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Brief paper

# All stabilizing PID controllers for time delay systems☆

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ABSTRACT

This paper presents a method to compute the entire set of stabilizing PID controller parameters for an arbitrary (including unstable) linear time delay system. The main contribution is to handle the infinite number of stability boundaries in the $(k_d, k_i)$-plane for a fixed proportional gain $k_p$. For retarded open loops, it is shown that the stable region in the $(k_d, k_i)$-plane consists of convex polygons. Concerning neutral loops, a new phenomenon is introduced. For certain systems and certain $k_p$, the exact stable region in the $(k_d, k_i)$-plane can be described by the limit of a sequence of polygons with an infinite number of vertices. This sequence may be well approximated by convex polygons. Moreover, the paper describes a necessary condition for $k_p$-intervals potentially having a stable region in the $(k_d, k_i)$-plane. Thus, the set of stabilizing controller parameters can be calculated after gridding $k_p$ in these intervals. A Matlab tool implementing the presented method is available.



## 1. Introduction

The PID controller is undoubtedly the most common controller in industrial practice, and PID control pertains to the oldest control concepts. However, basic stability questions even for linear systems are still a field of investigation in control theory. This holds especially true, if the system includes a time delay. This paper treats the problem of calculating the entire set of stabilizing PID controller parameters $(k_i, k_p, k_d)$ of the closed loop given in Fig. 1. Its characteristic function (c.f.) is

$$P(s) := (k_i + k_p s + k_d s^2)A(s) + B(s)e^{Ls}, \tag{1}$$

with $B(s) = sR(s)$, the time delay value $L > 0$ and the polynomials

$$A(s) := a_0 + a_1 s + \cdots + a_m s^m, \quad a_m \neq 0,$$
$$B(s) := b_1 s + \cdots + b_n s^n, \quad b_n \neq 0.$$

The degree difference of $B(s)$ and $A(s)$ is $l := n - m$ and $\hat{B}(s) := B(s)e^{Ls}$. It is assumed that $A(s)$ and $B(s)$ are coprime. The c.f. (1) is a so-called quasipolynomial possessing an infinite number of roots. To yield a stable closed loop, all roots must lie in the open left half plane (LHP) and the roots far from the origin must not approach the imaginary axis asymptotically (Bellman & Cooke, 1963). The *principal term condition* (Pontryagin, 1955) requires for stability that $l \geq 2$ holds, which is assumed in the sequel.

A key idea to understand the stabilizing controller parameter space was that the stable region in the $(k_d, k_i)$-plane for fixed $k_p$ consists of convex polygons for delay-free systems, which was first shown in Ho, Datta, and Bhattacharayya (1998) by an extension of the Hermite–Biehler theorem. Silva, Datta, and Bhattacharyya (2002) apply the approach to first-order delay systems and report that the convex polygon property extends to this case. Hwang and Hwang (2003); Ou, Zhang, and Gu (2006); Wang (2007a,b) focus on second-order, $n$th order, integrating and unstable first-order delay systems, respectively. Xu, Datta, and Bhattacharayya (2003) expand the Hermite–Biehler approach to arbitrary linear delay systems, but the formulated problem is slightly different and needs an additional grid on a frequency variable $\omega$. Therein, the set of controller parameters is searched, which stabilizes systems with a constant time delay equal or *smaller* than a number $L_0$. The papers (Bajcinca, 2005a; Bajcinca, Koeppe, & Ackermann, 2002; Hohenbichler & Ackermann, 2003) extend a D-decomposition approach to the case of arbitrary linear delay systems and find out that for a fixed $k_p$ the stability boundaries in the $(k_d, k_i)$-plane are straight lines and the number of stability boundaries is infinite. In Section 2, the present paper reformulates this approach in a compact way. The most important contributions of this paper are to be found in Section 3. It is proven that the exact stable region can be described by a finite number of boundaries if $l > 2$, yielding a set of convex polygons as the stable region in the $(k_d, k_i)$-plane. This is due to the fact that the roots far from the origin behave quite regularly and lie asymptotically on so-called root chains, see Bellman and Cooke (1963). If $l > 2$, (1) is of retarded type and possesses a retarded

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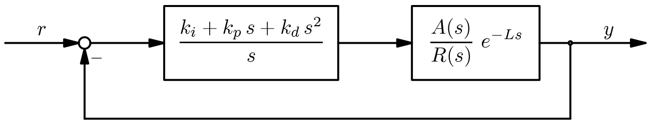


**Fig. 1.** A PID-controlled time delay system.

root chain, which means that the large roots move deep into the LHP. Consequently, the large roots do not contribute to the stability question. Only if $l = 2$ the very large roots have to be considered. In this case, (1) is of neutral type, and there is a neutral root chain, where the roots far from the origin lie on a vertical line parallel to the imaginary axis, which may influence the stable region. For neutral systems, there are cases, in which the stable region in the $(k_d, k_i)$-plane no longer consists of polygons (having a finite number of vertices), but can be described by the limit of a sequence of polygons (having an infinite number of vertices). Section 3 renders these questions of the large roots and also suggests a polygonal approximation of the latter stable region.

To apply the mentioned results, a grid on $k_p$ is needed. Concerning delay-free systems, Söylemez, Munro, and Baki (2003) introduced a theorem describing necessary conditions for $k_p$-intervals potentially leading to stable regions in the $(k_d, k_i)$-plane. Bajcinca (2005b,c) suggest theorems including the case of arbitrary delay systems. However, these theorems do not convince in every detail, especially if there are roots of $A(s)$ on the imaginary axis. Section 4 picks up the idea of the proof and presents a revised theorem with a simpler proof. Therefore, the grid on $k_p$ can be reduced to the $k_p$-intervals fulfilling the conditions of the theorem to calculate the entire set of stabilizing $(k_i, k_p, k_d)$.

