Exercise (4.6.). Show that two different parallel lines are either projected onto parallel lines or on two points by the projection $p_{\pi,d}$ for $\pi \not\parallel d$, where:

$$\pi : Ax + By + Cz + D = 0$$
$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

PROOF. From $\pi \not\parallel d$ we deduce that $Ap + Bq + Cr \neq 0$. Suppose that our two parallel lines are given by the parametric equations:

$$d_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \qquad d_2 : \begin{cases} x = x_2 + sv_x \\ y = y_2 + sv_y \\ z = z_2 + sv_z \end{cases}$$

Comment: The reason I chose to give different names for the parameters of these equations is pedagogical. I wanted it to be clear that these parameters are not necessarily the same. In general, there is no harm in denoting the parameters of two lines by the same letter.

We now make the notations:

$$F(x, y, z) := Ax + By + Cz + D$$
$$\overrightarrow{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \qquad \overrightarrow{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

For every $P(x, y, z) \in d_1$ we have:

$$p_{\pi,d}(P) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{F(x, y, z)}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$
$$p_{\pi,d}(P) = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\overrightarrow{v} - \frac{A(x_1 + tv_x) + B(y_1 + tv_y) + C(z_1 + tv_z) + D}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

We see that in this formula we have free terms and terms that depend on the parameter t. We then separate them:

$$p_{\pi,d}(P) = \begin{pmatrix} x_1 - p\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - q\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - r\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix} + t \cdot \left(\overrightarrow{v} - \frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right)$$

This is the parametric equation of a line d_1' that contains the point P_1' with coordinate vector:

$$\begin{pmatrix} x_1 - p\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - q\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - r\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix}$$

and director vector:

$$\overrightarrow{w} = \overrightarrow{v} - \frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

By performing the exact same computations for the projection of the line d_2 we obtain the equation of a line d_2' that contains the point P_2' with coordinate vector:

$$\begin{pmatrix} x_2 - p\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ y_2 - q\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ z_2 - r\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \end{pmatrix}$$

and director vector:

$$\overrightarrow{w} = \overrightarrow{v} - \frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

The director vector is the same, therefore, the two lines d_1 and d_2 are projected onto the parallel lines d_1' and d_2' if w is nonzero or onto the points P_1' and P_2' if w is the zero vector.

Remark. We see that the lines d_1 and d_2 are projected onto points if and only if we have

$$\overrightarrow{v} = \frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d},$$

which can be rewritten to

$$\overrightarrow{v} = \frac{\overrightarrow{n_\pi} \cdot \overrightarrow{v}}{\overrightarrow{n_\pi} \cdot \overrightarrow{d}} \cdot \overrightarrow{d},$$

This is equivalent to $\overrightarrow{v} \parallel \overrightarrow{d}$, which corresponds to the geometrical intuition that a line is projected onto a plane as a single point if and only if it is parallel to the direction of the projection.

$\square$