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Chapter 6. Linear difference equations and systems[1]

We start with an example. We all heard about or even studied the Fibonacci sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots .$$

We will present first another application of this sequence in a problem regarding a computer network. Consider that at the initial moment $k = 0$ one computer is getting infected with some virus. Such a virus has two properties: it becomes active at the next time after it infected some computer, and it infects another computer at the next time after it became active. Denoting x_k the number of active viruses at time k, we deduce that the following relations must hold:

$$x_0 = 0,\ x_1 = 1,\ x_{k+2} = x_{k+1} + x_k,\ k \geq 0, \tag{1}$$

i.e. the sequence $(x_k)_{k\geq 0}$ is the Fibonacci sequence.

The problem (1) can be seen, from theoretical point of view, as *an initial value problem (IVP, for short) for a second order difference equation*. Moreover, the difference equation $x_{k+2} - x_{k+1} - x_k = 0$ is linear and homogeneous with constant coefficients. The first aim of this chapter is to give the fundamental results for the class of linear difference equations of arbitrary order, denoted n (where $n \geq 1$ is a natural number). The general form of such an equation is

$$x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k = f_k, \quad k \geq 0, \tag{2}$$

where $(a_{1,k})_{k\geq 0}$, $(a_{2,k})_{k\geq 0}$, ..., $(a_{n,k})_{k\geq 0}$, $(f_k)_{k\geq 0}$ are sequences of real numbers.

A *solution* of (2) is a sequence $(x_k)_{k\geq 0}$ of real numbers. We denote the set of all sequences of real numbers by $\mathbb{R}^\infty$. This set has a structure of linear space, where addition between two such sequences and multiplication with a real constant are defined in a natural way. From now on, writing $x \in \mathbb{R}^\infty$ will mean that $(x_k)_{k\geq 0}$ is a sequence of real numbers. Note that the sequence $(x_k)_{k\geq 0}$ can be also seen as a scalar function of discrete variable $x : \mathbb{N} \to \mathbb{R}$.

The sequences denoted by a are called *the coefficients*, and the sequence denoted by f is called *the non-homogeneous part* or *the force* of equation (2). When $f_k \equiv 0$ we say that (2) is *linear homogeneous* or *unforced*, otherwise we say that (2) is *linear nonhomogeneous* or *forced*. When all the coefficients are constant sequences, we say that (1) is *a linear difference equation with constant coefficients*.

Examples.

1) $x_{k+3} + x_k = 0$ is a third order linear homogeneous difference equation with constant coefficients.

2) $x_{k+2} + k\, x_k = 0$ is a second order linear homogeneous difference equation, but the coefficients are not all constant.

3) Let λ be a real parameter. The equation $x_{k+2} + \lambda x_k = 2\,(-3)^k - k^2$ is a second order linear nonhomogeneous difference equation with constant coefficients. The nonhomogeneous part is $f_k = 2\,(-3)^k - k^2$.

4) The equation $x_{k+2} - 2x_{k+1} + x_k^2 = 0$ is a second order *non-linear* difference equation. Indeed, it has one non-linear term, x_k^2.

It is worth to note that, given $\eta_1,\ \eta_2,\ ...\eta_n \in \mathbb{R}$, *the following IVP has a unique solution.*

$$(3) \qquad \begin{aligned} & x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k = f_k, \quad k \geq 0 \\ & x_0 = \eta_1, \quad x_1 = \eta_2, \quad ... \quad x_{n-1} = \eta_n. \end{aligned}$$

For further reference we write below the form of a linear homogeneous difference equation.

$$(4) \qquad x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k = 0, \quad k \geq 0.$$

When equation (2) is linear nonhomogeneous, we say that (4) is *the linear homogeneous difference equation associated* to it.

The fundamental theorem for linear homogeneous difference equations follows.

Theorem 1 *Let $x_1,\ x_2, ..., x_n \in \mathbb{R}^\infty$ be n linearly independent solutions of* (4). *Then the general solution of* (4) *is*

$$x_k = c_1\, x_{1,k} + ... + c_n\, x_{n,k}, \quad c_1, ..., c_n \in \mathbb{R}.$$

The fundamental theorem for linear nonhomogeneous difference equations follows.