## 2. The root boundaries in the $(k_d, k_i)$-plane

Following the D-decomposition approach, the stable region in the $(k_d, k_i)$-plane for a fixed $k_p$ is calculated by finding the *root boundaries* in the $(k_d, k_i)$-plane, where a root crosses the imaginary axis.

**Definition 1** (*Finite root boundaries*). A *finite root boundary* in the $(k_d, k_i)$-plane for a $k_p \in \mathbb{R}$ is the locus, where (1) possesses a root $s = \mathrm{j}\omega_\eta$ on the imaginary axis with a finite $\omega_\eta \in \mathbb{R}$. The root boundary is called a *real root boundary (RRB)* if $\omega_\eta = 0$, and a *complex root boundary (CRB)* if $\omega_\eta \neq 0$. If the root boundary is crossed starting from its so-called *unstable side*, the corresponding root moves from the the right half plane (RHP) to the LHP.

A root boundary will further be given as an inequality, which is fulfilled by points opposite to its unstable side. Note that the opposite side of the unstable side—the 'more stable' side—is not necessarily stable. Roots may also cross the imaginary axis at infinity, leading to an *infinite root boundary (IRB)*. All large frequency effects will be summarized in Section 3.

The *singular frequencies* play a crucial role in the description of the CRB. Their name originates from the fact that the CRB condition $P(\mathrm{j}\omega) = 0$ is singular in this case, see Bajcinca (2005a), Bajcinca et al. (2002), and Hohenbichler and Ackermann (2003).

**Definition 2** (*Singular frequencies*). Given a $k_p \in \mathbb{R}$ which is not an extremal value of (2) in $0 \le \omega < \infty$. If $\mathrm{j}\omega_j$ with $\omega_j \in \mathbb{R}^+$ is a root of $A(s)$ then also $k_p \neq \lim_{\omega\to\omega_j} f(\omega)$ must hold. The *singular frequencies* $\omega_\eta$ are the real positive solutions of the equation $f(\omega) = k_p$ with

$$f(\omega) := f_1(\omega)\sin(\omega L) + f_2(\omega)\cos(\omega L) \qquad (2)$$

and

$$f_1(\omega) := \frac{-R_A R_B - I_A I_B}{\omega(R_A^2 + I_A^2)}, \qquad f_2(\omega) := \frac{I_A R_B - R_A I_B}{\omega(R_A^2 + I_A^2)},$$

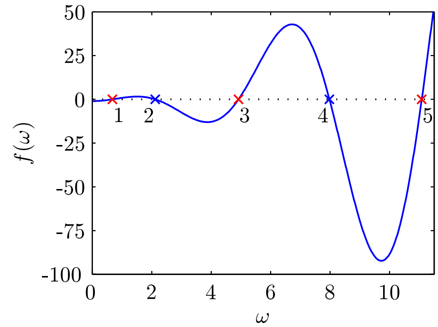


**Fig. 2.** $f(\omega)$ and the singular frequencies $\omega_\eta$, $\eta = 1, 2, \ldots, 5$ for Example 3 and $k_p = 0$.

where $R$ and $I$ denote the real and imaginary parts of $A$ and $B$, respectively, at $s = \mathrm{j}\omega$. The singular frequencies may be grouped w.r.t. to the sign of $f'(\omega)$ as

$$\Omega^+ := \{\omega_\eta \in \mathbb{R}^+ \mid f(\omega_\eta) - k_p = 0 \wedge f'(\omega_\eta) > 0\},$$

$$\Omega^- := \{\omega_\eta \in \mathbb{R}^+ \mid f(\omega_\eta) - k_p = 0 \wedge f'(\omega_\eta) < 0\}.$$

$\Omega := \Omega^+ \cup \Omega^-$ is the set of all singular frequencies.

The singular frequencies and their relation to function $f(\omega)$ are illustrated by the following example.

**Example 3.**

$A(s) := 1,$

$\hat{B}(s) := (s^3 + s^2 + s)e^s.$

The singular frequencies for $k_p = 0$ (see Fig. 2) are

$\Omega^+ = \{0.6763, 4.9212, 11.0863, \ldots\},$

$\Omega^- = \{2.1171, 7.9806, \ldots\}.$

Considering the graphical solution of $f(\omega) = k_p$ (e.g. see Fig. 2), the continuity of $f(\omega)$, the conditions on the choice of $k_p$ in Definition 2 and the fact that $|f(\omega)|$ exceeds all bounds at large frequencies (see (4)), the following properties of the singular frequencies apply.

**Corollary 4** (*Singular frequencies' properties*). *The following statements hold.*

(1) *The sets $\Omega$, $\Omega^+$ and $\Omega^-$ can be sorted as* $\{\omega_1, \omega_2, \ldots, \omega_\eta, \ldots\}$ *with* $\omega_{\eta+1} > \omega_\eta$. *(In the sequel, it is assumed that the sets are sorted in ascending order.)*
(2) *The singular frequencies in $\Omega^+$ and $\Omega^-$ interlace. If $\Omega = \{\ldots, \omega_\eta, \omega_{\eta+1}, \ldots\}$ and $\omega_\eta \in \Omega^+$ then $\omega_{\eta+1} \in \Omega^-$, and vice versa.*
(3) *The sets $\Omega$, $\Omega^+$ and $\Omega^-$ are infinite.*
(4) *If $s = \mathrm{j}\omega_j$ with $\omega_j \in \mathbb{R}^+$ is a root of $A(s)$, then $\omega_j \notin \Omega$, and vice versa.*

Having the singular frequencies defined and their properties described, all finite root boundaries in the $(k_d, k_i)$-plane can be set up by the following theorem.

