta+2\gamma = 0} \end{cases}$$

$$\begin{array}{l} \begin{cases} \alpha+2\beta+\gamma = 0 \\ \alpha+\beta+2\gamma = 0 \end{cases} \\ \hline \diagup \quad \beta - \gamma = 0 \end{array} \Rightarrow \begin{cases} \beta = \gamma \\ \alpha = -3\gamma \end{cases}$$

In particolare, se scelgo $\gamma = 1 \Rightarrow \beta = 1, \alpha = -3$

$$-3 u_1 + u_2 + u_3 = 0 \Rightarrow \begin{cases} u_1 = \frac{1}{3} u_2 + \frac{1}{3} u_3 \\ u_2 = 3 u_1 - u_3 \\ u_3 = 3 u_1 - u_2 \end{cases}$$

$\Rightarrow U = \langle 1 + x + x^2 + x^3, \; 2 + 2x + x^2 + x^3 \rangle$ tali vettori sono lin. ind. poiché non sono uno multiplo dell'altro $\Rightarrow$

$\Rightarrow \dim_F U = 2.$

__ES.__ $F^2$

$$U = \langle \overset{u_1}{(1,0)}, \overset{u_2}{(2,0)}, \overset{u_3}{\underline{\underline{(0,1)}}} \rangle \qquad u_2 = 2 u_1$$

ES. $V = M_2(F)$ $\quad W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; d = 2a-2b-c \right\}$

1) W è sottosp. vett.: a) $\forall A, B \in W,\ A+B \in W$?

b) $\forall A \in W$ e $\forall \alpha \in F$, $\alpha A \in W$?

a) $\forall A, B \in W$, $A = \begin{pmatrix} a & b \\ c & 2a-2b-c \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & 2x-2y-z \end{pmatrix}$

$$A+B = \begin{pmatrix} a+x & b+y \\ c+z & 2a-2b-c+2x-2y-z \end{pmatrix} \in W$$

$$2a-2b-c+2x-2y-z = 2(a+x)-2(b+y)-(c+z) = 2a+2x-2b-2y-c-z$$

b) $\forall A \in W$, $\forall \alpha \in F$, $A = \begin{pmatrix} a & b \\ c & 2a-2b-c \end{pmatrix} \Rightarrow \alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & 2\alpha a - 2 \end{pmatrix}$

$$\Rightarrow \alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \underbrace{2\alpha a - 2\alpha b - \alpha c}_{\substack{\parallel \\ 2(\alpha a) - 2(\alpha b) - \alpha c}} \end{pmatrix} \in W.$$

b) $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \right\}$ base di $W$ su $F$?

$\dim_F W = 3$ basta dimostrare che i vettori di $\mathcal{B}$ sono lin. indip.

$$\alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \alpha+\beta & \beta-\gamma \\ \alpha & \alpha+2\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha+\beta=0 \\ \beta-\gamma=0 \\ \alpha=0 \\ \alpha+2\gamma=0 \end{cases} \Rightarrow$$

$$\begin{cases} \alpha=0 \\ \beta=0 \\ \gamma=0 \end{cases}$$ $\Rightarrow$ i vettori sono lin. indip. $\Rightarrow$ $\mathcal{B}$ è una base di $W$ su $F$.

c) $$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} = \underbrace{\alpha\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \beta\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \alpha+\beta & \beta-\gamma \\ \alpha & \alpha+2\gamma \end{pmatrix} \Rightarrow \begin{matrix} \alpha+\beta=1 \\ \beta-\gamma=-1 \\ \alpha=3 \end{matrix}$$

$$\begin{cases} \alpha = 3 \\ \beta = -2 \\ \gamma = -1 \end{cases} \qquad \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & 2a-2b-c \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \beta\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{cases} \alpha + \beta = a \\ \beta - \gamma = b \\ \alpha = c \\ \alpha + 2\gamma = 2a - 2b - c \end{cases} \qquad \begin{cases} \alpha = c \\ \beta = a - c \\ \gamma = a - b - c \end{cases}$$

ES. $V = M_2(F) \qquad U = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x, y \in F \right\} ; \; W = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\}$

a) $\dim_F U = 2 \qquad \mathcal{B}_U = \left\{ \overset{u_1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \right\}$

b) $\dim_F W = 2 \qquad \mathcal{B}_W = \left\{ \underset{w_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

c) Notiamo che $u_1 = w_1 + w_2 \Rightarrow u_1 \in W \Rightarrow u_1 \in U \cap W \Rightarrow$

$\Rightarrow \dim_F (U \cap W) \geq 1$. Ma se $\dim_F (U \cap W) = 2 \Rightarrow U \cap W = U$ poiché $U \cap W \subseteq U$ ed hanno la stessa dimensione. Ma osserviamo che $u_2 \notin \langle w_1, w_2 \rangle$ poiché l'antidiagonale di $u_2$ é $\neq 0$ mentre in $w_1$ e $w_2$ é $= 0 \Rightarrow u_2 \notin U \cap W \Rightarrow U \cap W \neq U$