Theorem 2 *Let $x_h \in \mathbb{R}^\infty$ be the general solution of the linear homogeneous differential equation associated to* (2) *and let $x_p \in \mathbb{R}^\infty$ be a particular solution of* (2)*. Then the general solution of* (2) *is*

$$x_k = x_{h,k} + x_{p,k}.$$

The superposition principle holds also in this case. Before writing its statement, we make a notation associated to equation (2):

$$\mathcal{L} : \mathbb{R}^\infty \to \mathbb{R}^\infty, \quad (\mathcal{L}x)_k = x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k,\ k \geq 0.$$

It can be proved that $\mathcal{L}$ is a linear map, and this property immediately gives the superposition principle which follows.

Theorem 3 *Let $f_1,\ f_2 \in \mathbb{R}^\infty$ and $\alpha \in \mathbb{R}$. Suppose that $x_{p1} \in \mathbb{R}^\infty$ is a particular solution of $\mathcal{L}x = f_1$ and x_{p2} is a particular solution of $\mathcal{L}x = f_2$.*

Then $x_p = x_{p1} + x_{p2}$ is a particular solution of $\mathcal{L}x = f_1 + f_2$ and $\tilde{x}_p = \alpha\, x_{p1}$ is a particular solution of $\mathcal{L}x = \alpha f_1$.

Making a summary of the fundamental theorems we can describe the main steps of a *method for finding the general solution* of a linear nonhomogeneous equation of the form (2), i.e.

$$x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k = f_k. \tag{2}$$

Step 1. Write the linear homogeneous difference equation associated $x_{k+n} + a_{1,k}\, x_{k+n-1} + a_{2,k}\, x_{k+n-2} + \cdots + a_{n,k}\, x_k = 0$ and find its general solution. Denote it by x_h. For this it is sufficient to find n linearly independent solutions, denote them by $x_1, ..., x_n$. Hence,

$$x_h = c_1\, x_1 + ... + c_n\, x_n, \quad c_1, ..., c_n \in \mathbb{R}.$$

Step 2. Find a particular solution of the linear nonhomogeneous equation (2). Denote it by x_p.

Step 3. Write the general solution of (2) as

$$x = x_h + x_p.$$

Example-exercise. Given $a \in \mathbb{R} \setminus \{0,\ 1\}$ and $b \in \mathbb{R}$, find the general solution of

$$x_{k+1} - a\, x_k = b, , \quad k \geq 0.$$

First we notice that this is a first order linear nonhomogeneous difference equation. We follow the steps of the method presented above.

Step 1. The linear homogeneous difference equation associated is

$$x_{k+1} - a\, x_k = 0.$$

In order to find its general solution it is sufficient if we find a non-null solution. We notice that $x_{1,k} = a^k$ verifies $x_{k+1} = a\, x_k$, hence it is a nonnull solution. Then

$$x_{h,k} = c\, a^k, \quad c \in \mathbb{R}.$$

Step 2. We notice that the constant sequence $x_p = \frac{b}{1-a}$ verifies $x_{k+1} - a\, x_k = b$.

Step 3. The general solution of $x_{k+1} - a\, x_k = b$ is

$$x_k = c\, a^k + \frac{b}{1-a}, \quad c \in \mathbb{R}.$$

Exercise. Given $b \in \mathbb{R}$ find the general solution of $x_{k+1} - \ x_k = b, \quad k \geq 0$ and of $x_{k+1} = b, \quad k \geq 0$.

Exercise. Given $a \in \mathbb{R}^\infty$ find the general solution of $x_{k+1} - a_k\, x_k = 0, \quad k \geq 0$.

Linear difference equations with constant coefficients. In this special case there is a method, called *the characteristic equation method* to find the n linearly independent solutions of an nth order linear homogeneous equation.

We write now a linear homogeneous difference equation with constant coefficients denoted $a_1, ..., a_n \in \mathbb{R}$.

$$x_{n+k} + a_1\, x_{n+k-1} + \cdots + a_{n-1}\, x_{k+1} + a_n\, x_k = 0, \tag{5}$$

and consider again the linear map $\mathcal{L}$ (defined above) corresponding to (5). From now on we assume that $a_n \neq 0$.

We start by noticing that, when looking for solutions of (5) of the form

$$x_k = r^k$$

(with $r \in \mathbb{R}^*$ that has to be found), we obtain that r must be a root of the nth degree polynomial equation with real coefficients

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \tag{6}$$

More precisely, we have that

$$\mathcal{L}(r^k) = r^k \, l(r),$$

where

$$l(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

Then every real root $\alpha \in \mathbb{R}$ of (6) provides a solution $(\alpha^k)_{k \geq 0}$ of (5). But we know that, in general, not all the roots of an algebraic equation are real. However, we will show how the roots of the algebraic equation (6) provide all the n linearly independent solutions of (5) needed to obtain its general solution.

