Dynamical Systems. Seminar 5.

First integrals for planar autonomous systems

We consider the planar autonomous system

$$(1)\quad \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1-function.

<u>Definition 1</u> Let $U \subset \mathbb{R}^2$ be an open nonempty set. We say that $H : U \to \mathbb{R}$ is a first integral in U of system (1) if H is a non-constant C^1 function and $H(\varphi(t,\eta)) = H(\eta)$, for all $t \in I_\eta$, $\eta \in U$.

As usual, $\varphi(t,\eta)$ denotes the flow of (1).

<u>Remark.</u> If H is a first integral then the orbits lie on the level curves of H; $H(x,y) = c$, $c \in \mathbb{R}$.

<u>Examples</u> 1) The function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = xy$ is a first integral of $\dot{x} = x$, $\dot{y} = -y$.

In order to check this we need the expression of the flow of the system. For each $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$ we consider the ~~initial~~ IVP $\dot{x} = +x$, $\dot{y} = -y$, $x(0) = \eta_1$, $y(0) = \eta_2$.

It is easy to see that its unique solution is $x = \eta_1 e^{t}$, $y = \eta_2 e^{-t}$, hence the flow $\varphi : \mathbb{R}^2 \to \mathbb{R}$ has the expression $\varphi(t,\eta_1,\eta_2) = (\eta_1 e^{t}, \eta_2 e^{-t})$.

We have $H(\varphi(t,\eta)) = \eta_1 e^{t} \cdot \eta_2 e^{-t} = \eta_1 \eta_2 = H(\eta)$ for all $(t,\eta) \in \mathbb{R}^2$. The conclusion follows by Def 1. $\square$

2) The function $H:\mathbb{R}^2 \to \mathbb{R}$, $H(x,y)=x^2+y^2$ is a first integral of the system $\dot{x}=-y$, $\dot{y}=x$.

Again, we need the expression of the flow. For each $\eta=\binom{\eta_1}{\eta_2}\in\mathbb{R}^2$ we consider the IVP $\dot{x}=-y$, $\dot{y}=x$, $x(0)=\eta_1$, $y(0)=\eta_2$. We find the general solution of the system reducing it to a second order d.e. We have $\ddot{x}=-\dot{y}=-x$. Thus $\ddot{x}+x=0$. Its characteristic equation is $r^2+1=0$, whose roots are $r_{1,2}=\pm i \longmapsto \cos t, \sin t$. Hence

$x=c_1\cos t+c_2\sin t$, $c_1,c_2\in\mathbb{R}$ and

$y=-\dot{x}=c_1\sin t-c_2\cos t$. Moreover, we get

$x(0)=c_1$ and $y(0)=-c_2$. Then $c_1=\eta_1$, $c_2=-\eta_2$

and the flow $\varphi(t,\eta)=(\eta_1\cos t-\eta_2\sin t,\ \eta_1\sin t+\eta_2\cos t)$.

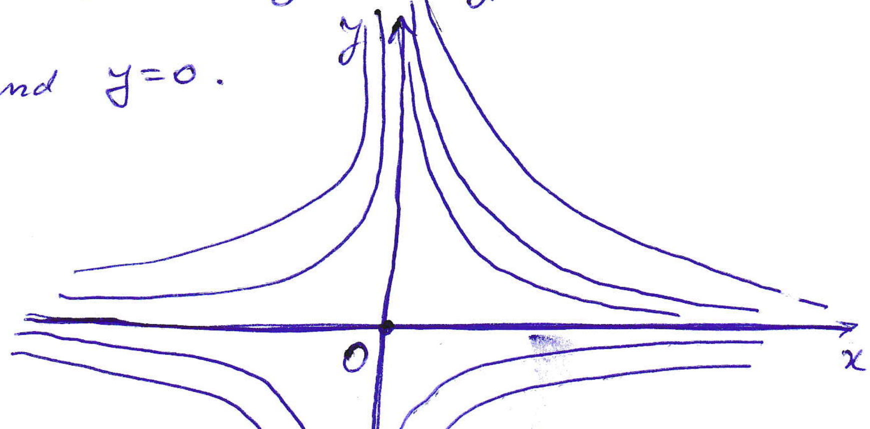
We have $\underline{H(\varphi(t,\eta))}=(\eta_1\cos t-\eta_2\sin t)^2+(\eta_1\sin t+\eta_2\cos t)^2$

$=\eta_1^2\cos^2 t-\cancel{2\eta_1\eta_2\cos t\sin t}+\eta_2^2\sin^2 t+\eta_1^2\sin^2 t+$