**Theorem 5** (*Finite root boundaries*). *Given a system with the c.f. (1) and a $k_p$ fulfilling the conditions in Definition 2, the finite root boundaries in the $(k_d, k_i)$-plane for a fixed value of $k_p$ are the following lines*

(1) *$k_i > 0$, if $a_0 \neq 0$ and $k_p > f_0$ (RRB),*
(2) *$k_i < 0$, if $a_0 \neq 0$ and $k_p < f_0$ (RRB),*
(3) *$k_i < \omega_\eta^2 k_d + g(\omega_\eta)$ for all $\omega_\eta \in \Omega^+$ (CRB),*

(4) *$k_i > \omega_\eta^2 k_d + g(\omega_\eta)$ for all $\omega_\eta \in \Omega^-$ (CRB)*

*with*

$$g(\omega) = \omega\left(-f_2(\omega)\sin(\omega L) + f_1(\omega)\cos(\omega L)\right) \tag{3}$$

*and $f_0 = \lim_{\omega\to 0} f(\omega)$, which exists for $a_0 \neq 0$.*

The proof can be found in Bajcinca (2005a), Bajcinca et al. (2002), and Hohenbichler and Ackermann (2003). The theorem formulates all finite root boundaries in the $(k_d, k_i)$-plane as an infinite set of straight lines. Assuming that only a finite number of the root boundaries contributes to the stable region (which will be analysed in detail in the following sections), the $(k_d, k_i)$-plane is devided into a set of convex polygons. Identifying *inner polygons*, whose boundaries' unstable sides all point to the polygons' outside, helps to reduce the number of stability tests. Thus, only each inner polygon has to be checked for stability at one point from the polygon's inside to find the stable region in the $(k_d, k_i)$-plane.

**Remark 6.** Theorem 5 shows that there is a one-by-one relation between a singular frequency and a CRB. The CRB belonging to a singular frequency $\omega_\eta$ will henceforth be shortly referred to as 'CRB $\omega_\eta$'.

## 3. Large frequency behaviour

This section deals with the question to handle only a finite number of root boundaries when searching for the exact stable region in the $(k_d, k_i)$-plane. Note that $f(\omega)$, $g(\omega)$ can be evaluated at large frequencies by

$$f(\omega) = \begin{cases} -\dfrac{b_n}{a_m} j^l \omega^{l-1}\sin(\omega L) + \mathcal{O}(\omega^{l-2}), & \text{if } l \text{ even} \\ -\dfrac{b_n}{a_m} j^{l-1}\omega^{l-1}\cos(\omega L) + \mathcal{O}(\omega^{l-2}), & \text{if } l \text{ odd} \end{cases} \tag{4}$$

$$g(\omega) = \begin{cases} -\dfrac{b_n}{a_m} j^l \omega^{l}\cos(\omega L) + \mathcal{O}(\omega^{l-1}), & \text{if } l \text{ even} \\ \dfrac{b_n}{a_m} j^{l-1}\omega^{l}\sin(\omega L) + \mathcal{O}(\omega^{l-1}), & \text{if } l \text{ odd.} \end{cases} \tag{5}$$

The first important result is that large singular frequencies approach a regular periodicity of $\frac{\pi}{L}$, as described in detail by the following lemma. This regularity has been first described in Hohenbichler and Ackermann (2003); here a revised proof is added.

**Lemma 7** (*Large singular frequencies*). *The singular frequencies $\omega_\eta$ can be expressed as*

$$\omega_\eta = \begin{cases} \dfrac{\kappa(\eta)\pi}{L} + \mathcal{O}\left(\dfrac{1}{\kappa(\eta)}\right), & \text{if } l \text{ even,} \\ \dfrac{(2\kappa(\eta)+1)\pi}{2L} + \mathcal{O}\left(\dfrac{1}{\kappa(\eta)}\right), & \text{if } l \text{ odd,} \end{cases} \tag{6}$$

*with $\kappa(\eta) := \kappa_0 + \eta$ and an appropriate $\kappa_0 \in \mathbb{N}$. Consequently, the large singular frequencies approach $\frac{\kappa(\eta)\pi}{L}$ or $\frac{(2\kappa(\eta)+1)\pi}{2L}$, respectively.*

**Proof.** The singular frequencies are the real positive solutions of (2). By differentiating (2) implicitly w.r.t. to $k_p$, a Taylor series of

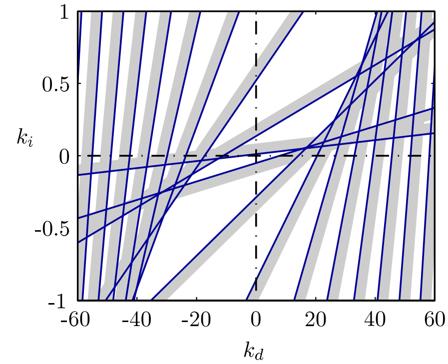


**Fig. 3.** The large CRBs $\omega_\eta$, $\eta = 3, 4, \ldots$ of Example 3, $k_p = 0$. The unstable sides are shaded.

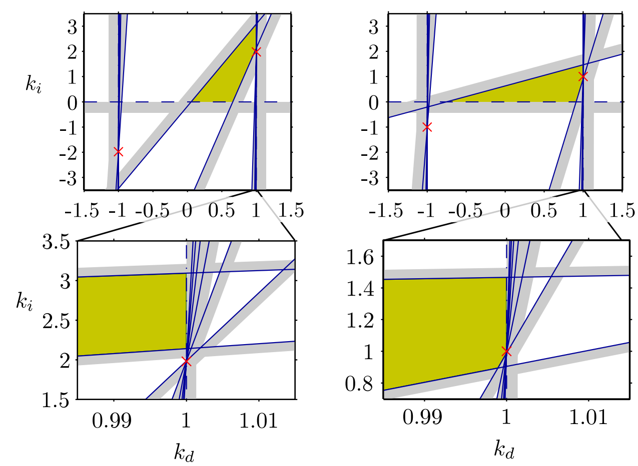


**Fig. 4.** Root boundaries with shaded unstable sides and inner polygon of Example 14 for singular frequencies $\omega_\eta < 40$ at $k_p = 1.4$ (left column) and at $k_p = 0$ (right column). The lower images show a magnification around the IRB junction point (1, 1.98) (left) or (1, 1) (right).

