EXERCICE 6: Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in V$ be vectors ~~where~~ such that $\vec{v}_1 \parallel d_1$, $\vec{v}_2 \parallel d_2$, $\vec{v}_3 \parallel d_3$, $\vec{v}_4 \parallel d_4$.

Then $d_{ij} \parallel \vec{v}_i \times \vec{v}_j$

$d_{12} \perp d_{34} \Leftrightarrow (\vec{v}_1 \times \vec{v}_2) \cdot (\vec{v}_3 \times \vec{v}_4) = 0$

$d_{14} \perp d_{23} \Leftrightarrow (\vec{v}_1 \times \vec{v}_4) \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

$d_{13} \perp d_{24} \Leftrightarrow (\vec{v}_1 \times \vec{v}_3) \cdot (\vec{v}_2 \times \vec{v}_4) = 0$

We need to show that if $(\vec{v}_1 \times \vec{v}_2) \cdot (\vec{v}_3 \times \vec{v}_4) = 0$ and $(\vec{v}_1 \times \vec{v}_3) \cdot (\vec{v}_2 \times \vec{v}_4) = 0$ then $(\vec{v}_1 \times \vec{v}_4) \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

- We know that
$$\left.\begin{aligned} &(\vec{v}_1 \times \vec{v}_2) \cdot (\vec{v}_3 \times \vec{v}_4) = (\vec{v}_1 \cdot \vec{v}_3)(\vec{v}_2 \cdot \vec{v}_4) - (\vec{v}_1 \cdot \vec{v}_4)(\vec{v}_2 \cdot \vec{v}_3) \\ &(\vec{v}_1 \times \vec{v}_2) \cdot (\vec{v}_3 \times \vec{v}_4) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (\vec{v}_1 \cdot \vec{v}_3)(\vec{v}_2 \cdot \vec{v}_4) = (\vec{v}_1 \cdot \vec{v}_4)(\vec{v}_2 \cdot \vec{v}_3) \quad (1)$$

- We know that
$$\left.\begin{aligned} &(\vec{v}_1 \times \vec{v}_3) \cdot (\vec{v}_2 \times \vec{v}_4) = (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_3 \cdot \vec{v}_4) - (\vec{v}_1 \cdot \vec{v}_4)(\vec{v}_3 \cdot \vec{v}_2) \\ &(\vec{v}_1 \times \vec{v}_3) \cdot (\vec{v}_2 \times \vec{v}_4) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_3 \cdot \vec{v}_4) = (\vec{v}_1 \cdot \vec{v}_4)(\vec{v}_3 \cdot \vec{v}_2) \quad (2)$$

From (1) and (2) $\Rightarrow (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_3 \cdot \vec{v}_4) = (\vec{v}_1 \cdot \vec{v}_3)(\vec{v}_2 \cdot \vec{v}_4) \Leftrightarrow$

$$\left.\begin{aligned} &\Leftrightarrow (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_3 \cdot \vec{v}_4) - (\vec{v}_1 \cdot \vec{v}_3)(\vec{v}_2 \cdot \vec{v}_4) = 0 \\ &\vec{v}_3 \cdot \vec{v}_4 = \vec{v}_4 \cdot \vec{v}_3 \\ &\vec{v}_2 \cdot \vec{v}_4 = \vec{v}_4 \cdot \vec{v}_2 \end{aligned}\right\} \Leftrightarrow (\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_4 \cdot \vec{v}_3) - (\vec{v}_1 \cdot \vec{v}_3)(\vec{v}_4 \cdot \vec{v}_2) = 0$$

$\Leftrightarrow (\vec{v}_1 \times \vec{v}_4) \cdot (\vec{v}_2 \times \vec{v}_3) = 0 \Leftrightarrow d_{14} \perp d_{23} \Rightarrow$

$\Rightarrow$ if $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$ then $d_{14} \perp d_{23}$ TRUE