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1. S cannot be empty because there exists at least one solution to $x_{k+2} - (\sin k) x_k = k$, say we know $x_0 = x_1 = 0$.

Then $x_2 = \sin 2 \cdot 0 + 2 = 2$

$x_3 = \sin(3) \cdot 2 + 3 = 3 + 2 \sin 3$

$x_4 = \sin 4 (3 + 2 \sin 3) + 4 = 2 \sin 3 \sin 4 + 3 \sin 4 + 4$

$\vdots$

$x_k = 2 \sin 3 \sin 4 \dots \sin k + 3 \sin 4 \dots \sin k + k.$

$\vdots$

We also know that for any $x_0 = y_1, x_1 = y_2$ there exists a unique solution because the IVP theorem holds. So we have proven T is injective because any solution is generated by one IVP. To prove surjectivity we need to show that any IVP can generate a solution from S, ~~so~~ there is a one to one correspondence between the IVP and the solution. Thus we have bijectivity

S is indeed a linear space of finite, in this case dimension 2. We know because any equation of the form

$x_{k+n} + a_{1,k} x_{k+n-1} + \dots + a_{n,k} x_k = f_k, \ k \geq 0$ has solutions which form a linear space where addition and multiplication are defined in a natural way. (from the lecture)

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2. $x''+2tx'=0$

~~we can integrate the~~

Denote $y=x'$, the system becomes $y'+2ty=0$

Remember from lecture 3 the solution of the equation above is $c\cdot e^{-A(t)}$, where $A(t)$ is the integral of $2t$. So $c\cdot e^{-t^2}$ is the solution, indeed $(c\cdot e^{-t^2})'+2t\cdot ce^{-t^2}=0$,

$$c\cdot -2t\cdot e^{-t^2}+2t\cdot c\cdot e^{-t^2}=0$$

$$0=0.$$

Integrating $c\cdot e^{-t^2}$ we get $\frac{c\sqrt{\pi}}{2}\operatorname{erf}(t)+c_2$ which we may simply write as $c_1\cdot\operatorname{erf}(t)+c_2$ where erf is the error function.

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3. $\dot{x} = -y^2 - yx$

$\dot{y} = xy + x^2$

a) The equilibrium points are the solutions of $\begin{cases} -y^2 - yx = 0 \\ xy + x^2 = 0 \end{cases}$

$$\begin{aligned} -y(y+x) &= 0 \\ x(y+x) &= 0 \end{aligned} \Rightarrow \text{either } x = y = 0, \text{ or } x = -y.$$ So there are an infinity of equilibria.

Now we may find a first integral by separating the variables and integrating.

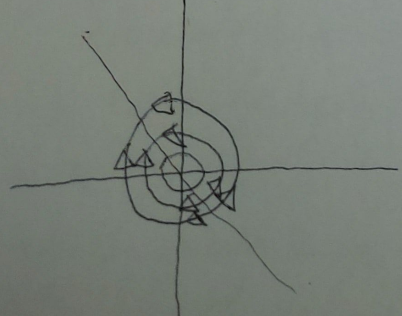
$$\frac{x'}{y'} = \frac{-y\cancel{(y+x)}}{x\cancel{(y+x)}}$$

$$x'x = -y'y$$

$$\int x'x = \int -y'y$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + C$$

$x^2 + y^2 = C \Rightarrow$ The level curves of the first integral are circles which do not pass through $y = -x$



The straight line is the line which contains all equilibria

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3. b) Looking at the phase portrait we notice arrows on the circles point to the upper ~~sie~~ left equilibria, so $\lim_{t\to\infty} \varphi(t,3,0) = \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$ because that point is the intersection of the circle and the line $x+y=0$ at the upper left point.

Similarly $\lim_{t\to\infty} \varphi(t,1,0) = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$

c) These two functions are not periodic because they have a limit towards ∞.

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4. $\dot{x} = -\frac{1}{20}(x-10)$

$$\dot{x} + \frac{1}{20}x - \frac{1}{2} = 0$$

First we solve the homogeneous part $\dot{x} + \frac{1}{20}x = 0$

characteristic polynomial $r + \frac{1}{20} = 0 \Rightarrow r = -\frac{1}{20} \Rightarrow c \cdot e^{-\frac{1}{20}t}$ is a solution to the homogeneous system.