For our purpose we need to see that the concept of real-valued solution for (5) can be extended to that of complex-valued solution. Denoting a complex-valued sequence by $\gamma \in \mathbb{C}^\infty$, its real part by $u \in \mathbb{R}^\infty$ and its imaginary part by $v \in \mathbb{R}^\infty$ we have $\gamma_k = u_k + i\, v_k$, for all $k \geq 0$. The sequence γ can be identified with a vectorial sequence with two real components u and v.

Since we defined a solution to be a sequence of real numbers, we will only say that a complex-valued sequence *verifies* or not a difference equation. With respect to the linear homogeneous difference equation with constant real coefficients (5) we have the following result.

Proposition 1 *Assume that the complex-valued sequence $\gamma \in \mathbb{C}^\infty$ verifies* (5). *Then, both its real part u and its imaginary part v are solutions of* (5).

Proof. In order to shorten the presentation, we use again the notation of the linear map $\mathcal{L}$ as presented before. Thus equation (5) can be written equivalently as $\mathcal{L}x = 0$ It is not difficult to see that $\mathcal{L}(\gamma) = \mathcal{L}(u + iv) = \mathcal{L}u + i\mathcal{L}v$, where, of course, $\mathcal{L}u$ and $\mathcal{L}v$ are real-valued sequences. By hypothesis we have that $\mathcal{L}(\gamma) = 0$. Thus $\mathcal{L}u = 0$ and $\mathcal{L}v = 0$, which give the conclusion. □

We need to work with the sequence of complex numbers

$$\gamma_k = (\alpha + i\beta)^k, \quad k \geq 0,$$

where $\alpha, \beta \in \mathbb{R}$ are fixed real numbers. In order to easily find the real part and the imaginary part of γ_k we propose to write the complex number $\alpha + i\beta$ in its polar form. It is known that for $\rho = \sqrt{\alpha^2 + \beta^2} \geq 0$ and $\theta \in [0, 2\pi)$ satisfying $\tan\theta = \beta/\alpha$, we have

$$\alpha + i\beta = \rho(\cos\theta + i\sin\theta),$$

and, further, for any $k \geq 0$,

$$\gamma_k = (\alpha + i\beta)^k = \rho^k(\cos\theta + i\sin\theta)^k = \rho^k(\cos k\theta + i\sin k\theta).$$

We thus have

Proposition 2 *If $r = \alpha + i\beta = \rho(\cos\theta + i\sin\theta)$ with $\beta \neq 0$, is a root of* (6)*, then $(\rho^k \cos k\theta)_{k\geq 0}$ and $(\rho^k \sin k\theta)_{k\geq 0}$ are solutions of* (5)*.*

We notice that, since the polynomial $l(r)$ has real coefficients, in the case that $r = \alpha + i\beta$ with $\beta \neq 0$ is a root of l, we have that its conjugate, $r = \alpha - i\beta$ is a root, too. According to the previous proposition, this gives that $(\rho^k \cos k\theta)_{k\geq 0}$ and $(-\rho^k \sin k\theta)_{k\geq 0}$ are solutions of (5). But this is no new information. In fact, it is usually said that the two solutions indicated in the proposition comes from the two roots $\alpha \pm i\beta$.

Thus, we have seen that any complex root provides a solution of (5). But still there is the possibility that the solutions obtained are not enough, since we know by the Fundamental Theorem of Algebra that a polynomial of degree n has indeed n roots, but counted with their multiplicities. We will show that

Proposition 3 *If $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l, then $(k^{(j)}\, r^k)_{k\geq 0}$ verifies* (5) *for any $j \in \{0, 1, 2, ..., m-1\}$. Here we made the notation*

$$k^{(j)} = k(k-1)...(k-j+1) = \frac{k!}{(k-j)!}.$$

Proof. We recall first that $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l if and only if

$$l(r) = l'(r) = ... = l^{(m-1)}(r) = 0.$$

By direct calculations we obtain for each $j \in \{0, 1, 2, ..., m-1\}$ that

$$\mathcal{L}(k^j\, r^k) = r^k \sum_{s=0}^{j} C_j^s\, k^{(j-s)}\, r^s\, l^{(s)}(r) = 0. \qquad \square$$

We describe now **The characteristic equation method** for the linear homogeneous difference equation with constant coefficients (5).