the explicit solution can be calculated at $\varpi(\eta)$

$$\omega_\eta = \varpi + \frac{1}{f'(\varpi)}(k_p - f(\varpi)) - \frac{f''(\varpi)}{f'(\varpi)^3}(k_p - f(\varpi))^2 + \cdots \tag{7}$$

with $\varpi := \varpi(\eta)$ and $\varpi(\eta) := \frac{\kappa(\eta)\pi}{L}$ if $l$ is even and $\varpi(\eta) := \frac{(2\kappa(\eta)+1)\pi}{2L}$ if $l$ is odd. Introducing $f(\varpi(\eta))$ and its derivatives and summarizing all terms of $\kappa(\eta)$ with order less than zero in $\mathcal{O}\left(\frac{1}{\kappa(\eta)}\right)$ leads to the statement. □

To provide a short formulation, a set of *large singular frequencies* $\Omega_{\omega_l}$ is defined which depends on the choice of a certain frequency limit $\omega_l$:

**Definition 8.** The *set of all large singular frequencies* $\Omega_{\omega_l}$ is defined as the set of all singular frequencies larger than $\omega_l$

$$\Omega_{\omega_l} := \left\{\omega_\eta \in \Omega \mid \omega_\eta > \omega_l\right\}$$

and is divided into the interlacing subsets $\Omega^+_{\omega_l}$ and $\Omega^-_{\omega_l}$ like in Definition 2.

Note that $\Omega_{\omega_l}$ is an infinite set, while $\Omega \setminus \Omega_{\omega_l}$ is finite. The regularity of the large singular frequencies also extends to the corresponding CRBs. The following new lemma highlights the $k_d$-axis intersections of the large CRBs (see Figs. 3 and 4 for examples).

**Lemma 9** (*$k_d$-axis intersections of large CRBs*). *There is an $\omega_l$ such that for the intersections of the large CRBs $\omega_\eta \in \Omega_{\omega_l}$ with the $k_d$-axis applies: The sequence of the absolute values of their $k_d$-coordinates tends strictly monotonically to $\infty$, if $l > 2$, or to $\left|\frac{b_n}{a_m}\right|$, if $l = 2$. The CRBs in $\Omega^-_{\omega_l}$ cross the $k_d$-axis to the right, the CRBs in $\Omega^+_{\omega_l}$ cross the $k_d$-axis to the left of the origin. The more stable sides of all CRBs in $\Omega_{\omega_l}$ point to the origin of the $(k_d, k_i)$-plane.*

**Proof.** From Theorem 5, the $k_d$-coordinate of the intersection of a CRB with the $k_d$-axis results as $-\frac{g(\omega_\eta)}{\omega_\eta^2}$. Using (5), (6) leads to the following sequence

$$\left\langle (-1)^{\kappa(\eta)}\frac{b_n}{a_m}\mathrm{j}^{\lambda}\left(\varpi(\eta)+\mathcal{O}\left(\frac{1}{\kappa(\eta)}\right)\right)^{l-2}+\mathcal{O}\left(\kappa(\eta)^{l-3}\right)\right\rangle \qquad (8)$$

with $\varpi(\eta) := \frac{\kappa(\eta)\pi}{L}$, $\lambda := l$ if $l$ is even, $\varpi(\eta) := \frac{(2\kappa(\eta)+1)\pi}{2L}$, $\lambda := l+1$ if $l$ is odd, and $\kappa(\eta) := \kappa_0 + \eta$ with an appropriate $\kappa_0 \in \mathbb{N}$. Evaluating the absolute value of (8) at a large $\kappa(\eta)$ leads directly to the first statement of the lemma. Using (4), the sequence of $f'(\omega_\eta)$ can be expressed as

$$\left\langle -\frac{b_n L(l-1)\mathrm{j}^{\lambda}}{(-1)^{\kappa(\eta)}a_m}\left(\varpi(\eta)+\mathcal{O}\left(\frac{1}{\kappa(\eta)}\right)\right)^{l-2}+\mathcal{O}\left(\kappa(\eta)^{l-3}\right)\right\rangle. \qquad (9)$$

The sign of (8) is changing from an element $\eta$ to the next element $\eta+1$ for sufficiently large $\eta$. Furthermore, the sign of an element $\eta$ in (8) is opposite to the sign in (9), as $L > 0$ and $l \geq 2$. It can be concluded from the Theorem 5 that the $k_d$-axis intersections of all CRBs in $\Omega^-_{\omega_l}$ with $f'(\omega_\eta) < 0$ are positive and that their more stable sides point to the origin. The same argumentation also holds for the CRBs in $\Omega^+_{\omega_l}$. □

### 3.1. Retarded system

By these results, one of the main results can be established, which allows to describe the exact stable region by only a finite number of root boundaries in the case of a retarded closed loop ($l > 2$). For the proof, the following lemma is needed.

