Chapter 1

Differential Equations. Forms and Solutions

Forms.[1] We will study *differential equations in the vectorial form*

$$x' = f(t, x) \tag{1}$$

where the function $f : D \to \mathbb{R}^n$ is continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$. The natural number $n \geq 1$ is called *the dimension of the equation.* The unknown of the differential equation (1) is *a function* $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$. The variable of the function x is denoted by t. It is also said that t is *the independent variable* of (1), while x is *the dependent variable.* The symbol x' in (1) denotes *the first order derivative* of x with respect to t.

When $n = 1$ the equation is said to be *scalar.* Note that for $n \geq 2$ we can say that (1) is a system of n scalar differential equations with n scalar unknowns. More precisely, denoting the components of the vectorial functions x and f by $x_1, x_2, ..., x_n$ and $f_1, f_2, ..., f_n$, respectively (note that we consider x and f as column vectors), we can write equation (1) as

$$\begin{aligned} x_1' &= f_1(t, x_1, ..., x_n) \\ x_2' &= f_2(t, x_1, ..., x_n) \\ &\dots \\ x_n' &= f_n(t, x_1, ..., x_n). \end{aligned}$$

When first presenting differential equations, one can say, roughly speaking, that a differential equation is a relation involving the derivatives of some unknown function up to a given order. This means that it is a scalar equation of the form

$$x^{(n)} = g(t, x, x', ..., x^{(n-1)}). \tag{2}$$

Here we consider $g : D \to \mathbb{R}$ a continuous function on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$. The natural number $n \geq 1$ is called *the order* of the differential equation (2). The unknown is a scalar *function* $x : I \to \mathbb{R}$ defined on $I \subset \mathbb{R}$ and whose variable is denoted by t. The symbol $x^{(k)}$ in (2) denotes *the k-th order derivative* of x with respect to t, for any $k = 1, ..., n$.

We will show in the sequel that an equation of the form (2) can be put into the form (1). First note that $x(t)$ in (2) is a scalar, while $x(t)$ in (1) is a vector. Hence we use a new notation, X, for a vectorial function, such that we have to arrive to the n-dimensional system

$$X' = f(t, X). \tag{3}$$

This also clarifies that we need to introduce $(n-1)$ scalar unknowns beside the scalar unknown x of our equation (2). These new unknowns will be the derivatives of x up to order $(n-1)$, that is

$$X_1 = x, \quad X_2 = x', \quad X_3 = x'', \quad ... \quad X_n = x^{(n-1)}. \tag{4}$$

From (2) and (4) we obtain that $X_1, ..., X_n$ satisfy

$$\begin{array}{lcl} X_1' & = & X_2 \\ X_2' & = & X_3 \\ & ... & \\ X_{n-1}' & = & X_n \\ X_n' & = & g(t, X_1, X_2, ..., X_n). \end{array}$$

The final step in seeing that this system can be put into the vectorial form (3) is to identify the components of the vectorial function f. Of course, these are

$$f_1(t, X_1, ..., X_n) = X_2, \quad f_2(t, X_1, ..., X_n) = X_3, \quad ... \quad f_n(t, X_1, ..., X_n) = g(t, X_1, ..., X_n).$$

In the sequel we provide some examples.

1) $x' = 2t + \sin t$, $x' = x$, $x' = tx$, $x' = \sin(t^2 x)$ are scalar first order differential equations. For each of them the unknown is the function x of variable t.

2) The same equations can be written using other notations for the variables. For example, when we denote the unknown function by u and let the independent variable be t, we have $u' = 2t + \sin t$, $u' = u$, $u' = tu$, $u' = \sin(t^2 u)$.

3) We write now the same equations as in 1) and 2) as $y' = 2x + \sin x$, $y' = y$, $y' = xy$, $y' = \sin(x^2 y)$. For each of them the unknown is the function y of variable x.

4) $x''' = t$, $x''' = 3\cos t + e^t - 5x' + 7x'x''$ are scalar third order differential equations. For each of them the unknown is the function x of variable t.

5) The following is a 2-dimensional differential system, or, in other words, a system of 2 (scalar) differential equations with two unknowns, the functions x_1, x_2 of variable t.

$$\begin{aligned} x_1' &= tx_1 + \sin x_2 \\ x_2' &= -\sin(2t)x_1. \end{aligned}$$

By denoting the unknowns as x, y of variable t, the same system can be written as

$$\begin{aligned} x' &= tx + \sin y \\ y' &= -\sin(2t)x. \end{aligned}$$

Solutions. Now we intend to present the precise notion of solution for a differential equation. We give also some examples.

Definition 1.1 *We say that a vectorial function $\varphi : I \to \mathbb{R}^n$ is a solution of the differential equation* (1) *if*

(i) *$I \subset \mathbb{R}$ is an open interval, $\varphi \in C^1(I, \mathbb{R}^n)$,*

(ii) *$(t, \varphi(t)) \in D$, for all $t \in I$,*

(iii) *$\varphi'(t) = f(t, \varphi(t))$, for all $t \in I$.*

In particular, for the nth order differential equation (2) the notion of solution is as follows.

Definition 1.2 *We say that scalar a function* $\varphi : I \to \mathbb{R}$ *is a solution of the nth order differential equation* (2) *if*

(i) $I \subset \mathbb{R}$ *is an open interval,* $\varphi \in C^n(I)$,

(ii) $(t, \varphi(t), \varphi'(t), ..., \varphi^{(n-1)}(t)) \in D$, *for all* $t \in I$,

(iii) $\varphi^{(n)}(t) = g(t, \varphi(t), \varphi'(t), ..., \varphi^{(n-1)}(t))$, *for all* $t \in I$.

We present now some examples in the form of exercises (this means that you have to check them).

1) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^2 - \cos t + 34$ is a solution of the scalar first order differential equation $x' = 2t + \sin t$.

2) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = -23\, e^t$ is a solution of the scalar first order differential equation $x' = x$.

3) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = 987\, e^{t^2/2}$ is a solution of the scalar first order differential equation $x' = tx$.

4) Let the functions $\varphi_1, \varphi_2, \varphi_3 : (-2, \infty) \to \mathbb{R}$ be given by the expressions $\varphi_1(t) = 1 + t$, $\varphi_2(t) = 1 + 2t$, $\varphi_3(t) = 1$. For the scalar first order differential equation

$$x' = \frac{x^2 - 1}{t^2 + 2t}$$

we have that φ_1 and φ_3 are solutions, while φ_2 is not a solution.

We remark that, in general, a differential equation has *many* solutions, where *many* does not mean 2 or 3, not even ten thousands. For example, for the scalar first order differential equation $x' = 2t + 1$ the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^2 + t + c$ is a solution for an arbitrary constant $c \in \mathbb{R}$. Hence, this differential equation has as many solutions as real numbers. In other words, the cardinal of the set of the solutions of $x' = 2t + 1$ is $\aleph$.

When talking about equations, one is used to say that he wants to "solve" it. To "solve" a differential equation means to find the whole family of solutions, which will be represented in a formula depending on one or more arbitrary constants. This formula is also called *the general solution* of the differential equation. For example, we say that $x' = 2t+1$ has the general solution $x = t^2+t+c$, for arbitrary $c \in \mathbb{R}$.

It is worth to say that one (human or computer) can not find the general solution of any differential equation. It is proved that the general solution of most of the differential equations can not be written as a finite combination of elementary functions.

The Initial Value Problem. When adding Initial Conditions to a differential equation, we say that an Initial Value Problem (IVP, for short) is formulated. These type of problems are also called Cauchy Problems, after the French mathematician Augustin-Louis Cauchy (1789-1857). More precisely, the IVP for (1) is

$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= \eta, \end{aligned}$$

where $f : D \to \mathbb{R}^n$ continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, \eta) \in D$ are all given. Note that t_0 is called *the initial time* while η is called *the initial value* or *the initial position*. In the particular case $n = 2$ we have

$$\begin{aligned} x_1' &= f_1(t, x) \\ x_2' &= f_2(t, x) \\ x_1(t_0) &= \eta_1, \\ x_2(t_0) &= \eta_2. \end{aligned}$$

The IVP for (2) is

$$\begin{aligned} x^{(n)} &= g(t, x, x', ..., x^{(n-1)}) \\ x(t_0) &= \eta_1 \\ x'(t_0) &= \eta_2 \\ ... & \\ x^{(n-1)}(t_0) &= \eta_n, \end{aligned}$$

where $g : D \to \mathbb{R}$ continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, \eta_1, ..., \eta_n) \in D$ are all given. In the particular case $n = 2$ we have

$$\begin{aligned} x'' &= g(t, x, x') \\ x(t_0) &= \eta_1 \\ x'(t_0) &= \eta_2. \end{aligned}$$

In this case η_1 is called *the initial position* while η_2 is *the initial velocity.*

It is worth to say that, in general, an IVP has a unique solution.

Examples of problems which are not correctly-defined IVPs.

1) $x' = tx - 1, \quad x(0) = 0, \quad x'(0) = 2$. It is not correct since it has an extra condition, $x'(0) = 2$, while the scalar differential equation is of first order. A correctly-defined IVP is $x' = tx - 1, \quad x(0) = 0$.

2) $x'' = tx - 1, \quad x(2) = 5, \quad x'(0) = -6$. It is not correct because there are two different "initial times", $t_0 = 2$ and also $t_0 = 0$. Two correctly-defined IVPs are $x'' = tx - 1, \quad x(0) = 5, \quad x'(0) = -6$ and $x'' = tx - 1, \quad x(2) = 5, \quad x'(2) = -6$.

3) $x' = 2x + \sin t - 5t^2y, \quad y' = xy - 3tx^2y^3, \quad x(0) = 1, \quad x'(0) = 2$. It is not correct because for this first order differential system appears a condition for the first order derivative of one of the unknowns, i.e. $x'(0) = 2$. A correctly-defined IVP is $\quad x' = 2x + \sin t - 5t^2y, \quad y' = xy - 3tx^2y^3, \quad x(0) = 1, \quad y(0) = 2$.

4) $x' = (x^2 - 1)/(t^2 + 2t), \quad x(-2) = 1$ is not correctly defined because the right-hand side of the differential equation is not defined for $t = -2$ (which is the initial time). More precisely, in the notations used here, consider $f(t, x) = (x^2 - 1)/(t^2 + 2t)$ and notice that it is not defined for $t \in \{-2, 0\}$. Hence, f it is defined only in $D = (-\infty, -2) \times \mathbb{R} \cup (-2, 0) \times \mathbb{R} \cup (0, \infty) \times \mathbb{R}$, but $(-2, 1) \notin D$ as it is required (see again the above definition where (t_0, η) must be in D).

Here are some exercises.

1) Knowing that the initial value problem $\quad x' = 1 - x^2, \quad x(0) = 1 \quad$ has a unique solution, find it among the following objects:

(a) *solution*; (b) the unit circle; (c) the constant function $x = 1$;
(d) the constant function $x = -1$; (e) the derivative.

2) Knowing that the initial value problem $\quad x' = 3x, \quad x(0) = 1 \quad$ has a unique solution, find it among the following functions:

(a) $x = e^t$; (b) $x = 1$; (c) $x = 1/3$; (d) $x = t$; (e) $x = e^{3t}$.

3) Knowing that the initial value problem $x' = x - 3, \quad x(0) = 1$ has a unique solution, find it among the functions of the form $x = 3 + c\,e^t$, with $c \in \mathbb{R}$.

4) Knowing that the initial value problem $x' = x - e^t, \quad x(0) = 1$ has a unique solution, find it among the functions of the form $x = (at+b)\,e^t$, with $a, b \in \mathbb{R}$.

5) Check that, for any $c \geq 0$, the function $\varphi_c : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_c(t) = \begin{cases} 0 & , \quad t \leq c \\ \left[\frac{2}{3}(t-c)\right]^{3/2} & , \quad t > c \end{cases}$$

is a solution of the IVP $x' = x^{1/3}, \quad x(0) = 0.$

Chapter 2

Scalar Linear Differential Equations

2.1 The Existence and Uniqueness Theorem for scalar linear differential equations

[1] Let $n \geq 1$ be a natural number. In the previous lecture we saw the general form of an nth order scalar differential equation. In this lecture we begin the study of a particular case of such equations, namely the class of nth order scalar linear differential equations which have the form

$$(1) \qquad x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)x' + a_n(t)x = f(t),$$

where $a_1, ..., a_n, f \in C(I)$, $I \subset \mathbb{R}$ being a nonempty open interval.

