C.2.4. Find the locus of points on the hyperbolic paraboloid $(P_h): y^2 - z^2 = 2x$, through which the rectilinear generatrices are perpendicular.

<u>Solution</u>: $d_\lambda : \begin{cases} y - z = \lambda \\ \lambda(y+z) = 2x \end{cases}$ $\quad d'_\mu : \begin{cases} y+z = \mu \\ \mu(y-z) = 2x \end{cases}$

$$d_\lambda : \begin{cases} y = z + \lambda \\ \lambda(y+z) = 2x \end{cases} \Leftrightarrow \begin{cases} y = z + \lambda \\ \lambda(2z+\lambda) = 2x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = z + \lambda \\ x = \lambda z + \frac{\lambda^2}{2} \end{cases} \Leftrightarrow \frac{x - \frac{\lambda^2}{2}}{\lambda} = \frac{y-\lambda}{1} = \frac{z-0}{1}$$

$$\Rightarrow \vec{d_\lambda} = (\lambda, 1, 1)$$

$$d'_\mu : \begin{cases} y = \mu - z \\ \mu(y-z) = 2x \end{cases} \Leftrightarrow \begin{cases} y = \mu - z \\ \mu(\mu - 2z) = 2x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \mu - z \\ x = -\mu z + \frac{\mu^2}{2} \end{cases} \Leftrightarrow \frac{x - \frac{\mu^2}{2}}{-\mu} = \frac{y-\mu}{-1} = \frac{z-0}{1}$$

$$\Rightarrow \vec{d'_\mu} = (-\mu, -1, 1)$$

$d_\lambda \perp d'_\mu \iff \vec{d_\lambda} \cdot \vec{d'_\mu} = 0 \iff -\lambda\mu - 1 + 1 = 0 \iff$

$\iff \lambda = 0$ or $\mu = 0$

Thus; we need to find the locus of the intersection points $P_{\lambda,\mu}$ of d_λ and d'_μ, when $\lambda = 0$ or $\mu = 0$

We fix $\lambda = 0$ and we let μ vary

$$P_{0,\mu} : \begin{cases} d_0 \\ d'_\mu \end{cases} \iff \begin{cases} \dfrac{y-0}{1} = \dfrac{z-0}{1} \text{ and } x = \dfrac{\lambda^2}{2} = 0 \\ \dfrac{x - \frac{\mu^2}{2}}{-\mu} = \dfrac{y-\mu}{-1} = \dfrac{z-0}{1} \end{cases} \iff$$

$$\iff \begin{cases} x = 0 \\ y = z \\ \dfrac{x - \frac{\mu^2}{2}}{-\mu} = \dfrac{y-\mu}{-1} = z \end{cases} \Rightarrow y - \mu = -y \Rightarrow \frac{\mu^2}{2\mu} = y$$

$\Rightarrow P_{0,\mu}\left(0, \frac{\mu}{2}, \frac{\mu}{2}\right) \Rightarrow$ this part of the locus is a line, the first bisector in (yOz)

In the same way we get (by fixing $\mu = 0$ and letting λ vary)

$$P_{\lambda,0} : \begin{cases} d_\lambda \\ d'_0 \end{cases} \iff P_{\lambda,0}\left(0, \frac{\lambda}{2}, -\frac{\lambda}{2}\right)$$

$\Rightarrow$ the locus is the Second bisector in (yOz)