Pertanto $\dim_F (U \cap W) \neq 2 \Rightarrow \dim_F (U \cap W) = 1$

$$B_{U\cap W} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$U \cap W = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in F \right\} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \in F \right\}.$$

d) $\dim_F (U + W) = \dim_F U + \dim_F W - \dim_F (U \cap W) = 2 + 2 - 1 = 3$

$U + W \neq V.$

$$B_{U\cap W} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \longrightarrow B_{U+W} = \underline{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}}.$$

$A \in U + W, \quad A = \alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \beta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \gamma\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \qquad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$$= \begin{pmatrix}\alpha & \beta\\ \beta & \alpha+\gamma\end{pmatrix}$$

$$U + W = \left\{ \begin{pmatrix}a & b\\ b & c\end{pmatrix} \;\middle|\; a, b, c \in F \right\} = S$$

**ES.** $V = F^4$

$$U = \langle \underset{u_1}{(1,0,0,0)}, \underset{u_2}{(3,-2,-2,0)}, \underset{u_3}{(0,1,1,0)} \rangle$$

$$W = \langle \underset{w_1}{(0,0,1,1)}, \underset{w_2}{(1,0,1,1)}, \underset{w_3}{(2,0,1,1)} \rangle$$

- $\underline{U}$: $u_2 = 3u_1 - 2u_3$

$U = \langle (1,0,0,0),\ (0,1,1,0) \rangle$ tali vettori sono lin. ind

perché non sono multipli $\Rightarrow \dim_F U = 2$, $\mathcal{B}_U = \{ u_1, u_3 \}$.

$v \in U$, $v = \alpha u_1 + \beta u_3 = \alpha (1,0,0,0) + \beta (0,1,1,0) =$

$= (\alpha, \beta, \beta, 0)$.

$U = \{ (\alpha, \beta, \beta, 0) \mid \alpha, \beta \in F \}$.

- <u>W</u>: $\quad w_3 = 2w_2 - w_1$

$W = \langle (0,0,1,1),\ (1,0,1,1) \rangle$ lin. ind. $\Rightarrow \dim_F W = 2$

e $\mathcal{B}_W = \{ (0,0,1,1),\ (1,0,1,1) \}$.

$v \in W$, $v = \alpha w_1 + \beta w_2 = \alpha (0,0,1,1) + \beta (1,0,1,1) =$

$= (\beta, 0, \alpha + \beta, \alpha + \beta)$

$W = \{(a, 0, b, b) \mid a, b \in F\}$.

- $\underline{U \cap W}$: Notiamo che $u_1 = w_2 - w_1 \Rightarrow u_1 \in U \cap W \Rightarrow$
$\Rightarrow \dim_F (U \cap W) \geq 1$. Ma $u_3 \notin W \Rightarrow U \cap W \neq U \Rightarrow$
$\Rightarrow \dim_F (U \cap W) \neq 2$ e quindi $\dim_F (U \cap W) = 1$ e
$B_{U \cap W} = \{u_1\}$. $U \cap W = \{(a, 0, 0, 0) \mid a \in F\}$

- $\underline{U+W}$ : $\dim_F U+W = \dim_F U + \dim_F W - \dim_F U\cap W = 2+2-1 = 3$

$\Rightarrow U+W \neq V$

$B_{U\cap W} = \{(1,0,0,0)\} \longrightarrow B_{U+W} = \{(1,0,0,0), (0,1,1,0), (0,0,1,1)\}$

$v \in U+W$, $v = \alpha(1,0,0,0) + \beta(0,1,1,0) + \gamma(0,0,1,1) =$

$= (\alpha, \beta, \beta+\gamma, \gamma)$

$U+W = \{(\alpha, \beta, \beta+\gamma, \gamma) \mid \alpha, \beta, \gamma \in F\}.$

# MATRICI E SISTEMI LINEARI

RICORDI: 1) $A \in M_{m \times n}(F)$, $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

2) $A \in M_{m \times n}(F)$, $A_i$ = i-esima riga; $A^j$ = j-esima colonna

3) $M_{m \times 1}(F)$ vettori colonna ; $M_{1 \times n}(F)$ vettori riga .

