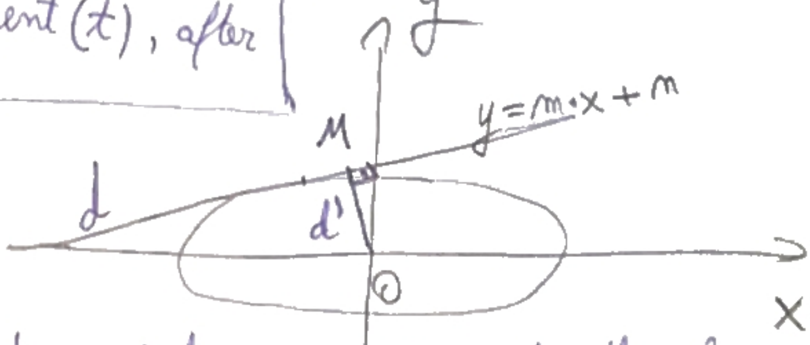
Curilă Sebastian
Group: 912

S9 - Exercises

9.10)

Find the locus of the orthogonal projection of the center $O(0,0)$ of the lipse $E: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on its tangents.

OBS: (d) - becomes the tangent (t), after $\Delta=0$.



Let it be $d: y=m\cdot x+n$

We compute the intersection between (d) and the elipse (E):

$$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ y=m\cdot x+n \;|(\,)^2\end{cases} \Leftrightarrow \begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \;|\cdot a^2b^2 \\ y^2=m^2x^2+2mnx+n^2\end{cases} \Leftrightarrow \begin{cases}x^2b^2+y^2a^2=a^2b^2 \\ y^2=m^2x^2+2mnx+n^2\end{cases} \Rightarrow$$

$\Rightarrow x^2b^2+(m^2x^2+2mnx+n^2)\cdot a^2=a^2b^2 \Longleftrightarrow$ We make computations and we get,

$\Rightarrow x^2\cdot(b^2+m^2\cdot a^2)+2mna^2\cdot x+a^2(n^2-b^2)=0$

$\Delta=(2mna^2)^2-4(b^2+m^2a^2)(n^2-b^2)\cdot a^2$

$\Delta=4a^4m^2b^2-4a^2b^2n^2+4a^2b^4$

d is tangent to the elipse (E) $\Leftrightarrow \Delta=0 \Rightarrow$

$\Rightarrow 4a^4m^2b^2-4a^2b^2n^2+4a^2b^4=0 \;|:4a^2b^2;\; a,b\neq 0$

$a^2m^2-n^2+b^2=0 \Rightarrow n^2=a^2m^2+b^2 \Rightarrow n=\pm\sqrt{a^2m^2+b^2}$

~~We g~~

The equation for the tangent ~~t~~ is:

$(t): y=m\cdot x\pm\sqrt{a^2m^2+b^2}$ (hence we have 2 tangents to the elipse (E), having the slope m.)

Let it be $d' \perp t \Rightarrow m_{d'} = -\frac{1}{m_t}$
$\left.\begin{matrix} m_{d'} = -\frac{1}{m_t} \\ m_t = m \end{matrix}\right\} \Rightarrow m_{d'} = -\frac{1}{m} \Rightarrow$

The equation of the line (d') that goes throw $O(0,0)$ is:

(d'): $y = -\frac{1}{m}x$.

The locus fa is $\{M\} = d' \cap t$.

$$\begin{cases} y = -\frac{x}{m} \\ y = m \cdot x \pm \sqrt{a^2m^2+b^2} \end{cases} \Leftrightarrow \begin{cases} m = -\frac{x}{y} \\ y = -\frac{x^2}{y} \pm \sqrt{a^2 \frac{x^2}{y^2} + b^2} \end{cases} \Big| \cdot y \Rightarrow$$

$$\Rightarrow y^2 + x^2 = \pm\sqrt{a^2 \cdot \frac{x^2}{1} + b^2} \quad \Big| \; ()^2$$

$$\cancel{(y^2+x^2)^2 = (a^2x^2+b^2y^2)}$$

$$(x^2+y^2)^2 = a^2x^2 + b^2y^2 \text{ — q.e.d.}$$

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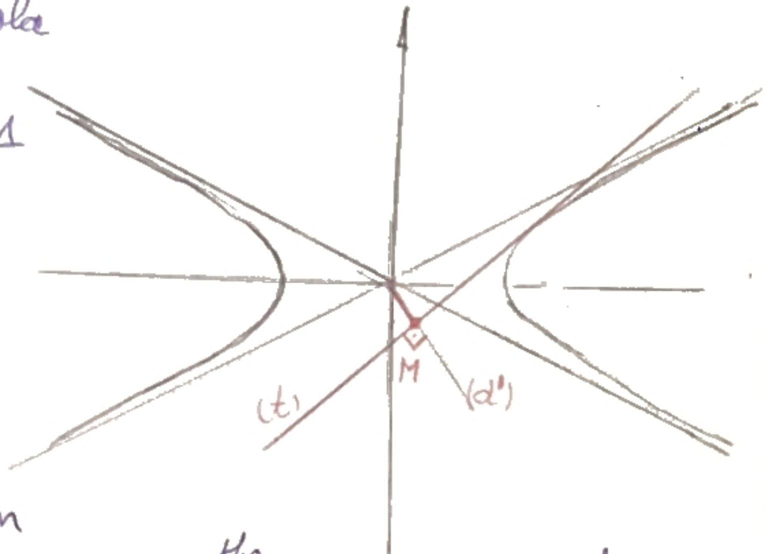
9.11.)

Find the locus of the orthogonal projections of the center $O(0,0)$ of the hyperbola

$$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

on its tangents.



Let it be $(d): y = m \cdot x + n$

We compute the intersection between the line (d) and the hyperbola $(H): \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = m \cdot x + n \,|\, (\,)^2 \end{cases} \Leftrightarrow \begin{cases} b^2x^2 - a^2y^2 = a^2b^2 \\ y^2 = m^2x^2 + 2mnx + n^2 \end{cases}$$

We compute the system likewise in ex. 9.10 $\Rightarrow$

$\Rightarrow$ the equation: $(b^2 - a^2m^2)x^2 - 2a^2mnx - a^2 \cdot (n^2 + b^2) = 0$

for $\Delta = 0 \Rightarrow d$ become the tangent (t) to (H).

by computing the eq. $\Delta = 0 \Rightarrow n = \pm\sqrt{a^2m^2 - b^2}$

$\Rightarrow (t): y = m \cdot x \pm \sqrt{a^2m^2 - b^2}$

Let it be $d' \perp t \Rightarrow \left.\begin{matrix} m_{d'} \cdot m_t = -1 \\ m_t = m \end{matrix}\right\} \Rightarrow m_{d'} = -\frac{1}{m} \Rightarrow$

OBS. (d') goes through $O(0,0)$

$\Rightarrow (d'): y = -\frac{1}{m} \cdot x$

Let it be $\{M\} = d' \cap t$: $\begin{cases} y = -\frac{1}{m} \cdot x \\ y = m \cdot x \pm \sqrt{a^2 m^2 - b^2} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} m = -\frac{x}{y} \\ y = mx \pm \sqrt{a^2 m^2 - b^2} \end{cases} \Rightarrow y = -\frac{x^2}{y^2} \pm \sqrt{a^2 \cdot \frac{x^2}{y^2} - b^2} \quad \Big| \cdot y$

$(x^2 + y^2)^2 = a^2 x^2 - b^2 y^2 \to$ the locus of the orthogonal ~~operation~~ projection of the center $O(0,0)$ of the elipse.