Curs 7 19.11.2019

Chapter 3. MATRICES AND LINEAR SYSTEMS

1. Elementary operations

Def: Let V be a k vector space. Then by an elementary operation we mean any of the following functions

1) $\varepsilon_{ij} : V^n \to V^n$ ($n \in \mathbb{N}^*$ fixed)
$\varepsilon_{ij}(v_1, \dots, v_i, \dots, v_j, \dots, v_n) =$
$= (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$

2) $\varepsilon_{i\alpha} : V^n \to V^n$, $\alpha \in k^*$
$\varepsilon_{i\alpha}(v_1, \dots, v_i, \dots, v_n) = (v_1, \dots, \alpha v_i, \dots v_n)$

3) $\varepsilon_{ij\alpha} : V^n \to V^n$ $\alpha \in k^*$
$\varepsilon_{ij\alpha} = (v_1, \dots, v_i, \dots, v_j, \dots, v_n) =$
$= (v_1, \dots, v_i, \dots, \alpha v_i + v_j, \dots, v_n)$

$\forall\, i, j \in \{1, \dots, n\}$

Lemma: Let V be a k vector space and $n \in \mathbb{N}^*$. Then V^n is a k vector space with respect to the operations
$$\begin{cases} (v_1, \dots, v_n) + (v_1', \dots, v_n') = (v_1 + v_1', \dots, v_n + v_n') \\ k(v_1, \dots, v_n) = (kv_1, \dots, kv_n) \end{cases}$$
$\forall\, (v_1, \dots, v_n), (v_1', \dots, v_n') \in V^n$
$\forall\, k \in k$

Theorem: With the previous notation, $\varepsilon_{ij}, \varepsilon_{i\alpha}, \varepsilon_{ij\alpha} \in Aut(V^n)$

Definition: Let $X = (v_1, \dots, v_n)$, $X' = (v_1', \dots, v_n')$ be fields of vectors in a k vector space V. Then X and X' are called equivalent (not. $X \sim X'$) if one of them can be obtained ~~by a~~ from the other one by a finite number of operations.

Remarks.

a) $X \sim X' \Leftrightarrow X' \sim X$

b) "$\sim$" is an equivalent relation on lists of vectors

Theorem: Let V be a k-vector space and X, X' are lists such that $X \sim X'$ (with n vectors) then:

1) X is linearly independent in $V \Leftrightarrow$ so is X'

2) X is a system of generators for $V \Leftrightarrow$ so is X'

3) X is a basis of $V \Leftrightarrow$ so is X'

Let $A \in M_{m,n}(k)$, say $A = (a_{ij})$

We may view A as a list of column vectors $(a^1, a^2, \dots, a^n)$, where

$$a^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \quad j = \overline{1,n}$$

Also we may view A as a list of row vectors $(a_1, a_2, \dots, a_m)$, where

$$a_i = (a_{i1}, a_{i2}, \dots, a_{in}), \quad i = \overline{1,m}$$

Say $A = (a^1, \dots, a^n)$. Then the elementary operations on the list A become

- interchange 2 columns of A
- multiply a column by $\alpha \in k^*$
- multiply a column by $\alpha \in k$ and add it to another column

Theorem: The value of an elementary operation applied on $A = (a_{ij}) \in M_{m,n}(k)$, seen as a list of column vectors $(a^1, \dots, a^n)$ is equal to A multiplied on the right hand side by the matrix obtained from the identity ~~element~~ matrix I_n, also seen as a list of column-vectors by applying the same elementary operations.