4) $A \in M_{m \times n}(F)$, $A = (a_{ij})_{i,j} \Rightarrow A^t = (a_{ji})_{i,j}$ trasposta

5) $A \in M_n(F)$, $\begin{cases} \text{se } A = A^t \Rightarrow A \text{ é simmetrica} \\ \text{se } A = -A^t \Rightarrow A \text{ é antisimmetrica} \end{cases}$

6) Se $A = (a_{ij})_{i,j} \in M_n(F)$ è t.c. $a_{ij} = 0 \quad \forall\, i > j \Rightarrow A$ é detta triangolare superiore

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$$

7) Se $A = (a_{ij}) \in M_n(F)$ é t.c. $a_{ij} = 0 \quad \forall\, i < j \Rightarrow A$ é detta triangolare inferiore

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ & & a_{33} & \\ & * & \ddots & 0 \\ & & & a_{nn} \end{pmatrix}$$

8) $A \in M_n(F)$ é detta diagonale se $a_{ij} = 0 \quad \forall\, i \neq j$

$$A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$$

9) $A \in M_n(F)$ è detta scalare se è diagonale e tutti gli elementi sulla diagonale coincidono, $A = \begin{pmatrix} a & & 0 \\ & a & \\ & \ddots & \\ 0 & & a \end{pmatrix}$.

**PRODOTTO TRA MATRICI:** $A \in M_{m\times n}(F)$, $B \in M_{n\times t}(F)$

$\Rightarrow$ è possibile calcolare $A\cdot B$. Se $t \neq m$ non si può calcolare $B \cdot A$.

Se $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ e $B = (b_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,t}}$ allora

$$A\cdot B = \left( \sum_{k=1}^{n} a_{ik} b_{kj} \right)_{i,j}$$

**ES.** $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \in M_{2\times 3}(F)$ ; $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -1 \end{pmatrix} \in M_{3\times 2}$

$$A \cdot B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \overset{(1,1)}{1 \cdot 1 + 1 \cdot 0 + 2 \cdot 3} & 1 \cdot 0 + 1 \cdot 1 + 2 \cdot (\ldots \\ \underset{(2,1)}{0 \cdot 1 + (-1) \cdot 0 + 1 \cdot 3} & \underset{(2,2)}{0 \cdot 0 + (-1) \cdot 1 + 1 \cdot (\ldots} \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -1 \\ 3 & -2 \end{pmatrix} \in M_2(F)$$

$$BA \in M_3(F)$$

**OSS.** Se $A, B \in M_n(F)$, possiamo eseguire sia $AB$ che $BA$. Ma in genere $AB \neq BA$, ad esempio: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ e $B = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix}$

$$\Rightarrow \quad A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\#$$

$$B \cdot A = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}$$

$$xy = 0 \iff x = 0 \text{ opp. } y = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$I_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$\underset{\substack{\neq \\ 0}}{X} \; \underset{\substack{\neq \\ 0}}{Y} = 0$$

**PROPRIETÀ:** $\forall A, B \in M_{m \times n}(F)$ e $\forall C, D \in M_{n \times t}(F)$:

1) $(A+B)C = AC + BC$

2) $A(C+D) = AC + AD$

3) $\forall \alpha \in F, \quad \alpha(AC) = A(\alpha C) = (\alpha A)C$

4) $A \cdot I_n = I_m \cdot A = A \; ; \quad I_n \cdot C = C \cdot I_t = C$

5) $(AC)E = A(CE), \quad E \in M_{t \times q}(F)$

6) $(A+B)^t = A^t + B^t$

7) $(AC)^t = C^t A^t$

Dunque se $V = M_n(F) \Rightarrow I_n$ è l'elemento neutro del prodotto tra matrici: $V$ è un gruppo rispetto al prodotto?

NO, infatti per esempio in $M_2(F)$ se si considera la matrice $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, allora $A$ non è invertibile:

se fosse invertibile, $\exists A^{-1} \in M_2(F)$ t.c. $AA^{-1} = I_2$.

$$A^{-1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$0 = 1$ ASSURDO!

DEF. Si definisce gruppo generale lineare di ordine $n$, l'insieme

$$GL_n(F) = \{ A \in M_n(F) \mid A \text{ è invertibile} \}.$$

OSS. $(GL_n(F), \cdot)$ è un gruppo.

1) $\forall A, B, C \in GL_n(F)$, $(AB)C = A(BC)$ vera perché deriva dalla propr.

associativa delle matrici.

2) $\forall A, B \in GL_n(F)$, $AB \in GL_n(F)$ ?

Ma $\forall A, B \in GL_n(F)$, $\exists A^{-1}, B^{-1} \in M_n(F)$ t.c. $AA^{-1} = I_n$, $BB^{-1} = I_n$

Ma allora $(AB)^{-1} = B^{-1} \cdot A^{-1}$ infatti: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} =$

$= (A I_n)A^{-1} = A \cdot A^{-1} = I_n \Rightarrow B^{-1}A^{-1} = (AB)^{-1} \Rightarrow AB$ è invertibile

$\Rightarrow AB \in GL_n(F)$.

3) Elem. neutro $= I_n$.

4) $\forall A \in GL_n(F)$, per costruzione $\exists A^{-1} \in GL_n(F) \Rightarrow A$ é invertibile $\Rightarrow$
$\Rightarrow$ in $GL_n(F)$ esistono gli inversi.

OSS. $(GL_n(F), +)$ non é un gruppo

Difatti: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(F)$; $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(F)$ $(B^{-1} = B)$

$$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin GL_2(F).$$