Crăciun Ioan-Flaviu, 912

1. $\dot{x} = -2(x-5)$

$\dot{x}+2x=10$

First we solve the homogeneous part

$\dot{x}+2x=0$

$r+2=0$

$r=-2 \Rightarrow e^{-2t}$ is a solution to the homogeneous part

Now we ~~solve~~ find a particular solution

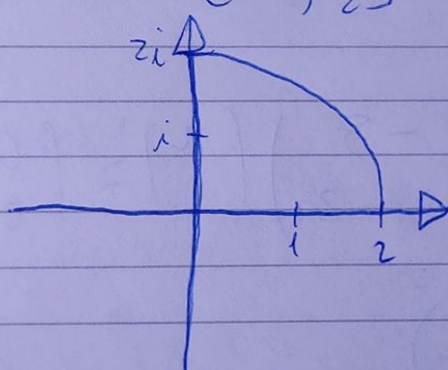
Consider constant solutions

$2x_p=10$

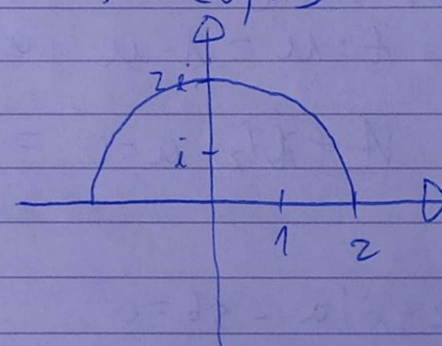
$x_p=5$

$$\left.\begin{aligned} & x = x_h + x_p \Rightarrow x = c\cdot e^{-2t}+5 \\ & x(0)=\eta \\ & c\cdot e^{-2\cdot 0}+5=\eta \Rightarrow c=\eta-5 \end{aligned}\right\} \Rightarrow \text{The flow is } \varphi(t,\eta)=(\eta-5)e^{-2t}+5$$

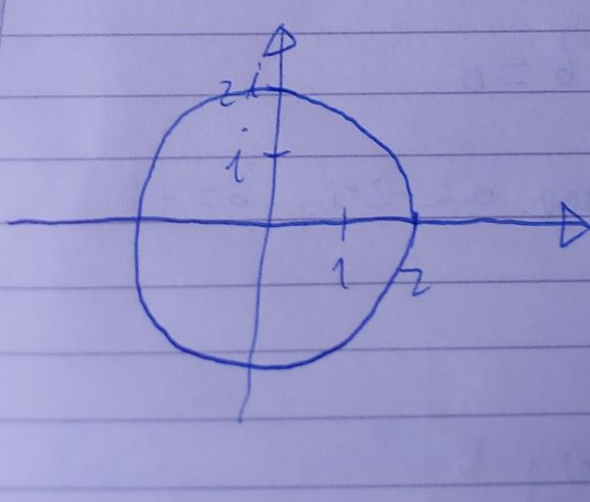
2. $2e^{it}: t\in[0,\frac{\pi}{2}]$



$t\in[0,\pi]$



$t\in[0,2\pi]$



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3\. a) First we find the eigenvalues of A

$\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-2-\lambda) + 5 = 0$

$-4 + \cancel{2\lambda} - \cancel{2\lambda} + \lambda^2 + 5 = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

We found 2 distinct eigenvalues, so A is diagonalizable

For $\lambda = i$

$A \cdot u = i \cdot u$, $u = ?$

$(A + i I_2) u = 0 \Rightarrow \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{array}{l} (2+i)a - 5b = 0 \\ a + (-2+i)b = 0 \,/\cdot(2+i) \\ (2+i)a + (2+i)(-2+i)b = 0 \end{array}\right\} -$$

