Crăciun Ioan-Flaviu
912
8 iulie 2020

1. order 7 => the difference equation has a polynomial of degree 7 => 7 roots in $\mathbb{C}$

a) $t^2\cos 2t$ is a solution => $t\cos 2t$, $\cos 2t$, $t^2\sin 2t$, $t\sin 2t$, $\sin 2t$ are solutions

$r_1 = (1 \pm 2i)^3$

e^{7t} - sol => $r_2 = 7$

we found 7 roots, they could be roots of difference equation

b) $t^2\cos 2t$ => $t^2\cos 2t$, $t\cos 2t$, $\cos 2t$, $t^2\sin 2t$, $t\sin 2t$, $\sin 2t$ are solutions so $r_1 = (1 \pm 2i)^3$

$t\sin 7t$ => $t\sin 7t$, $\sin 7t$, $t\cos 7t$, $\cos 7t$

Too many roots, cannot be difference equation

c) $t^2\cos 2t$ => $t^2\cos 2t$, $t\cos 2t$, $\cos 2t$, $t^2\sin 2t$, $t\sin 2t$, $\sin 2t$

~~already~~ 6

We have 6 solutions, a difference equation of order 7 would need one more solution, but there are such equations.

Crăciun Ioan-Flavin
912 Page 2
8 iulie 2020

2. a) $x'' + 9x = 0$

char poly $r^2 + 9 = 0 \Rightarrow r_{12} = \pm 3i \Rightarrow$ solutions are $\cos 3t, \sin 3t$

The general solution is $c_1 \cos 3t + c_2 \sin 3t$

$c_1 \neq 0 \Leftrightarrow x = c_1\left(\cos 3t + \frac{c_2}{c_1} \sin 3t\right)$

Let $\frac{c_2}{c_1} = tg\,\varphi,\ \varphi \in (0, 2\pi)$

$x = c_1(\cos 3t + tg\,\varphi \cdot \sin 3t)$

$x = c_1\left(3t + \frac{\sin\varphi}{\cos\varphi} \sin 3t\right)$

$x = \frac{c_1}{\cos\varphi}(\cos 3t \cos\varphi + \sin\varphi \sin 3t)$

$x = \frac{c_1}{\cos\varphi}(\cos(3t - \varphi))$

$A_0 = \frac{c_1}{\cos\varphi} \Rightarrow x = A_0 \cos(3t - \varphi_0)$ is the general solution

b) $(at \sin 3t)'' + 9at \sin 3t = A_1 \cos 3t$

$(+3at \cos 3t + a \sin 3t)' + 9at \sin 3t = A_1 \cos 3t$

$+3a\,\cancel{\cos 3t} + (+3a\,\cancel{\cos 3t}) - 9t\,\cancel{\sin 3t} + 9\cancel{at}\sin 3t = A_1\,\cancel{\cos 3t}$

