8. $\mathcal{R} = (O, b)$, $b = [\vec{u}, \vec{v}, \vec{w}]$

$\pi: Ax + By + Cz + D = 0$

$d: \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$

Show that:

a) $\overrightarrow{p_{\pi,d}(M)\, p_{\pi,d}(N)} = p(\overrightarrow{MN})$, $\forall M, N \in \mathcal{V}$, where

$p: \mathcal{V} \to \mathcal{V}$ is a lin. transf. with the matrix:

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

Proof: $\overrightarrow{p_{\pi,d}(M)\, p_{\pi,d}(N)} = \overrightarrow{Op_{\pi,d}(N)} - \overrightarrow{Op_{\pi,d}(M)} \overset{4.12}{=}$

$$= \overrightarrow{ON} - \frac{F(N)}{Ap + Bq + Cr} \cdot \vec{d} - \left(\overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \cdot \vec{d}\right)$$

$$= \overrightarrow{ON} + \overrightarrow{MO} - \frac{F(N) - F(M)}{Ap + Bq + Cr} \cdot \vec{d}$$

$$= \overrightarrow{MN} - \frac{F(N) - F(M)}{Ap + Bq + Cr} \cdot \vec{d}. \quad (1)$$

We have to prove that expr. (1) is equal to $p(\overrightarrow{MN})$.

Denote $M(x_M, y_M, z_M)$, $N(x_N, y_N, z_N)$ and $\begin{cases} \alpha = x_N - x_M \\ \beta = y_N - y_M \\ \gamma = z_N - z_M \end{cases}$

• $\overrightarrow{MN} = (x_N - x_M) \cdot \vec{u} + (y_N - y_M) \cdot \vec{v} + (z_N - z_M) \cdot \vec{w}$

$\Rightarrow \overrightarrow{MN}(\alpha, \beta, \gamma)$.

For a vector $\vec{v}$, we have that $p(\vec{v}) = [p]_b \cdot \vec{v}$.

In our case, the vector $\vec{v}$ is (α, β, γ).

$$\Rightarrow p(\overrightarrow{MN}) = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} =$$

$$= \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq\alpha + Cr\alpha - Bp\beta - Cp\gamma \\ -Aq\alpha + Ap\beta + Cr\beta - Cq\gamma \\ -Ar\alpha - Br\beta + Ap\gamma + Bq\gamma \end{pmatrix}$$

$$\Rightarrow p(\overrightarrow{MN}) = \frac{1}{Ap+Bq+Cr}\cdot(Bq\alpha+Cr\alpha-Bp\beta-Cp\gamma)\cdot\vec{u} + \frac{1}{Ap+Bq+Cr}\cdot$$

$$\cdot(-Aq\alpha+Ap\beta+Cr\beta-Cq\gamma)\cdot\vec{v} + \frac{1}{Ap+Bq+Cr}(-Ar\alpha-Br\beta+Ap\gamma+Bq\gamma)\cdot\vec{w} =$$

$$= \frac{Ap\alpha+Bq\alpha+Cr\alpha-Ap\alpha-Bp\beta-Cp\gamma}{Ap+Bq+Cr}\cdot\vec{u} + \frac{Ap\beta+Cr\beta+Bq\beta-Bq\beta-Aq\alpha-Cq\gamma}{Ap+Bq+Cr}\cdot\vec{v}$$

$$+ \frac{Ap\gamma+Bq\gamma+Cr\gamma-Cr\gamma-Ar\alpha-Br\beta}{Ap+Bq+Cr}\cdot\vec{w} =$$

$$= \left(\alpha\cdot\underbrace{\frac{Ap+Bq+Cr}{Ap+Bq+Cr}}_{=1} - p\cdot\frac{A\alpha+B\beta+C\gamma}{Ap+Bq+Cr}\right)\cdot\vec{u} + \left(\beta - q\cdot\frac{A\alpha+B\beta+C\gamma}{Ap+Bq+Cr}\right)\cdot\vec{v} +$$

$$+ \left(\gamma - r\cdot\frac{A\alpha+B\beta-C\gamma}{Ap+Bq+Cr}\right)\cdot\vec{w}.$$

$$= \underbrace{\alpha\cdot\vec{u}+\beta\cdot\vec{v}+\gamma\cdot\vec{w}}_{=\overrightarrow{MN}} - \frac{A\alpha+B\beta+C\gamma}{Ap+Bq+Cr}\underbrace{(p\vec{u}+q\vec{v}+r\vec{w})}_{=\vec{d}}$$

$$= \overrightarrow{MN} - \frac{Ax_N+By_N+Cz_N+D-Ax_M-By_M-Cz_M-D}{Ap+Bq+Cr}\cdot\vec{d}$$

$$= \overrightarrow{MN} - \frac{F(N)-F(M)}{Ap+Bq+Cr}\cdot\vec{d} \quad (1)$$

So $\overrightarrow{p_{\pi,d}(M)\,p_{\pi,d}(N)} = \cancel{p(}\overrightarrow{MN} - \frac{F(N)-F(M)}{Ap+Bq+Cr}\cdot\vec{d} = p(\overrightarrow{MN})$.