$(-4 + \cancel{2i} - \cancel{2i} + i^2 + 5) \cdot b = 0$

$b = \dfrac{(2+i)a}{-5}$, we chose $a = -2+i$, $b = +1$

For $\lambda = -i$

$A \cdot u = i \cdot u$

$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-i)a - 5b = 0$

$b = \dfrac{2-i}{+5} a$, we chose $a = 2+i$, $b = 1$

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The general solution of $x' = Ax$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \cdot e^{it} \cdot \begin{pmatrix} -2+i \\ 1 \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\begin{cases} x_1 = c_1 \cdot e^{it}(2+i) + c_2 e^{-it}(2+i) \\ x_2 = c_1 \cdot e^{it} + c_2 \cdot e^{-it} \end{cases}$$

IVP

$x(0)=1 \Rightarrow c_1(-2+i) + c_2(2+i) = 1$

$y(0)=0 \qquad c_1 + c_2 = 0$

$$(-c_2)(-2+i) + c_2(2+i) = 1$$

$$4c_2 + \cancel{c_2 i} - \cancel{c_2 i} = 1$$

$$c_2 = \frac{1}{4}$$

$$c_1 = -\frac{1}{4}$$

IVP

$x(0)=0 \Rightarrow c_1(-2+i) + c_2(2+i) = 0$

$y(0)=1 \qquad c_1 + c_2 = 1$

$$(1-c_2)(-2+i) + 2c_2 + c_2 i = 0$$

$$-2 + 2c_2 + i - i\cancel{c_2} + 2c_2 + \cancel{c_2} i = 0$$

$$4c_2 = 2 - i$$

$$c_2 = \frac{2-i}{4} = \frac{1}{2} - \frac{i}{4}$$

$$c_1 = 1 - \frac{2-i}{4} = \frac{1}{2} + \frac{i}{4}$$

~~Principal matrix~~ $U(t) = \begin{pmatrix} -\frac{1}{4} \cdot e^{it}(-2+i) & \cancel{(\frac{1}{2} + \frac{i}{4}) e^{it}(2+i)} \\ \cancel{\frac{1}{4} e^{-it}(2+i)} & \cancel{(\frac{1}{2} - \frac{i}{4}) e^{it}(} \end{pmatrix}$

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Principal matrix

$$U(t)=\begin{pmatrix} -\frac{1}{4} e^{it}(-2+i)+\frac{1}{4} e^{-it}(2+i) & \left(\frac{1}{2}+\frac{i}{4}\right)e^{it}(-2+i)+\left(\frac{1}{2}-\frac{i}{4}\right)e^{-it}(2+i) \\ -\frac{1}{4} e^{+it}+\frac{1}{4}\cdot e^{-it} & \left(\frac{1}{2}+\frac{i}{4}\right)e^{it}+\left(\frac{1}{2}-\frac{i}{4}\right)e^{-it} \end{pmatrix}$$

b) From theory we know e^{At} is equal to $U(t)$, so $e^{At}=U(t)$

We can also simplify the above by expanding e^{it} with $\cos(t)+i\cdot\sin(t)$ and e^{-it} with $\cos(t)-i\sin(t)$ and we get

$$U(t)=e^{At}=\begin{pmatrix} 2\sin(t)+\cos(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{pmatrix}$$

c) $H(x,y)=x^2+ay^2+bxy$

$$\frac{\partial H}{\partial x}(x,y)\cdot(2x-5y)+\frac{\partial H}{\partial y}(x,y)(x-2y)=0$$

$$(2x+by)(2x-5y)+(2ay+bx)(x-2y)=0$$

$$4x^2+2bxy-10xy-5by^2+$$
$$bx^2+2axy-2bxy-4ay^2=0$$

$$(4+b)x^2+(\cancel{2b}-10+2a-\cancel{2b})xy-(5b+4a)y^2=0$$

$4+b=0 \Rightarrow$ $\boxed{b=-4 \quad a=5}$

$2a-10=0$

$5b+4a=0$

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4. a) $x'' + t^2 x = 0$

$x(0) = 1$

$x'(0) = 1$

Has an infinity of solutions because the IVP requires 3 fixed initial values and the third is not fixed, so for any $x''(0) = y$ there exists a solution, hence an infinity

b) $x''' + t^2 x = 0$

$x(0) = 1$

$x'(0) = 1$

$x''(0) = 1$

Has one solution, it's the IVP