$6a = A_1 \Rightarrow a = \frac{A_1}{6}$

$x_p = \frac{A_1}{6} t \sin 3t$

Crăciun Ioan-Flaviu
912
8 iulie 2020

c) $x'' + 9x = A_1 \cos 3t$

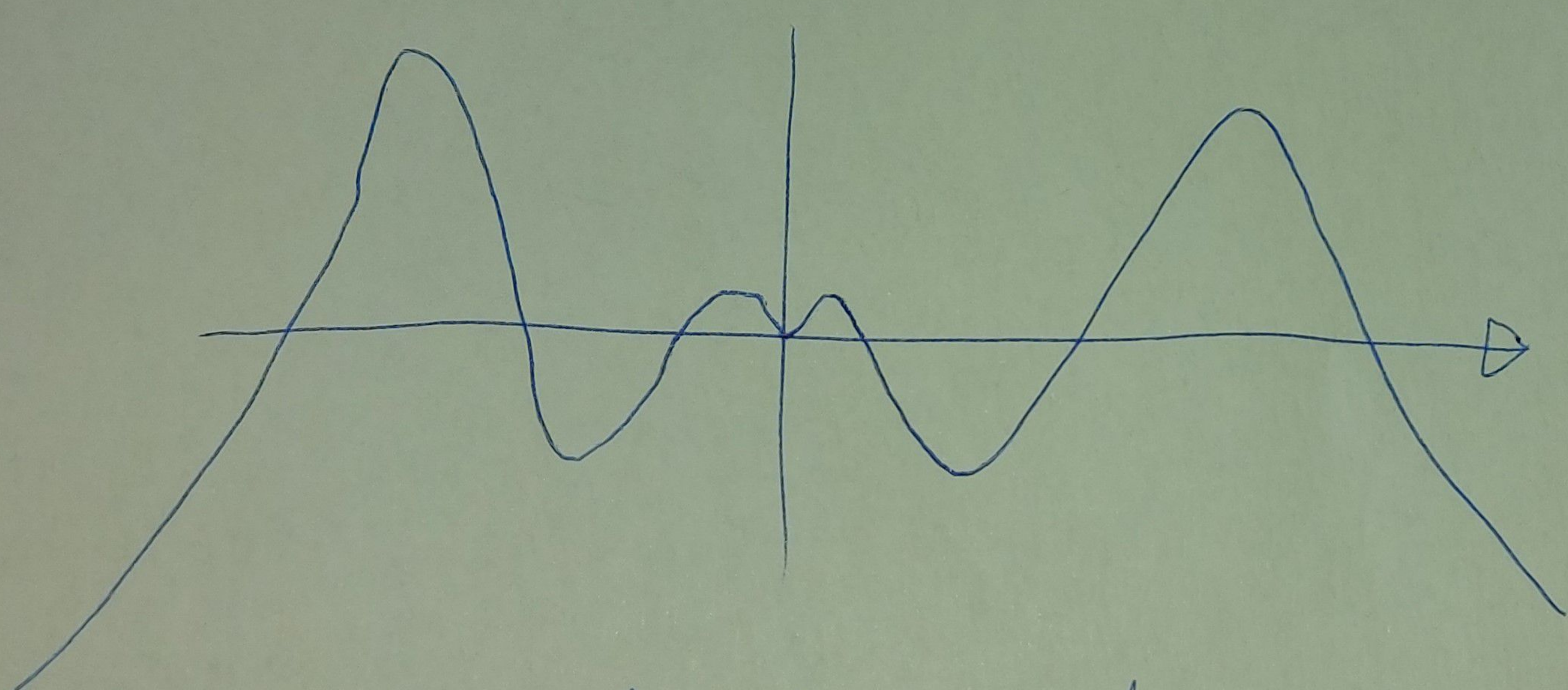
$x(0) = 0$

$x'(0) = 0$

Notice the partial solution found at b) $x_p = \frac{A_1}{6} t \sin 3t$ is the unique solution of the IVP, $\frac{A_1}{6} \cdot 0 \cdot \sin 0 = 0$ and

$\frac{A_1}{6} \sin 3 \cdot 0 + \frac{A_1}{2} 0 \cos 3 \cdot 0 = 0$

$\theta(t) = \frac{A_1}{6} t \sin 3t$ is the angle between the rod and the vertical at time t, so plotting it it looks like



And it looks like $\theta(t)$ will go up and down faster and faster, so it's like the pendulum has an external force applying on it.