Step 1. Write the *characteristic equation* (6). Note that it is a polynomial equation of degree n (equal to the order of the difference equation) and with the same coefficients as the difference equation.

Step 2. Find all the n roots in $\mathbb{C}$ of (6), counted with their multiplicities.

Step 3. Associate n sequences obeying the following rules.

For $r = \alpha$ a real root of order m we take m sequences:

$$\alpha^k, \quad k\,\alpha^k, \quad k(k-1)\,\alpha^k, \quad \dots \quad , \quad k^{(m-1)}\alpha^k.$$

For $r = \alpha + i\beta$ and $r = \alpha - i\beta$ with $\beta > 0$ roots of order m we take $2m$ sequences

$$\mathrm{Re}(\alpha+i\beta)^k, \quad \mathrm{Im}(\alpha+i\beta)^k, \quad \dots \quad , \quad k^{(m-1)}\mathrm{Re}(\alpha+i\beta)^k, \quad k^{(m-1)}\mathrm{Im}(\alpha+i\beta)^k.$$

The following useful result holds true.

Theorem 4 *The n sequences found by applying the characteristic equation method are n linearly independent solutions of* (5)*.*

In the discussion before the presentation of this method we proved that the n functions are solutions of (5). The proof of the above theorem would be completed by showing that they are linearly independent. But this is beyond the aim of these lectures.

Linear difference systems with constant coefficients. Let $n \in \mathbb{N}^*$, $A \in \mathcal{M}_n(\mathbb{R})$ and consider the difference system

$$X_{k+1} = AX_k, \quad k \geq 0. \tag{7}$$

We say that n is the *dimension* of the system while A is the *coefficients matrix* of the system. A *solution* of (7) is a sequence $(X_k)_{k\geq 0}$ of vectors in $\mathbb{R}^n$. We denote the set of all sequences of n-dimensional vectors by $\mathbb{R}^{n,\infty}$. Such a sequence can be also seen as a function of discrete variable $X : \mathbb{N} \to \mathbb{R}^n$.

We remark that any nth order linear homogeneous difference equation with constant coefficients can be written in the form (7). Indeed, consider

$$x_{n+k} + \alpha_1\, x_{n+k-1} + \cdots + \alpha_{n-1}\, x_{k+1} + \alpha_n\, x_k = 0,$$

where the coefficients are real constants. The n scalar unknowns of the system are

$$x_{1,k} = x_k, \quad x_{2,k} = x_{k+1}, \quad \dots \quad , \quad x_{n,k} = x_{n+k-1},$$

such that they satisfy the following system

$$\begin{aligned} x_{1,k+1} &= x_{2,k} \\ x_{2,k+1} &= x_{3,k} \\ &\dots \\ x_{n,k+1} &= -\alpha_n x_{1,k} - \alpha_{n-1} x_{2,k} + \dots - \alpha_1 x_{n,k}. \end{aligned} \tag{8}$$

We make the notations

$$X_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \\ \dots \\ x_{n,k} \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 \end{pmatrix},$$

such that system (8) can be written in the form (7).

Theorem 5 *Let $\eta \in \mathbb{R}^n$ be fixed. The unique solution of the IVP*

$$X_{k+1} = A\, X_k, \quad k \geq 0, \quad X_0 = \eta$$

is

$$X_k = A^k\, \eta, \quad k \geq 0.$$

The general solution of the difference system $X_{k+1} = A\, X_k$ is

$$X_k = A^k\, C, \quad k \geq 0 \quad , C \in \mathbb{R}^n.$$

Exercise. Let $A \in \mathcal{M}_2(\mathbb{R})$ be a matrix with two real distinct eigenvalues denoted by λ_1 and λ_2. Assume that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Prove that any solution of the difference system $X_{k+1} = AX_k$ satisfies $\lim\limits_{k\to\infty} X_k = 0$.

Exercise. We consider the difference equation $x_{k+2} = x_{k+1} + x_k$ and the corresponding IVP with the initial values $x_0 = 0$, $x_1 = 1$.

(i) Find the general solution of the equation using the characteristic equation method. Than find the solution of the IVP.

(ii) Write the 2-dimensional system equivalent to this equation and the corresponding IVP. Find the general solution of the system. Than find the solution of the IVP.

(iii) Compare the results obtained at (i) and, respectively, (ii).