18. Find the locus of the points whose distances to two ortogonal lines have the same ratio.

Solution

Let $R = (O, \vec{i}, \vec{j})$ the orthonormal reference of the plane and (d_1, d_2) two orthogonal lines in the plane, and consider $\{E\} = d_1 \cap d_2$, and let S be the locus of the points whose distances to d_1 and d_2 have the same ratio.

Denote $T = \{S, d_1, d_2, E\}$ the system determined by the orthogonal lines, their intersection and S and denote $\theta = (\widehat{\vec{d_1}, \vec{i}})$ the oriented angle determined by the oriented directions $d_1$and i(which orientation we choose for d_1 does not influence the result.). Applying to the system T the translation by the vector $\overrightarrow{EO}$ and the rotation of center O and angle $-\theta$ we overlap the lines d_1, d_2 and the point E over the axes of the plane. Now because those 2 trasformations preserved all the distances in the system T, it is enough to determine the locus S' that resulted after the transformations.

That being said, we are going to determine the locus S' of points whose distances to the axes Ox and Oy have the same ratio.

Let $P(x, y) \in S'$ and deote $k \in \mathbb{R}$ the ratio. $\Rightarrow \left|\frac{x}{y}\right| = k \Rightarrow x = ky$,and therefore we obtain the equation of 2 lines:

$$l_1 : x - ky = 0 \tag{1}$$

$$l_2 : x + ky = 0 \tag{2}$$

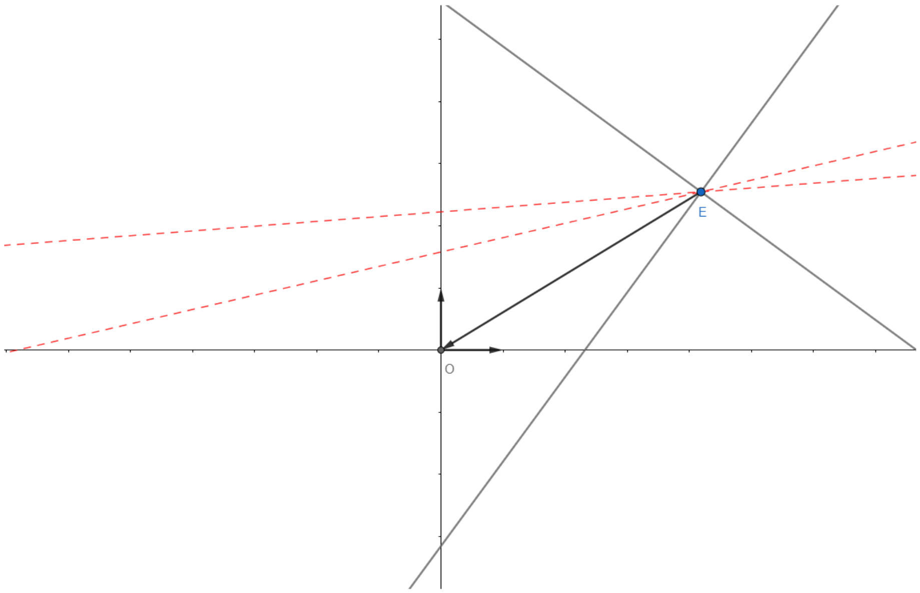
We do not take into consideration the ratio $\frac{y}{x}$; that would make no sense since it would inverse the order of the lines. However, all that it would have changed is an extra line belonging to S.

Because $\arctan\left(\frac{1}{k}\right)$ (k $=$ 0 treated separately) is the oriented angle of lines $\vec{l_1}$ and $\vec{i}$ by reversing the rotation we get the oriented angle $\arctan\left(\frac{1}{k}\right) + \theta$ for the original direction and the Ox axis, so $m_1 = \tan\left(\arctan\left(\frac{1}{k}\right) + \theta\right)$ is the original slope of one of the lines in S and $m_2 = \tan\left(\arctan\left(-\frac{1}{k}\right) + \theta\right)$ Now because the original lines go through point $E(e_1, e_2)$ and therefore we can compute theor original equations:

$$l'_1 : (y - e_1) = m_1(x - e_2) \tag{3}$$

$$l'_2 : (y - e_1) = m_2(x - e_2) \tag{4}$$

In conclusion, the set S is constituted by 2 straight lines passing through E that are symmetrical with respect to the first line of the orthogonal lines that we consider the distance to.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