Ex: Take $\mathcal{E}_{12}$: $\mathcal{E}_{12}(a^1, \dots, a^n)$
$= (a^2, a^1, a^3, \dots, a^n) = \begin{pmatrix} a_{12} & a_{11} & \dots & a_{1m} \\ a_{22} & a_{21} & \dots & a_{2m} \\ \vdots & & \dots & \\ a_{n2} & a_{n1} & \dots & a_{nm} \end{pmatrix}$

$\cdot \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = A \cdot \mathcal{E}_{12}(I_n)$

Definition: With the previous notation $\mathcal{E}_{ij}(I_n)$, $\mathcal{E}_{i\alpha}(I_n)$, $\mathcal{E}_{ij\alpha}(I_n)$ are called elementary matrices.
Note that they are invertible!

Definition: We say that $A \in M_{m,n}(k)$ is in echelon form with r non-zero rows if:
1) the first r ~~for~~ rows of A are non zero
2) $0 \le N(1) < N(2) < \dots$
Where $N(i)$ denotes the number of zero elements from the beginning of row i $(i = \overline{1,m})$

Theorem: Every $A \neq O_{m,n}$ is equivalent to a matrix in echelon form.

Example: $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & -2 & 6 \\ -1 & 1 & 1 & 0 \end{pmatrix} \underset{R_3+R_1}{\overset{(-3)R_1+R_2}{\sim}}$

$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \overset{R_3+2R_2}{\sim} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

② Applications of elementary operations

Theorem: Let $A \in M_{m,n}(k)$. Then $\operatorname{rank}(A) = \dim_k \langle a^1, \dots, a^n \rangle =$
$= \dim_k \langle a_1, \dots, a_m \rangle$

Theorem: Let $A \in \mathcal{M}_{m,n}(k)$ having an echalon form C with r non-zero rows. Then $\text{rank}(A) = \text{rank}(C) = r$

Example: The rank of A from the previous example is 3 (the number of non zero rows of an echalon form of A)

Theorem: Let $A \in \mathcal{M}_{n,n}(k)$ with $\det(A) \neq 0$. Then A is equivalent to I_n and A^{-1} is obtained from I_n by applying the same elementary operations as one does to obtain I_n from A

Example: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ $\det(A) = 1 \neq 0$

$\Rightarrow \exists A^{-1}$

$$\left(\underbrace{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{matrix}}_{A} \;\vdots\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right) \overset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{matrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \;\vdots\; \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}\right)$$

$$\overset{R_2 - R_1}{\sim} \left(\begin{matrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{matrix} \;\vdots\; \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{matrix}\right) \overset{R_3 + R_2}{\sim}$$

$$\sim \left(\begin{matrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\vdots\; \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{matrix}\right) \overset{R_1 - R_3}{\sim}$$

$$\sim \left(\begin{matrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\vdots\; \begin{matrix} -1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{matrix}\right) \overset{R_1 + 2R}{\sim}$$

$$\sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\vdots\; \begin{matrix} -1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{matrix}\right) \overset{R_2 \leftarrow -R_2}{\sim}$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\underbrace{\quad\quad}_{I_3} \qquad \underbrace{\quad\quad}_{A^{-1}}$$

$$A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

19.11.2019

Seminar 7

8.2. Using the Kronecker-Capelli Theorem decide if the following system is compatible and if so, solve it.

a) $\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = -1 \neq 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} \begin{matrix} C_2 - C_1 \\ = \\ C_3 - C_1 \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 2 & -5 & -1 \end{vmatrix} =$$

$$= 1 - 20 = -19 \neq 0$$

$\Rightarrow$ rank $A = 3$, rank $\bar{A} = 3$ $\Rightarrow$ system is compatible

~~$\bar{A} = ($~~

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{pmatrix}$$

$(S+T) = \langle (-2,-2,8),(7,3,4),(-1,1,-4) \rangle$

$$\dim S\cap T = \dim S + \dim T - \dim S+T$$
$$= 2+2-2 = 2$$

28.11.2019

Curs 8

Definition. Let V be a k vector space $B = (v_1, \dots, v_n)$ a basis of V, and $X = (u_1, \dots, u_n)$ a list of vectors in V. Then we may write (uniquely)

$$\begin{cases} u_1 = a_{11}v_1 + \dots + a_{1n}v_n \\ \dots \\ u_n = a_{m1}v_1 + \dots + a_{mn}v_n \end{cases}$$

Then we denote
$[X]_B = (a_{ij}) \in M_{m,n}(k)$ and we call it the matrix of X in the basis B.