Definition 2.1 *A solution of* (1) *is a function* $\varphi \in C^n(I)$ *that satisfies* (1), *i.e.* $\varphi^{(n)}(t) + a_1(t)\varphi^{(n-1)}(t) + \ldots + a_n(t)\varphi(t) = f(t)$, *for all* $t \in I$.

The functions $a_1, ..., a_n$ are called *the coefficients* and the function f is called *the nonhomogeneous part* or *the force* of equation (1). When $f \equiv 0$ we say that (1) is *linear homogeneous* or *unforced*, otherwise we say that (1) is *linear nonhomogeneous* or *forced*. When all the coefficients are constant functions, we say that (1) is *a linear differential equation with constant coefficients.*

Examples of linear and nonlinear differential equations are presented in the sequel.

1) $x''' + x = 0$ is a third order linear homogeneous differential equation with constant coefficients.

2) $x'' + tx = 0$ is a second order linear homogeneous differential equation, but the coefficients are not all constant.

3) Let λ be a real parameter. The equation $x'' + \lambda x = 2\sin(3t) - t^2$ is a second order linear nonhomogeneous differential equation with constant coefficients. The nonhomogeneous part is $f(t) = 2\sin(3t) - t^2$.

4) The equation $x'' - 2x' + x^2 = 0$ is a second order *non-linear* differential equation. Indeed, it has one non-linear term, x^2.

In the conditions described at the beginning of this section we have the following important result. Its proof will be postponed by the end of the semester.

Theorem 2.2 *Let $t_0 \in I$ and $\eta_1, ..., \eta_n \in \mathbb{R}$ be given. We have that the IVP*

$$(2)\quad x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = f(t), \quad x(t_0) = \eta_1, \quad \ldots\,, \quad x^{(n-1)}(t_0) = \eta_n.$$

has a unique solution which is defined on the whole interval I.

For further reference we write below the form of a linear homogeneous differential equation.

$$x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)x' + a_n(t)x = 0. \tag{3}$$

When equation (1) is linear nonhomogeneous, we say (3) is *the linear homogeneous differential equation associated* to it.

2.2 The fundamental theorems for linear differential equations

[2] These theorems give the structure of the set of solutions of such equations. Their proofs relies on Linear Algebra. The key that opens the door of this theory

is to associate a linear map to a linear differential equation. First we note that the sets $C(I)$ and $C^n(I)$ have a linear structure when considering the usual operations of addition between real functions and multiplication of a real function with a real number. For each function $x \in C^n(I)$ we define a new function, denoted $\mathcal{L}x$, by

$$\mathcal{L}x(t) = x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t), \quad \text{for all } t \in I.$$

It is not difficult to see that $\mathcal{L}x \in C(I)$. In this way we obtain a map between the linear spaces $C^n(I)$ and $C(I)$, i.e.

$$\mathcal{L} : C^n(I) \to C(I).$$

Proposition 2.3 *The map $\mathcal{L}$ is linear, that is we have*

$$\mathcal{L}(\alpha x + \beta y) = \alpha\mathcal{L}x + \beta\mathcal{L}y, \text{ for all } x, y \in C^n(I) \text{ and all } \alpha, \beta \in \mathbb{R}.$$

Proof. HW □

Direct consequences of the linearity of $\mathcal{L}$ are the following.

Lemma 2.4 *The linear homogeneous differential equation* (3) *can be written equivalently*

$$\mathcal{L}x = 0,$$

while the linear nonhomogeneous differential equation (1) *can be written equivalently*

$$\mathcal{L}x = f.$$

Thus, the set of solutions of the differential equation (3) *is the linear space* $\ker \mathcal{L}$.

Proposition 2.5 (The linearity principle) *Let $m \geq 1$ and $x_1, x_2, ..., x_m$ be m solutions of the linear homogeneous differential equation* (3)*. Then, for any $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{R}$ we have that $(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m)$ is also a solution of* (3).

Proposition 2.6 (The superposition principle) *Let $m \geq 1$, $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{R}$, $f_1, f_2, ..., f_m \in C(I)$ be given. Let $f \in C(I)$ be defined by*

$$f = \alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_m f_m.$$

If, for each $k \in \overline{1,m}$ we have that x_{pk} is a particular solution of

$$\mathcal{L}x = f_k$$

then

$$x_p = \alpha_1 x_{p1} + \alpha_2 x_{p2} + ... + \alpha_m x_{pm}$$

is a particular solution of

$$\mathcal{L}x = f.$$

The fundamental theorem for linear homogeneous differential equations follows.

Theorem 2.7 (The fundamental theorem for linear homogeneous DEs) *The set of solutions of the linear homogeneous n-th order equation* (3) *is a linear space of dimension n. Thus, there exist $x_1, ..., x_n$ linearly independent solutions of* (3) *and its general solution can be described as*

$$x = c_1 x_1 + ... + c_n x_n, \quad c_1, ..., c_n \in \mathbb{R}.$$

Proof. We recall that the set of solutions of the linear homogeneous differential equation (3) is $\ker \mathcal{L}$. Since $\mathcal{L}$ is linear, $\ker \mathcal{L}$ is a linear subspace of $C^n(I)$. With all these in mind, note that, in order to complete the proof of our theorem it remains to prove that the linear space $\ker \mathcal{L}$ has dimension n. We know that the Euclidean space $\mathbb{R}^n$ has dimension n. We intend to find an isomorphism between $\ker \mathcal{L}$ and $\mathbb{R}^n$, because, as we know from Linear Algebra, an isomorphism between linear spaces preserves the dimension. Let $t_0 \in I$ be fixed. We define the map

$$\Phi : \ker \mathcal{L} \to \mathbb{R}^n, \quad \Phi(\varphi) = \left(\varphi(t_0), \varphi'(t_0), ..., \varphi^{(n-1)}(t_0)\right).$$

The existence and uniqueness Theorem 2.2 assures that the map Φ is bijective. It is easy to see that Φ is also a linear map.

As we discussed in the beginning of this proof, we can conclude now that the set of solutions of the linear homogeneous differential equation (3), which coincides with $\ker \mathcal{L}$, is a linear space of dimension n. The hypothesis of our theorem is that $x_1, ..., x_n$ are linearly independent solutions of (3), which, in other words, means that $\{x_1, ..., x_n\}$ is a basis of $\ker \mathcal{L}$. Then

$$\ker \mathcal{L} = \{c_1 x_1 + ... + c_n x_n, \quad c_1, ..., c_n \in \mathbb{R}\},$$

which gives the conclusion of our theorem. □

We note that, in order to prove that n functions $\{x_1, ..., x_n\}$ of $C^n(I)$ are linearly independent, one can uses, of course, the definition. An alternative way is by using the following notion and result.

Definition 2.8 *Let $m \geq 1$. The Wronskian of the functions $x_1, ..., x_m \in C^{m-1}(I)$ has the following expression*

$$W(x_1, ..., x_m)(t) = \begin{vmatrix} x_1(t) & \dots & x_m(t) \\ x_1'(t) & \dots & x_m'(t) \\ \vdots & \ddots & \vdots \\ x_1^{(m-1)}(t) & \dots & x_m^{(m-1)}(t) \end{vmatrix}, \quad \text{for all } t \in I.$$

Theorem 2.9 *Let $x_1, ..., x_n \in C^n(I)$ be n solutions of the linear homogeneous DE* (3)*. Then the conditions below are equivalent.*

(i) $x_1, ..., x_n$ are linearly independent;

(ii) $W(x_1, ..., x_n)(t) \neq 0$ for all $t \in I$;

(iii) there exists $t_0 \in I$ such that $W(x_1, ..., x_n)(t_0) \neq 0$.

Proof. We prove first the implication $(i) \Rightarrow (ii)$. Assume, by contradiction, that there exists $t_0 \in I$ such that

$$W(x_1, ..., x_n)(t_0) = 0.$$

Consider the linear algebraic system

$$\begin{pmatrix} x_1(t_0) & \dots & x_n(t_0) \\ x_1'(t_0) & \dots & x_n'(t_0) \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t_0) & \dots & x_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

whose determinant is $W(x_1, ..., x_n)(t_0) = 0$. Then, it has a non null solution $(\alpha_1, \dots \alpha_n) \in \mathbb{R}^n$. Consider now the function

$$\varphi = \alpha_1 x_1 + \dots + \alpha_n x_n.$$

According to Proposition 2.5 and the definition of $(\alpha_1, \dots \alpha_n)$, one can see that φ is (the unique, according to Theorem 2.2) solution of the IVP

$$\mathcal{L}x = 0, \quad x(t_0) = 0, \quad \dots, \quad x^{(n-1)}(t_0) = 0.$$

Then $\varphi = 0$, which assures that $x_1, ..., x_n$ are linearly dependent. This contradicts the hypothesis.

The implication $(ii) \Rightarrow (iii)$ is obvious. It remains to prove the implication $(iii) \Rightarrow (i)$.

Assume, by contradiction, that $x_1, ..., x_m$ are linearly dependent. We have that their derivatives of a certain order are linearly dependent, too. Hence, the columns of the determinant in the definition of their Wronskian are linearly dependent, which implies that $W(x_1, ..., x_m)(t) = 0$ for all $t \in I$. This, of course, contradicts the hypothesis. □

The hypothesis that the n functions are solutions of (3) is essential.

For example, let $x_1, x_2 \in C^1(\mathbb{R})$ be defined by $x_1(t) = 2t^2$, $x_2(t) = |t|t$ for $t \in \mathbb{R}$. One can check that $W(x_1, x_2)(t) = 0$ for all $t \in \mathbb{R}$. Using the definition, one can also check that x_1, x_2 are linearly independent.

In general, for linearly dependent functions $x_1, \ldots, x_n \in C^{(n-1)}(I)$ it can be easily seen that $W(x_1, \ldots, x_n)(t) = 0$ for all $t \in I$.
The fundamental theorem for linear nonhomogeneous differential equations follows.

Theorem 2.10 (The fundamental theorem for linear nonhomogeneous DEs) *Let x_h be the general solution of the linear homogeneous differential equation associated to* (1) *and let x_p be a particular solution of* (1). *Then the general solution of* (1) *is*

$$x = x_h + x_p.$$

Proof. The set of solutions of (1) coincides with the set of solutions of $\mathcal{L}x = f$. By Linear Algebra we know that the set of solutions of $\mathcal{L}x = f$ is $\ker \mathcal{L} + \{x_p\}$. With this the proof is finished. □

Making a summary of the fundamental theorems we can describe the main steps of a *method for finding the general solution* of a linear nonhomogeneous equation of the form (1), i.e.

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = f(t). \tag{5}$$

Step 1. Write the linear homogeneous differential equation associated $x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0$ and find its general solution. Denote it by x_h. For this it is sufficient to find n linearly independent solutions, denote them by $x_1, ..., x_n$. Hence,

$$x_h = c_1 x_1 + ... + c_n x_n, \quad c_1, ..., c_n \in \mathbb{R}.$$

Step 2. Find a particular solution of the linear nonhomogeneous equation (1). Denote it by x_p.

Step 3. Write the general solution of (1) as

$$x = x_h + x_p.$$

Exercise 2.11 *Find the general solution of $x' - 2tx = 0$.*

Solution. First we notice that this is a first order linear homogeneous differential equation. In order to find its general solution it is sufficient if we find a non-null solution. We notice that $x_1 = e^{t^2}$ verifies $x' = 2tx$, hence it is a nonnull solution. Then

$$x = c\, e^{t^2}, \quad c \in \mathbb{R}.$$

Exercise 2.12 *Find the general solution of $x' - x = -5$.*

Solution. First we notice that this is a first order linear nonhomogeneous differential equation. We follow the steps of the method presented above.

Step 1. The linear homogeneous differential equation associated is

$$x' - x = 0.$$

In order to find its general solution it is sufficient if we find a non-null solution. We notice that $x_1 = e^t$ verifies $x' = x$, hence it is a nonnull solution. Then

$$x_h = c\, e^t, \quad c \in \mathbb{R}.$$

Step 2. We notice that $x_p = 5$ verifies $x' - x = -5$.

Step 3. The general solution of $x' - x = -5$ is

$$x = c\, e^t + 5, \quad c \in \mathbb{R}.$$

Exercise 2.13 *Find the general solution of $x'' - x = 2$.*

Solution. First we notice that this is a second order linear nonhomogeneous differential equation. We follow the steps of the method presented above.

