Discrete dynamical systems

We consider the non-linear difference equation in $\mathbb{R}^n$

(1) $x_{k+1} = f(x_k)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous

Rmk. 1) For an arbitrary $\eta \in \mathbb{R}^n$, the ivp $x_{k+1} = f(x_k)$, $x_0 = \eta$ has a unique solution $\eta, f(\eta), f(f(\eta)), \dots, f^k(\eta), \dots$, where $f^k = \underbrace{f \circ f \circ \dots \circ f}_{k \text{ times}} \to$ the k iterate of f

2) The flow of (1) is $f: \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(k, \eta) = f^k(\eta)$

Def. 1) $\mathbb{R}^n$ is called the state space, η - initial state, $f^k(\eta)$ - the state at time k.

2) The positive orbit of η is $\gamma^+_\eta = \{\eta, f(\eta), \dots, f^k(\eta), \dots\}$

In the case that f is invertible, we define the orbit of η

$\gamma_\eta = \{\dots, f^{-k}(\eta), \dots f^{-1}(\eta), \eta, f(\eta), \dots, f^k(\eta), \dots\}$, where f^{-1} is the inverse of f and $f^{-k} = (f^{-1})^k$.

3) We say that $\eta^* \in \mathbb{R}$ is a fixed point of f is $f(\eta^*) = \eta^*$.

4) Let η^* be a fixed point of f. We say that η^* is stable when, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. whenever $\|\eta - \eta^*\| < \delta$ we have that $\|f^k(\eta) - \eta^*\| < \varepsilon$, $\forall k \geq 0$.

We say that η^* is an attractor when $\exists r > 0$ s.t. whenever $\|\eta - \eta^*\| < r$ we have that $\lim_{k \to \infty} f^k(\eta) = \eta^*$.

The basin of attraction of an attractor η^* is $A_{\eta^*} = \{\eta \in \mathbb{R}^n : \lim_{k \to \infty} f^k(\eta) = \eta^*\}$

Rmk. 1) Let η^* be a fixed point of f. Then η^* is a fixed point of f^k, $\forall k \geq 1$

Proof by induction: 1) η^* is a fixed point of $f \iff f(\eta^*) = \eta^*$

I $\Rightarrow f(f(\eta^*)) = f(\eta^*) \Rightarrow f^2(\eta^*) = \eta^* \Rightarrow \eta^*$ fixed point of f^2

II Assume that $f^k(\eta^*) = \eta^*$ and prove that $f^{k+1}(\eta^*) = \eta^*$

2) The positive orbit of η^* is $\gamma_{\eta^*} = \{\eta^*\}$.

Proposition 3) The unique solution of the ivp $\begin{cases} x_k = f(x_k) \\ x_0 = \eta^* \end{cases}$ is the sequence

$x_k = f^k(\eta^*) = \eta^*$, $\forall k \geq 0$.

If x_k is a solution of (1), which is a convergent sequence, then its limit is a fixed point of f.

Proof. Let $\eta^* \in \mathbb{R}^n$ be s.t. $\lim_{k\to\infty} x_k = \eta^*$.

By hypothesis, we have that $x_{k+1} = f(x_k)$, $\forall k \geq 0$.

$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(k) \Rightarrow \eta^* = f(\eta^*)$ q.e.d.

Def. Let $\eta^* \in \mathbb{R}^n$ and $p \in \mathbb{N}$, $p \geq 2$. We say that η^* is a p-periodic point of f when $f^p(\eta^*) = \eta^*$ and η^* is not a fixed point of $f, f^2, \dots, f^{p-1}$.

Rmk: 1) Let η^* be a p-periodic point of f. Then $\gamma^+_{\eta^*} = \{\eta^*, f(\eta^*), \dots f^{p-1}(\eta^*)\} \rightarrow$ p-periodic cycle

The unique solution of the IVP $x_{k+1} = f(x_k)$; $x_0 = \eta^*$ is $\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*), \dots \eta^*, \dots, f(\eta^*), \dots$

2) η^* is a p-periodic point $\Rightarrow$ $f(\eta^*) \dots f^{p-1}(\eta^*)$ are also p-periodic points.