Example: We consider the canonical v.s. $\mathbb{R}^4$ and $X = (u_1, u_2, u_3)$ where
$$u_1 = (1,2,3,4)$$
$$u_2 = (5,6,7,8)$$
$$u_3 = (9,10,11,12)$$

$$[X]_E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Theorem: Let V be a k-vector space, B a basis of V and X a list of vectors in V. Then $\dim\langle X\rangle = \operatorname{rank}([X]_B)$ and the basis of $\langle X \rangle$ consists of ~~the number~~ the non-zero rows from the echelon form of $[X]_B$

③ Matrix of a linear map

Definition: Let V be a k-vector space $B = (v_1, \dots, v_n)$ be a basis of V and

$v \in V$. Then we may uniquely write
$v = k_1 v_1 + \dots + k_n v_n$
Then we denote $[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in M_{n,1}(k)$
and we call it the matrix of v in the basis B.

Definition: Let $f: V \to V'$ be a k-linear map, let $B = (v_1, \dots, v_n)$ be a basis of V, let $B' = (v_1', \dots, v_m')$ be a basis of V'.
Then we may uniquely write

$$\begin{cases} f(v_1) = a_{11} v_1' + a_{21} v_2' + \dots + a_{m1} v_m' \\ f(v_2) = a_{21} v_1' + a_{22} v_2' + \dots + a_{m2} v_m' \\ \dots \\ f(v_n) = a_{1n} v_1' + a_{2n} v_2' + \dots + a_{mn} v_m' \end{cases}$$

Then we denote by

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{m1} \\ a_{21} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$
$$\begin{array}{ccc} f(v_1) & f(v_2) & f(v_n) \end{array}$$

$= (a_{ij}) \in M_{mn}(k)$
and we call it the matrix of f in the basis $B\ B'$.

If $V = V'$ and $B = B'$, then we denote $[f]_B = [f]_{BB}$

$$f(v_j) = \sum_{i=1}^{n} a_{ij} v_i' \quad , \forall j = \overline{1,n}$$

Example: Let $f: \mathbb{R}^4 \to \mathbb{R}^3$
$f(x,y,z,t) = (x+y+z, y+z+t, z+t+x)$.
This is an $\mathbb{R}$-linear map. Consider the canonical bases E of $\mathbb{R}^4$ and E' of $\mathbb{R}^3$. Let us compute $[f]_{EE'}$

$f(e_1) = f(1,0,0,0) = (1,0,1) = e_1' + e_3'$
$f(e_2) = f(0,1,0,0) = (1,1,0) = e_1' + e_2'$
$f(e_3) = f(0,0,1,0) = (1,1,1) = e_1' + e_2' + e_3'$
$f(e_4) = f(0,0,0,1) = (0,1,1) = e_2' + e_3'$

$$\Rightarrow [f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Conversely, given $[f]_{EE'}$, one may recover the definition of f

$\forall\, v = (x,y,z,t) \in \mathbb{R}^4$
$f(v) = f(x e_1 + y e_2 + z e_3 + t e_4)$
$= x f(e_1) + y f(e_2) + z f(e_3) + t f(e_4)$
$= x(1,0,1) + y(1,1,0) + z(1,1,1) + t(0,1,1)$
$= (x+y+z,\ y+z+t,\ z+t+x)$

Theorem: Let $f: V \to V$ be a k linear map, $B = (v_1, \dots, v_n)$ be a basis of V, $B' = (v_1', \dots, v_m')$ a basis of V' and $v \in V$. Then $[f(v)]_{B'} = [f]_{BB'} [v]_B$