Step 1. The linear homogeneous differential equation associated is

$$x'' - x = 0.$$

In order to find its general solution it is sufficient if we find two linearly independent solutions. We notice that $x_1 = e^t$ and $x_2 = e^{-t}$ verifies $x'' = x$, hence they are solutions. It is easy to check that they are linearly independent. Then

$$x_h = c_1\, e^t + c_2\, e^{-t}, \quad c_1, c_2 \in \mathbb{R}.$$

Step 2. We notice that $x_p = -2$ verifies $x'' - x = 2$.

Step 3. The general solution of $x'' - x = 2$ is

$$x = c_1\, e^t + c_2\, e^{-t} - 2, \quad c_1, c_2 \in \mathbb{R}.$$

2.3 The general solution of a first order linear differential equation

[3] Take

$$x' + a(t)x = f(t) \tag{4}$$

where $a, f \in C(I)$ and write also the linear homogeneous equation associated,

$$x' + a(t)x = 0. \tag{5}$$

Let $t_0 \in I$ be fixed and denote by A a primitive of a, i.e.,

$$A(t) = \int_{t_0}^{t} a(s)ds.$$

It is not difficult to check the following result on (5).

Proposition 2.14 *We have that $x_1 = e^{-A(t)}$ is a solution of* (5)*. Hence, the general solution of this differential equation is $x = ce^{-A(t)}$, $c \in \mathbb{R}$.*

In particular, when a is a constant function, i.e, $a(t) = \lambda$ for all $t \in I$ and for a $\lambda \in \mathbb{R}$, then $x_1 = e^{-\lambda t}$ is a solution of $x' + \lambda x = 0$. Hence, the general solution of the differential equation $x' + \lambda x = 0$ is $x = ce^{-\lambda t}$, $c \in \mathbb{R}$.

Let us now deduce *qualitative* properties of the solutions of (5).

Proposition 2.15 (i) *Let $\varphi : I \to \mathbb{R}$ be a solution of* (5)*. Then either $\varphi(t) = 0$ for all $t \in I$, or $\varphi(t) \neq 0$ for all $t \in I$.*

(ii) *Assume that $a(t) \neq 0$ for all $t \in I$ and let $\varphi : I \to \mathbb{R}$ be a non-null solution of* (5)*. Then φ is strictly monotone.*

Proof. (i) In this situation the most handful way to prove this, is to use that $x = ce^{-A(t)}$, $c \in \mathbb{R}$ is the general solution of (5). Indeed, we deduce that there exists $\tilde{c} \in \mathbb{R}$ such that $\varphi(t) = \tilde{c}\, e^{-A(t)}$ for all $t \in I$. Then either $\tilde{c} = 0$ or $\tilde{c} \neq 0$. Using that the exponential function is always positive, we obtain the conclusion.

We comment that there is another proof that uses the Existence and Uniqueness Theorem. Indeed, let φ be a solution of (5) such that $\varphi(t_0) = 0$ for some $t_0 \in I$. Then φ is a solution of the IVP

$$\begin{aligned} &x' + a(t)x = 0 \\ &x(t_0) = 0. \end{aligned}$$

As one can easily see the null function is also a solution of this IVP, which, by Theorem 2.2, has a unique solution. Hence $\varphi \equiv 0$.

(ii) Since a is a continuous function on I, the hypothesis $a(t) \neq 0$ for all $t \in I$ assures that, either $a(t) > 0$ for all $t \in I$ or $a(t) < 0$ for all $t \in I$. Applying (i) we deduce that a similar result holds for φ. Hence

$$\text{either } a(t)\varphi(t) > 0 \text{ for all } t \in I \text{ or } a(t)\varphi(t) < 0 \text{ for all } t \in I. \tag{6}$$

We are interested to study the sign of φ'. Since φ is a solution of (5), we have that $\varphi'(t) = -a(t)\varphi(t)$ for all $t \in I$. Using (6) we obtain that φ' has a definite sign on I, hence φ is strictly monotone on I. □

An alternative method to find the general solution of (5) is *the separation of variables method*, which we present below.

Step 1. We notice that $x = 0$ is always a solution of (5).

Step 2. We look now for the non-null solutions of (5) which we write as $x'(t) = -a(t)x(t)$. We write this equation in the form (we "separate" the dependent variable x from the independent variable t)

$$\frac{x'(t)}{x(t)} = -a(t).$$

Step 3. We integrate the above equation, that is we look for primitives of each side of the equation. Note that a primitive for the left-hand side is $\ln|x(t)|$, while a primitive for the right-hand side is $-A(t)$. Hence we obtain

$$\ln|x(t)| = -A(t) + c, \quad c \in \mathbb{R}.$$

Step 4. We write the solution explicitly. We have $|x(t)| = e^{-A(t)+c}$, hence $x(t) = \pm e^c\, e^{-A(t)}$ for an arbitrary constant $c \in \mathbb{R}$. Now we note that $\{\pm e^c \ : \ c \in \mathbb{R}\} = \mathbb{R}^*$. Then we can write equivalently $x(t) = c\, e^{-A(t)}$ for an arbitrary constant $c \in \mathbb{R}^*$.

Step 5. The solution $x = 0$ found at *Step 1* and the family of solutions $x(t) = c\, e^{-A(t)}$, $c \in \mathbb{R}^*$ found at *Step 4* can be written together into the formula

$$x(t) = c\, e^{-A(t)}. \quad c \in \mathbb{R}.$$

Now we present **the Lagrange method**, also called *the variation of the constant method* used to find a particular solution of the first order linear nonhomogeneous differential equation (4). This consists in looking for a function $\varphi \in C^1(I)$ with the property that

$$x_p = \varphi(t)\, e^{-A(t)} \ \text{ is a solution of } (4)\ .$$

After replacing this form of x_p in (4) we obtain $\varphi'(t) = e^{A(t)} f(t)$. Hence, a function φ can be written as $\varphi(t) = \int_{t_0}^{t} e^{A(s)} f(s) ds$. Consequently we found a particular solution of (4)

$$x_p = \int_{t_0}^{t} e^{-A(t)+A(s)} f(s) ds.$$

The next result follows now applying the Fundamental Theorem for linear nonhomogeneous differential equations.

Proposition 2.16 *The general solution of the first order linear nonhomogeneous differential equation* (4) *is*

$$x(t) = c\, e^{-A(t)} + \int_{t_0}^{t} e^{-A(t)+A(s)} f(s) ds, \quad c \in \mathbb{R}.$$

We mention that, in practice, the separation of variables method and, respectively, the Lagrange method are widely used. An alternative way to solve both (5) and (4) is using the next result.

Proposition 2.17 *The function $\mu(t) = e^{A(t)}$ is an integrating factor for* (4).

Proof. We will show that, after multiplying (4) with the function $\mu(t)$ given in the statement, it is possible to integrate it, thus finding its general solution. Indeed, after multiplying (4) with $e^{A(t)}$ we obtain

$$x'(t)e^{A(t)} + x(t)\,a(t)e^{A(t)} = f(t)e^{A(t)},$$

that, further can be written as $(x(t)\,e^{A(t)})' = f(t)e^{A(t)}$. Of course, a primitive of the left-hand side is $x(t)\,e^{A(t)}$, and a primitive of the right-hand side is $\int_{t_0}^{t} e^{A(s)}f(s)ds$. We thus obtain

$$x(t)\,e^{A(t)} = \int_{t_0}^{t} e^{A(s)}f(s)ds + c, \quad c \in \mathbb{R}.$$

Writing explicitly the unknown $x(t)$ we obtain the same expression of the general solution as in Proposition 2.16. □

Exercise 2.18 *Find the general solution of the first order linear differential equation $x' - 2tx = t$ using all the methods presented above.*

Solution. First we notice that this is a first order linear nonhomogeneous differential equation with variable coefficient $a(t) = -2t$. The nonhomogeneous part is $f(t) = t$. We can take $I = \mathbb{R}$. Let $A(t) = \int_0^t a(s)ds = -t^2$.

Method 1: The integrating factor method. Then $\mu(t) = e^{A(t)} = e^{-t^2}$ is an integrating factor of the given equation. Thus, after multiplying it with e^{-t^2}, and noticing that $(x\,e^{-t^2})' = e^{-t^2}(x' - 2tx)$, the given equation can be written

$$(x\,e^{-t^2})' = te^{-t^2}.$$

Integrating with respect to t the above equation we obtain

$$x\,e^{-t^2} = -\frac{1}{2}e^{-t^2} + c, \quad c \in \mathbb{R}.$$

Hence the general solution of the given equation is

$$x = ce^{t^2} - \frac{1}{2}, \quad c \in \mathbb{R}.$$

Method 2: The separation of variables method plus the Lagrange method. We write the linear homogeneous equation associated

$$x' - 2tx = 0.$$

For this, $x = 0$ is a solution. We look for the non-null solutions by separating the variables. We have

$$\frac{dx}{x} = 2t\,dt,$$

which, after integration, gives $\ln|x| = t^2 + c$. Then $x = \pm e^c e^{t^2}$, $c \in \mathbb{R}$. Recall that $x = 0$ is another solution. Then the general solution is

$$x_h = c\,e^{t^2}, \quad c \in \mathbb{R}.$$

Here we make a parenthesis. Note that, in the case that one reminds (or simply "guess") that $e^{-A(t)} = e^{t^2}$ is a solution of $x' - 2tx = 0$, x_h can be obtained immediately by Theorem 2.7 (as in Proposition 2.14).

Now we apply the Lagrange method to find a particular solution, denoted x_p, of the given equation, i.e. we look for a function $\varphi \in C^1(\mathbb{R})$ such that

$$x_p = \varphi(t)\,e^{t^2}.$$

After replacing we obtain that $\varphi'(t) = te^{-t^2}$. Such a function is $\varphi(t) = -\frac{1}{2}e^{-t^2}$. Thus

$$x_p = -\frac{1}{2}.$$

We finally deduce that the general solution is

$$x = c\,e^{t^2} - \frac{1}{2}, \quad c \in \mathbb{R}.$$

Method 3: Direct application of Proposition 2.16. Remind that $A(t) = -t^2$, $f(t) = t$, $I = \mathbb{R}$. We can choose $t_0 = 0$. Then, by Proposition 2.16, we have that the general solution is

$$x(t) = c\,e^{-A(t)} + \int_{t_0}^{t} e^{-A(t)+A(s)} f(s)ds = c\,e^{t^2} + \int_0^t e^{t^2-s^2} s\,ds = c\,e^{t^2} - \frac{1}{2}(1 - e^{t^2}),$$

where $c \in \mathbb{R}$ is an arbitrary constant. Hence we can also write the general solution as before

$$x = d\,e^{t^2} - \frac{1}{2}, \quad d \in \mathbb{R}.$$

Exercise 2.19 *Find the general solution of the first order linear differential equation* $x' - 2tx = \frac{2}{\sqrt{\pi}}$ *in terms of the special function erf:* $\mathbb{R} \to \mathbb{R}$*, called the Gauss error function and defined by*

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds, \text{ for all } t \in \mathbb{R}.$$

.

Solution. In this case $A(t) = -t^2$, $f(t) = 2/\sqrt{\pi}$. Then $I = \mathbb{R}$ and we can choose $t_0 = 0$. by Proposition 2.16, we have that the general solution is

$$x(t) = c\, e^{-A(t)} + \int_{t_0}^t e^{-A(t)+A(s)} f(s) ds = c\, e^{t^2} + \frac{2}{\sqrt{\pi}} \int_0^t e^{t^2 - s^2}\, ds,$$

where $c \in \mathbb{R}$ is an arbitrary constant. Then, the general solution is

$$x = c\, e^{t^2} + \text{erf}(t) e^{t^2}, \quad c \in \mathbb{R}.$$

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2.4 Reduction of order in second order linear equations

[4] Consider the second order linear differential equation

$$x'' + a_1(t)x' + a_2(t)x = f(t), \tag{7}$$

where $a_1, a_2, f \in C(I)$, $I \subset \mathbb{R}$ is an open, nonempty, interval.

Case 1. Assume that $a_2(t) = 0$ for all $t \in I$. Making the change of variable $y = x'$, we arrive to the first order equation $y' + a_1(t)y = f(t)$.

Exercise 2.20 *Find the general solution of* $x'' - x' = -5$.