**Lemma 10.** *Given three real numbers a, b, and c with $c > b > a > 0$. Then, it holds for $l \geq 3$, $l \in \mathbb{N}$*

$$\frac{c^l-b^l}{c^2-b^2} > \frac{b^l-a^l}{b^2-a^2}.$$

**Proof.** The proof can be done using the substitution $u := a^2$, $v := b^2$, $w := c^2$, $p := \frac{l}{2}$ and applying the mean value theorem to the function $f(x) := x^p$. □

**Theorem 11** (*Retarded systems' stable region*). *Given a system with the c.f. (1), $l > 2$ and a $k_p$ fulfilling the conditions in Definition 2. Then, the CRBs from a certain singular frequency on do not contribute to the stable region. If a stable region in the $(k_d, k_i)$-plane exists, it is a set of convex polygons.*

**Proof.** The intersections between two adjacent large CRBs $\omega_\eta$ and $\omega_{\eta+2}$ (which pertain to the same subset $\Omega^+_{\omega_l}$, or $\Omega^-_{\omega_l}$ respectively) is calculated. The intersection's $k_d$-coordinate is $\frac{-g(\omega_{\eta+2})+g(\omega_\eta)}{\omega_{\eta+2}^2-\omega_\eta^2}$ and can be expressed, using (5) and (6), by the sequence

$$\left\langle \frac{(-1)^{\kappa(\eta)}b_n}{a_m\mathrm{j}^{-l}}\left(\frac{\pi}{L}\right)^{l-2}\frac{(\kappa(\eta)+2)^l-\kappa(\eta)^l}{(\kappa(\eta)+2)^2-\kappa(\eta)^2}+\mathcal{O}\left(k^{1-l}\right)\right\rangle,$$

if $l$ is even, or

$$\left\langle \frac{(-1)^{\kappa(\eta)}b_n}{a_m\mathrm{j}^{-l-1}}\left(\frac{\pi}{2L}\right)^{l-2}\frac{(2\kappa(\eta)+3)^l-(2\kappa(\eta)+1)^l}{(2\kappa(\eta)+3)^2-(2\kappa(\eta)+1)^2}+\mathcal{O}\left(k^{1-l}\right)\right\rangle,$$

if $l$ is odd, and $\kappa(\eta) := \kappa_0 + \eta$ with an appropriate $\kappa_0 \in \mathbb{N}$. By Lemma 10, it can be stated that this sequence is strictly monotonic at sufficiently large $\kappa$. Furthermore, as the sequence of the slopes of the CRBs strictly monotonically increases (see Theorem 5), the sequence of the $k_i$-coordinates must also be strictly monotonic. Taking Lemma 9 into consideration, the $k_d$- and $k_i$-coordinates tend to $\pm\infty$ for $\eta \to \infty$. Therefore, the intersections in $\Omega^-_{\omega_l}$, or $\Omega^+_{\omega_l}$ respectively, tend strictly monotonic to $(\infty, \infty)$ or $(-\infty, -\infty)$, depending on the sign of $\frac{b_n}{a_m}$. Now consider the inner polygon of a number of large CRBs in $\Omega_{\omega_l}$ up to a sufficiently large singular frequency. Because of the fact that the sequence of the intersections of the adjacent large CRBs is strictly monotonic and not bounded, these intersections will leave the inner polygon (e.g. see Fig. 3). This means that all CRBs in the sequence of the large CRBs in $\Omega_{\omega_l}$ beyond a certain singular frequency do not contribute to the inner polygon of the large CRBs, and therefore to the inner polygon of all root boundaries. The inner polygon of all root boundaries may be affected additionally by a finite number of lower CRBs and the RRB (if it exists), but the number of root boundaries limiting the inner polygon stays finite. A finite number of root boundaries results in a set of convex polygons and each stable polygon is also an inner polygon. □

### 3.2. Neutral system

In the case of neutral systems ($l = 2$), a necessary condition on the $k_d$-value of a stabilizing controller can be stated, which results from the neutral root chain of the roots far from the origin (see also Hohenbichler and Ackermann (2003) and Xu et al. (2003)).

**Lemma 12** (*Neutral loop's IRB*). *Given a delay system with the c.f. (1) and $l = 2$. If $(k_i, k_p, k_d)$ is a stabilizing controller parameter set, then $k_d$ satisfies*

$$|k_d| < \left|\frac{b_n}{a_m}\right|$$

**Proof.** If $l = 2$, (1) tends to

$$\tilde{P}(s) = k_d a_m s^n + b_n s^n e^{sL}$$

at $|s| \to \infty$. The real part of the roots far from the origin $p_\infty$ satisfies

$$\mathfrak{Re}\{p_\infty\} = \frac{1}{L}\ln\left(\left|\frac{k_d a_m}{b_n}\right|\right).$$

The rest of the proof is done by contradiction. Assume a stable controller parameter set $(k_i, k_p, k_d)$ with $k_d$ violating the stated condition by $|k_d| \geq |\frac{b_n}{a_m}|$. In this case, it holds $\mathfrak{Re}\{p_\infty\} \geq 0$ implying an unstable closed loop. □

Lemma 9 reveals that the large CRBs approach the lines $k_d = \pm\frac{b_n}{a_m}$ for $l = 2$. The next lemma describes the approaching behaviour more precisely, stating that the $k_i$-coordinates of the intersections of the large CRBs with the lines $k_d = \pm\frac{b_n}{a_m}$ converge to a certain value.

**Lemma 13** (*IRB junction points*). *Given is a fixed $k_p$ and $l = 2$. Then, there is an $\omega_l$ such that the sequence of the $k_i$-coordinates of the intersections of the large CRBs in $\Omega^-_{\omega_l}$ with the line $k_d = \frac{b_n}{a_m}$ converges monotonically to $k_i^\infty$ and the sequence of $k_i$-coordinates of the intersections of the large CRBs in $\Omega^+_{\omega_l}$ with the line $k_d = -\frac{b_n}{a_m}$ converges monotonically to $-k_i^\infty$, where*

$$k_i^\infty := \frac{a_{m-1}^2 b_n^2 - a_m^2 b_{n-1}^2 - 2a_m a_{m-2} b_n^2 + 2a_m^2 b_n b_{n-2} + k_p^2 a_m^4}{2a_m^3 b_n}$$

*and $a_{-i} := b_{-i} := b_0 := 0$ for $i \in \mathbb{N}$. The points $\left(\frac{b_n}{a_m}, k_i^\infty\right)$ and $\left(-\frac{b_n}{a_m}, -k_i^\infty\right)$ in the $(k_d, k_i)$-plane will be called* IRB junction points.