Proof: Let $[f]_{BB'} = (a_{ij}) \in M_{m,n}(k)$
$\forall\, j = \overline{1,n} \quad f(v_j) = \sum_{i=1}^{m} a_{ij} v_i' \qquad (1)$

Let $v = \sum_{j=1}^{n} k_j v_j,\ k_j \in k \qquad (2)$

$f(v) = \sum_{i=1}^{m} k_i' v_i',\ k_i' \in k \qquad (3)$

On the other hand we have
$f(v) \overset{2}{=} f\left(\sum_{j=1}^{n} k_j v_j\right) = \sum_{j=1}^{n} k_j f(v_j)$

$\overset{1}{=} \sum_{j=1}^{n} k_j \left(\sum_{i=1}^{m} a_{ij} v_i'\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} k_j\right) v_i' \qquad (4)$

(3) and (4) are writings of the same

$f(v)$ as linear coordinates of the vectors in the basis B. But such writing is unique!

$\Rightarrow k_i' = \sum_{j=1}^{n} a_{ij} k_j, \ i = \overline{1,n}$

$$\begin{pmatrix} k_1' \\ k_2' \\ \vdots \\ k_n' \end{pmatrix} = (a_{ij}) \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

$$[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$$

Theorem. Let $f: V \to V'$ be a k-linear map. Then $\dim(\operatorname{Im} f) = \operatorname{rank}([f]_{BB'})$ in any pair of bases B, B' of V, V' respectively.

$\dim(\operatorname{Im} f)$ is also denoted by $\operatorname{rank}(f)$ and called the rank of f.

Theorem: Let V, V', V'' be k-vector spaces, $B = (v_1, \dots, v_n)$ a basis of V, $B' = (v_1', v_2', \dots, v_n')$ a basis of V', $B'' = (v_1'', v_2'', \dots, v_n'')$ a basis of V''. Then $\forall f, g \in \operatorname{Hom}_k(V, V')$
$\{f : V \to V' \mid f$ is a k-linear map$\}$, $\forall h \in \operatorname{Hom}_k(V', V'')$, we have

$$\begin{cases} [f+g]_{BB'} = [f]_{BB'} + [g]_{BB'} \\ [k \cdot f]_{BB'} = k[f]_{BB'} \\ [h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'} \end{cases}$$

and $\operatorname{Hom}_k(V, V')$ is a k-v.s.

Proof: Denote

$$[f]_{BB'} = (a_{ij}) \in M_{m,n}(k)$$

$\forall j = \overline{1,n}, \ f(v_j) = \sum_{i=1}^{m} a_{ij} \cdot v_i'$

$[g]_{BB'} = (b_{ij}) \, M_{m,n}(k)$

$\forall j \in \overline{1,n} \ g(v_j) = \sum_{i=1}^{m} b_{ij} \cdot v_i'$

$[h]_{B'B''} = (c_{ij}) \in M_{pm}(k)$

$\forall i = \overline{1,m} \quad h(v_i') = \sum_{k=1}^{p} c_{ki} v_k''$

We have $(f+g)(v_j) = f(v_j) + g(v_j)$

$= \sum_{i=1}^{m} a_{ij} v_i' + \sum_{i=1}^{m} b_{ij} \cdot v_i' =$

$= \sum_{i=1}^{m} (a_{ij} + b_{ij}) v_i' \quad , \forall j = \overline{1,n}$

$\Rightarrow [f+g]_{BB'} = (a_{ij} + b_{ij}) = [f]_{BB'} + [g]_{BB'}$

$(kf)(v_j) = k f(v_j) = k \sum_{i=1}^{m} a_{ij} v_i' = \sum_{i}^{m} k a_{ij} v_i'$