Crăciun Ioan-Flaviu
912

8 iulie 2020

d) $x'' + 9x = A_2$

First solve $x'' + 9x = 0$

We found $x_h = A_0 \cos(3t - \varphi_0)$ at a)

Look for a constant $9x_p = A_2 \Rightarrow x_p = \frac{A_2}{9}$

$x = x_h + x_p = A_0 \cos(3t - \varphi_0) + \frac{A_2}{9}$

e) We can apply the superposition principle and add the 2 particular solutions found at b) and d)

$x = A_0 \cos(3t - \varphi_0) + \frac{A_1}{6} t \sin 3t + \frac{A_2}{9}$

Crăciun Ioan-Flaviu
912 Page 5
8 iulie 2020

3. a) $y' = y$
$y(0) = 1$
$y = e^t$ is IVP, $(e^t)' = e^t$, $e^0 = 1$

b) $y_{k+1} = y_k + (x_{k+1} - x_k) f(x_k, y_k)$

$y_{k+1} = y_k + \frac{1}{100} \cdot y_k$

$y_{k+1} = y_k (1 + \frac{1}{100})$ for $y_0 = 0$, $k \in \overline{1,100}$

c) Notice $y_{k+1} = y_k (1 + \frac{1}{100})$, so $y_{k+2} = y_k (1 + \frac{1}{100})^2$,

$y_{100} = y_0 (1 + \frac{1}{100})^{100} \Rightarrow y_{100} = (1 + \frac{1}{100})^{100}$.

Plugging $(1 + \frac{1}{100})^{100}$ into a calculator we get 2.704813...
but we know $e = 2.71828...$ so it's not precisely e but a good approximation. In fact $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.

Crăciun Dan-Flavius
912
8 iulie 2020

4. a) $\dot{x} = -y\sqrt{3} + x(9 - x^2 - 3y^2)$

$\dot{y} = \frac{x}{\sqrt{3}} + y(9 - x^2 - 3y^2)$

The Jacobian $Jf(x,y) = \begin{pmatrix} 9 - 3x^2 - 3y^2 & -\sqrt{3} - 6xy \\ \frac{1}{\sqrt{3}} - 2yx & 9 - x^2 - 9y^2 \end{pmatrix}$

At $(0,0)$ $Jf(0,0) = \begin{pmatrix} 9 & -\sqrt{3} \\ \frac{1}{\sqrt{3}} & 9 \end{pmatrix}$

$\begin{vmatrix} 9-\lambda & -\sqrt{3} \\ \frac{1}{\sqrt{3}} & 9-\lambda \end{vmatrix} = (9-\lambda)^2 + 1 = 0$

$\lambda^2 - 18\lambda + 82 = 0$

$\lambda_{1,2} = \frac{18 \pm \sqrt{18^2 - 4 \cdot 82}}{2} = 9 \pm i$

$\lambda_1 = 9 + i$

$\lambda_2 = 9 - i$

The eigenvalues have real part $\neq 0 \Rightarrow (0,0)$ is hyperbolic.

The system is a focus and $(0,0)$ is a global repellor

Crăciun Ioan-Flaviu
912 Page 7
8 iulie 2020

b) To show that $\varphi(t,3,0) = (3\cos t, \sqrt{3}\sin t)$ we need to show that is is a solution to the dynamical system and it also respects the IVP $x(0) = 3$, $y(0) = 0$.

$$(3\cos t)' = -3\sin t + 3\cos t(9 \underbrace{- 9\cos^2 t - 9\sin^2 t}_{-9})$$

$$-3\sin t = -3\sin t$$

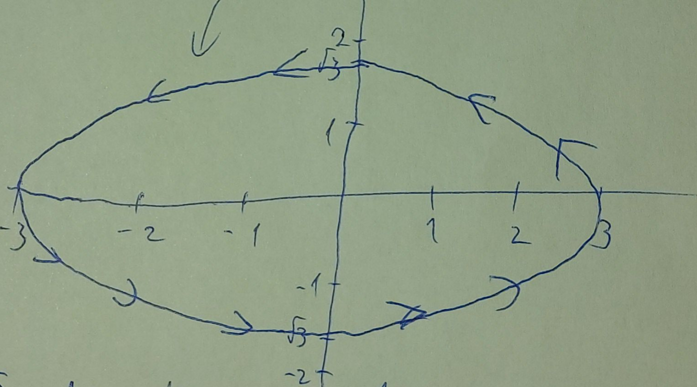
$$(\sqrt{3}\sin t)' = \frac{3\cos t}{\sqrt{3}} + \sqrt{3}\sin t(9 \underbrace{- 9\cos^2 t - 9\sin^2 t}_{-9})$$

