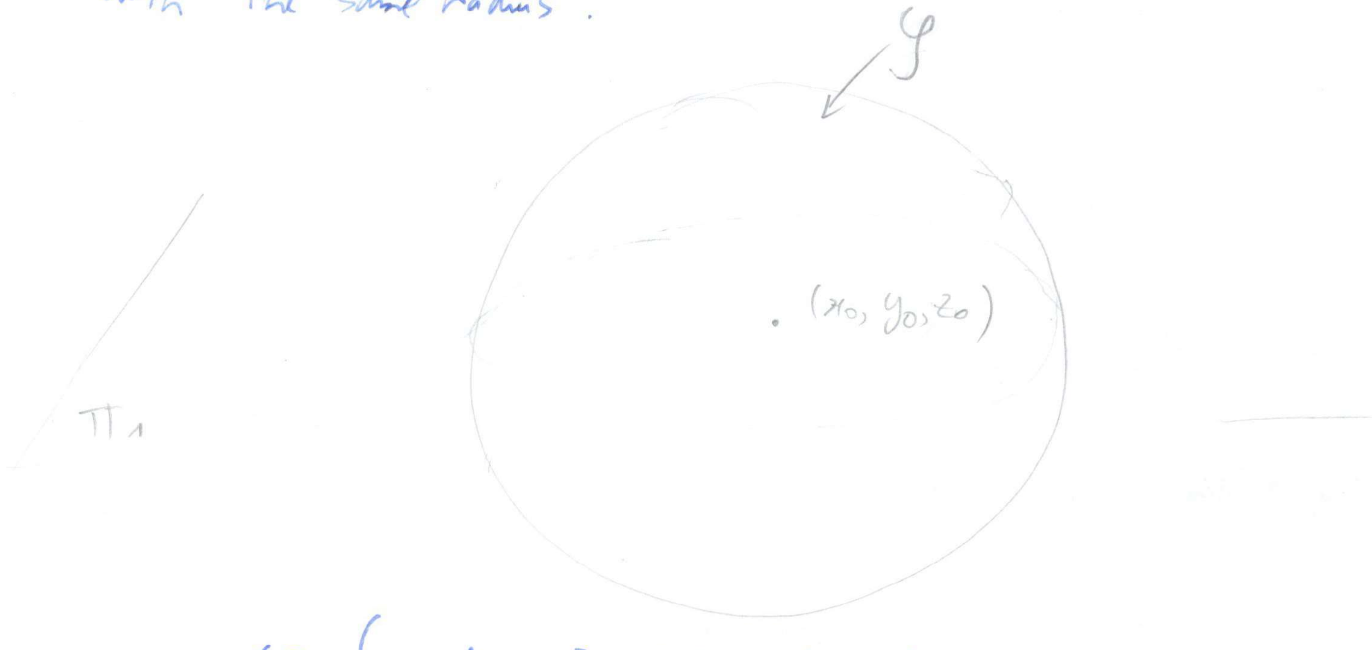
C.3.3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface C, generated by a variable line (Δ) which intersects the line (d) and the circle $(\mathscr{C})$ and remains orthogonal to (d). (The Willis conoid)

Solution:

A circle in 3D is given by the intersection between a plane Π_1 and a sphere $\mathscr{S}$ centered in the center of the circle with the same radius.



$$\mathscr{C}: \begin{cases} \Pi_1: Ax + By + Cz + D = 0 \\ \mathscr{S}: (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \end{cases}$$

(14)

The line d has the equation of the form:

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C} \Leftrightarrow \begin{cases} B(x-x_1)-A(y-y_1)=0 \\ C(x-x_1)-A(z-z_1)=0 \end{cases}$$

(because $d \parallel \Pi_1$)

We use theorem 3.6. from the notes.

The director plane Π of the surface C is parallel to the variable line (Δ), which is orthogonal to (d), thus Π is orthogonal to (d), so $\Pi: Ax + By + Cz + D_1 = 0 \Rightarrow \Pi \parallel \Pi_1$

An arbitrary generatrix of the conoidal surface is contained in a plane parallel to Π (whose equation is of the form $\Pi = \lambda$) and, on the other hand, comes from the bundle of planes containing d

$\Rightarrow$ a generatrix is given by

$$d_{\lambda\mu}: \begin{cases} Ax+By+Cz+D = \lambda \\ \mu\left(B(x-x_1)-A(y-y_1)\right) + C(x-x_1)-A(z-z_1)=0 \end{cases}$$

The condition is that $d_{\lambda\mu} \cap \mathcal{C} \neq \emptyset$, so the following system must be compatible:

$$\begin{cases} Ax+By+Cz+D=\lambda \\ \mu\left(B(x-x_1)-A(y-y_1)\right)+C(x-x_1)-A(z-z_1)=0 \\ Ax+By+Cz+D=0 \\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} Ax+By+Cz+D=0 \\ \lambda=0 \\ \mu=\dfrac{A(z-z_1)-C(x-x_1)}{B(x-x_1)-A(y-y_1)} \\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z=z_1+\dfrac{\mu B(x-x_1)-\mu A(y-y_1)+C(x-x_1)}{A} \\ Ax+By+Cz_1+\dfrac{C}{A}\left(\mu B(x-x_1)-\mu A(y-y_1)+C(x-x_1)\right)+D=0 \\ (x-x_0)^2+(y-y_0)^2+\left(z_1+\dfrac{\cdots}{A}-z_0\right)^2=r^2 \end{cases}$$

And then you solve this system and you find some condition on μ of the form $\varphi(\lambda,\mu)=0$ and the equation of the surface is obtained from

$$\varphi\left(0, \frac{A(z-z_1)-C(x-x_1)}{B(x-x_1)-A(y-y_1)}\right)=0$$