$\Rightarrow [kf]_{BB'} = k [f]_{BB'}$

$(h \circ f)(v_j) = h\left(\sum_{i=1}^{m} a_{ij} v_i' \right) =$

$= \sum_{i=1}^{m} a_{ij} h(v_i') = \sum_{i=1}^{m} a_{ij} h(v_i') =$

$= \sum_{i=1}^{m} a_{ij} \left(\sum_{k=1}^{p} c_{ki} v_k'' \right) = \sum_{k=1}^{p} \left(\sum_{i=1}^{m} c_{ki} a_{ij} \right) \cdot v_k''$

$\Rightarrow [h \circ f]_{BB''} = \left(\sum_{i=1}^{m} c_{ki} a_{ij} \right) = (c_{ki}) \cdot (a_{ij})$

$= [h]_{B'B''} \cdot [f]_{BB'}$

Corollary Let V and V' be k v.s with the $\dim_k V = n$ and $\dim_k V' = m$. Then the map $\varphi : \mathrm{Hom}_k(V, V') \to M_{m,n}(k)$

$\varphi(f) = [f]_{BB'}$, when B, B' are bases of V, V' respectively, is an isomorphism of k-vector spaces.

- φ bij
- $\forall f, g \in \mathrm{Hom}_k(V, V')$

$\varphi(f+g) = \varphi(f) + \varphi(g)$
$\forall k \in k \quad \forall f \in Hom_k(V, V')$
$\varphi(kf) = k\,\varphi(f)$

Corollary: Let V be a k v.s with $\dim_k V = n$. Then
$$\varphi: End_k(V) \to M_n(k)$$
$\varphi(f) = [f]_B$ where B is a basis of V, is an isomorphism of k-v.s and an isomorphism between the rings
$(End_k(V), +, \circ)$ and $(M_n(k), +, \cdot)$

Corollary: Let V be a k-vector space with $\dim_k V = n$ and $f \in End(V)$. Then $f \in Aut_k(V) \iff f$ is invertible in the ring $(End_k(V), +, \circ)$
$\iff \varphi(f) = [f]_B$ is invertible in the ring $(M_n(k), +, \cdot)$
$\iff \det[f]_B \neq 0$

2.12.2019

Seminar 10

Th: $f: V \to V'$ linear map

$B = (v_1, \dots, v_n)$ basis for V
$B' = (v_1', \dots, v_m')$ basis for V'

$$[f]_{BB'} = \left([f(v_1)]_{B'} \dots [f(v_m)]_{B'} \right)$$

$$V_B \xrightarrow{f} V'_{B'} \xrightarrow{g} V_{B''}$$

$$[g \circ f]_{BB''} = [g]_{B'B''} \cdot [f]_{BB'}$$

$$f: V_B \to V_{B_1', B_2'}$$

$$[f]_{BB_1'} = [id_V]_{B_1 B_1'} \cdot [f]_{B_2 B_2'} \cdot [id_V]_{B B_2}$$

$$-\begin{cases} a+b-3c+2d=0 \\ a+b+c+4d=0 \\ 2a+b-5c+d=0 \\ a+2b-4c+5d=0 \end{cases}$$

$$\begin{cases} a+b-3c+2d=0 \\ b-c+3d=0 \end{cases}$$

$$\begin{cases} a+b=3\alpha-2\beta \\ b=\alpha-3\beta \\ c=\alpha \\ d=\beta \end{cases}$$

$$a=3\alpha-2\beta-\alpha+3\beta$$

$$a=2\alpha+\beta$$

$$\ker f=\{(2c+d,\ c-3d,\ c,\ d)\ /\ c,d\in\mathbb{R}\}$$

$$=\langle(2,1,1,0),(1,-3,0,1)\rangle$$

$((2,1,1,0),(1,-3,0,1))$ a basis

$$\dim_{\ker f}=2$$

6.12.2019

~~Seminar~~ Curs 11

4. Change of bases

Def: Let V be a k-vector space,
$B=\{v_1,\dots,v_n\}$
$B'=\{v_1',\dots,v_n'\}$ are bases of V
Then we may write uniquely

$$\begin{cases} v_1'=t_{11}v_1+t_{21}v_2+\dots+t_{n1}v_n \\ v_2'=t_{12}v_1+t_{22}v_2+\dots+t_{n2}v_n \\ \dots \\ v_n'=t_{1n}v_1+t_{2n}v_2+\dots+t_{nn}v_n \end{cases}$$

Then $T_{BB'}=(t_{ij})\in M_n(k)$ is called the change matrix from B to B'.

