Exercise 15.

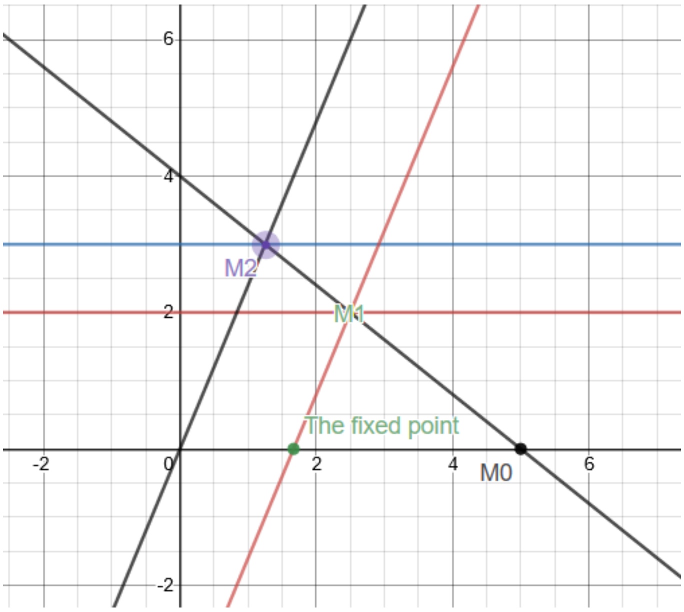
Let β be the bundle of vertex $M_0(5,0)$. An arbitrary line from β intersects the lines $d_1: y-2=0$, $d_2: y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

Let $l: r(x-x_0)+s(y-y_0)=0, \forall (r,s) \in \mathbb{R}^2\{(0,0)\}$

$$l: r(x-5)+sy=0, \forall (r,s) \in \mathbb{R}^2\{(0,0)\}$$

$$\begin{cases} y-2=0 \\ r(x-5)+sy=0 \end{cases} \Rightarrow \begin{cases} y=2 \\ r(x-5)+2s=0 \end{cases} \Rightarrow \begin{cases} y=2 \\ x=5-\dfrac{2s}{r} \end{cases} \Rightarrow M_1(5-\frac{2s}{r},2)$$

$$\begin{cases} y-3=0 \\ r(x-5)+sy=0 \end{cases} \Rightarrow \begin{cases} y=3 \\ r(x-5)+3s=0 \end{cases} \Rightarrow \begin{cases} y=3 \\ x=5-\dfrac{3s}{r} \end{cases} \Rightarrow M_2(5-\frac{3s}{r},3)$$



$$OM_2: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 5-\dfrac{3s}{r} & 3 & 1 \end{vmatrix} = 0$$

$$OM_2: \left(5-\frac{3s}{r}\right)y-3x=0 \Rightarrow m=\frac{3}{5-\frac{3s}{r}}=\frac{3r}{5r-3s}$$

$d: y-y_0=m(x-x_0)$, d – parallel with OM_2 going through M_1.

$$d: y-2=\frac{3r}{5r-3s}\left(x-5+\frac{2s}{r}\right)$$

$$d: y-2=\frac{3r}{5r-3s}x-\frac{15r}{5r-3s}+\frac{6s}{5r-3s} \mid *(5r-3s)$$

$$d: (5r-3s)y-10r+6s=3rx-15r+6s$$

$$d: 3rx+(3s-5r)y-5r=0$$

Let A be a possible fixed point. If it does exist, we can find it at the intersection of 2 different lines d, generated by the bundle.

$$r=1, s=0: 3x-5y-5=0$$

$$r=0, s=1: 3y=0$$

$$\begin{cases} 3x-5y-5=0 \\ 3y=0 \end{cases} \Rightarrow \begin{cases} 3x-5=0 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=\dfrac{5}{3} \\ y=0 \end{cases} \Rightarrow A\left(\frac{5}{3},0\right)$$

If A is a fixed point then $A \in d, \forall (r,s) \in \mathbb{R}^2\{(0,0)\}$.

$3r*\frac{5}{3}+(3s-5r)*0-5r=5r-5r=0 \Rightarrow A$ is a fixed point on d.