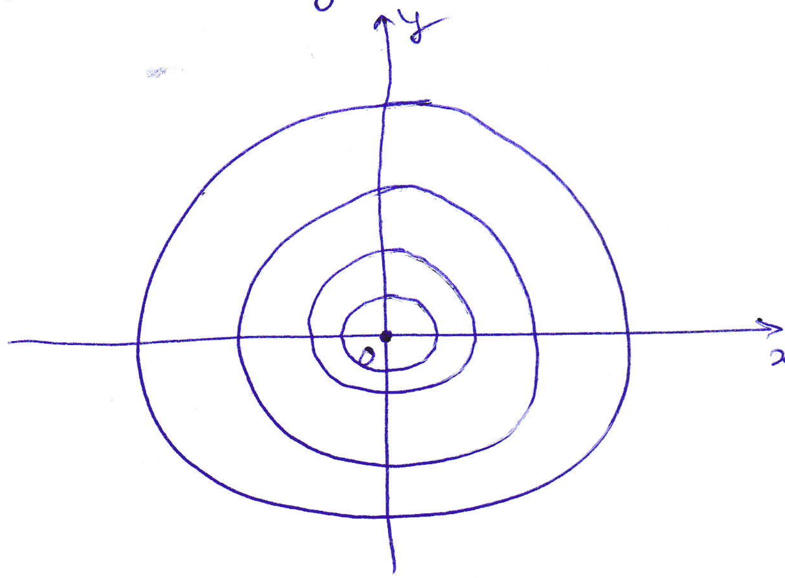
$+\cancel{2\eta_1\eta_2\sin t\cos t}+\eta_2^2\cos^2 t=\eta_1^2+\eta_2^2=\underline{H(\eta)}$.

The conclusion follows again by Def 1. $\square$

<u>Remarks</u> 1) The level curves of $H(x,y)=xy$ have the ~~lin~~ equations $xy=c$, $c\in\mathbb{R}$ or $y=\frac{c}{x}$, $c\in\mathbb{R}$. They are hyperbolas with asymptotes $x=0$ and $y=0$.



2) The level curves of $H(x,y) = x^2 + y^2$ have the equations $x^2+y^2 = c$, $c \in \mathbb{R}$. They are circles centered in the origin of coordinates.



Now we intend to present a method to find a first integral of (1), without the explicit knowledge of the flow.

<u>The cartesian differential equation of the orbits of (1) is</u>

$$(2) \quad \frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$$

(In this notation x and y are not functions of t).

In the case we are able to integrate this d.e. and to write its general solution in the form $H(x,y) = c$, $c \in \mathbb{R}$, then H is a first integral of (1).

We will integrate (2) only in the case that (2) is a <u>separable d.e.</u>, more exactly it has the form

$$(3) \quad \frac{dy}{dx} = a(x)\, g(y) \, , \quad a, g \text{ are continuous.}$$

In order to integrate (3) we first separate the variables and write $\frac{dy}{g(y)} = a(x)\,dx$. Then integrate:

$$\int \frac{dy}{g(y)} = \int a(x)dx \quad \text{and obtain} \quad G(y) = A(x) + c,\ c\in\mathbb{R}.$$

Exercises

Find a first integral and represent the phase portrait of the following linear systems (without finding explicitely the flow).

1) $\begin{cases} \dot{x} = x \\ \dot{y} = -3y \end{cases}$ 2) $\begin{cases} \dot{x} = 2y \\ \dot{y} = -3x \end{cases}$ 3) $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$

Solutions: 1) The cartesian differential equation of the orbits is $\frac{dy}{dx} = \frac{-3y}{x}$ which is separable.

Then we have $\int \frac{dy}{y} = -3\int \frac{dx}{x}$ and, moreover,

$\ln|y| = -3\ln|x| + c,\ c\in\mathbb{R}$

At this stage we obtain a first integral

$$H_1(x,y) = \ln|y| + 3\ln|x|,\quad x\neq 0,\ y\neq 0,$$

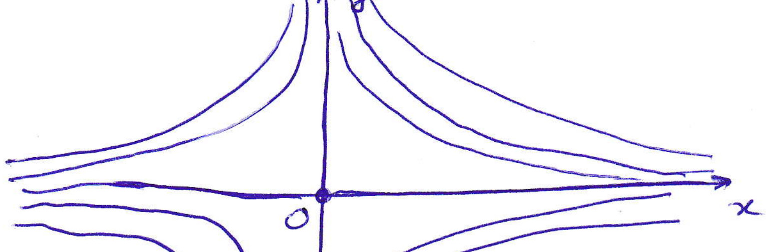
but we will see that we can obtain another one which has a simpler form.