Solution. Making the change of variable $y = x'$, we arrive to the first order equation $y' - y = -5$, whose general solution is $y = 5 + c_1 e^t$. Since $y = x'$, we find x by integrating the expression of y, thus $x = 5t + c_1 e^t + c_2$, $c_1, c_2 \in \mathbb{R}$. □

Case 2. Assume that $f(t) = 0$ for all $t \in I$ and that $\varphi \in C^2(I)$ is a known non null solution of equation (7). Making the change of variable $x = \varphi(t)y$, we arrive to a second order equation which fits into Case 1. The justification follows. We compute $x' = \varphi'(t)y + \varphi(t)y'$, $x'' = \varphi''(t)y + 2\varphi'(t)y' + \varphi(t)y''$, and introduce in (7). Using that $\varphi''(t) + a_1(t)\varphi'(t) + a_2(t)\varphi(t) = 0$ for all $t \in I$, we arrive to

$$\varphi(t)y'' + (2\varphi'(t) + a_t(t)\varphi(t))y' = 0.$$

As indicated in Case 1, we make the change of variable $u = y'$ to arrive to the first order linear equation

$$\varphi(t)u' + (2\varphi'(t) + a_t(t)\varphi(t))u = 0.$$

Exercise 2.21 *We consider the equation* $t^2x'' + tx' - 4x = 0$ *on the interval* $(0, \infty)$ *and the function* $\varphi(t) = t^2$. *Check that the given equation has* φ *as a solution, and find its general solution.*

Solution. First note that the given equation is not in the normal form (7), thus we have to divide by t^2. Then, the coefficients of the equation are $a_1(t) = 1/t$ and $a_2(t) = -4/t^2$, which are not defined in $t = 0$. This is reason why one can not choose the whole interval $I = \mathbb{R}$ for applying the theory of this chapter to the given equation. But the interval $I = (0, \infty)$ is convenient.

It is easy to check that φ is a solution of $t^2x'' + tx' - 4x = 0$. We make the change of variable

$$x = t^2y.$$

We have $x' = 2ty + t^2y'$, $x'' = 2y + 4ty' + t^2y''$. Introducing in the given equation we obtain $2t^2y + 4t^3y' + t^4y'' + 2t^2y + t^3y' - 4t^2y = 0$ and, further, $t^4y'' + 5t^3y' = 0$, which gives

$$ty'' + 5y' = 0.$$

We make a second change of variable

$$u = y'$$

and arrive to

$$tu' + 5u = 0,$$

whose general solution is

$$u = c_1 e^{-5\ln t} = c_1/t^5, \quad c_1 \in \mathbb{R}.$$

Since $u = y'$, we find y by integrating u. Hence

$$y = -\frac{c_1}{4}t^{-4} + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

Using $x = t^2 y$ we find

$$x = k_1 t^{-2} + k_2 t^2, \quad k_1, k_2 \in \mathbb{R}.$$

2.5 Lagrange method for second order linear differential equations

[5] We assume that $x_1, x_2 \in C^2(I)$ are two known linearly independent solutions of the second order linear homogeneous equation

$$x'' + a_1(t)x' + a_2(t)x = 0.$$

Our aim is to present the Lagrange method (also called the variation of the constants method) to find a particular solution (hence, finally, the general solution) of the linear nonhomogeneous equation (7).

We start by writing the general solution of the linear homogeneous equation:

$$x_h = c_1 x_1(t) + c_2 x_2(t), \quad c_1, c_2 \in \mathbb{R}.$$

The Lagrange method consists in looking for a particular solution of (7) having the form

$$x_p = \varphi_1(t)x_1(t) + \varphi_2(t)x_2(t),$$

where the unknown functions φ_1 and φ_2 satisfy

$$\varphi_1' x_1 + \varphi_2' x_2 = 0.$$

We compute $x_p' = \varphi_1' x_1 + \varphi_1 x_1' + \varphi_2' x_2 + \varphi_2 x_2' = \varphi_1 x_1' + \varphi_2 x_2'$ and $x_p'' = \varphi_1' x_1' + \varphi_1 x_1'' + \varphi_2' x_2' + \varphi_2 x_2''$. After replacing these expressions in equation (7), using also that $x_1'' + a_1 x_1' + a_2 x_1 = 0$ and $x_2'' + a_1 x_2' + a_2 x_2 = 0$, one obtains the relation $\varphi_1' x_1' + \varphi_2' x_2' = f$. Then $\varphi_1'(t)$ and $\varphi_2'(t)$ must satisfy the system

$$\begin{aligned} \varphi_1'(t)x_1(t) + \varphi_2'(t)x_2(t) &= 0 \\ \varphi_1'(t)x_1'(t) + \varphi_2'(t)x_2'(t) &= f(t). \end{aligned}$$

Now we see that $\varphi_1'(t)$ and $\varphi_2'(t)$ are solutions of the above linear algebraic whose discriminant is $W(x_1, x_2)(t)$. Since, by hypothesis, x_1, x_2 are two linearly independent solutions of $x'' + a_1(t)x' + a_2(t)x = 0$, we have that $W(x_1, x_2)(t) \neq 0$ for all $t \in I$. Then the above algebraic system has the unique solution

$$\varphi_1' = \frac{1}{W(x_1, x_2)} \begin{vmatrix} 0 & x_2 \\ f & x_2' \end{vmatrix}, \quad \varphi_2' = \frac{1}{W(x_1, x_2)} \begin{vmatrix} x_1 & 0 \\ x_1' & f \end{vmatrix}.$$

Integrating the above expressions, one can find some functions φ_1 and φ_2, thus, finally, a particular solution $x_p = \varphi_1 x_1 + \varphi_2 x_2$.

Exercise 2.22 *Find the general solution of* $x'' + x = \cos t$.

Solution. It can be easily checked that two linearly independent solutions of $x'' + x = 0$ are $\cos t$ and $\sin t$, thus its general solution is $x_h = c_1 \cos t + c_2 \sin t$, $c_1, c_2 \in \mathbb{R}$. Now we look for the functions φ_1, φ_2 such that $x_p = \varphi_1 \cos t + \varphi_2 \sin t$ is a solution of $x'' + x = \cos t$, and $\varphi_1' \cos t + \varphi_2' \sin t = 0$. After replacing x_p in $x'' + x = \cos t$ we obtain the system

$$\begin{aligned} \varphi_1'(t) \cos t + \varphi_2'(t) \sin t &= 0 \\ -\varphi_1'(t) \sin t + \varphi_2'(t) \cos t &= \cos t. \end{aligned}$$

The discriminant of this system is 1, and its unique solution is

$$\varphi_1' = \begin{vmatrix} 0 & \sin t \\ \cos t & \cos t \end{vmatrix} = -\sin t \cos t, \quad \varphi_2' = \begin{vmatrix} \cos t & 0 \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t.$$

Then $\varphi_1(t) = \frac{1}{2}\cos^2 t$ and $\varphi_2(t) = \frac{1}{2}t + \frac{1}{4}\sin 2t$. Thus $x_p = \frac{1}{2}\cos^3 t + \frac{1}{2}t \sin t + \frac{1}{4}\sin 2t \sin t = \frac{1}{2}\cos^3 t + \frac{1}{2}\sin^2 t \cos t + \frac{1}{2}t \sin t = \frac{1}{2}\cos t + \frac{1}{2}t \sin t$. Then the general solution of $x'' + x = \cos t$ is

$$x = k_1 \cos t + k_2 \sin t + \frac{1}{2} t \sin t, \quad k_1, k_2 \in \mathbb{R}.$$

Remark 2.23 *The Lagrange method is available for linear homogeneous equations of arbitrary order.*

To describe it, we assume that $x_1, \ \ldots \ , x_n \in C^n(I)$ are n known linearly independent solutions of the nth order linear homogeneous equation

$$x^{(n)} + a_1(t)x^{(n-1)} + \ \ldots \ + a_n(t)x = 0.$$

We again start by writing the general solution of this linear homogeneous equation:

$$x_h = c_1x_1(t) + \ \ldots \ + c_nx_n(t), \quad c_1, \ldots, c_n \in \mathbb{R}.$$

The Lagrange method consists in looking for a particular solution of (4) having the form

$$x_p = \varphi_1(t)x_1(t) + \ \ldots \ + \varphi_n(t)x_n(t),$$

where the unknown functions $\varphi_1, \ldots, \varphi_n$ satisfy

$$\varphi_1'x_1 + \ldots + \varphi_n'x_n = 0, \quad \varphi_1'x_1' + \ldots + \varphi_n'x_n' = 0, \ \ldots \ , \quad \varphi_1'x_1^{(n-2)} + \ldots + \varphi_n'x_n^{(n-2)} = 0.$$

We note that, making the calculations after introducing in (4) the above expression of x_p, and using the above relations, one arrives to $\varphi_1'x_1^{(n-1)} + \ldots + \varphi_n'x_n^{(n-1)} = f(t)$. Thus, the problem is reducing to the one of finding the vector $\varphi_1'(t), \ldots, \varphi_n'(t)$ as a solution of an $n \times n$ linear algebraic system whose discriminant is $W(x_1, \ldots, x_n)(t)$. $\square$

2.6 Linear differential equations with constant coefficients

[6] In this special case there is a method, called *the characteristic equation method* to find the n linearly independent solutions of an nth order linear homogeneous equation. We will also present here *the undetermined coefficients method* to find a particular solution for such kind of equations when the nonhomogeneous part has some special forms.

We write now a linear homogeneous differential equation with constant coefficients denoted $a_1, ..., a_n \in \mathbb{R}$.

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x = 0, \tag{8}$$

and consider again the linear map $\mathcal{L}$ (defined in the beginning) corresponding to (8).

We start by noticing that, when looking for solutions of (8) of the form

$$x = e^{rt}$$

(with $r \in \mathbb{R}$ that has to be found), we obtain that r must be a root of the nth degree algebraic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \tag{9}$$

More precisely, we have that

$$\mathcal{L}(e^{rt}) = e^{rt}\, l(r),$$

where

$$l(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

Then every real root of (9) provides a solution of (8). But we know that, in general, not all the roots of an algebraic equation are real. However, we will show how the complex roots of the algebraic equation (9) provide all the n linearly independent solutions of (8) needed to obtain its general solution.

For our purpose we need to see that the concept of real-valued solution for (8) can be extended to that of complex-valued solution. Denoting a complex-valued function by $\gamma : \mathbb{R} \to \mathbb{C}$, its real part by $u : \mathbb{R} \to \mathbb{R}$, and its imaginary part by $v : \mathbb{R} \to \mathbb{R}$, we have $\gamma(t) = u(t) + i\, v(t)$, for all $t \in \mathbb{R}$. The function γ can be identified with a vector-valued function of one real variable t, and with two real components u and v. Hence properties of u and v (as, for example, continuity or differentiability) are transferred to γ and viceversa.

Since we defined a solution to be a real-valued function, we will only say that a complex-valued function *verifies* or not a differential equation. With respect to the linear homogeneous differential equation with constant real coefficients (8) we have the following result.

Proposition 2.24 *Assume that the complex-valued function $\gamma \in C^n(\mathbb{R}, \mathbb{C})$ verifies* (8)*. Then, both its real part u and its imaginary part v are solutions of* (8)*.*

Proof. In order to shorten the presentation, we use again the notation of the linear map $\mathcal{L}$ as presented in the beginning of this lecture. Thus equation (8) can be written equivalently as $\mathcal{L}x = 0$. It is not difficult to see that $\mathcal{L}(\gamma) = \mathcal{L}(u + iv) = \mathcal{L}u + i\mathcal{L}v$, where, of course, $\mathcal{L}u$ and $\mathcal{L}v$ are real-valued functions. By hypothesis we have that $\mathcal{L}(\gamma) = 0$. Thus $\mathcal{L}u = 0$ and $\mathcal{L}v = 0$, which give the conclusion. $\square$

We need to work with the complex-valued function of real variable

$$\gamma(t) = e^{(\alpha+i\beta)t}, \quad t \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$ are fixed real numbers. Using Euler's formula we know that its real and, respectively, imaginary parts are

$$u(t) = e^{\alpha\, t} \cos \beta\, t, \quad v(t) = e^{\alpha\, t} \sin \beta\, t.$$

Using that $\gamma'(t) = u'(t) + iv'(t)$, one can check that

$$\gamma'(t) = (\alpha + i\beta)e^{(\alpha+i\beta)t}, \quad t \in \mathbb{R}.$$

This last formula tells us that the derivatives of the function e^{rt}, where $r \in \mathbb{C}$ is fixed, are computed using the same rules as when $r \in \mathbb{R}$. Hence in the case that $r = \alpha + i\beta$ is a root of (9), the complex-valued function e^{rt} verifies (8). We thus have

Proposition 2.25 *If $r = \alpha + i\beta$ with $\beta \neq 0$, is a root of* (9)*, then $e^{\alpha\, t} \cos \beta\, t$ and $e^{\alpha\, t} \sin \beta\, t$ are solutions of* (8)*.*

We notice that, since the polynomial $l(r)$ has real coefficients, in the case that $r = \alpha + i\beta$ with $\beta \neq 0$ is a root of l, we have that its conjugate, $r = \alpha - i\beta$ is a root, too. According to the previous proposition, this gives that $e^{\alpha\, t} \cos \beta\, t$ and $-e^{\alpha\, t} \sin \beta\, t$ are solutions of (8). But this is no new information. In fact, it is usually said that the two solutions indicated in the proposition comes from the two roots $\alpha \pm i\beta$.

