EXERCICE 10 Show that the trace of the parametrized differential curve $\alpha : \mathbb{R} \to \mathbb{R}^3, \alpha(t) = (e^t \cos t, e^t \sin t, 2t)$ is contained in the regular surface of equation $z = \ln(x^2+y^2)$ and write the equation of the tangent plane of the surface at the points $\alpha(t)$, $t \in \mathbb{R}$.

$\alpha(t)$ is contained in $\overset{z=\ln(x^2+y^2)}{\varphi}$ $\Leftrightarrow$ ~~Replacing the coordinates of $\alpha(t)$~~

$\Leftrightarrow$ The coordinates of $\alpha(t)$ satisfy $z = \ln(x^2+y^2)$

$2t = \ln((e^t\cos t)^2 + (e^t \sin t)^2) \Leftrightarrow 2t = \ln(e^{2t}\cdot\cos^2 t + e^{2t}\cdot\sin^2 t) \Leftrightarrow$

$\Leftrightarrow 2t = \ln(e^{2t}(\cos^2 t + \sin^2 t)) \Leftrightarrow 2t = \ln e^{2t} \Leftrightarrow 2t = 2t \Rightarrow$ True $\Rightarrow$

$\Rightarrow$ $\alpha(t)$ is contained in the $\underbrace{z=\ln(x^2+y^2)}_{\varphi}$ regular surface equation

Let $\varphi(x,y) = \ln(x^2+y^2)$

$$z = z_\alpha + \frac{\partial \varphi}{\partial x_\alpha}(x_\alpha, y_\alpha)\cdot(x-x_\alpha) + \frac{\partial \varphi}{\partial y_\alpha}(x_\alpha, y_\alpha)\cdot(y-y_\alpha)$$

$$\frac{\partial \varphi}{\partial x}(x,y) = \frac{2x}{x^2+y^2} \qquad \frac{\partial \varphi}{\partial y}(x,y) = \frac{2y}{x^2+y^2}$$

$$\Rightarrow z = z_\alpha + \frac{2x_\alpha(x-x_\alpha)}{x_\alpha^2+y_\alpha^2} + \frac{2y_\alpha(y-y_\alpha)}{x_\alpha^2+y_\alpha^2} \Rightarrow z = z_\alpha + \frac{2x\cdot x_\alpha - 2x_\alpha^2}{x_\alpha^2+y_\alpha^2} + \frac{2y_\alpha y - 2y_\alpha^2}{x_\alpha^2+y_\alpha^2}$$

$$\Rightarrow z = 2t + \frac{2x\cdot e^t\cos t - 2e^{2t}\cos^2 t}{e^{2t}} + \frac{2y\cdot e^t \sin t - 2e^{2t}\sin^2 t}{e^{2t}} \Rightarrow$$

$$\Rightarrow z = 2t + \frac{2\cancel{e^t}(x\cos t + y \sin t)}{e^{\cancel{2}t}} + \frac{-2\cancel{e^{2t}}(\sin^2 t + \cos^2 t)}{\cancel{e^{2t}}} \Rightarrow$$

$$\Rightarrow z = \frac{x\cos t + y\sin t}{e^t} - 2 + 2t$$

$$\Rightarrow \frac{x\cos t + y \sin t}{e^t} - z - 2 + 2t = 0$$

$\to$ equation of the tangent plane of the $z=\ln(x^2+y^2)$ surface at the points $\alpha(t)$, $t\in\mathbb{R}$