Remark: We put the sets of coordinates on the columns of $T_{BB'}$

Theorem: With the above notation $T_{BB'}$ is invertible and $T_{BB'}^{-1} = T_{B'B}$

Proof:

$$v'_j = \sum_{i=1}^{n} t_{ij} \cdot v_i \quad \forall j = \overline{1,n} \quad (1)$$

Denote $S = T_{B'B} = (s_{ki}) \in M_n(K)$

$$v_i = \sum_{k=1}^{n} s_{ki} \cdot v'_k \quad \forall i = \overline{1,n} \quad (2)$$

$(1),(2) \Rightarrow v'_j = \sum_{i=1}^{n} t_{ij} \left(\sum_{k=1}^{n} s_{ki} v'_k \right)$

$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} s_{ki} t_{ij} \right) v'_k$

$v'_j = 1 \cdot v'_j$ unique writing $\Big\} \Rightarrow$

$$\Rightarrow \sum_{i=1}^{n} s_{ki} t_{ij} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \quad j, k = \overline{1,n}$$

$\Rightarrow S \cdot T_{BB'} = I_n$

$\Rightarrow T_{BB'}$ has S as a left inverse

Similarly S is also a ~~left~~ right inverse of $T_{BB'}$

$\Rightarrow T_{BB'}^{-1} = S = T_{B'B}$

Theorem: with the above notation we have $[v]_B = T_{BB'} \cdot [v]_{B'} \quad \forall v \in V$

Proof: $T_{BB'} = (t_{ij}) \in M_n(K)$

$(1)\ v'_j = \sum_{i=1}^{n} t_{ij} v_i \ , \ j = \overline{1,n}$

$(2)\ v = \sum_{i=1}^{n} k_i v_i$

$(3)\ v = \sum_{j=1}^{n} k'_j v'_j$

Replace (1) into (3) to get

$$v = \sum_{j=1}^{n} k'_j \left(\sum_{i=1}^{n} t_{ij} v_i \right) =$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} t_{ij} k_j' \right) v_i \quad (4)$$

(2) & (4) and the unique writing of v as a linear combination of the vectors of B

$$\Rightarrow k_i = \sum_{j=1}^{n} t_{ij} k_j', \quad i = \overline{1,n}$$

$$\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = (t_{ij}) \begin{pmatrix} k_1' \\ \vdots \\ k_n' \end{pmatrix}$$

$$\Rightarrow [v]_B = T_{BB'} [v]_{B'}$$

Theorem: Let f ~~be an~~ $\in End_K(V)$, B, B' be bases of V. Then

$$[f]_{B'} = T_{BB'}^{-1} \cdot [f]_B \cdot T_{BB'}$$

Example. In the canonical real vector space $_{\mathbb{R}}\mathbb{R}^3$ consider the bases

$E = (e_1, e_2, e_3)$ - the canonical basis

$B = (v_1, v_2, v_3)$ where

$v_1 = (0, 1, 1)$

$v_2 = (1, 1, 2)$ $\quad T_{BE} = ?$

$v_3 = (1, 1, 1)$ $\quad T_{EB} = ?$

$$\begin{cases} v_1 = e_2 + e_3 \\ v_2 = e_1 + e_2 + 2e_3 \\ v_3 = e_1 + e_2 + e_3 \end{cases}$$

$$T_{EB} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$T_{BE} = T_{EB}^{-1} = \ldots$ OR

express the vectors of E as linear combinations of the vectors of B.

