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1. a) The probability is modelled by a poisson distribution with $\lambda = 10$ (10 messages per hour). Knowing λ ~~the~~ we need to compute

$$\frac{10^5}{5!} \cdot e^{-10} + \frac{10^6}{6!} \cdot e^{-10} + \dots = 1 - \text{poisscdf}(4, 10) = 0.97075$$

b) $P(X \geq 8 \mid X \geq 6)$?

$$P(X \geq 8 \mid X \geq 6) = \frac{P(X \geq 8 \cap X \geq 6)}{P(X \geq 6)} = \frac{P(X \geq 8)}{P(X \geq 6)}$$

$$= \frac{1 - \text{Poisscdf}(7, 10)}{1 - \text{Poisscdf}(5, 10)} = \frac{0.77978}{0.93291} = 0.83585$$

c) For a ~~poisson~~ ~~dis~~ variable with poisson distribution the expected value is equal to its parameter λ which models the average number of events in a time interval.

If 1 message arrives in 6 minutes then 15 messages should arrive in 90 minutes.

$E(X) = 15$ where X – poiss with $\lambda = 15$

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2. $X \in N(0,1)$

$Y = 2X - 7$

$g(x) = 2x - 7$ is a strictly increasing function and also differentiable.

$g'(x) = 2$

$g^{-1}(x) = \frac{x+7}{2}$

$g(\mathbb{R}) = \mathbb{R}$

$f_X = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$

$$\boxed{f_Y = \frac{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\left(\frac{x+7}{2}\right)^2}{2}}}{2} = \frac{1}{\sqrt{8\pi}} \cdot e^{-\frac{(x+7)^2}{8}}}$$

By applying $f_Y = \begin{cases} \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, & y \in g(\mathbb{R}) \\ 0, & y \notin g(\mathbb{R}) \end{cases}$

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3. a) ~~θ_1~~ we solve $\theta_1 = \bar{\theta}_1$

where $\theta_1 = E(x)$ and $\bar{\theta}_1$ is the mean of all samples.

$$E(x) = \bar{x}$$

$$3\mu = \bar{x}$$

$$\hat{\mu} = \frac{\bar{x}}{3}$$ — estimator from method of moments

b) Only one unknown => only one equation

$$\frac{\partial \ln L(x_1, \dots x_n, \mu)}{\partial \mu} = 0$$

$$L(x_1, x_2 \dots x_n, \mu) = \prod_{i=1}^{n} f(x_i; \mu)$$

$$= \prod_{i=1}^{n} \frac{1}{3\mu} \cdot e^{-\frac{x_i}{3\mu}}$$

$$= (3\mu)^{-n} \cdot e^{\sum_{i=1}^{n} -\frac{x_i}{3\mu}}$$

$$\ln L = -n \cdot \ln 3\mu + \sum_{i=1}^{n} \frac{-x_i}{3\mu} \cdot \ln e$$

$$= -n \cdot \ln 3\mu - \frac{1}{3\mu} \cdot \sum_{i=1}^{n} x_i$$

$$\frac{\partial \ln L}{\partial \mu} = -n \cdot \frac{1}{\mu} + \frac{1}{3\mu^2} \cdot \sum_{i=1}^{n} x_i$$

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$$\frac{\partial \ln L}{\partial \mu} = 0 \Rightarrow \frac{-n}{\mu} + \frac{\cancel{3}}{3\mu^2} \cdot \bar{x} = 0 \quad | \cdot \mu^2$$

$$-n\mu + \frac{1}{3}\bar{x} = 0$$

$$\bar{\mu} = \frac{\cancel{3}\bar{x}}{3n} \text{ - estimator for max likelihood}$$

c) $\bar{\mu}$ is absolutely correct if

$$E(\bar{\mu}) = \mu \quad \text{and} \quad \lim_{n \to \infty} V(\bar{\mu}) = 0$$

$$E(\bar{\mu}) = E\left(\frac{\bar{x}}{3n}\right) = \frac{1}{3n} E(\bar{x}) = \frac{1}{3n} \cdot E(x) = \frac{1}{3n} \cdot 3\mu$$

$$= \frac{\mu}{n}$$

$$V(\hat{\mu}) = V\left(\frac{\bar{x}}{3n}\right) = \left(\frac{1}{3n}\right)^2 \cdot V(\bar{x}) = \frac{1}{9n} \cdot \frac{V(x)}{n}$$

$$= \frac{1}{\cancel{9}n} \cdot \frac{\cancel{9}\mu^2}{n}$$

$$= \frac{\mu^2}{n^2} \to 0$$

Only one requirement is met $\Rightarrow$ $\bar{\mu}$ is not an absolutely correct estimator

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d) $f(x,\mu) = \begin{cases} \frac{1}{3\mu} e^{-\frac{x}{3\mu}} \end{cases}$

We will use the Neyman-Pearson Lemma

$H_0: \mu = 1$

$H_1: \mu = 2$

$$RR = \left\{ \frac{L(\mu_1)}{L(\mu_0)} \geq k_a \right\}$$

$$\frac{L(\mu_1)}{L(\mu_0)} = \frac{(3\mu_1)^{-n} \cdot e^{-\sum_{i=1}^{n} \frac{x_i}{3\mu_1}}}{(3\mu_0)^{-n} \cdot e^{-\sum_{i=1}^{n} \frac{x_i}{3\mu_0}}}$$

$$= \left(\frac{\mu_1}{\mu_0}\right)^{-n} \cdot e^{\sum_{i=1}^{n} \frac{x_i}{3\mu_0} - \frac{x_i}{3\mu_1}}$$

$$= \left(\frac{2}{1}\right)^{-n} \cdot e^{\sum_{i=1}^{n} \frac{\mu_1 - \mu_0}{3\mu_0\mu_1} \cdot x_i}$$

$$= 2^{-n} \cdot e^{\frac{2-1}{3\cdot 2} \cdot \sum_{i=1}^{n} x_i}$$

$$= 2^{-n} \cdot e^{\frac{\overline{x}}{6}}$$

$$RR = \left\{ 2^{-n} \cdot e^{\frac{\overline{x}}{6}} \geq k_a \right\} = \left\{ \overline{x} \geq K_a \right\}$$

$$K_a = 6 \ln(k_a \cdot 2^n)$$

$a = P(\bar{X}_1 \in RR \mid H_0) = P(\bar{x} \geq k_a \mid \mu = 1)$ Crăciun Ioan-Flaviu

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By C.L.T. $\dfrac{\bar{x} - 3\mu}{\sqrt{n} \cdot 3\mu} \to N(0,1)$

$$P(\bar{x} \geq k_a \mid \mu = 1) = P\left(\frac{\bar{x} - 3\mu_0}{\sqrt{n} \cdot 3\mu_0} \geq \frac{k_a - 3\mu_0}{\sqrt{n} \cdot 3\mu_0} \mid \mu = 1 \right)$$

$$= P\left(Z \geq \frac{k_a - 3\mu_0}{\sqrt{n} \cdot 3\mu_0} \mid \mu = 1 \right)$$

$$= P\left(Z \geq \frac{k_a - 3}{3\sqrt{n}} \right)$$

$$= \int_{\frac{k_a - 3}{3\sqrt{n}}}^{\infty} N(0,1)\, dx$$

$$= \int_{\frac{6\ln(k_a \cdot 2^{\mu}) - 3}{3\sqrt{n}}}^{\infty} N(0,1)\, dx$$

Now write k_a as a function of a and you have a rejection region for a most powerful test