**Proof.** For this proof, (7) is evaluated up to the first-order term for $l = 2$, reading

$$\omega_\eta = \frac{\kappa(\eta)\pi}{L} + \frac{(-1)^{\kappa(\eta)} a_m^2 k_p + a_m b_{n-1} - a_{m-1} b_n}{a_m b_n \kappa(\eta)\pi} + \mathcal{O}\left(\frac{1}{\kappa(\eta)^3}\right) \tag{10}$$

with $\kappa(\eta) := \kappa_0 + \eta$ and an appropriate $\kappa_0 \in \mathbb{N}$. The $k_i$-value of the intersection of a CRB with the line $k_d = \frac{b_n}{a_m}$ is $k_i^\eta(\omega_\eta) = \frac{b_n}{a_m}\omega_\eta^2 + g(\omega_\eta)$, while $g(\omega_\eta)$ can be expressed in the case $l = 2$ as

$$g(\omega_\eta) = \left(\frac{b_n}{a_m}\omega_\eta^2 + c_1 + \mathcal{O}\left(\frac{1}{\omega_\eta^2}\right)\right)\cos(\omega_\eta L) - \left(c_2\omega_\eta + \mathcal{O}\left(\frac{1}{\omega_\eta}\right)\right)\sin(\omega_\eta L) \tag{11}$$

with

$$c_1 := \frac{a_m a_{m-1} b_{n-1} + a_m a_{m-2} b_n - a_m^2 b_{n-2} - a_{m-1}^2 b_n}{a_m^3},$$

$$c_2 := \frac{a_{m-1} b_n - a_m b_{n-1}}{a_m^2}.$$

Eq. (8) shows that odd $\kappa(\eta)$ lead to $k_d$-coordinates of $k_d$ – axis intersections converging to $\frac{b_n}{a_m}$. Now (10) is used with odd $\kappa(\eta)$ as approximation for large $\omega_\eta \in \Omega_{\omega_l}^-$. After substituting $h := \frac{1}{\kappa(\eta)\pi}$ and inserting (10) and (11), it can be shown that $\lim_{h\to 0} k_i^\eta(h) = k_i^\infty$. As the argument of the sin- and cos–function in $k_i^\eta(h)$ tends to zero for $h \to 0$, the convergence is monotonic. The same procedure can be applied to the intersection with the line $k_d = -\frac{b_n}{a_m}$, now using an even $\kappa(\eta)$ in (10), which leads to a monotonic convergence to $-k_i^\infty$. □

The following neutral loop example demonstrates the proposition of the last lemma.

**Example 14.**

$A(s) := s + 1,$
$\hat{B}(s) := (s^3 + s^2 + s)e^s.$

For $k_p = 1.4$, it holds $k_i^\infty = 1.98$, and for $k_p = 0$, it holds $k_i^\infty = 1$ (see Fig. 4). Considering the root boundaries up to a large singular frequency, at $k_p = 1.4$, the IRB junction points lie outside the inner polygon. At $k_p = 0$, the IRB junction point $(1, 1)$ lies on an edge of the inner polygon.

Now, the next main result can be established, which indicates, in which cases the exact stable region can be described by a finite number of boundary lines for neutral systems. If the number of relevant boundary lines is infinite, the theorem states a conservative approximation and a superset of the stable region as set of polygons, allowing the approximation error to be estimated.

**Theorem 15** (*Neutral systems' stable region*). *Given a closed loop delay system with the c.f. (1), $l = 2$ and a finite $k_p$ fulfilling the conditions in Definition 2. A sufficiently large $\omega_l$ and an $\omega^* > \omega_l$ are chosen such that the properties of the large CRBs in Lemmata 9 and 13 apply and that in the interval $(\omega_l, \omega^*)$ at least two singular frequencies exist. $\Pi$ is the set of all inner polygons containing a stable point with respect to the RRB (if it exists), the IRBs and all CRBs in the interval $(0, \omega^*)$. The set of all points lying on the edges of the polygons in $\Pi$ excluding the vertices is named as $\Lambda$.*

*Then it holds: $\Pi$ exactly represents the stable region, if neither of the two IRB junction points $\left(\frac{b_n}{a_m}, k_i^\infty\right)$ and $\left(-\frac{b_n}{a_m}, -k_i^\infty\right)$ are in $\Lambda$. The stable region is composed of the limit of a sequence of convex polygons and $\Pi$ represents a superset of the stable region if one of the points is in $\Lambda$. Then, a conservative approximation of the stable region as a set of convex polygons is the intersection of the following boundary lines with $\Pi$*

- *$k_i > \omega_i^2\left(k_d - \frac{b_n}{a_m}\right) + k_i^\infty$, if $\left(\frac{b_n}{a_m}, k_i^\infty\right)$ is in $\Lambda$, and*
- *$k_i < \omega_j^2\left(k_d + \frac{b_n}{a_m}\right) - k_i^\infty$, if $\left(-\frac{b_n}{a_m}, -k_i^\infty\right)$ is in $\Lambda$,*

*where i and j are the indices of two CRBs in $\Omega^-$ and $\Omega^+$, respectively, with $\omega_{i/j} \in (\omega_l, \omega^*)$.*

**Proof.** There are the following cases.

(1) No IRB junction point is in $\Lambda$ (e.g., see Fig. 4, left column). The large CRBs in $\Omega_l$ leave the inner polygon as the convergence property described by Lemma 13 is monotonic. Clearly, $\Pi$ represents the exact stable region.
(2) The IRB junction point $\left(\frac{b_n}{a_m}, k_i^\infty\right)$ is in $\Lambda$ (e.g., see Fig. 4, right column). As the convergence property described by Lemma 13 is monotonic and the CRB slope is monotonically increasing (Theorem 5), each unevaluated neighbouring CRB $\omega_{i+2k} \in \Omega_{\omega_l}^-$, $k \in \mathbb{N}$, will cut off a piece with a constantly decreasing area of the inner polygon. If a copy of the CRB $\omega_\eta$ is translated onto the IRB junction point maintaining its slope and if the new inner polygon is determined, all subsequent CRBs will not affect the new inner polygon. The translated CRB is described by the stated theorem.
(3) The IRB junction point $\left(-\frac{b_n}{a_m}, -k_i^\infty\right)$ is in $\Lambda$. This case is analogous to the case (2) for $\omega_j \in \Omega_{\omega_l}^+$. □

