

Laboratory 6: Errors ... where they come from?

1. Round-off errors.

Using the command **evalf** evaluate $25^{1/8}$. Raise the result to the power eight. What do you expect to obtain?

Theory. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ we define the sequence $(x_k)_{k \geq 0}$ of iterations of f starting from $x_0 \in \mathbb{R}$ as satisfying the recurrence

$$x_{k+1} = f(x_k), \text{ for any } k \geq 0.$$

A fixed point of f is a number $\eta^* \in \mathbb{R}$ such that $f(\eta^*) = \eta^*$. Thus, in order to find the fixed points of f one has to solve the equation

$$f(x) = x.$$

2. The logistic map: from order to chaos.

For each $\lambda \in \mathbb{R}$ we consider $f_\lambda : [0, 1] \rightarrow [0, 1]$, $f_\lambda(x) = \lambda x(1 - x)$.

For $\lambda = 1$, consider the function f_1 having the expression $x(1 - x)$. Find the fixed points of f_1 . Then compute the first 200 iterations of f_1 starting from $x_0 = 0.5$ and represent them using **pointplot**. Interpret what you see and describe in your notebook your observations. Do the same for the initial values $x_0 = 0$ and, respectively, $x_0 = 0.7$. You can try also with other initial values if you want.

Repeat what you have done before for $\lambda = 2$, $\lambda = 3.1$, $\lambda = 3.5$, $\lambda = 3.55$, $\lambda = 3.6$ and, respectively, $\lambda = 3.8$. It is important to consider them all!

3. Sensitivity with respect to small errors is a feature of chaos.

Consider two functions $f(x) = 4x(1 - x)$ and, respectively, $\tilde{f}(x) = 4x - 4x^2$. Of course $f = \tilde{f}$. Note that f maps the interval $[0, 1]$ into itself. In particular this implies that each of the iterations of f starting with $x_0 = 0.67$ is in the interval $[0, 1]$.

What you have to do is to compute (better in the same **for**) the first 40 iterations of f and, respectively, of \tilde{f} starting with $x_0 = 0.67$. What do you expect and what you obtain?

Theory. We consider the IVP

$$y' = f(x, y), \quad x \in [x_0, x^*], \quad y(x_0) = y_0.$$

The Euler method formula with constant step size $h = (x^* - x_0)/n$ is

$$y_{k+1} = y_k + h f(x_k, y_k), \quad x_{k+1} = x_k + h, \quad k = \overline{0, n-1}.$$

The improved Euler method formula with constant step size $h = (x^* - x_0)/n$ is

$$y_{k+1} = y_k + \frac{h}{2} f(x_k, y_k) + \frac{h}{2} f(x_k + h, y_k + h f(x_k, y_k)), \quad x_{k+1} = x_k + h, \quad k = \overline{0, n-1}.$$

If we denote by φ the exact solution of the given IVP, then the value y_k is an approximation of $\varphi(x_k)$.

4. For the IVP $y' = 2xy, \quad x \in [0, 1], \quad y(0) = 1$ first find its exact solution and then use

(a) Euler's method with step size $h = 0.1$;

(b) the improved Euler's method with step size $h = 0.1$;

to find approximate values of the solution in the interval $[0, 1]$.

Compute and write in your notebooks the absolute value of the difference between the correct value and the approximate one at $x = 0.5$ and, respectively, $x = 1$. Formulate a conclusion.

5. For the IVP $y' = y^2 + x^2, \quad y(0) = 0$, apply the two numerical methods in the interval $[0, 2]$ with step size $h = 0.1$.

Use *DEplot* to represent the direction field of the differential equation and the graph of the solution of the IVP. Note that this is the graph of an approximate solution, which is found with a Runge-Kutta type numerical method.

For what reason are these results so different when you approach 2? It seems that on the intervals $[0, 1]$, or even $[0, 1.5]$ the approximations are quite good!

6. What is a stiff equation? For the IVP $y' = -250y, \quad y(0) = 1$, apply the Euler's method and the improved Euler's method in the interval $[0, 1]$ with step size $h = 0.1$. Compare the approximate values of the solution with the exact one. Write the error in each case in your notebook. Have you ever seen such a huge error?