Hence we have seen that any complex root provides a solution of (8). But still there is the possibility that the solutions obtained are not enough, since we know by the Fundamental Theorem of Algebra that a polynomial of degree n has indeed n roots, but counted with their multiplicity. We will show that

Proposition 2.26 *If $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l, then $t^k\, e^{rt}$ verifies* (8) *for any $k \in \{0, 1, 2, ..., m-1\}$.*

Proof. We recall first that $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l if and only if

$$l(r) = l'(r) = ... = l^{(m-1)}(r) = 0.$$

By direct calculations we obtain for each $k \in \{0, 1, 2, ..., m-1\}$ that

$$\mathcal{L}(t^k\, e^{rt}) = e^{rt} \sum_{j=0}^{k} C_k^j\, t^{k-j}\, l^{(j)}(r),$$

which, in the case that $r \in \mathbb{C}$ is a root of multiplicity m gives that $\mathcal{L}(t^k\, e^{rt}) = 0$. □

We describe now **The characteristic equation method** for the linear homogeneous differential equation with constant coefficients (8).

Step 1. Write the *characteristic equation* (9). Note that it is an algebraic equation of degree n (equal to the order of the differential equation) and with the same coefficients as the differential equation.

Step 2. Find all the n roots in $\mathbb{C}$ of (9), counted with their multiplicity.

Step 3. Associate n functions obeying the following rules.

For $r = \alpha$ a real root of order m we take m functions:

$$e^{\alpha t},\ te^{\alpha t}, \dots,\ t^{m-1}e^{\alpha t}.$$

For $r = \alpha + i\beta$ and $r = \alpha - i\beta$ roots of order m we take $2m$ functions

$$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t, \dots,\ t^{m-1}e^{\alpha t}\cos\beta t,\ t^{m-1}e^{\alpha t}\sin\beta t.$$

The following useful result holds true.

Theorem 2.27 *The n functions found by applying the characteristic equation method are n linearly independent solutions of* (8).

In the discussion before the presentation of this method we proved that the n functions are solutions of (8). The proof of the above theorem would be completed by showing that they are linearly independent. But this is beyond the aim of these lectures.

We present now **the undetermined coefficients method** to find a particular solution for a linear nonhomogeneous differential equation

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = f(t), \tag{10}$$

with constant coefficients $a_1, ..., a_n$ and with $f \in C(\mathbb{R})$ of special form. Denote again the characteristic polynomial $l(r) = r^n + a_1 r^{(n-1)} + ... + a_n$.

We consider those functions f which can be solutions to some linear homogeneous differential equation with constant coefficients. More exactly, the function f can be either of the form $P_k(t)e^{\alpha t}$ or $P_k(t)e^{\alpha t}\cos\beta t + \tilde{P}_k(t)e^{\alpha t}\sin\beta t$, where $P_k(t)$ and $\tilde{P}_k(t)$ denote some polynomials in t of degree at most k. Roughly speaking, the idea behind this method is that (10) has a particular solution of the same form as $f(t)$. The following rules apply.

Assume that $f(t) = P_k(t)e^{\alpha t}$.

In the case that $r = \alpha$ is not a root of the characteristic polynomial $l(r)$, then $x_p = Q_k(t)e^{\alpha t}$, for a polynomial $Q_k(t)$ of degree at most k whose coefficients have to be determined.

In the case that $r = \alpha$ is a root of multiplicity m of the characteristic polynomial $l(r)$, then $x_p = t^m Q_k(t)e^{\alpha t}$, for a polynomial $Q_k(t)$ of degree at most k whose coefficients have to be determined.

Assume now that $f(t) = P_k(t)e^{\alpha t}\cos\beta t + \tilde{P}_k(t)e^{\alpha t}\sin\beta t$.

In the case that $r = \alpha + i\beta$ is not a root of the characteristic polynomial $l(r)$, then $x_p = Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t$, for some polynomials $Q_k(t)$ and $\tilde{Q}_k(t)$ of degree at most k whose coefficients have to be determined.

In the case that $r = \alpha + i\beta$ is a root of multiplicity m of the characteristic polynomial $l(r)$, then $x_p = t^m[Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t]$, for some polynomials $Q_k(t)$ and $\tilde{Q}_k(t)$ of degree at most k whose coefficients have to be determined.

We present now some examples to understand the rules of the *undetermined coefficients method*. For simplicity, we take equations with the same homogeneous part. This will be $\mathcal{L}x = x'' - 4x$, whose characteristic polynomial $l(r) = r^2 - 4$ has the real simple roots $r_1 = -2$ and $r_2 = 2$.

1) For $x'' - 4x = 1$ we have $f(t) = 1$, which is a polynomial of degree 0. We have to check whether $r = 0$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = a$, where $a \in \mathbb{R}$ has to be determined.

2) For $x'' - 4x = 2t^2$ we have $f(t) = 2t^2$, which is a polynomial of degree 2. We have to check whether $r = 0$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = at^2 + bt + c$, where $a, b, c \in \mathbb{R}$ have to be determined.

3) For $x'' - 4x = -5e^{3t}$ we have $f(t) = -5e^{3t}$. We have to check whether $r = 3$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = ae^{3t}$, where $a \in \mathbb{R}$

has to be determined.

4) For $x'' - 4x = -5te^{3t}$ we have $f(t) = -5te^{3t}$. We have to check whether $r = 3$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = (at + b)e^{3t}$, where $a, b \in \mathbb{R}$ have to be determined.

5) For $x'' - 4x = -5e^{2t}$ we have $f(t) = -5e^{2t}$. We have to check whether $r = 2$ is a root of $l(r)$. It is a simple root. Then we look for $x_p = ate^{2t}$, where $a \in \mathbb{R}$ has to be determined.

6) For $x'' - 4x = -5\sin 2t$ we have $f(t) = -5\sin 2t$. We have to check whether $r = 2i$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = a\sin 2t + b\cos 2t$, where $a, b \in \mathbb{R}$ have to be determined.

2.7 A real-world application: the Spring-Mass System.

We have a mass sliding on an horizontal wall and attached to a vertical wall by an elastic spring. The position of the mass when the spring is relaxed is called *the equilibrium position* and, on a real axe Ox we identify it with the origin O. When it is not in the equilibrium position, we denote by $x(t)$ the distance from the equilibrium at time t. Using Newton's second law and Hooke's law from Physics, one can find that the function $t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}$ satisfies the differential equation

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = f(t). \tag{11}$$

Here $m > 0$ is the mass, $\nu > 0$ is the friction (also called damping) coefficient, $k > 0$ is the elasticity constant (also called stiffness) of the spring, and $f(t)$ is the measure of an external force. Recall that $x'(t)$ is the instantaneous velocity, $x''(t)$ is the instantaneous acceleration. The forces that produce the movement of the mass are *the restoring force* of the spring, given by the Hooke's law,

$$F_r = -kx(t), \text{ for a constant } k > 0,$$

the damping force

$$F_f = -\nu x'(t), \text{ for a constant } \nu \geq 0,$$

and *an external force*

$$F_e = f(t), \text{ where } f \in C(\mathbb{R}).$$

Thus, the total force is

$$F = -kx(t) - \nu x'(t) + f(t).$$

Recall the Newton's second law

$$F = m\, x''(t).$$

It is easy to see that the last two relations give the differential equation (11).

As we know, this mathematical model is a second order linear differential equation with constant coefficients. We will study the following three cases:

Case 1. Undamped motion without external force:

$$x'' + \frac{k}{m}x = 0.$$

Case 2. Damped motion without external force:

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = 0.$$

Case 3. Undamped motion with an external force:

$$x'' + \frac{k}{m}x = A\cos\omega t.$$

We consider now each case. We will find the general solution using mainly the characteristic equation method.

Case 1. Undamped motion without external force:

$$x'' + \frac{k}{m}x = 0. \tag{12}$$

The general solution of (12),

$$x = c_1\cos\omega_0 t + c_2\sin\omega_0 t, \quad c_1, c_2 \in \mathbb{R},$$

where we used the notation

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Each of these non-null functions is periodic with minimal period $T = 2\pi/\omega_0$. Moreover, note that for $A_0 \geq 0$ and $\varphi_0 \in [0, 2\pi)$ such that $c_1 = A_0\cos\varphi_0$ and $c_2 = A_0\sin\varphi_0$, the general solution can be written equivalently as

$$x(t) = A_0\cos(\omega_0 t - \varphi_0), \quad A_0 \geq 0,\ \varphi_0 \in [0, 2\pi).$$

Hence *the mass oscillates with constant amplitude* A_0 *around the equilibrium position.*

Case 2. Damped motion without external force ($\gamma > 0$):

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = 0. \tag{13}$$

Using the characteristic equation method one can see that we have to consider three subcases, which correspond to the sign of the discriminant of the characteristic equation.

Case 2.1. The motion is underdamped: $\nu < \sqrt{4mk}$.
In this case the general solution is

$$x = e^{-\frac{\nu}{2m}t}(c_1 \cos\tilde{\omega}_0 t + c_2 \sin\tilde{\omega}_0 t), \quad c_1, c_2 \in \mathbb{R},$$

or, equivalently,

$$x(t) = A_0 e^{-\frac{\nu}{2m}t}\cos(\tilde{\omega}_0 t - \varphi_0), \quad A_0 \geq 0, \ \varphi_0 \in [0, 2\pi),$$

where we have used the notation

$$\tilde{\omega}_0 = \frac{\sqrt{-\nu^2 + 4mk}}{2m}.$$

Hence *the mass oscillates around the equilibrium position, but the amplitude* $A_0 e^{-\frac{\nu}{2m}t}$ *decreases exponentially to* 0.

Case 2.2. The motion is critically damped: $\nu = \sqrt{4mk}$.
In this case the general solution is

$$x = e^{-\frac{\nu}{2m}t}(c_1 + c_2 t), \quad c_1, c_2 \in \mathbb{R}.$$

Hence *there are no oscillations and the mass goes rapidly to the equilibrium position.*

Case 2.3. The motion is overdamped: $\nu > \sqrt{4mk}$.
In this case the general solution is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad c_1, c_2 \in \mathbb{R},$$

where

$$r_1 = \frac{-\nu + \sqrt{\nu^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-\nu - \sqrt{\nu^2 - 4mk}}{2m}.$$

Note that $r_1 < 0$ and $r_2 < 0$.
Hence *there are no oscillations and the mass goes rapidly to the equilibrium position.*

Case 3. Undamped motion with an external force:

$$x'' + \frac{k}{m}x = A\cos\omega t. \tag{14}$$

This is a linear nonhomogeneous differential equation whose homogeneous part is like in Case 1. Then

$$x_h = c_1\cos\omega_0 t + c_2\sin\omega_0 t, \quad c_1, c_2 \in \mathbb{R},$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

In order to write a particular solution we need to consider two subcases.

Case 3.1: $\omega \neq \omega_0$. A particular solution is

$$x_p = -\frac{A}{\omega^2 - \omega_0^2}\cos\omega t,$$

then the general solution is

$$x = c_1\cos\omega_0 t + c_2\sin\omega_0 t - \frac{A}{\omega^2 - \omega_0^2}\cos\omega t, \quad c_1, c_2 \in \mathbb{R}.$$

In this case *oscillations occur with bounded amplitude, but the motion is periodic if and only if ω and ω_0 are commensurable, i.e., $\omega/\omega_0 \in \mathbb{Q}$.* However, in practice, since the oscillations are bounded in any case, the periodic motion can not be distinguished between the non-periodic one, especially when the period is very big. Other interesting phenomena appear in this case, like oscillations with modulated amplitude, also called *beats*. It is not our aim to treat this.

Case 3.2: $\omega = \omega_0$. A particular solution is

$$x_p = \frac{1}{2\omega_0}t\sin\omega_0 t,$$

then the general solution is

$$x = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{1}{2\omega_0}t\sin\omega_0 t, \quad c_1, c_2 \in \mathbb{R}.$$

Any function of the above form is unbounded. In this case *oscillations occur with an amplitude that increases to ∞.* This phenomenon is called *resonance.*

Chapter 3

Linear differential equations in $\mathbb{R}^n$

Let $n \in \mathbb{N}$, $n \geq 1$. First we note that a (first order) linear differential equation in $\mathbb{R}^n$ is also called a linear differential system of dimension n. We saw in Chapter 1 that any nth order scalar differential equation can be written equivalently as a first order differential system of dimension n. The linearity is preserved by this change. Then, the theory presented in the previous chapter can be obtained as a particular case of the theory presented in this chapter.

3.1 The existence and uniqueness theorem

We consider the class of n-dimensional linear differential systems

$$
(1) \qquad \begin{array}{l}
x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + \ldots a_{1n}(t)\, x_n + f_1(t) \\
x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + \ldots a_{2n}(t)\, x_n + f_2(t) \\
\vdots \\
x_n' = a_{n1}(t)\, x_1 + a_{n2}(t)\, x_2 + \ldots a_{nn}(t)\, x_n + f_n(t),
\end{array}
$$

where $a_{ij}, f_i \in C(I, \mathbb{R})$, for all $i, j \in \{1, ..., n\}$, and $I \subset \mathbb{R}$ is an open nonempty interval. The function $a_{ij}(t)$ is said to be the *coefficient* of the unknown x_j in the i-th equation, while the function $f_i(t)$ is said to be the *force* or the *non-homogeneous part* in the i-th equation.

With the notations

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & & & \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \dots \\ f_n(t) \end{pmatrix}, \quad X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}$$

system (1) can be equivalently written in vectorial form as

$$X' = A(t)\,X + f(t). \tag{2}$$

Note that $A \in C(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ is the matrix of the coefficients, and $f \in C(I, \mathbb{R}^n)$ is the nonhomogeneous part. When $f \equiv 0$, we say that system (2) is linear homogeneous, otherwise we say that system (2) is linear nonhomogeneous. When A is a constant matrix function, we say that (2) is a linear system with constant coefficients.

Definition 3.1 *A solution of system* (2) *is a function* $\phi \in C^1(I, \mathbb{R}^n)$ *that satisfies* $\phi'(t) = A(t)\,\phi(t) + f(t)$ *for all* $t \in I$.

An important result is the existence and uniqueness theorem for such systems. Its proof is postponed by the end of the semester.

Theorem 3.2 *Let* $t_0 \in I$ *and* $\eta \in \mathbb{R}^n$. *The IVP*

$$X' = A(t)\,X + f(t), \quad X(t_0) = \eta \tag{3}$$

has a unique solution denoted $\phi^* \in C^1(I, \mathbb{R}^n)$.

Now we consider the class of n-th order scalar linear differential equations

$$x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \dots + a_{n-1}(t)x' + a_n(t)x = f(t), \tag{4}$$

where $a_1, ..., a_n, f \in C(I)$. Associated to this equation we write the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_n' &= -a_n(t)\,x_1 - a_{n-1}(t)\,x_2 - \dots - a_1(t)\,x_n + f(t). \end{aligned} \tag{5}$$

Proposition 3.3 *Let $\varphi \in C^n(I)$. We have that φ is a solution of the n-th order equation* (4) *if and only if $(\varphi, \varphi', \dots, \varphi^{(n-1)}) \in C^1(I, \mathbb{R}^n)$ is a solution of the n-dimensional system* (5).

As consequence of Theorem 3.2 we obtain

Theorem 3.4 *Let $t_0 \in I$ and $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$ be given. We have that the IVP*

$$(6) \quad x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = f(t), \quad x(t_0) = \eta_1, \quad \dots , \quad x^{(n-1)}(t_0) = \eta_n.$$

has a unique solution denoted $\varphi \in C^n(I)$.

3.2 The fundamental theorems for linear differential systems

We associate to system (2) the following map

$$\mathcal{L} : C^1(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n), \quad \mathcal{L}(X)(t) = X'(t) - A(t)\,X(t), \quad t \in I.$$

We leave as an exercise to prove that $\mathcal{L}$ is a linear map between the two (real) vector spaces. An important remark is that the kernel of this linear map, $\operatorname{Ker}\mathcal{L}$ *is the set of solutions of the linear homogeneous system $X' = A(t)\,X$*, and $\operatorname{Ker}\mathcal{L} + \{X_p\}$ *is the set of solutions of the linear non-homogeneous system $X' = A(t)\,X + f(t)$*, where X_p is one of its solutions. In the Linear Algebra language, the map $\mathcal{L} - f$ (when $\mathcal{L}$ is a linear map) is said to be an *affine map*. That is why the system $X' = A(t)\,X + f(t)$ is sometimes called *affine*.

We denote the linear homogeneous system

$$(7) \qquad X' = A(t)\,X.$$

Definition 3.5 *A matrix solution of system $X' = A(t)\,X$ is a function $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ that satisfies $U'(t) = A(t)\,U(t)$ for all $t \in I$.*

Remark 3.6 *$U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ is a matrix solution of system $X' = A(t)\,X$ if and only if its columns are solutions of this system.*

Definition 3.7 *A fundamental matrix solution of system $X' = A(t)\,X$ is a matrix solution $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ whose columns are linearly independent.*

Proposition 3.8 *Let $t_0 \in I$ be fixed. For any matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ there exists a unique matrix solution $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ of system (7) satisfying $U(t_0) = M$. If, in addition, M is an invertible matrix (i.e. $\det M \neq 0$), then U is a fundamental matrix solution.*

Proof. For each $i = \overline{1,n}$ let us denote by $m_i \in \mathbb{R}^n$ the i-th column of the matrix M. According to The Existence and Uniqueness Theorem, there exists a unique solution, denoted $u_i \in C^1(I, \mathbb{R}^n)$, of system (7) satisfying $u_i(t_0) = m_i$. In conclusion, there exists a unique matrix solution U, whose columns are u_1, ..., u_n, satisfying $U(t_0) = M$.

Assume now that $\det M \neq 0$ and assume, by contradiction, that U is not a fundamental matrix solution, i.e. its columns are linearly dependent. Thus there exists a non-null vector $\xi \in \mathbb{R}^n$ such that $U(t)\xi = 0$ for all $t \in I$. This implies that $\det U(t) = 0$ for all $t \in I$. Thus, in particular, we have $\det U(t_0) = \det M = 0$, which contradicts the hypothesis. □

Theorem 3.9 *Let $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ be a matrix solution of system (7). We have that U is a fundamental matrix solution if and only if $\det U(t) \neq 0$ for all $t \in I$.*

Proof. First assume that U is a fundamental matrix solution. Denote by $u_1, \dots u_n \in C^1(I, \mathbb{R}^n)$ the columns of U, which, by definition, are linearly independent solutions of system (7). Assume, by contradiction, that there exists $t_0 \in I$ such that $\det U(t_0) = 0$. Then there exists $\xi \in \mathbb{R}^n$, $\xi \neq 0$ such that $U(t_0)\,\xi = 0$. Let $\varphi : I \to \mathbb{R}^n$ be defined by $\varphi(t) = U(t)\,\xi$. It is not difficult to see that φ is a solution of the IVP

$$X' = A(t)\,X, \quad X(t_0) = 0,$$

which, of course, has also the null solution. The Existence and Uniqueness Theorem assures that $\varphi(t) = 0$ for all $t \in I$, i.e. $U(t)\,\xi = 0$ for all $t \in I$. Since $\xi \neq 0$, this implies that the columns of U are linearly dependent functions. This contradicts the hypotheses that U is a fundamental matrix solution. Thus $\det U(t) \neq 0$ for all $t \in I$.

The reversed implication follows by the previous Proposition. □

Proposition 3.10 *Let U be a fundamental matrix solution of* (7)*. Then*

(i) $V(t) = U(t)\,M$ *is another fundamental matrix solution, when $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is an invertible (constant) matrix.*

(ii) $M(t) = [U(t)]^{-1}V(t)$ *is a constant matrix, when V is another fundamental matrix solution.*

Proof. (i) Note that $V'(t) = U'(t)\,M = A(t)\,U(t)\,M = A(t)\,V(t)$ for all $t \in I$. Thus $V(t)$ is a matrix solution of (7). We have that, for any $t \in I$, $\det V(t) \neq 0$ since $V(t)$ is a product of two invertible matrices.

(ii) Taking the derivative of $V(t) = U(t)\,M(t)$ we obtain $V'(t) = U'(t)\,M(t) + U(t)\,M'(t) = A(t)\,U(t)\,M(t) + U(t)\,M'(t) = A(t)\,V(t) + U(t)\,M'(t)$. Since $V'(t) = A(t)\,V(t)$ we have $U(t)\,M'(t) = 0$ for all $t \in I$. Since $U(t)$ is invertible, we have $M'(t) = 0$ for all $t \in I$, hence M is constant on the interval I. □

Definition 3.11 *Let $t_0 \in I$ be fixed. The principal matrix solution at t_0 of system* (7) *is the unique matrix solution E satisfying $E(t_0) = I_n$.*

Remark 3.12 *Given a fundamental matrix solution $U(t)$, the principal matrix solution at t_0 is $E(t) = U(t)[U(t_0)]^{-1}$.*

Theorem 3.13 (The fundamental theorem for linear homogeneous systems) *The set of solutions of system* (7) *is a linear space of dimension n. Consequently, given n linearly independent solutions u_1, ..., u_n, its general solution can be described as*

$$X = c_1 u_1 + \ldots c_n u_n, \quad c_1, \ldots, c_n \in \mathbb{R},$$

or, given U a fundamental matrix solution, its general solution can be described as

$$X = U(t)C, \quad C \in \mathbb{R}^n.$$

Proof. It is sufficient to prove that the vector space $\operatorname{Ker}\mathcal{L}$ is isomorphic to $\mathbb{R}^n$. Fix $t_0 \in I$ and define

$$\Phi : \operatorname{Ker}\mathcal{L} \to \mathbb{R}^n, \quad \Phi(X) = X(t_0).$$

It is not difficult to see that Φ is a linear map and, by Theorem 3.2, Φ is bijective. □

Theorem 3.14 (The fundamental theorem for linear nonhomogeneous systems) *Let $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ be a fundamental matrix solution of (7). The general solution of system (2) is*

$$X = U(t)\, C + \int_{t_0}^{t} U(t)[U(s)]^{-1} f(s) ds, \quad C \in \mathbb{R}^n.$$

Let $t_0 \in I$ and $\eta \in \mathbb{R}^n$. The unique solution of the IVP

$$X' = A(t)\, X + f(t), \quad X(t_0) = \eta$$

is

$$X(t) = U(t)[U(t_0)]^{-1}\, \eta + \int_{t_0}^{t} U(t)[U(s)]^{-1} f(s) ds.$$

Proof. We present the variation of constants method to find a solution of (2). It consists in looking for a function $\varphi \in C^1(I, \mathbb{R}^n)$ such that $X = U(t)\, \varphi(t)$ to be a solution of (2), i.e. to satisfy $U'\, \varphi + U\, \varphi' = A\, U\, \varphi + f$. Using that $U' = A\, U$ we thus have $\varphi' = U^{-1} f$. Then, for a $t_0 \in I$, we can write

$$\varphi(t) = \int_{t_0}^{t} [U(s)]^{-1} f(s) ds, \quad t \in I$$

hence a particular solution of (2) is

$$X_p(t) = U(t) \int_{t_0}^{t} [U(s)]^{-1} f(s) ds.$$

The rest is left as an exercise. □

3.3 More on the fundamental matrix solution

As a consequence of Theorem 3.9, we have that, for a matrix solution U, either $\det U(t) \neq 0$ for all $t \in I$ or $\det U(t) = 0$ for all $t \in I$. This can also be seen as a consequence of the next result due to the French mathematician Joseph Liouville (1809-1882).

Theorem 3.15 (Liouville) *Let $t_0 \in I$ and $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ be a matrix solution of system (7). Then*

$$\det U(t) = \det U(t_0)\, \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s) ds \right) \quad \textit{for all } \; t \in I.$$

Proof. Denote by

$$W(t) = \det U(t) = \begin{vmatrix} u_{11}(t) & u_{12}(t) & \dots & u_{1n}(t) \\ u_{21}(t) & u_{22}(t) & \dots & u_{2n}(t) \\ \dots & & & \\ u_{n1}(t) & u_{n2}(t) & \dots & u_{nn}(t) \end{vmatrix} \quad \text{for all } t \in I.$$

It is known that $W \in C^1(I, \mathbb{R})$ and its derivative can be computed as

$$W'(t) = \sum_{i=1}^{n} \begin{vmatrix} u_{11}(t) & u_{12}(t) & \dots & u_{1n}(t) \\ \dots & & & \\ u'_{i1}(t) & u'_{i2}(t) & \dots & u'_{in}(t) \\ \dots & & & \\ u_{n1}(t) & u_{n2}(t) & \dots & u_{nn}(t) \end{vmatrix}.$$

Remind that, for each k, $u_{ik}(t)$ is the i-th component of a solution of system (7), thus it satisfies the i-th equation of this system, $u'_{ik}(t) = \sum_{j=1}^{n} a_{ij}(t)\, u_{jk}$. Then $u'_{ik}(t) = a_{ii}(t)\, u_{ik}(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)\, u_{jk}$. Known properties of the determinants show that the determinant corresponding to i in the previous sum is $a_{ii}(t)\, W(t)$, thus

$$W'(t) = (\operatorname{tr} A(t))\; W(t)$$

since $\operatorname{tr} A(t) = \sum_{i=1}^{n} a_{ii}(t)$ for all $t \in I$. It is easy to deduce now the conclusion. □

3.4 Linear homogeneous systems with constant coefficients

Let $A \in \mathcal{M}_n(\mathbb{R})$. The aim of this section is to study systems of the form

$$X' = AX.$$

In this section, the principal matrix solution will be the principal matrix solution at $t_0 = 0$ and will be denoted by $E(t)$.

In the first subsection we will define the exponential function whose variable is a matrix and we will show that e^{At} is the principal matrix solution. In this way, the problem of finding the general solution of the system is replaced by the problem to compute the matrix exponential e^{At}, using mainly its definition. In general this is a very difficult problem that, for solving it, one needs Advanced Linear Algebra. Here we will present only few examples and ideas.

In the second subsection we will show how the eigenvalues and the eigenvectors of A can be used to construct solutions of the system. Recall that one needs n linearly independent solutions in order to find the general solution of the system. Again, the results presented here are not sufficient to find the n linearly independent solutions for any matrix A.

In the third subsection we consider the particular case when the matrix A is diagonalizable over $\mathbb{R}$ and we present several methods to find the general solution of any such system.

In the fourth subsection we consider the particular case $n = 2$ and we present several methods to find the general solution of any such system (which is called *planar*).

3.4.1 The exponential of a matrix

Let $A \in \mathcal{M}_n(\mathbb{R})$. The topology of $\mathcal{M}_n(\mathbb{R})$ is the topology of $\mathbb{R}^{n^2}$, thus the convergence of a sequence or a series of matrices is the componentwise convergence.

In the case that the series from the right-hand side is convergent, we define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{8}$$

Example 3.16 *Let $t \in \mathbb{R}$. We have that $e^{O_2} = I_2$; $e^{I_2 t} = I_2 e^t$; if $A^2 = O_2$ then $e^A = I_2 + A$;*

$$e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}; \quad e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

We will prove that the series from the right-hand side of (8) is convergent for any $A \in \mathcal{M}_n(\mathbb{R})$.

For each $\eta \in \mathbb{R}^n$ we denote by $\varphi(\cdot, \eta) : \mathbb{R} \to \mathbb{R}^n$ the unique solution of the IVP $\dot{X} = AX$, $X(0) = \eta$. We have the result whose proof will be postponed by the end of the semester.

Theorem 3.17 *Let $\eta \in \mathbb{R}^n$. Consider the sequence $(\phi_n(\cdot,\eta))_{n\geq 0}$ from $C^1(\mathbb{R},\mathbb{R}^n)$, defined by*

$$\phi_0(t,\eta)=\eta, \quad \phi_{k+1}(t,\eta)=\eta+A\int_{t_0}^{t}\phi_k(s,\eta)ds, \quad t\in\mathbb{R}, \quad k\geq 0.$$

Then, for each $t\in\mathbb{R}$, we have $\phi_k(t,\eta)\to\varphi(t,\eta)$ as $k\to\infty$.

The sequence that appears in the previous theorem is called Picard sequence (or the sequence of successive approximations) for this IVP. One can find by induction the expression of ϕ_k, which is given in the next result.

Proposition 3.18 *For $k\in\mathbb{N}$,*

$$\phi_k(t,\eta)=[I_2+At+\frac{1}{2!}(At)^2+\cdots+\frac{1}{k!}(At)^k]\eta, \quad t\in\mathbb{R}, \quad k\geq 0.$$

Theorem 3.19 *We have.*

(i) The series $\sum_{k=0}^{\infty}\frac{1}{k!}(At)^k$ is convergent for any $t\in\mathbb{R}$. In particular, the series $\sum_{k=0}^{\infty}\frac{1}{k!}A^k$ is convergent.

(ii) $\frac{d}{dt}(e^{At})=Ae^{At}$ and $e^{At}|_{t=0}=I_n$, i.e. e^{At} is the principal matrix solution of $X'=AX$.

Proof. We denote by $(U_k)_{k\geq 0}$ the sequence of partial sums of the given series, i.e. $U_k:\mathbb{R}\to\mathcal{M}_n(\mathbb{R})$, $U_k(t)=I_n+At+\frac{1}{2!}(At)^2+\cdots+\frac{1}{k!}(At)^k$. Writing $U_k=U_kI_n$ we notice that the i-th column of $U_k(t)$ is $\phi_k(t,e_i)$, where, as usual, e_i is the i-th column of the identity matrix I_n. By Theorem 3.17 and Proposition 3.18 we have $\phi_k(t,e_i)\to\varphi(t,e_i)$ as $k\to\infty$. Then the sequence of matrices $(U_k(t))_{k\geq 0}$ is convergent for any $t\in\mathbb{R}$.

By definition (8), for any $t\in\mathbb{R}$, $U_k(t)\to e^{At}$ as $k\to\infty$. Then the i-th column of e^{At} is $\varphi(t,e_i)$. From the definition of φ we deduce that e^{At} is the principal matrix solution of system $X'=AX$. □

Remark 3.20 *The unique solution of the IVP*

$$X'=AX, \quad X(0)=\eta$$

is

$$\varphi(t,\eta)=e^{At}\eta.$$

Lemma 3.21 *Let $A, B \in \mathcal{M}_n(\mathbb{R})$ with $AB = BA$. Then $e^{At}B = Be^{At}$, $e^{(A+B)t} = e^{At}e^{Bt}$ and $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.*

Proof. First we prove that $e^{At}B = Be^{At}$ for all $t \in \mathbb{R}$. Let us denote $U(t) = e^{At}B$ and $V(t) = Be^{At}$. By Theorem 3.19 we have $\dot{U} = AU$ and $U(0) = B$. Using that A and B commutes, we infer that $\dot{V} = AV$ and $V(0) = B$. The existence and uniqueness theorem assures that $U(t) = V(t)$.

Now we prove that $e^{(A+B)t} = e^{At}e^{Bt}$ for all $t \in \mathbb{R}$. Again we denote $U(t) = e^{(A+B)t}$ and $V(t) = e^{At}e^{Bt}$. By Theorem 3.19 we have $\dot{U} = (A+B)U$, $U(0) = I_2$. We intend to show that V satisfies also these relations. In order to compute the derivative of V we will use the rule: The derivative of a product of matrix functions is computed by the rule of derivation of the product of scalar functions. We will also use that $e^{At}B = Be^{At}$. We have $\dot{V} = Ae^{At}e^{Bt} + e^{At}Be^{Bt} = (A+B)V$. It is easy to see that $V(0) = I_2$. Again, the existence and uniqueness theorem assures that $U(t) = V(t)$.

Now we prove that $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$. In the relation $e^{(A+B)t} = e^{At}e^{Bt}$ we take $B = -A$ and deduce that $I_n = e^{At}e^{-At} = e^{-At}e^{At}$. Hence e^{-At} is the matrix inverse of e^{At}. □

Example 3.22 *Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Then we have that $AB \neq BA$ and $e^{(A+B)t} \neq e^{At}e^{Bt}$.*

Proof. It is to compute and to obtain that $A^2 = B^2 = O_2$. Then $e^{At} = I_2 + At$, $e^{Bt} = I_2 + Bt$, and

$$e^{At}e^{Bt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} = \begin{pmatrix} 1-t^2 & t \\ -t & 1 \end{pmatrix}.$$

On the other hand, we have $A + B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$e^{(A+B)t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Example 3.23 *Let $\lambda \in \mathbb{R}$ be fixed and $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Compute e^{At}.*

Proof. First we write $A = B + C$, where $B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We notice that $BC = CB$, thus $e^{At} = e^{Bt}e^{Ct}$. We know that $e^{Bt} = I_2 e^{\lambda t}$, and, since $C^2 = O_2$, $e^{Ct} = I_2 + Ct$. Then

$$e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Example 3.24 *Let $\alpha, \beta \in \mathbb{R}$ be fixed and $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Compute e^{At}.*

Proof. First we write $A = B + C$, where $B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and $C = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$. We notice that $BC = CB$, thus $e^{At} = e^{Bt}e^{Ct}$. We know that $e^{Bt} = I_2 e^{\alpha t}$, and $e^{Ct} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \beta t}$. Then

$$e^{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} t} = \begin{pmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ -e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix}.$$

Lemma 3.25 *Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be such that there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ with $A = PBP^{-1}$. Then $e^{At} = Pe^{Bt}P^{-1}$ for all $t \in \mathbb{R}$.*

Proof. We denote $U(t) = e^{At}P$ and $V(t) = Pe^{Bt}$. We have $\dot{U}(t) = AU(t)$, $U(0) = P$ and $\dot{V}(t) = PBe^{Bt} = APe^{Bt} = AV(t)$, $V(0) = P$. By the existence and uniqueness theorem we deduce that $U = V$. □

3.4.2 Construction of solutions using the eigenvalues and the eigenvectors of the system's matrix

First we recall some notions from Linear Algebra. We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of the matrix A if there exists $v \in \mathbb{C}^n$, $v \neq 0$ such that $Av = \lambda v$. The vector v is said to be an *eigenvector* of the matrix A corresponding to the eigenvalue λ.

Note that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A if and only if it is a root of the (characteristic) polynomial of degree n,

$$p(\lambda) = \det(A - \lambda I_n).$$

Note that the polynomial equation

$$\det(A - \lambda I_n) = 0, \tag{9}$$

is also called *the characteristic equation* of the linear system $X' = AX$.

The *algebraic multiplicity* of the eigenvalue $\lambda \in \mathbb{C}$ of A is its order as a root of the characteristic polynomial $p(\lambda)$.

The *geometric multiplicity* of the eigenvalue $\lambda \in \mathbb{C}$ of A is the dimension of the kernel of the matrix $(A - \lambda I_n)$. Thus, the geometric multiplicity of the eigenvalue λ is the maximum number of linearly independent eigenvectors corresponding to λ.

A *generalized eigenvector* of the eigenvalue $\lambda \in \mathbb{C}$ of A is any vector that belongs to the kernel of $(A - \lambda I_n)^p$ for some natural number $p \geq 1$.

Note that any eigenvector is a generalized eigenvector (choose $p = 1$ in the previous definition).

Let λ be an eigenvalue of A with algebraic multiplicity 2 and geometric multiplicity 1. Let $v_1 \neq 0$ be an eigenvector corresponding to λ. It is known from Linear Algebra that there exists a vector v_2 such that $(A - \lambda I_n)v_2 = v_1$ and, in addition, v_1, v_2 are linearly independent. Moreover, $(A - \lambda I_n)^2 v_2 = 0$, i.e. v_2 is a generalized eigenvector.

Proposition 3.26 *Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $v \in \mathbb{C}^n$ be a corresponding eigenvector. Then we have.*

(i) *The function $\varphi(t) = e^{\lambda t} v$ satisfies $X' = AX$.*

(ii) *When $\lambda \in \mathbb{R}$ we have that $v \in \mathbb{R}^n$ and the function $\varphi(t) = e^{\lambda t} v$ is a solution of $X' = AX$.*

(iii) *When $\lambda = \alpha + i\beta \in \mathbb{C}$ and $v = u + iw \in \mathbb{C}^n$ then*

$$\begin{aligned} \varphi_1(t) &= Re(e^{\lambda t} v) = e^{\alpha t} \cos(\beta t)\, u - e^{\alpha t} \sin(\beta t)\, w, \\ \varphi_2(t) &= Im(e^{\lambda t} v) = e^{\alpha t} \sin(\beta t)\, u + e^{\alpha t} \cos(\beta t)\, w \end{aligned}$$

are solutions of $X' = AX$.