$$\sqrt{3}\cos t = \sqrt{3}\cos t$$

arrows in trigonometric direction

$3\cos 0 = 3$

$\sqrt{3}\sin 0 = 0$



Notice that $(3\cos t, \sqrt{3}\sin t)$ is the parametrization of an ellipse centered at $(0,0)$ with height $2\sqrt{3}$ and width 6.

Crăciun Ioan-Flaviu
912
Page 8
8 iulie 2020

c) $(x,y) \to (r,\varphi)$

$$\begin{cases} \frac{x}{\sqrt{3}} = r\cos\varphi \\ y = r\sin\varphi \end{cases} \to \begin{cases} r^2 = \frac{x^2}{3} + y^2 \\ \tan\varphi = \frac{y}{\left(\frac{x}{\sqrt{3}}\right)} \end{cases}$$

$$\begin{cases} r r' = \frac{x}{\sqrt{3}}\left(\frac{x}{\sqrt{3}}\right)' + y y' \\ \frac{\varphi'}{\cos^2\varphi} = \frac{y'\left(\frac{x}{\sqrt{3}}\right) - y\left(\frac{x}{\sqrt{3}}\right)'}{\left(\frac{x}{\sqrt{3}}\right)^2} \end{cases}$$

$$\cancel{\begin{cases} r r' = \frac{-xy\sqrt{3} + x^2(9-x^2-3y^2)}{\sqrt{3}^2} + \frac{yx}{\sqrt{3}} + y^2(9-x^2-3y^2) \\ \frac{\varphi'}{\cos^2\varphi} = \frac{\frac{x^2}{\sqrt{3}} + xy(9-x^2-3y^2) - (y^2\sqrt{3} + xy(9-x^2-3y^2))}{x^2} \end{cases}}$$

$$\begin{cases} r r' = \frac{x}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\left(-\cancel{y\sqrt{3}} + x(9-x^2-3y^2)\right) + \cancel{\frac{yx}{\sqrt{3}}} + y^2(9-x^2-3y^2) \\ \frac{\varphi'}{\cos^2\varphi} = \frac{\frac{x}{\sqrt{3}}\left(\frac{x}{\sqrt{3}} + \cancel{y(9-x^2-3y^2)}\right) - \frac{y}{\sqrt{3}}\left(-y\sqrt{3} + \cancel{x(9-x^2-3y^2)}\right)}{\frac{x^2}{3}} \end{cases}$$

$$\begin{cases} r r' = \left(\frac{x^2}{3} + y^2\right)(9 - x^2 - 3y^2) \\ \frac{\varphi'}{\cos^2\varphi} = \frac{\frac{x^2}{3} + y^2}{\frac{x^2}{3}} \end{cases}$$

substitute $\frac{x^2}{3} + y^2$ with r^2

substitute $\frac{x^2}{3}$ with $r^2\cos^2\varphi$

Crăciun Ioan-Flaviu
912
8 iulie 2020

Page 9

c) $\begin{cases} \cancel{r} r' = r^{\cancel{2}}(9-3r^2) \\ \dfrac{\varphi'}{\cancel{\cos^2\varphi}} = \dfrac{\cancel{r^2}}{\cancel{r^2}\cancel{\cos^2\varphi}} \end{cases}$

$\begin{cases} r' = -3r^3 + 9r \\ \varphi' = 1 \end{cases}$

d) we know $\varphi(t,3,0)$ is the ellipse we found, also notice in the new system when r gets large r' is negative and when ~~r is positive r' is~~ r is small r' is positive. So the phase portrait will look like: