②

$$\Rightarrow t^2\left(\frac{1}{4}+\frac{3}{4}+1\right)+t(1+3+2)+1+3+1=1$$

$$\Rightarrow 2t^2+6t+4=0 \Rightarrow t^2+3t+2=0 \Rightarrow$$

$$\Rightarrow t_{1,2}=\frac{-3\pm\sqrt{9-8}}{2} \Rightarrow t\in\{-2,-1\}$$

So the intersection points are $(0,0,2)$ and $(2,-3,0)$.

C.2.2. Find the rectilinear generatrices of the quadric

$$4x^2-9y^2=36z$$

which pass through the point $P(3\sqrt{2},2,1)$

<u>Lecture revision</u>:

There are some quadrics that (in spite of their curvy aspect) contain some families of lines.

They are, as follows:

The hyperboloid of one sheet

The equation is $\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (*)$

so $\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$

By just looking at this equation we can see that the points satisfying

$$d_\lambda : \begin{cases} \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 + \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right) \end{cases} \quad \forall \lambda \in \mathbb{R}$$

$$d'_\mu : \begin{cases} \mu\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right) \end{cases} \quad \forall \mu \in \mathbb{R}$$

also satisfy the equation $(*)$ and, thus, are points of $\mathcal{H}_1$.

Clearly d_λ and d'_μ are equations of lines.

We can choose λ and μ arbitrarily and every choice gives a line $\Rightarrow$ $(d_\lambda)_{\lambda\in\mathbb{R}}$ and $(d'_\mu)_{\mu\in\mathbb{R}}$ are families of lines on $\mathcal{H}_1$

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The hyperbolic paraboloid

$$P_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z \quad , \quad p, q > 0$$

$$\Leftrightarrow \left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} \right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} \right) = 2z$$

So P_h contains the families of lines:

$$d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = \lambda \\ \lambda\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z \end{cases} , \quad \lambda \in \mathbb{R}$$

$$d'_\mu : \begin{cases} \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = \mu \\ \mu\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = 2z \end{cases} , \quad \mu \in \mathbb{R}$$

Other surfaces also contain lines (e.g. the elliptic cylinder) but these two are what interests us for now.

Remark: We can see that in the two cases outlined above, every point of the quadric belongs to a line in one of the families.

This means that these lines generate the quadrics above (hence the name "generatrix")

Solution for c.2.2.:

$4x^2 - 9y^2 = 36z$ this is a hyperbolic paraboloid

$d_\lambda : \begin{cases} 2x - 3y = \lambda \\ \lambda(2x+3y) = 36z \end{cases}$ $d'_\mu : \begin{cases} 2x+3y = \mu \\ \mu(2x-3y) = 36z \end{cases}$

$P \in d_\lambda \Rightarrow \lambda = 2\cdot(3\sqrt{2}) - 6$, $\mu = 2(3\sqrt{2}) - 6$

For d_λ we have: $\begin{cases} 2x - 3y = 6\sqrt{2} - 6 \\ (6\sqrt{2}-6)(2x+3y) = 36z \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = \dfrac{3y + 6\sqrt{2} - 6}{2} \\ (6\sqrt{2}-6)\,(3y + 6\sqrt{2} - 6 + 3y) = 36z \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = \frac{3}{2}y + 3\sqrt{2} - 3 \\ z = (\sqrt{2}-1)\cdot(y + \sqrt{2} - 1) = (\sqrt{2}-1)y + 3 - 2\sqrt{2} \end{cases}$

Thus, the equation of the line in the family $(d_\lambda)_{\lambda \in \mathbb{R}}$ that passes through P is:

$$\frac{x - (3\sqrt{2} - 3)}{3/2} = \frac{y - 0}{1} = \frac{z - (3 - 2\sqrt{2})}{\sqrt{2} - 1}$$

We compute the equation of the line in the family $(d'_\mu)_{\mu \in \mathbb{R}}$ in the exact same way.

C.2.3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1) \quad \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane $(\Pi)\ x + y + z = 0$

Solution: $d_\lambda : \begin{cases} \lambda\left(\frac{x}{6} + \frac{z}{2}\right) = 1 + \frac{y}{3} \\ \frac{x}{6} - \frac{z}{2} = \lambda\left(1 - \frac{y}{3}\right) \end{cases}$

$d_\mu : \begin{cases} \mu\left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3} \\ \frac{x}{6} - \frac{z}{2} = \mu\left(1 + \frac{y}{3}\right) \end{cases}$

You solve the systems, get $\vec{d_\lambda}$ and $\vec{d_\mu}$ and choose λ, μ s.t. $\vec{d_\lambda} \cdot \vec{n_\Pi} = \vec{d_\mu} \cdot \vec{n_\Pi} = 0$