Example: Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. Find it's fixed points and its 2-periodic points.

fixed points: $f(x) = x$, $x = ?$, $x \in \mathbb{R}$

$\Leftrightarrow 1 - 2x^2 = x \Leftrightarrow -2x^2 - x + 1 = 0 \Leftrightarrow -1$ and $\frac{1}{2}$

2-periodic points: $f^2(x) = x$, $x \in \mathbb{R}$, $f(x) \neq x$

$f^2(x) = (f \circ f)(x) = f(f(x)) = 1 - 2[f(x)]^2$

$= 1 - 2(1 - 2x^2)^2 = 1 - 2 + 8x^2 - 8x^4 = -8x^4 + 8x^2 - 1$

$8x^4 - 8x^2 + x + 1 = 0$

We know $-1, \frac{1}{2}$ are fixed points of $f \Rightarrow$ also fixed points of $f^2 \Rightarrow$ they are roots of $8x^4 - 8x^2 + x + 1 = 0$.

$8x^4 - 8x^2 + x + 1 = \underbrace{(2x^2 + x - 1)}_{\text{f.p.}}(4x^2 - 2x - 1)$

$4x^2-2x-1=0 \Rightarrow$ f.p.: $\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}$

Def. Let η^* be a p-periodic point of f. We say that $\gamma^+_{\eta^*}$ is stable is stable / attractor / unstable when η^* is stable / attractor / unstable fixed point of f^p.

Rmk Let η^* be a p-periodic point of f which is an attractor $\Rightarrow$ $\exists r>0$ s.t. whenever $\|\eta-\eta^*\|<r$ we have $\lim_{k\to 0} (f^p)^k(\eta)=\eta^*$.

$\gamma^*_\eta = \{f^k(\eta) : \forall k\geq 0\} = \{\eta, f(\eta) \dots f^{p-1}(\eta), f^p(\eta) \dots\}$

$$\underset{\substack{\downarrow k\to\infty \\ \eta^*}}{f^{kp}(\eta)}, \quad \underset{\substack{\downarrow k\to\infty \\ f(\eta^*)}}{f(f^{kp}(\eta))} \quad \dots \quad f^{p-1}(\eta^*)$$

This $(f^k(\eta))_{k\geq 0}$ has p convergent subsequences with the limits $\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)$.

The linearization method to study the stability of the fixed points of scalar maps

Let η^* be a fixed point of $f:\mathbb{R}\to\mathbb{R}$, which is C^1.

If $|f'(\eta^*)|<1$ then η^* is an attractor.

If $|f'(\eta^*)|>1$ then η^* is unstable.

Exercise: Study the stability of the fixed points and the 2-periodic orbit of $f(x)=1-2x^2$.

fixed points of f: -1 and $\frac{1}{2}$

$f'(x)=-4x$

$f'(-1)=4 \Rightarrow |f'(-1)|>1 \Rightarrow -1$ is unstable f.p.

$f'(\frac{1}{2})=-2 \Rightarrow |f'(\frac{1}{2})|>1 \Rightarrow \frac{1}{2}$ unstable f.p.

f has 2-periodic orbit $\left\{ \frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4} \right\}$

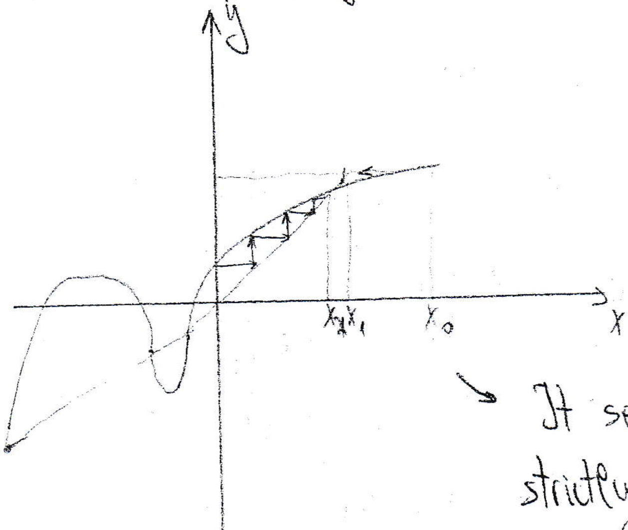
$f^2(x) = -8x^4 + 8x^2 - 1$

$g = f^2$

$g'(x) = -32x^2 + 16x$

$g'\left(\frac{1-\sqrt{5}}{4}\right) = \ldots$

The cob-web or stair step diagram to study the dynamic of a scalar map of f.



