Exercise (4.8.). Let $\mathcal{P}$ be the three-dimensional Euclidean space and $\mathcal{V}$ the space of vectors in $\mathcal{P}$. Let $R = (O, b)$ with $b = (u, v, w)$ is the Cartesian reference system for $\mathcal{P}$ and we take the plane π and the line d with the equations:

$$\begin{aligned}\pi :& Ax + By + Cz + D = 0\\ d :& \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}\end{aligned}$$

If $\pi \not\parallel d$, show that:

(a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = p(\overrightarrow{MN})$, for all points $M, N \in \mathcal{P}$, where $p : \mathcal{V} \to \mathcal{V}$ is the linear transformation whose matrix representation is

$$[p]_b = \frac{1}{Ap + Bq + Cr}\begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}$$

(b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = s(\overrightarrow{MN})$, for all points $M, N \in \mathcal{P}$, where $s : \mathcal{V} \to \mathcal{V}$ is the linear transformation whose matrix representation is

$$[s]_b = \frac{1}{Ap + Bq + Cr}\begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix}$$

Proof. We take the notations:

$$\begin{aligned}F(x, y, z) &:= Ax + By + Cz + D\\ \overrightarrow{d} &= \begin{pmatrix} p \\ q \\ r \end{pmatrix}\end{aligned}$$

(a)

$$[\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)}]_b = [\overrightarrow{r_{p_{\pi,d}(N)}}]_b - [\overrightarrow{r_{p_{\pi,d}(M)}}]_b =$$

$$= \left(\begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix} - \frac{F(x_N, y_N, z_N)}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right) - \left(\begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} - \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right) =$$

$$= \begin{pmatrix} x_N - x_M \\ y_N - y_M \\ z_N - z_M \end{pmatrix} - \frac{1}{Ap + Bq + Cr} \cdot (F(x_N, y_N, z_N) - F(x_M, y_M, z_M)) \cdot \overrightarrow{d} =$$

$$= [\overrightarrow{MN}]_b - \frac{1}{Ap + Bq + Cr} \big(A(x_N - x_M) + B(y_N - y_M) + C(z_N - z_M) \big) \cdot \overrightarrow{v} =$$

$$= [\overrightarrow{MN}]_b - \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Ap(x_N - x_M) + Bp(y_N - y_M) + Cp(z_N - z_M) \\ Aq(x_N - x_M) + Bq(y_N - y_M) + Cq(z_N - z_M) \\ Ar(x_N - x_M) + Br(y_N - y_M) + Cr(z_N - z_M) \end{pmatrix} =$$

$$= [\overrightarrow{MN}]_b - \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\overrightarrow{MN}]_b - \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap + Bq + Cr} \left(\begin{pmatrix} Ap + Bq + Cr & 0 & 0 \\ 0 & Ap + Bq + Cr & 0 \\ 0 & 0 & Ap + Bq + Cr \end{pmatrix} - \begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix} \right) \cdot [\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & Bp & Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= [p]_b \cdot [\overrightarrow{MN}]_b =$$

$$= [p(\overrightarrow{MN})]_b$$

We have thus shown that $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = p(\overrightarrow{MN})$.

Comment: This is due to the following easy linear algebra fact: two elements v, w in a vector space V are the same if and only if their coordinate vectors in a basis B of V are the same: $[v]_B = [w]_B$.

(b)

$$[\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)}]_b = [\overrightarrow{r_{s_{\pi,d}(N)}}]_b - [\overrightarrow{r_{s_{\pi,d}(M)}}]_b =$$

$$= \left(\begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix} - 2\frac{F(x_N, y_N, z_N)}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right) - \left(\begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} - 2\frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \cdot \overrightarrow{d} \right) =$$

$$= \begin{pmatrix} x_N - x_M \\ y_N - y_M \\ z_N - z_M \end{pmatrix} - \frac{2}{Ap + Bq + Cr} \cdot (F(x_N, y_N, z_N) - F(x_M, y_M, z_M)) \cdot \overrightarrow{d} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap + Bq + Cr} \big(A(x_N - x_M) + B(y_N - y_M) + C(z_N - z_M) \big) \cdot \overrightarrow{v} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap + Bq + Cr} \begin{pmatrix} Ap(x_N - x_M) + Bp(y_N - y_M) + Cp(z_N - z_M) \\ Aq(x_N - x_M) + Bq(y_N - y_M) + Cq(z_N - z_M) \\ Ar(x_N - x_M) + Br(y_N - y_M) + Cr(z_N - z_M) \end{pmatrix} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap + Bq + Cr} \begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\overrightarrow{MN}]_b - \frac{2}{Ap + Bq + Cr} \begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap + Bq + Cr} \left(\begin{pmatrix} Ap + Bq + Cr & 0 & 0 \\ 0 & Ap + Bq + Cr & 0 \\ 0 & 0 & Ap + Bq + Cr \end{pmatrix} - \begin{pmatrix} 2Ap & 2Bp & 2Cp \\ 2Aq & 2Bq & 2Cq \\ 2Ar & 2Br & 2Cr \end{pmatrix} \right) \cdot [\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap + Bq + Cr} \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix} \cdot [\overrightarrow{MN}]_b =$$

$$= [s]_b \cdot [\overrightarrow{MN}]_b =$$

$$= [s(\overrightarrow{MN})]_b$$

We have thus shown that $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = s(\overrightarrow{MN})$.

□