So, we write $\ln|y| + 3\ln|x| = c$, $\ln|yx^3| = c$,

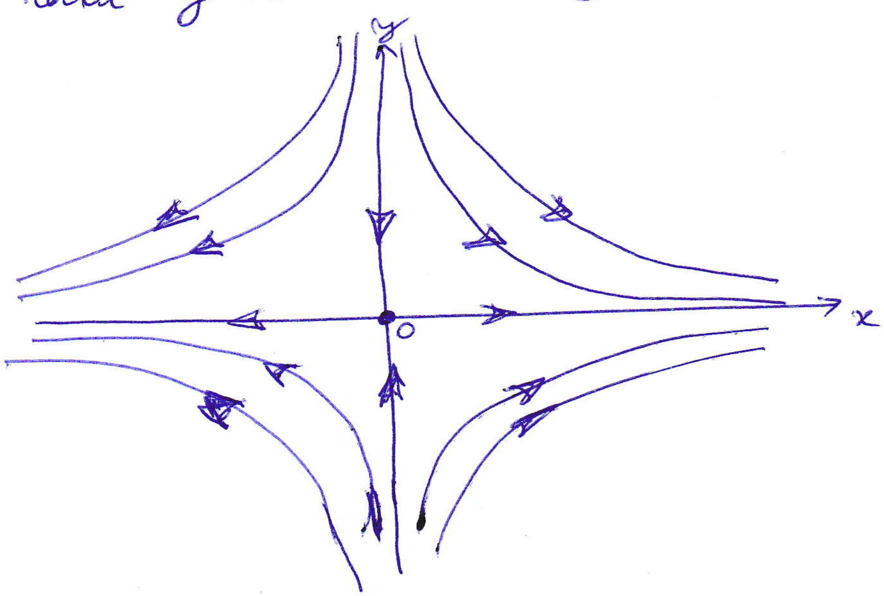
$\boxed{yx^3 = k}$, $k\in\mathbb{R}$ or $y = \frac{k}{x^3}$, $k\in\mathbb{R}$.

Then $H_2(x,y) = x^3y$, $H_2 : \mathbb{R}^2 \to \mathbb{R}$ is another f.i.

The level curves of H_2 looks like



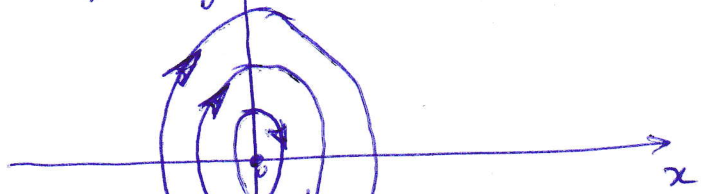
These are also the orbits of the planar system. In order to complete its phase portrait we must insert arrows on each orbit. In the upper semiplane, $y>0$, we have that $\dot{y}=-3y<0$. Thus the arrows on the orbits from the upper semiplane must indicate that y is decreasing. In the lower semiplane, $y<0$, we have that $\dot{y}=-3y>0$. Thus the arrows on the orbits from the lower semiplane must indicate that y is increasing. Here is the phase portrait



2) $\dot{x}=2y$, $\dot{y}=-3x$

The cartesian d.e. of the orbits is $\frac{dy}{dx}=\frac{-3x}{2y}$ which is separable. $\int 2y\,dy=-\int 3x\,dx$ $\quad y^2=-\frac{3}{2}x^2+c$, $c\in\mathbb{R}$

Then $H(x,y)=y^2+\frac{3}{2}x^2$ is a first integral in $\mathbb{R}^2$. Its level curves ~~all~~ are ellipses. Hence the phase portrait of the planar system looks like.



Note that this linear system is of center type.

Find a first integral of the following nonlinear planar systems.

1) $\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \sin x \end{cases}$ 2) $\begin{cases} \dot{x} = N_1 x - xy \\ \dot{y} = -N_2 y + xy \end{cases}$

where $\omega, N_1, N_2 > 0$ are real parameters.

<u>Solutions</u> 1) $\dfrac{dy}{dx} = -\dfrac{\omega^2 \sin x}{y}$ separable

$\int y\, dy = -\int \omega^2 \sin x\, dx$ $\dfrac{y^2}{2} = \omega^2 \cos x + c$, $c \in \mathbb{R}$

Then $H(x,y) = \dfrac{y^2}{2} - \omega^2 \cos x$ is a first integral in $\mathbb{R}^2$.

2) $\dfrac{dy}{dx} = \dfrac{-N_2 y + xy}{N_1 x - xy} \Leftrightarrow \dfrac{dy}{dx} = \dfrac{y(x-N_2)}{-x(y-N_1)} \Leftrightarrow$

$\dfrac{dy}{dx} = \dfrac{x-N_2}{-x} \cdot \dfrac{y}{y-N_1}$ which is separable.

$\int \dfrac{y-N_1}{y}\, dy = \int \dfrac{x-N_2}{-x}\, dx \Leftrightarrow \int \left(1 - N_1 \cdot \dfrac{1}{y}\right) dy = \int \left(-1 + N_2 \cdot \dfrac{1}{x}\right) dx$

$\Leftrightarrow y - N_1 \ln|y| = -x + N_2 \ln|x| + c$, $c \in \mathbb{R}$

Then $H(x,y) = x - N_2 \ln x + y - N_1 \ln y$ is a first integral in $U = (0,\infty) \times (0,\infty)$. $\square$