A particular solution is a constant x so $\frac{1}{20}x = \frac{1}{2} \Rightarrow x = 10$

$c \cdot e^{-\frac{1}{20}t} + 10$ – the general solution

IVP: $c \cdot e^{-\frac{1}{20} \cdot 0} + 10 = \eta$

$$c + 10 = \eta$$

$$c = \eta - 10$$

The flow is $(\eta - 10)e^{-\frac{1}{20}t} + 10$

To cool from $80°$ to $40°$ we solve

$$(80-10)e^{-\frac{1}{20}t} + 10 = 40$$

$$e^{-\frac{1}{20}t} = \frac{3\cancel{0}}{7\cancel{0}}$$

$$-\frac{1}{20}t = \ln\frac{3}{7}$$

$$t = 20\ln\frac{7}{3}$$

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5. a) $x_{k+1} = -x_k + 3y_k$

$y_{k+1} = -3x_k - y_k$

$y_k = \frac{x_{k+1} + x_k}{3}$

$\frac{x_{k+2} + x_{k+1}}{3} = -3x_k - \frac{x_{k+1} + x_k}{3} \quad | \cdot 3$

$x_{k+2} + x_{k+1} + x_{k+1} + 10x_k = 0$

$x_{k+2} + 2x_{k+1} + 10x_k = 0$

$x_0 = 0$

$y_0 = 2 \Rightarrow x_1 = 0 + 6 = 6$

$r^2 + 2r + 10 = 0$

$r_{1,2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm i3$

$r_{1,2} = 2(\cos \pm 120^\circ + i \sin \pm 120^\circ)$

$= 2(\cos \pm \frac{2\pi}{3} \pm i \sin \pm \frac{2\pi}{3})$

$x_k = c_1 \cdot 2^k \cos \frac{2k\pi}{3} + c_2 \cdot 2^k \cdot \sin \frac{2k\pi}{3}$

$x_0 = 0 \Rightarrow c_1 = 0$

$x_1 = 6 \Rightarrow c_2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = 6 \Rightarrow c_2 = 2\sqrt{3}$

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5. a) So $x_k = 2\sqrt{3} \cdot 2^k \cdot \sin\frac{2k\pi}{3}$

$$x_k = \frac{y_{k+1}+y_k}{-3}$$

$$\frac{y_{k+2}+y_{k+1}}{-3} = \frac{y_{k+1}+y_k}{3} + 3y_k \quad | \cdot -3$$

$$y_{k+2}+y_{k+1}+y_{k+1}+y_k+9y_k=0$$

$$y_{k+2}+2y_{k+1}+10y_k=0$$

$$y_k = c_1 \cdot 2^k \cos\frac{2k\pi}{3} + c_2 \cdot 2^k \sin\frac{2k\pi}{3}$$

$$y_0 = 2 \Rightarrow c_1 = 2$$

$$y_1 = -3 \cdot 0 - 2 = -2 \Rightarrow 2 \cdot 2\cos\frac{2\pi}{3} + c_2 \cdot 2 \cdot \sin\frac{2\pi}{3} = -2$$

$$-2 + c_2 \cdot 2\sin\frac{2\pi}{3} = -2 \Rightarrow \boxed{c_2 = 0}$$

$$\begin{cases} y_k = 2 \cdot 2^k \cdot \cos\frac{2k\pi}{3} \\ x_k = 2\sqrt{3}\, 2^k \sin\frac{2k\pi}{3} \end{cases}$$ solution of the IVP

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S.6)

$$A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{10} & \frac{-3}{10} \\ \frac{3}{10} & \frac{-1}{10} \end{pmatrix}$$

$$A^{-2} = \begin{pmatrix} \frac{-2}{25} & \frac{3}{50} \\ \frac{-3}{50} & \frac{-2}{25} \end{pmatrix}$$

$\vdots$

Notice the terms keep getting smaller, approacting 0 because they are smaller than 1 by a lot and multiplying 2 terms smaller than 1 we get even smaller numbers.

So $\lim_{n\to\infty} A^{-n} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Another solution is noticing $A^{-n}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is the solution for the IVP calculated at a) for $k = -n$, so

$$\begin{pmatrix} \lim_{n\to-\infty} 2\cdot 2^{n} \cos\frac{2n\pi}{3} \\ \lim_{n\to+\infty} 2\sqrt{3}\cdot 2^{n} \sin\frac{2n\pi}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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6. ~~$f(2) = -5$~~

Suppose we start at $x = 2$

$f(2) = -5$. Now we are at $x = -5$

$f(-5) = 2$. We notice it is a cycle because 2 goes right back to -5 and -5 to 2 and so on.

We have established $\{-5, 2\}$ is a cycle. We also know a cycle is an attractor if $|(f \circ f)'(x)| < 1$.

$x = 2$

$$|(f \circ f)'(2)| = |f'(f(2))\, f'(2)| = |f'(-5)\, f'(2)|$$

$x = -5$

$$|(f \circ f)'(-5)| = |f'(f(-5))\, f'(-5)| = |f'(2) \cdot f'(-5)|$$

We have shown now that $|(f \circ f)'(x)| < 1$ and $|f'(-5) f'(2)| < 1$ are equivalent.

for $x = 2$ or $x = -5$