$$\begin{cases} e_1 = -v_1 + v_3 \\ e_2 = v_1 - v_2 + v_3 \\ e_3 = v_2 - v_3 \end{cases}$$

$$T_{BE} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Now take a vector $u = (1,2,3)$

$$[u]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad [u]_B$$

$$[u]_B = T_{BE} \cdot [u]_E = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Now: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x,y,z) = (x+y,\ y-z,\ z+x)$$

$$f(e_1) = f(1,0,0) = (1,0,1)$$
$$f(e_2) = f(0,1,0) = (1,1,0)$$
$$f(e_3) = f(0,0,1) = (0,-1,1)$$

$$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$[f]_B = T_{EB}^{-1} [f]_E \cdot T_{EB} =$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \ldots = \begin{pmatrix} 1 & -3 & -2 \\ 1 & 4 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

5. Eigen vectors and eigen values (vectori şi valori proprii)

Definition. Let $f \in End_k(V)$. Then a non-zero $v \in V$ is called an eigenvector of f if $\exists\ \lambda \in k$ such that $f(v) = \lambda v$

Remark. Assume that $\exists\ \lambda, \lambda' \in k'$ such that $f(v) = \lambda v$ and $f(v) = \lambda' v$

$\Rightarrow \lambda v = \lambda' v$

$\Rightarrow (\lambda - \lambda') v = 0$

$\Rightarrow \lambda - \lambda' = 0 \Rightarrow \lambda = \lambda'$

λ is called the igen value of f corresponding to the eigenvector v.

For an eigenvalue $\lambda \in k$, denote $V(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$ the set consisting of 0 and all eigenvectors of v having eigenvalue λ.

Theorem: Let $f \in \mathrm{End}_k(V)$ and λ be an ~~eigenvector~~ eigenvalue of f.

Then $V(\lambda) \leq_k V$, called the eigenspace of λ or the characteristic subspace of λ

Proof. $0 \in V(\lambda) \neq \emptyset$

Let $k_1, k_2 \in K$ and $v_1, v_2 \in V(\lambda)$

We show that $k_1v_1 + k_2v_2 \in V(\lambda)$

$f(k_1v_1 + k_2v_2) = k_1 f(v_1) + k_2 f(v_2)$

$= k_1(\lambda v_1) + k_2(\lambda v_2) = \lambda(k_1v_1 + k_2v_2)$

$\Rightarrow k_1v_1 + k_2v_2 \in V(\lambda)$

Theorem: Let $f \in \mathrm{End}_k(V)$

$B = (v_1, v_2, \dots, v_n)$ be a basis of V

$A = [f]_B = (a_{ij}) \in M_n(k)$.

Then λ is an eigenvalue of f

$$\Leftrightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} \begin{array}{l} \text{char. equation} \\ = 0 \\ \\ \text{charact. det} \end{array}$$

$k \in \{\mathbb{R}, \mathbb{C}\}$

Remark: Note that the igen values of f do not depend of the choice of the basis B!

Proof of the theorem:

$\lambda \in k$ is an eigenvalue of $f \iff \exists$ $0 \neq v \in V$ such that $f(v) = \lambda \cdot v$

We have $f(v) = \lambda \cdot v \iff f(v) - \lambda v = 0$

$\iff [f(v) - \lambda v]_B = [0]_B$

$\iff [f(v)]_B - \lambda [v]_B = [0]_B$

$\iff [f]_B \cdot [v]_B - \lambda [v]_B = [0]_B$

$\iff ([f]_B - \lambda \cdot I_n) [v]_B = [0]_B$

$\iff (A - \lambda I_n) [v]_B = [0]_B$ (*)

We may uniquely write

$v = x_1 v_1 + \dots + x_n v_n$

for some $x_1, \dots, x_n \in k$

$\Rightarrow [v]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then (*) $\iff$

$\iff (A - \lambda I_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in M_{n,1}(k)$

