GHERMAN
CRISTIAN
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Exercise 7.3.

Prove the following identity:

$$\boxed{(\vec{u}\times\vec{v},\ \vec{v}\times\vec{w},\ \vec{w}\times\vec{u}) = (\vec{u},\vec{v},\vec{w})^2}$$

$$(\vec{u}\times\vec{v},\ \vec{v}\times\vec{w},\ \vec{w}\times\vec{u}) = ((\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w}))\cdot(\vec{w}\times\vec{u}) \quad (1.)$$

$$(\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w}) = ((\vec{u}\times\vec{v})\cdot\vec{w})\cdot\vec{v} - ((\vec{u}\times\vec{v})\cdot\vec{v})\cdot\vec{w}$$
$$= (\vec{u},\vec{v},\vec{w})\cdot\vec{v} - (\vec{u}\cdot(\vec{v}\times\vec{v}))\cdot\vec{w}$$

$$\vec{v}\times\vec{v} = \|\vec{v}\|^2\cdot\sin(\vec{v},\vec{v}) = \vec{0}$$

$$\Rightarrow (\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w}) = (\vec{u},\vec{v},\vec{w})\cdot\vec{v} \quad \text{From (1.)} \Rightarrow$$

$$\Rightarrow (\vec{u}\times\vec{v},\ \vec{v}\times\vec{w},\ \vec{w}\times\vec{u}) = ((\vec{u},\vec{v},\vec{w})\cdot\vec{v})\cdot(\vec{w}\times\vec{u}) =$$
$$= (\vec{u},\vec{v},\vec{w})\cdot\vec{v}\cdot(\vec{w}\times\vec{u}) = (\vec{u},\vec{v},\vec{w})\cdot(\vec{v}\cdot(\vec{w}\times\vec{u}))$$
$$= (\vec{u},\vec{v},\vec{w})\cdot((\vec{v}\times\vec{w})\cdot\vec{u}) = (\vec{u},\vec{v},\vec{w})\cdot(\vec{v},\vec{w},\vec{u}) \quad (2.)$$

- We know that $(\vec{x_1},\vec{x_2},\vec{x_3}) = \operatorname{sgn}(\sigma)\cdot(\vec{x_{\sigma(1)}},\vec{x_{\sigma(2)}},\vec{x_{\sigma(3)}})$
$\forall \vec{x_1},\vec{x_2},\vec{x_3}\in V,\ \sigma\in S_3 \Rightarrow$

$$\Rightarrow (\vec{v},\vec{w},\vec{u}) = (\vec{w},\vec{u},\vec{v}) = (\vec{u},\vec{v},\vec{w}) \quad \text{From (2.)} \Rightarrow$$

$$\Rightarrow \boxed{(\vec{u}\times\vec{v},\ \vec{v}\times\vec{w},\ \vec{w}\times\vec{u}) = (\vec{u},\vec{v},\vec{w})^2}$$