Consequently, if at least one IRB junction is in $\Lambda$, $\Pi$ represents a superset of the exact stable region, which is the limit of the sequence of polygons when $\eta \to \infty$. □

### 3.3. *Practical determination of the upper frequency limit*

The results in Theorems 11 and 15 concerning the stable region in the $(k_d, k_i)$-plane rely on the determination of a sufficiently large upper limit $\omega_l$, such that the CRBs in $\Omega_{\omega_l}$ possess the properties of Lemmata 9 and 13.

How can $\omega_l$ be determined practically? The error induced by neglecting the $\mathcal{O}$-terms in (4) and (5) can be used as orientation. It should be relatively small for all $\omega > \omega_l$ compared to the function values of (2) and (3). In practical applications, an error limit of 10% has been successfully used.

## 4. The stabilizing $k_p$-interval

The previous results depend on a grid of $k_p$-values to derive the stable region in the $(k_p, k_d, k_i)$-space. This section presents a theorem which returns $k_p$-intervals potentially having stable regions in the $(k_d, k_i)$-plane, such that gridding $k_p$ can be reduced to those intervals.

Note that the number of singular frequencies in a finite frequency interval depends on the actual value of $k_p$ (see Definition 2). At extrema of $f(\omega)$, the number of singular frequencies as solutions of $f(\omega) = k_p$ changes. The following revised theorem (see also

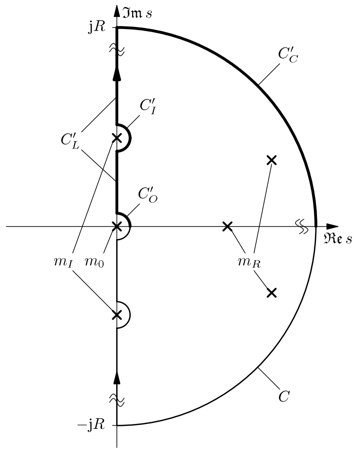


**Fig. 5.** The curves $C$, $C'_O$, $C'_C$, $C'_L$ and $C'_I$.

Bajcinca (2005b,c)) states a minimal number of singular frequencies in a certain frequency interval. This means that the extrema of $f(\omega)$ appear as limits of $k_p$-intervals, for which a stable region in the $(k_d, k_i)$-plane may exist.

**Theorem 16** ($k_p$-*Interval*). *Given a system with the c.f. (1) and a $k_p$ fulfilling the conditions in Definition 2. The number of singular frequencies in an interval $(0, R)$ is named as $z$. Then it holds: if there is a stable region in the $(k_d, k_i)$-plane then there is a $\kappa_l \in \mathbb{N}$ such that $z$ fulfills the inequality $z \geq z_{\min}$ for each $\kappa \geq \kappa_l$, $\kappa \in \mathbb{N}$ with $R := \frac{2\kappa+(l \bmod 2)-1}{2L}\pi$ and*

$$z_{\min} := \kappa + m_R + \frac{m_I - \hat{m}_I}{2} + \left\lceil \frac{l}{2} \right\rceil + \left\lceil \frac{m_O}{2} \right\rceil - 1,$$

*where $\lceil\cdot\rceil$ denotes the ceiling function, which returns the smallest integer not less than the argument. $m_R$ is the number of open RHP roots, $m_O$ is the number of roots in the origin and $m_I$ is the number of roots $s = \mathrm{j}\omega_j$ of $A(s)$ with $\omega_j \neq 0$. $\hat{m}_I$ of the $m_I$ roots of $A(s)$ possess an odd order and lead additionally to an existing limit $\lim_{\omega\to\omega_j} |f(\omega)|$.*

**Proof.** For the proof, the following function is defined

$$F(s) := \frac{P(s)}{A(s)} = (k_i + k_p s + k_d s^2) + \frac{B(s)}{A(s)} e^{sL}.$$

For large $|s|$ $F(s)$ can be evaluated conveniently by

$$F(s) = \begin{cases} \dfrac{b_n}{a_m} s^l e^{Ls} + \mathcal{O}\left(s^{l-1} e^{Ls}\right) & \text{if } l > 2, \\ s^l \left(\dfrac{b_n}{a_m} e^{Ls} + k_d\right) + \mathcal{O}\left(s^{l-1} e^{Ls}\right) & \text{if } l = 2. \end{cases} \tag{12}$$

Now consider a large semicircle $C$ on the right-hand side of the $s$-plane with center at the origin and a sufficiently large radius $R$, such that the half circle contains all $m_R$ roots of $A(s)$, and $F(s)$ can be well approximated for $|s| = R$ by (12) neglecting the $\mathcal{O}$-term. Any roots of $A(s)$ on the imaginary axis are circumvented by infinitesimal small semicircles to the RHP, see Fig. 5. Let's name the part of curve $C$ above the real axis as $C'$. $C'$ is composed of $C'_L$ on the positive imaginary axis, the small quarter circle $C'_O$ around the origin, $C'_I$ consisting of half circles around roots of $A(s)$ on the positive imaginary axis, and the large quarter circle $C'_C$. If $P(s)$ is stable, the large semicircle $C$ only contains the $m_R$ roots of $A(s)$ as poles of $F(s)$ and Cauchy's argument principle can be applied: $F(s)$ encircles the origin $m_R$ times in the counterclockwise direction, when $s$ traverses $C$ in the clockwise direction. Therefore, the change of argument of $F(s)$ equals $m_R\pi$, if $s$ traverses only $C'$.

