Exercise (5.7.). Consider the planes

$$(\pi_1): \; 2x + y - 3z - 5 = 0$$

$$(\pi_2): \; x + 3y + 2z + 1 = 0$$

Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained in the acute regions of the dihedral angles formed by the two planes.

Proof. The bisector planes can be seen as the loci (the sets) of points $P(x, y, z)$ in space for which we have

$$\text{dist}(P, \pi_1) = \text{dist}(P, \pi_2)$$

For these points we will then have

$$\frac{|2x + y - 3z - 5|}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{|x + 3y + 2z + 1|}{\sqrt{1^2 + 3^2 + 2^2}}$$

Which gives:

$$|2x + y - 3z - 5| = |x + 3y + 2z + 1|$$

Therefore $2x + y - 3z - 5 = \pm(x + 3y + 2z + 1)$ and every choice of the sign gives us the equation of a bisector plane. These will then be:

$$(b_1) \; 2x + y - 3z - 5 = x + 3y + 2z + 1$$

$$x - 2y - 5z - 6 = 0$$

$$(b_2) \; 2x + y - 3z - 5 = -x - 3y - 2z - 1$$

$$3x + 4y - z - 4 = 0$$

Now, in order to decide if (b_1) or (b_2) is the plane that we want, all we need to do is pick a particular point M on one of them, say on (b_1). Let us choose $M = (6, 3, 0)$. If this point is in the acute region, then the bisector plane (b_1) is in the acute region. If it's in the obtuse region, then the other bisector plane, (b_2), is in the acute region. If the point is in both regions, which can only happen if we chose a point that is on the common line of π_1 and π_2, then we have to pick a different point.

We have the following criterion (Exercise 5.6.) that enables us to decide if a point is in the acute region or the obtuse region of the dihedral angle of two planes:

For two planes given by the equations:

$$(p_1): \; F_1(x,y,z) = A_1x + B_1y + C_1z + D_1 = 0$$

$$(p_2): \; F_2(x,y,z) = A_2x + B_2y + C_2z + D_2 = 0$$

which are not parallel and not perpendicular. Then a point $M(x,y,z)$ belongs to the acute region of their dihedral angle if and only if

$$F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$$

Let us apply this criterion to $M(6,3,0)$:

$$F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) =$$

$$= (2 \cdot 6 + 1 \cdot 3 - 3 \cdot 0 - 5) \cdot (1 \cdot 6 + 3 \cdot 3 + 2 \cdot 0 + 1) \cdot (2 \cdot 1 + 1 \cdot 3 - 3 \cdot 2) =$$

$$= 10 \cdot 16 \cdot (-1) = -160 < 0$$

Therefore M belongs to the acute region and hence (b_1) is the bisector plane that is included in the acute region.

$\square$