$\iff \begin{cases} (a_{11} - \lambda) x_1 + a_{12} x_2 + \dots + a_{1n} x_n = 0 \\ a_{21} x_1 + (a_{22} - \lambda) x_2 + \dots + a_{2n} x_n = 0 \\ \quad - \quad - \quad - \\ a_{n1} x_1 + a_{n2} x_2 + \dots + (a_{nn} - \lambda) x_n = 0 \end{cases}$

characteristic system

Hence $\lambda \in k$ is an eigenvalue of f

$\iff$ (S) has a non-zero solution

$\iff \det(S) = 0 \iff \det(A - \lambda I_n) = 0$

Theorem: Let $f \in End_k(V)$ and $\dim_k V = n$ and assume f has n distinct eigen values $\lambda_1, \dots, \lambda_n$ Then

$$[f]_B = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

where B is the basis consisting of the eigenvectors of f.

Example: Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$

$f(x, y, z) = (2x, y + 2z, -y + 4z)$

$f(e_1) = (2, 0, 0)$
$f(e_2) = (0, 1, -1)$
$f(e_3) = (0, 2, 4)$

$$\Rightarrow A = [f]_E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix}$$

$\lambda \in \mathbb{R}$ is an eigenvalue of $f \iff$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$\iff \begin{cases} \lambda_1 = \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$$

I. $\lambda_1 = \lambda_2 = 2$. Then the eigenvectors of f having eigenvalues $\lambda_1 = \lambda_2 = 2$ are the solutions of the system

$$\begin{pmatrix} 2-\lambda_1 & 0 & 0 \\ 0 & 1-\lambda_1 & 0 \\ 0 & -1 & 4-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_2 + 2x_3 = 0 \\ -x_2 + 2x_3 = 0 \end{cases} \quad x_2 = 2x_3$$

The solutions are $(x_1, 2x_3, x_3)_{x_1, x_3 \in \mathbb{R}}$

$V(2) = \{(x_1, 2x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\}$
$= \langle (1,0,0), (0,2,1) \rangle$

II. $\lambda_3 = 3$. Homework

10.12.2019

Seminar 11
Eigenvalues and eigenvectors

$f \in End(V)$ ($f : V \to V$ linear map)
$v \in V \setminus \{0\}$ eigenvector for f, if $\exists$
$\lambda \in K$ (called an eigenvalue) s.t.

$$H = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) = (I_4 | P)$$

Syndrome	c 2
0000	0000000
0001	0001000
0010	0010000
0011	011000
0100	0100000
0101	000110
0111	000001
1000	1000000
1001	0000011
1010	000110
1011	0000100
1100	1100000
1100	1101000 — 0011010
1110	0000010
1111	0001010

~~Seminar~~ Curs 13 — 9.01.2020

Lynear systems of equations

Throughout k will be a field

$$S \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in k, \forall i = \overline{1,m}, j = \overline{1,n}$

$x_1, \dots, x_n \in k$ unknown

$A = (a_{ij}) \in M_{m,n}(k)$ is called the matrix of S

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & & - & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$ is

called the augmented (extended) matrix of (S)

Denote $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n,1}(k)$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M_{m,1}(k)$

$$(S) \quad Ax = b$$

We know that there is a k linear map associated to A.

$$A \in M_{m,n}(k) \longrightarrow f_A \in Hom_k(k^n, k^m)$$

~~We know~~ such that $[f_A]_{EE'} = A$ where E, E' are the canonical basses of k^n and k^m respectively.

Denote $x = (x_1, \ldots, x_n) \in k^n$
$b = (b_1, \ldots, b_m) \in k^m$

We have :

$$[f_A(x)]_{E'} = [f_A]_{EE'} \cdot [x]_E = \cancel{A}$$

$$= A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = [b]_{E'}$$

Hence: $(S) \; f_A(x) = b$

Denote by (S_0) the corresponding