$f(x) = x$

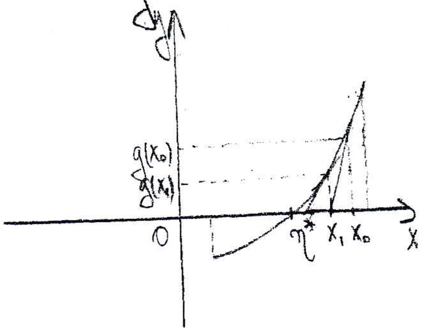
$x_1 = f(x_0)$

$x_2 = f(x_1)$

It seems that the sequense is strictly decreasing, convergent to the fixed point nearby.

The Newton-Raphson's method to approximate the zeros of scalar maps

Let $g: I \to \mathbb{R}$ be a C^2 map, where $I \subset \mathbb{R}$ is nonempty, open interval.



$\exists\, \eta^* \in I$ s.t. $g(\eta^*)=0$ and $g'(\eta^*) \neq 0$

the tangent line to the graph of g in $(x_0, g(x_0))$ has the eq: $y - g(x_0) = g'(x_0)(x - x_0)$

$\cap\, Ox \Rightarrow y = 0 \qquad -g(x_0) = g'(x_0)(x_1 - x_0) \Rightarrow x_1 = x_0 - \dfrac{g(x_0)}{g'(x_0)}$

We assume that $x_0 \in V$, where $V \subset I$ is s.t. $g'(x) \neq 0$, $\forall x \in V$ (for example, V is a sufficiently small neighborhood of η^*).

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad \forall k \geq 0,\ x_0 \in V \text{ fixed}}$$

We intend to prove that $\lim\limits_{k\to\infty} x_k = \eta^*$, for any x_0 sufficiently close to η^*.

We define $f: I \to \mathbb{R}$, $f(x) = x - \dfrac{g(x)}{g'(x)}$, $\forall x \in V$

We find the fixed points of f, i.e. $f(x) = x$, $x \in V$.

$x - \dfrac{g(x)}{g'(x)} = x \iff g(x) = 0 \Rightarrow \eta^*$ is the only fixed point of f in V

We want to prove that η^* is an attractor for f.

$f'(x) = 1 - \dfrac{g'(x)\cdot g'(x) - g(x)\cdot g''(x)}{[g'(x)]^2} \Rightarrow f'(\eta^*) = 1 - 1 = 0 \Rightarrow |f'(\eta^*)| < 1$

$\overset{L.M.}{\Longrightarrow} \eta^*$ is an attractor of $f \overset{def}{\Longleftrightarrow} \exists\, r > 0$ s.t. whenever $|\eta - \eta^*| < r$ we have $\lim\limits_{k\to\infty} f^k(\eta) = \eta^*$

$\eta = x_0$, $f^k(\eta) = x_k$

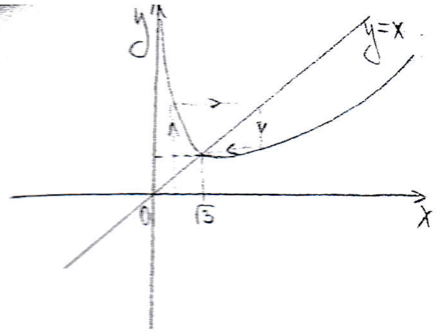
<u>A particular case</u>

$g: (0,\infty) \to \mathbb{R}$, $g(x) = x^2 - 3$ with $\eta^* = \sqrt{3}$, the only zero of g

We intend to estimate the basin of attraction of $\sqrt{3}$ in the Newton's method. (using the stair-step diagram).

