Exercise (4.7.). Show that two different parallel lines are projected onto parallel linesby the symmetry $s_{\pi,d}$ for $\pi \not\parallel d$, where:

$$\pi : Ax + By + Cz + D = 0$$
$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

PROOF. From $\pi \not\parallel d$ we deduce that $Ap + Bq + Cr \neq 0$. Suppose that our two parallel lines are given by the parametric equations:

$$d_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \qquad d_2 : \begin{cases} x = x_2 + sv_x \\ y = y_2 + sv_y \\ z = z_2 + sv_z \end{cases}$$

Comment: The reason I chose to give different names for the parameters of these equations is pedagogical. I wanted it to be clear that these parameters are not necessarily the same. In general, there is no harm in denoting the parameters of two lines by the same letter.

We now make the notations:

$$F(x, y, z) := Ax + By + Cz + D$$
$$\overrightarrow{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \qquad \overrightarrow{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

For every $P(x, y, z) \in d_1$ we have:

$$s_{\pi,d}(P) = 2 \cdot p_{\pi,d}(P) - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\frac{F(x, y, z)}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

$$s_{\pi,d}(P) = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\overrightarrow{v} - 2 \cdot \frac{A(x_1 + tv_x) + B(y_1 + tv_y) + C(z_1 + tv_z) + D}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

We see that in this formula we have free terms and terms that depend on the parameter t. We then separate them:

$$s_{\pi,d}(P) = \begin{pmatrix} x_1 - 2p\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - 2q\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - 2r\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix} + t \cdot \left(\overrightarrow{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right)$$

This is the parametric equation of a line d_1'' that contains the point P_1'' with coordinate vector:

$$\begin{pmatrix} x_1 - 2p\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - 2q\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - 2r\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix}$$

and director vector:

$$\overrightarrow{w} = \overrightarrow{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

By performing the exact same computations for the reflection of the line d_2 we obtain the equation of a line d_2'' that contains the point P_2'' with coordinate vector:

$$\begin{pmatrix} x_2 - 2p\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ y_2 - 2q\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ z_2 - 2r\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \end{pmatrix}$$

and director vector:

$$\overrightarrow{w} = \overrightarrow{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d}$$

The director vector is the same, therefore, the reflections d_1'' and d_2'' of the two lines d_1 and d_2 are parallel if w is nonzero.

Suppose now that w is zero, so we have

$$\overrightarrow{v} = 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \overrightarrow{d},$$

which can be rewritten to

$$\overrightarrow{v} = 2\frac{\overrightarrow{n_\pi} \cdot \overrightarrow{v}}{\overrightarrow{n_\pi} \cdot \overrightarrow{d}} \cdot \overrightarrow{d},$$

Because $\overrightarrow{v}$ and $\overrightarrow{d}$ are parallel, we have $\cos(\widehat{\overrightarrow{n_\pi}, \overrightarrow{v}}) = \cos(\widehat{\overrightarrow{n_\pi}, \overrightarrow{d}})$, so that

$$\frac{\overrightarrow{n_\pi} \cdot \overrightarrow{v}}{\overrightarrow{n_\pi} \cdot \overrightarrow{d}} = \frac{||\overrightarrow{n_\pi}|| \cdot ||\overrightarrow{v}|| \cos(\widehat{\overrightarrow{n_\pi}, \overrightarrow{v}})}{||\overrightarrow{n_\pi}|| \cdot ||\overrightarrow{d}|| \cos(\widehat{\overrightarrow{n_\pi}, \overrightarrow{d}})} = \frac{||\overrightarrow{v}||}{||\overrightarrow{d}||}$$

and this shows that

$$\overrightarrow{v} = 2\frac{||\overrightarrow{v}||}{||\overrightarrow{d}||} \cdot \overrightarrow{d},$$

which implies that

$$||\overrightarrow{v}|| = 2\frac{||\overrightarrow{v}||}{||\overrightarrow{d}||} \cdot ||\overrightarrow{d}||,$$

which can only be true if $\overrightarrow{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, case in which the lines d_1 and d_2 would be just points, a contradiction.

Therefore the only possibility is that the reflections of d_1 and d_2 are parallel. $\square$