

Exercise 7.3.

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Prove the following identity:

$$(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$$

$$(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = ((\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w})) \cdot (\vec{w} \times \vec{u}) \quad (1)$$

$$\begin{aligned} (\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w}) &= ((\vec{u} \times \vec{v}) \cdot \vec{w}) \cdot \vec{v} - ((\vec{u} \times \vec{v}) \cdot \vec{v}) \cdot \vec{w} \\ &= (\vec{u}, \vec{v}, \vec{w}) \cdot \vec{v} - (\vec{u}, \vec{v}, \vec{v}) \cdot \vec{w} \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \Rightarrow$$

$$\vec{v} \times \vec{v} = \|\vec{v}\|^2 \cdot \sin(\vec{v}, \vec{v}) = \vec{0}$$

$$\Rightarrow (\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w}) = (\vec{u}, \vec{v}, \vec{w}) \cdot \vec{v} \quad \left. \begin{aligned} & \\ & \text{From (1)} \end{aligned} \right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow (\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) &= ((\vec{u}, \vec{v}, \vec{w}) \cdot \vec{v}) \cdot (\vec{w} \times \vec{u}) = \\ &= (\vec{u}, \vec{v}, \vec{w}) \cdot \vec{v} \cdot (\vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w}) \cdot (\vec{v} \cdot (\vec{w} \times \vec{u})) \\ &= (\vec{u}, \vec{v}, \vec{w}) \cdot ((\vec{v} \times \vec{w}) \cdot \vec{u}) = (\vec{u}, \vec{v}, \vec{w}) \cdot (\vec{v}, \vec{w}, \vec{u}) \quad (2) \end{aligned}$$

We know that $(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \text{sgn}(\sigma) \cdot (\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)})$
 $\forall \vec{a}_1, \vec{a}_2, \vec{a}_3 \in V, \sigma \in S_3 \Rightarrow$

$$\Rightarrow (\vec{v}, \vec{w}, \vec{u}) = (\vec{w}, \vec{u}, \vec{v}) = (\vec{u}, \vec{v}, \vec{w}) \quad \left. \begin{aligned} & \\ & \text{From (2)} \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow (\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$$