$f: (0,\infty) \to \mathbb{R}$, $f(x) = x - \dfrac{x^2-3}{2x} = x - \dfrac{1}{2}x + \dfrac{3}{2}\cdot\dfrac{1}{x} = \dfrac{1}{2}x + \dfrac{3}{2}\cdot\dfrac{1}{x}$

$f(\sqrt{3}) = \sqrt{3}$



$f(x)>0, \ \forall x>0$

$\lim_{x\searrow 0} f(x)=+\infty, \quad \lim_{x\to\infty} f(x)=\infty$

$f'(x)=\frac{1}{2}-\frac{3}{2x^2}=\frac{1}{2}\cdot\frac{x^2-3}{x^2}$

x	0		$\sqrt{3}$		$+\infty$
f'		$- - -$	0	$+ \ +$	

$y=\frac{1}{2}x$ is an oblic asymptote

It seems that $A_{\sqrt{3}}=(0,\infty)$

We consider the map $T:[0,1]\to\mathbb{R}, \ T(x)=1-|2x-1|$.

a) Represent the graph of T. Find the fixed points of T.

b) Compute the orbits corresponding to the initial values $\eta=\frac{3}{2^n}, \ n\geq 2$.

c) Solve the equations $T(x)=0$; $T(x)=1$; $T(x)=\frac{1}{2}$; $T^2(x)=0$; $T^2(x)=1$; $T^2(x)=\frac{1}{2}$.

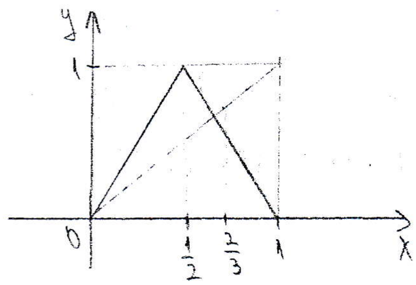
d) Compute T^2.

e) Represent the graphs of T^2 and T^3. How many fixed points they have?

f) T has a 2-periodic orbit? T has a 3-periodic orbit?

Fact: There is a result: "If a map has 3-periodic orbit, then it has a p-periodic orbit for any $p\in\mathbb{N}^*$".

a) $T(x)=\begin{cases}1-(2x-1), & 2x-1>0\\ 1+(2x-1), & 2x-1\leq 0\end{cases}=\begin{cases}2(1-x), & x\in(\frac{1}{2},1]\\ 2x, & x\in[0;\frac{1}{2}]\end{cases}$



$T(0)=0$
$T(\frac{1}{2})=1$
$T(1)=0$
$2(1-x)=x$
$2-2x=x$
$x=\frac{2}{3}$

the fixed points of T are: 0 and $\frac{2}{3}$

b) $T^n(\frac{3}{2^n})=?$ $\quad \gamma_{\frac{3}{2^n}}=?$

$n=2$: $\eta=\frac{3}{4}$, $T(\eta)=T(\frac{3}{4})=2(1-\frac{3}{4})=\frac{1}{2}$, $T^2(\eta)=T(\frac{1}{2})=1$, $T^3(\eta)=T(1)=0, 0, 0, \ldots$

$n=3$: $\eta=\frac{3}{8}$, $T(\eta)=T(\frac{3}{8})=2\cdot\frac{3}{8}=\frac{3}{4}$, $T^2(\eta)=T(\frac{3}{4})=\frac{1}{2}, 1, 0, 0, \ldots 0, \ldots$

$n\geq 3$: $T(\frac{3}{2^n})=\frac{3}{2^{n-1}}, \ldots, \frac{3}{8}, \frac{3}{4}, \frac{1}{2}, 1, 0, \ldots 0 \ldots$

Conclusion: $\frac{3}{2^n}$, $(n\geq 2)$ is eventually the fixed point 0

c) $T(x)=0 \iff x \in \{0, 1\}$

$T(x)=1 \iff x=\frac{1}{2}$

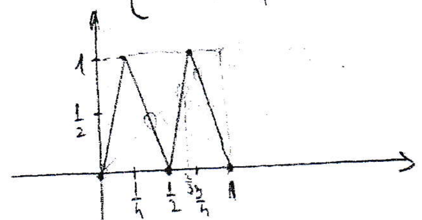
$T(x)=\frac{1}{2} \iff x \in \{\frac{1}{4}, \frac{3}{4}\}$

$T^2(x)=0 \iff T(T(x))=0 \iff T(x) \in \{0,1\} \iff x \in \{0, \frac{1}{2}, 1\}$

$T^2(x)=1 \iff T(T(x))=1 \iff T(x)=\frac{1}{2} \iff x \in \{\frac{1}{4}, \frac{3}{4}\}$

