A Formalisation that Z Property implies Confluence

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Abstract Rewriting Systems

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Definition (Z Property)

Let $(A, \to)$ be an abstract rewriting system (ARS). The system $(A, \to)$ has the Z property, if there exists a map $wb : A \to A$ such that:

$$\begin{tikzcd} a \arrow[r] & b \arrow[dl, dotted, two heads] \\ wb(a) \arrow[r, dotted, two heads] & wb(b) \end{tikzcd}$$

$\texttt{Definition}\ \mathit{Zprop}\ \{A{:}\texttt{Type}\}\ (R{:}\ \mathit{Rel}\ A) := \exists\ wb{:}A \to A, \forall\ a\ b,\ R\ a\ b \to ((\mathit{refltrans}\ R)\ b\ (wb\ a) \wedge (\mathit{refltrans}\ R)\ (wb\ a)\ (wb\ b)).$

Confluence and the Z Property

Definition (Confluence)

An ARS $(A, \to)$ is confluent if

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, ->>] \arrow[dr, ->>] & \\ b \arrow[dr, ->>, dotted] & & c \arrow[dl, ->>, dotted] \\ & d & \end{tikzcd}$$

$\texttt{Definition}$ *Confl* $\{A{:}\texttt{Type}\}$ $(R{:}\ \mathit{Rel}\ A) := \forall\ a\ b\ c,\ (\mathit{refltrans}\ R)\ a\ b \to (\mathit{refltrans}\ R)\ a\ c \to (\exists\ d,\ (\mathit{refltrans}\ R)\ b\ d \wedge (\mathit{refltrans}\ R)\ c\ d)$.

Confluence and the Z Property

$$\begin{tikzcd} a \arrow[r] & b \arrow[dl, dotted, two heads] \\ wb(a) \arrow[r, dotted, two heads] & wb(b) \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & a \arrow[dl, two heads] \arrow[dr, two heads] & \\ b \arrow[dr, dotted, two heads] & & c \arrow[dl, dotted, two heads] \\ & d & \end{tikzcd}$$

Theorem *Zprop_implies_Confl* $\{A$:Type$\}$: $\forall$ *R*: *Rel A*, *Zprop R* $\rightarrow$ *Confl R*.

Application: Explicit Substitutions

The set of free variables of a term

Definition

Let t be a term. The set of free variables of t, notation $\texttt{fv}(t)$, is inductively defined as:

- $\texttt{fv}(x) = \{x\}$

The λex-calculus

- Developed by Delia Kesner [1]

$$\mathcal{T} ::= x \mid \mathcal{T}\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}[x/\mathcal{T}]$$

$$\begin{array}{llll} t[x/u][y/v] & =_C & t[y/v][x/u] & \text{if } y \notin \texttt{fv}(u) \text{ and } x \notin \texttt{fv}(v) \\ (\lambda x.t)\ u & \to_{\mathtt{B}} & t[x/u] & \end{array}$$

Locally Nameless Representation

- Developed in Coq by Arthur Charguéraud.
- No need for α-conversion.
- Cofinite quantification is used to obtain strong induction principles.

D. Kesner.
A Theory of Explicit Substitutions with Safe and Full Composition.
5(3:1):1–29, 2009.