14) Given the bundle of lines of equations : $(1-t)x+(2-t)y+t-3=0$, $t\in\mathbb{R}$ and $x+y-1=0$, find:

a) the coordinates of the vertex of the bundle

The vertex of the bundle is a point $A(x_A, y_A)$ through which all the lines in the bundle pass. This means that:

$$\begin{cases} (1-t)x_A+(2-t)y_A+t-3=0, \forall t\in\mathbb{R} \\ x_A+y_A-1=0 \quad \Big| \cdot(-1) \end{cases} \Leftrightarrow \begin{cases} (-x_A-y_A+1)t+x_A+2y_A-3=0, \\ \qquad \forall t\in\mathbb{R} \\ -x_A-y_A+1=0 \end{cases}$$

$$\Rightarrow \begin{cases} -x_A-y_A+1=0 \\ x_A+2y_A-3=0 \end{cases} \oplus$$

$$/ \quad y_A=2 \Rightarrow x_A=1-y_A=1-2=-1 \rightarrow$$

$\Rightarrow A(-1,2)$ is the vertex of the bundle

b) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, s.t. $OM^2\cdot ON^2=4(OM^2+ON^2)$.

• Let's check if the line $d_1: x+y-1=0$ is a solution.

$d_1\cap Ox: y=0 \Rightarrow x+0-1=0 \Rightarrow x=1 \Rightarrow d_1\cap Ox=\{M_1(1,0)\}$

$d_1\cap Oy: x=0 \Rightarrow 0+y-1=0 \Rightarrow y=1 \Rightarrow d_1\cap Oy=\{N_1(0,1)\}$

$$OM_1^2=1,\ ON_1^2=1 \Rightarrow \left.\begin{array}{l} OM_1^2\cdot ON_1^2=1 \\ 4\cdot(OM_1^2+ON_1^2)=4\cdot 2=8 \end{array}\right\} \Rightarrow$$

$\Rightarrow OM_1^2\cdot ON_1^2 \neq 4\cdot(OM_1^2+ON_1^2) \Rightarrow d_1$ is not a solution.

• Therefore, the possible solution ~~+~~ we search for is a line of equation $d: (1-t)x+(2-t)y+t-3=0$, $t\in\mathbb{R}$.

Some remarks:

- If $1-t=0$ $(t=1)$, we obtain a line of equation $d: y-2=0$, which is parallel to (does not intersect) Ox.

- If $2-t=0$ $(t=2)$, we obtain a line of equation $d: -x-1=0$, which is parallel to Oy.

So we can put the conditions : $1-t\neq 0$ and $2-t\neq 0$.

Now, let $d \cap Ox = \{M(x_M, 0)\}$ and $d \cap Oy = \{N(0, y_N)\}$

$\Rightarrow OM = |x_M|$, $ON = |y_N| \Rightarrow OM^2 = x_M^2$ and $ON^2 = y_N^2$

$\Rightarrow x_M^2 \cdot y_N^2 = 4(x_M^2 + y_N^2)$. ①

$M(x_M, 0) \in d \Rightarrow (1-t)x_M + (2-t)\cdot 0 + t - 3 = 0$. ②

$N(0, y_N) \in d \Rightarrow (1-t)\cdot 0 + (2-t)y_N + t - 3 = 0$. ③

We have the system (formed by ①, ②, ③):

$$\begin{cases} (1-t)x_M + t - 3 = 0 \; \Big| : (1-t) \neq 0 \\ (2-t)y_N + t - 3 = 0 \; \Big| : (2-t) \neq 0 \\ x_M^2 \cdot y_N^2 = 4(x_M^2 + y_N^2) \end{cases} \Rightarrow \begin{cases} x_M = \dfrac{3-t}{1-t} \\ y_N = \dfrac{3-t}{2-t} \\ x_M^2 \cdot y_N^2 = 4(x_M^2 + y_N^2) \end{cases} \Rightarrow$$

$$\Rightarrow \frac{(3-t)^2}{(1-t)^2} \cdot \frac{(3-t)^2}{(2-t)^2} = 4 \cdot (3-t)^2 \cdot \left[\frac{1}{(1-t)^2} + \frac{1}{(2-t)^2}\right]$$

$$\Leftrightarrow \frac{(3-t)^4}{(1-t)^2(2-t)^2} = 4 \cdot (3-t)^2 \cdot \frac{(2-t)^2 + (1-t)^2}{(1-t)^2 \cdot (2-t)^2} \; \Big| \cdot (1-t)^2 \cdot (2-t)^2$$

$$\Leftrightarrow (3-t)^4 = 4 \cdot (3-t)^2 \cdot \left[(2-t)^2 + (1-t)^2\right]$$

$$\Leftrightarrow (3-t)^2 \cdot \left[(3-t)^2 - 4 \cdot \left((2-t)^2 + (1-t)^2\right)\right] = 0$$

$\Rightarrow$ I) $(3-t)^2 = 0 \Rightarrow t_1 = 3 \Rightarrow x_M = y_N = 0 \Rightarrow$

$\Rightarrow d \cap Ox = d \cap Oy = O(0,0)$

Then, $OM = ON = 0$ and $\boxed{d: -2x - y = 0.}$

II) $9 - 6t + t^2 - 4(4 - 4t + t^2 + 1 - 2t + t^2) = 0$

$\Leftrightarrow 9 - 6t + t^2 - 8t^2 + 24t - 20 = 0$

$\Leftrightarrow -7t^2 + 18t - 11 = 0$ $\Rightarrow t_2 = 1$ False

$\Rightarrow t_3 = \frac{11}{7}$. $\Rightarrow \begin{cases} x_M = \frac{10}{-4} = -\frac{5}{2} \\ y_N = \frac{10}{3} \end{cases}$

Then, $d': \left(1 - \frac{11}{7}\right)x + \left(2 - \frac{11}{7}\right)y + \frac{11}{7} - 3 = 0 \Rightarrow \boxed{d': -\frac{4}{7}x + \frac{3}{7}y - \frac{10}{7} = 0}$

So d and d' are solutions.