(iv) *When $\lambda \in \mathbb{R}$ is an eigenvalue of A with algebraic multiplicity 2 and geometric multiplicity 1, let v_1 be an eigenvector and v_2 be such that $(A - \lambda I_n)v_2 = v_1$. Then*

$$\varphi_1(t) = e^{\lambda t} v_1, \quad \textit{and} \quad \varphi_2(t) = e^{\lambda t}(t v_1 + v_2)$$

are two solutions of $X' = AX$.

Proof. (i) By direct computations and using that $Av = \lambda v$ we obtain $\varphi'(t) = e^{\lambda t}\lambda v = e^{\lambda t}Av = A\varphi(t)$ for all $t \in \mathbb{R}$.

(ii) By (i) this function satisfies the system and, since it is real-valued it is a solution in the sense of the definition given above.

(iii) By (i) the complex-valued function $\varphi(t) = e^{\lambda t}v$ satisfies the system. The Euler's formula helps us to find the real and, respectively, the imaginary part of this function,

$$\varphi(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(u + iw) = \varphi_1(t) + i\varphi_2(t).$$

Then $(\varphi_1 + i\varphi_2)' = A(\varphi_1 + i\varphi_2)$, which further gives $(\varphi_1' - A\varphi_1) + i(\varphi_2' - A\varphi_2) = 0$. Since both $(\varphi_1' - A\varphi_1)$ and $(\varphi_2' - A\varphi_2)$ are real, we deduce that they must be null. Hence φ_1 and φ_2 are solutions of the system $X' = AX$. The expressions of φ_1 and φ_2 can be found by direct computations.

(iv) HW. $\square$

3.4.3 The matrix of the system is diagonalizable

A very easy case is when the matrix A is diagonal, i.e there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

the system writes as

$$x_1' = \lambda_1 x_1, \quad x_2' = \lambda_2 x_2, \ \ldots\ , \quad x_n' = \lambda_n x_n.$$

It is very easy to see that the general solution of this system is

$$x_1 = c_1 e^{\lambda_1 t}, \quad x_2 = c_2 e^{\lambda_2 t}, \ \ldots\ , \quad x_n = c_n e^{\lambda_n t}, \quad c_1, \ldots, c_n \in \mathbb{R}.$$

It is easy to see that the principal matrix solution is

$$E(t) = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}.$$

We recall the following definitions.

Definition 3.27 *We say that the matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ are similar when there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ such that $A = PBP^{-1}$.*

A matrix A is said to be diagonalizable over $\mathbb{R}$ when it is similar to a diagonal matrix.

We recall the property that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable if and only if it has n linearly independent real eigenvectors. In this case, taking the transition matrix P such that its columns are the linearly independent eigenvectors of A, we have that the diagonal matrix B has on its diagonal the eigenvalues of A.

Exercise 3.28 *Prove that the matrix $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable. Find a transition matrix P and the corresponding diagonal matrix B.*

Solution. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -2$, while two linearly independent eigenvectors are

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where v_1 corresponds to λ_1, and v_2 corresponds to λ_2.

Thus the matrix A is diagonalizable. The corresponding diagonal matrix is $B = \operatorname{diag}(2, -2)$. The matrix P and its inverse are

$$P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}.$$

Finally, to be convinced, one must check, by direct calculations, that $A = PBP^{-1}$. $\diamond$

In the following we assume that A is diagonalizable and we present three methods to find the general solution of the system $X' = AX$. Denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ the eigenvalues of A (not necessarily distinct) and by $v_1, \ldots, v_n$ the corresponding linearly independent eigenvectors. Denote also $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and by $P = (v_1 \ \ldots \ v_n)$, the corresponding transition matrix.

Method 1: Compute e^{At}, the principal matrix solution.

Using the equality $e^{Bt} = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$, by Lemma 3.25 we have

$$e^{At} = P\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}.$$

Method 2: Find n linearly independent solutions.

We use Proposition 3.26 (ii) to justify that $e^{\lambda_1 t}v_1$, ..., $e^{\lambda_n t}v_n$ are solutions. Denote by $U(t) = (e^{\lambda_1 t}v_1 \ \dots \ e^{\lambda_n t}v_n)$ the matrix solution which have them as columns. In order to justify that the n solutions are linearly independent, we just note that $\det U(0) = \det P \neq 0$. So, we found that

$$e^{\lambda_1 t}v_1, \ \dots, \ e^{\lambda_n t}v_n \ \text{ are } n \text{ linearly independent solutions.}$$

Method 3: Do the linear change of variable $Y = P^{-1}X$ in the system $X' = AX$.

Computing $Y' = P^{-1}X' = P^{-1}AX = P^{-1}APY = BY$, we see that, doing the change $Y = P^{-1}X$, the system $X' = AX$ is transformed into the simpler one $Y' = BY$.

The general solution of $Y' = BY$ is $Y = (c_1 e^{\lambda_1}, \ \dots \ , c_n e^{\lambda_n})^{\mathrm{tr}}$ (a column vector). Then, the general solution of $X' = AX$ is $X = P(c_1 e^{\lambda_1}, \ \dots \ , c_n e^{\lambda_n})^{\mathrm{tr}}$, $c_1, ..., c_n \in \mathbb{R}$.

Exercise 3.29 *Prove that matrix of the next linear system is diagonalizable and find its general solution.*

$$x_1' = x_1 + 3x_2, \quad x_2' = x_1 - x_2.$$

Exercise 3.30 *Prove that matrix of the next linear system is diagonalizable and find its general solution.*

$$x_1' = x_2, \quad x_2' = x_1.$$

Exercise 3.31 *Prove that matrix of the next linear system is diagonalizable and find its general solution.*

$$x_1' = x_2, \quad x_2' = x_1, \quad x_3' = x_4, \quad x_4' = x_3.$$

Exercise 3.32 *Prove that matrix of the next linear system is diagonalizable and find its general solution.*

$$x_1' = 2x_2 - 3x_3, \quad x_2' = -2x_1 + 4x_2 - 3x_3, \quad x_3' = -2x_1 + 2x_2 - x_3.$$

3.4.4 Linear homogeneous systems with constant coefficients of dimension 2

The particular case $n = 2$ will be treated in detail. Consider

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{10}$$

whose matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Our aim is to find explicitly the general solution of this system. We will present two methods and, for a better understanding, we will discuss also on the similarities between them.

First we distinguish two classes of such systems: the *uncoupled* systems

$$\begin{aligned} x_1' &= a_{11}x_1 \\ x_2' &= a_{22}x_2 \end{aligned} \tag{11}$$

and, respectively, the *coupled* systems, which are the systems which are not uncoupled, that is either $a_{12} \neq 0$ or $a_{21} \neq 0$.

It is very easy to see that the uncoupled system (11) has the general solution

$$x_1 = c_1 e^{a_{11}t}, \quad x_2 = c_2 e^{a_{22}t}, \quad c_1, c_2 \in \mathbb{R}.$$

From now on we will study only coupled systems.

Method 1: Reduction of any coupled system to a second order LHDE with constant coefficients.

In the sequel we will see that for a coupled system with $a_{12} \neq 0$ the variable x_1 can be found as the solution of a second-order linear homogeneous differential equation. Similar property holds for the variable x_2 if $a_{22} \neq 0$.

Consider, for example, that $a_{12} \neq 0$. We use the first equation in (10) to write explicitly x_2 in function of x_1' and x_1,

$$x_2 = \frac{x_1' - a_{11}x_1}{a_{12}}, \tag{12}$$

and also to compute x_1'',

$$x_1'' = a_{11}x_1' + a_{12}x_2'.$$

Here we use the second equation in (10) to replace x_2' by $a_{21}x_1 + a_{22}x_2$. Now we use (12) to obtain x_1'' only in function of x_1' and x_1,

$$x_1'' = a_{11}x_1' + a_{12}a_{21}x_1 + a_{22}(x_1' - a_{11}x_1).$$

This last relation is the second order linear homogeneous equation

$$x_1'' - (a_{11} + a_{22})x_1' + (a_{11}a_{22} - a_{12}a_{21})x_1 = 0. \tag{13}$$

The general solution of system (10) can be found now in two steps. First find x_1 as the general solution of (13), then find x_2 using (12). This method is called *the method of reduction of the coupled system* (10) *to a second order differential equation.*

Method 2: The characteristic equation method.

The next method is based on the Fundamental Theorem and Proposition 3.26. We will present it like an algorithm.

Step 1. We write the characteristic equation (9), $\det(A - \lambda I_2) = 0$ which can be written explicitly as

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{14}$$

Step 2. We find the roots λ_1 and λ_2 of the characteristic equation. Remind that these are eigenvalues of the matrix A. We have to find also corresponding eigenvectors.

Step 3. According to the nature of the eigenvalues we attach two vector-valued functions following the rules.

When the roots are real and distinct consider

$$\varphi_1(t) = e^{\lambda_1 t}v_1 \text{ and } \varphi_2(t) = e^{\lambda_2 t}v_2,$$

where $(A - \lambda_1 I_2)v_1 = 0$, $v_1 \neq 0$ and $(A - \lambda_2 I_2)v_2 = 0$, $v_2 \neq 0$.

When λ_1 is a (real) double root consider

$$\varphi_1(t) = e^{\lambda_1 t}v_1 \text{ and } \varphi_2(t) = e^{\lambda_1 t}(tv_1 + v_2),$$

where $(A - \lambda_1 I_2)v_1 = 0$, $v_1 \neq 0$ and $(A - \lambda_1 I_2)v_2 = v_1$.

When $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$ consider

$$\begin{aligned} \varphi_1(t) &= \text{Re } (e^{(\alpha+i\beta)t}(v_1 + iv_2)) = e^{\alpha t}(\cos\beta t)\, v_1 - e^{\alpha t}(\sin\beta t)\, v_2 \\ \varphi_2(t) &= \text{Im } (e^{(\alpha+i\beta)t}(v_1 + iv_2)) = e^{\alpha t}(\sin\beta t)\, v_1 + e^{\alpha t}(\cos\beta t)\, v_2, \end{aligned}$$

where $(A - (\alpha + i\beta)I_2)(v_1 + iv_2) = 0$, $v_1 + iv_2 \neq 0$.

Step 4. The general solution of the coupled system (10) is

$$X = c_1\varphi_1(t) + c_2\varphi_2(t), \quad c_1, c_2 \in \mathbb{R}.$$

To complete the justification of the previous algorithm we need to prove that, in each case, the solutions φ_1 and φ_2 are linearly independent. For this, let $U = (\varphi_1 \ \ \varphi_2)$ be the matrix solution which have them as columns. Note that, in each case, $U(0) = (v_1 \ \ v_2)$. From Linear Algebra is known that the two vectors v_1, v_2 are linearly independent. Thus $\det U(0) \neq 0$, which assures that U is a fundamental matrix solution and, further, that the two solutions φ_1, φ_2 are linearly independent.

Note that the characteristic equation (14) of the coupled system (10) is also the characteristic equation of the second order differential equation (13). Moreover, note that $\mathrm{tr}A = a_{11} + a_{22}$ and $\det A = a_{11}a_{22} - a_{12}a_{21}$. Hence the characteristic equation can be written

$$\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0.$$

Method 3: Compute e^{At}.

From Linear Algebra it is known that any matrix $A \in \mathcal{M}_2(\mathbb{R})$ is similar to a matrix B that has one of the following forms

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

We already computed e^{Bt}, where B has one of the above forms. Computation of e^{At} follows applying the property that, when $A = PBP^{-1}$ we have $e^{At} = Pe^{Bt}P^{-1}$.

Note that $P = (v_1 \ \ v_2)$, where v_1, v_2 are as in the characteristic equation method.

3.4.5 The long-term behavior of the solutions of linear systems with constant coefficients

Theorem 3.33 *Let $n \geq 1$ and $A \in \mathcal{M}_n(\mathbb{R})$. We have that any solution of the system $X' = AX$ satisfies* $\lim_{t\to\infty} X(t) = 0$ *if and only if* $\mathrm{Re}\ (\lambda) < 0$ *for any eigenvalue $\lambda \in \mathbb{C}$ of A.*