Confluence of an Explicit Substitutions Calculus Formalized

Flávio L. C. de Moura[*], Leandro O. Rezende[†]
Departamento de Ciência da Computação
Universidadede Brasília, Brasília, Brazil

Abstract

Rewriting theory is a well established model of computation equivalent to the Turing machines. The most well known rewriting system is the λ-calculus, the theoretical foundation of the functional paradigm of programming. Confluence is an important property related to the determinism of the results given by a rewriting system. In this work, which is still in progress, we formalize the confluence property of an extension of the λ-calculus with explicit substitutions following the steps in [4, 5]. Confluence is obtained through the Z property [3]: we first formalized the fact that an abstract rewriting system, i.e. a binary relation over an arbitrary set, that satisfies the Z property is confluent. The formalization is done in the Coq proof assistant [7].

In the λ-calculus, terms that differ by the name of its bound variables are considered equal. This notion is known as α-equivalence, which is a costly computational equivalence relation. Alternatives to α-equivalence include the so called De Bruijn indexes [2], where variables are represented by natural numbers. In De Bruijn notation terms have a unique representation, and hence there is no need of α-equivalence. Nevertheless, defining a reduction in De Bruijn notation requires a non-trivial algebra for referencing and updating indexes. The Locally Nameless Representation [1] is a framework that takes the advantages of the two notations: bound variables are represented as De Bruijn indexes, while free variables uses names. The original framework uses classical logic and was built for representing λ-terms, therefore we decided to take some of its constructions (which are not based on classical logic) and define an operator for the substitution operation. Therefore, our framework is constructive, i.e. it does not rely on the law of excluded middle or on the proof by contradiction principles. This is important because one of the goals of this work is the generation of certified code via the extraction mechanism of Coq.

Our formalization is based on the paper [4], where the λex-calculus is defined. One of the challenging steps of this formalization is that the λex-calculus defines an equational theory based on the fact that reduction is done modulo permutation of independent substitutions. In order to avoid an explicit manipulation of permutation of independent substitutions, we use the generalized rewriting facilities of Coq [6]. Nevertheless, the generated equivalence relation needs to be defined over every expression, and not only over terms. In order to circumvent this problem, we proved that the reduction relation defined by the λex-calculus modulo permutations of independent substitutions is restricted to terms.

The formalization is available at `https://github.com/flaviodemoura/Zproperty.git` and is divided in two files:

1. The file `ZtoConfl.v` contains the proof that an abstract rewriting system R that satisfies the Z property is confluent;
2. The file `lex.v` contains the current status of the formalization showing that the λex-calculus in the locally nameless representation satisfies the Z-property, and hence is confluent.

References

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[*]flaviomoura@unb.br
[†]L-ordo.ab.chao@hotmail.com

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