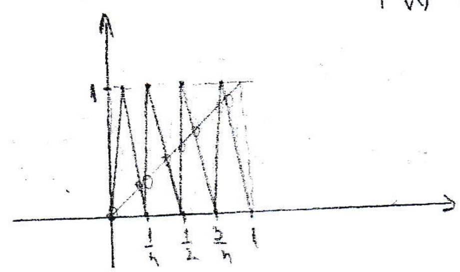
$T^2(x)=\frac{1}{2} \iff T(T(x))=\frac{1}{2} \iff T(x) \in \{\frac{1}{4}, \frac{3}{4}\} \iff x \in \{\frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{5}{8}\}$

d) $T^2(x) = \begin{cases} 2(1-T(x)), & T(x) \in (\frac{1}{2}, 1] \\ 2T(x), & T(x) \in [0, \frac{1}{2}] \end{cases} = \begin{cases} 2(1-2+2x), & x \in (\frac{1}{2}, 1] \wedge T(x) \in (\frac{1}{2}, 1] \\ & \end{cases}$



T^2 has 4 fixed points

$T^3(x)=0 \iff T(T^2(x))=0 \iff T^2(x) \in \{0,1\} \iff x \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

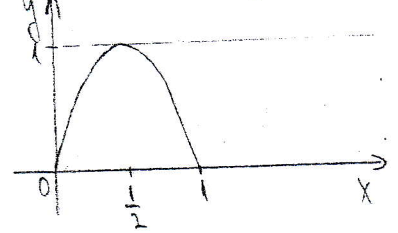


T^3 has 8 fixed points

f) T^2 has 4 fixed points. 2 of them are 0 and $\frac{2}{3}$, the fixed points of T. The other 2 fixed points of T^2 from a 2-periodic orbit of T. So, T has one 2-periodic orbit.

T^3 has 8 fixed points. 2 of them are 0 and $\frac{2}{3}$, the fixed points of T. The other 6 fixed points of T^3 form two 3-periodic orbits of T. So, T has two 3-periodic orbits.

Remark. Let $f: [0,1] \to \mathbb{R}$, $f(x) = 4x(1-x) = 4x - 4x^2$



$f'(x) = 4 - 8x = 4(1-2x)$

$f(\frac{1}{2}) = 1$

The Linearization Method

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map ($n \geq 1$, $n \in \mathbb{N}$) and $\eta^* \in \mathbb{R}^n$ be a fixed point of f.

If $|\lambda| < 1$ for any eigenvalue λ of $Jf(\eta^*)$ then η^* is an attractor.

If there exists an eigenvalue λ of $Jf(\eta^*)$ such that $|\lambda| > 1$ then η^* is unstable.

Newton's method

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2-map and $V \subset \mathbb{R}^n$ be open, nonempty such that $\exists\, \eta^* \in V$ with $g(\eta^*) = 0$, $g(\eta) \neq 0$, $\forall \eta \in V \setminus \{\eta^*\}$ and $Jf(\eta)$ is invertible $\forall \eta \in V$. We consider the sequence $(x_k)_{k \geq 0}$ such that

$x_{k+1} = x_k - [Jg(x_k)]^{-1} g(x_k)$, $\forall k \geq 0$ with $x_0 \in V$ fixed.

If V is sufficiently small, then $\lim_{k \to \infty} x_k = \eta^*$.

<u>Proof.</u> Let $f : V \to \mathbb{R}^n$, $f(x) = x - [Jg(x)]^{-1} g(x)$

We have to prove that: • η^* is the only fixed point of f: $f(x) = x \iff x - [Jg(x)]^{-1} g(x) = x \iff g(x) = 0$

• η^* is an attractor for f: $Jf(\eta^*) = I_n - [Jg(\eta^*)]^{-1} Jg(\eta^*) -$ "derivative of Jg" $\cdot\, g(\eta^*) \neq 0$

$\Rightarrow Jf(\eta^*) = O_n \Rightarrow Jf(\eta^*)$ has the eigenvalue 0 of multiplicity $n \overset{L.M.}{\Longrightarrow} \eta^*$ attractor