First, the case $l > 2$ is discussed and the change of argument of $F(s)$ for $s$ on $C'$ is calculated directly. The portion on the part $C'_L$ is named as $\Delta\phi_L$. The part $C'_O$ contributes a phase change of $-\frac{m_O\pi}{2}$ and each of the $\frac{m_I}{2}$ poles on the positive imaginary axis contributes a phase change of $-\pi$. For the part $C'_C$, the phase change can be approximated using (12) as

$$\Delta \arg\left(\frac{b_n}{a_m} s^l e^{Ls}\right) = -\frac{l}{2}\pi - RL. \tag{13}$$

Then, the whole change of argument of $F(s)$ on $C'$ is $\Delta\phi_L - \frac{l+m_I+m_O}{2}\pi - RL$. Note that if $P(s)$ is stable, the change of argument is $m_R\pi$. Therefore, a necessary stability condition is

$$\Delta\phi_L = \frac{l + 2m_R + m_I + m_O}{2}\pi + RL.$$

In the case $m_O = 0$, the Nyquist plot of $F(s)$ for $s$ on $C'_L$ starts on the real axis because $F(0)$ is real. In the case $m_O \neq 0$, the Nyquist plot starts at a phase of $-\frac{m_O\pi}{2}$ or $-\frac{m_O\pi}{2} + \pi$ and at infinite gain. At each of the $\frac{m_I}{2}$ roots $s = \mathrm{j}\omega_j$ of $A(s)$, the phase jumps about $-\pi$ at infinite gain. If this root is not counted in $\hat{m}_I$, either $|\mathfrak{Im}\{F(\mathrm{j}\omega)\}|$ tends to $\infty$, or the phase tends to 0 or $\pi$ when $\omega$ approaches $\omega_j$ from both directions. If this root is counted in $\hat{m}_I$, $|\mathfrak{Im}\{F(\mathrm{j}\omega)\}|$ tends to a constant value and the phase tends to $\pi$ when $\omega$ approaches $\omega_j$ from one direction and to $-\pi$ when approaching from the other direction. Additionally $\mathfrak{Im}\{F(\mathrm{j}R)\} \neq 0$ holds for a sufficiently large $R$. Note that all $s$ leading to an infinite gain of the Nyquist plot do not lie on $C'_L$. Consequently, the Nyquist plot crosses the real axis at least $z$ times if

$$\Delta\phi_L > z\pi - \frac{m_O \bmod 2}{2}\pi + \frac{\hat{m}_I}{2}\pi. \tag{14}$$

As $\mathfrak{Im}\{F(\mathrm{j}\omega)\} = \omega(k_p - f(\omega))$ and statement (4) of Corollary 4 holds, the Nyquist plot of $F(s)$ for $s$ on $C'_L$ crosses the real axis exactly at the singular frequencies $\omega_\eta \in \Omega$ (see Definition 2). Therefore, the minimal number of singular frequencies in $(0, R)$ for a stable $P(s)$ is

$$z_{\min} = \left\lceil \frac{\frac{l+2m_R+m_I+m_O+(m_O \bmod 2)-\hat{m}_I}{2}\pi + RL}{\pi} \right\rceil - 1 \tag{15}$$

which equals the expression in the stated theorem.

Finally, the case $l = 2$ is discussed. In this case, $2\kappa + (l \bmod 2) - 1$ is odd and the phase change on $C'_C$ can be estimated by

$$-RL - \frac{3}{2}\pi < \Delta \arg\left(s^2\left(\frac{b_n}{a_m}k_d + e^{Ls}\right)\right) < -RL - \frac{1}{2}\pi,$$

which replaces (13). Consequently, the number of singular frequencies must be in the interval

$$\left[\frac{2\kappa + (l \bmod 2) + 2m_R + m_I + m_O + (m_O \bmod 2) - \hat{m}_I}{2},\right.$$
$$\left.\frac{2\kappa + (l \bmod 2) + 2 + 2m_R + m_I + m_O + (m_O \bmod 2) - \hat{m}_I}{2}\right).$$

As the interval limits are neighbouring integers, the minimal number of singular frequencies can be determined at the interval's mean value which equals (15) for the case $l = 2$. □

The benefit of the last theorem will be demonstrated by the following example.

**Example 17** (*Continuation of* Example 3). It holds $l = 3$, $m_R = m_I = m_O = 0$. $\kappa$ is chosen as $\kappa = 8$, so $R = 12.5664$. Theorem 16 requires a minimal number of singular frequencies $z_{\min} = 5$ in the interval $(0, R)$. As seen from the Fig. 2, only $k_p \in (-1, 1.5849)$ fulfil the condition. Computing the stable region in the $(k_d, k_i)$-plane for all $k_p$ in the interval leads to the set of all stable controller parameters (see Fig. 6).

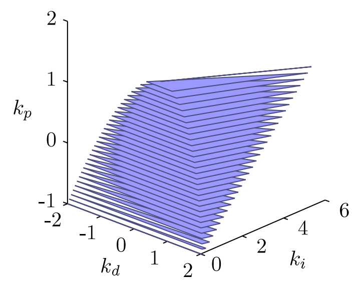


**Fig. 6.** The stable controller parameters for Example 17.

## 5. Conclusion and software tool

The paper presents a comprehensive method to compute the entire set of stabilizing PID controller parameters for arbitrary linear time delay systems. By Theorem 16, the $k_p$-axis can be gridded. For each $k_p$, the stable region in the $(k_d, k_i)$-plane is determined by Theorems 5, 11 and 15. For retarded systems, the stable region is a set of convex polygons, whereas for neutral systems the stable region can be described either by a set of convex polygons or by the limit of a sequence of convex polygons, which may be well approximated by convex polygons. A Matlab tool performing all the steps automatically is available for download at http://www.irt.rwth-aachen.de/pidrobust.

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