Lab 7: Newton's method

Given a C^2 real function of real variable g, we consider another function f(x)=x - g(x)/g'(x). It is not difficult to see that the zeros of g are the fixed points of f (that is, g(x)=0 iff f(x)=x).
Moreover, we have the following formula to compute the first derivative of f: f ' (x)= 1 - g'(x)/g'(x) - g(x)g"(x)/[g'(x)]^2= - g(x)g"(x)/[g'(x)]^2.
Thus, if eta* is a zero of g (that is, g(eta*)=0) we have that f ' (eta*)=0, implying that eta* is an attractor for the map f . As we know, it is useful to find the basin of attraction of an attractor.

For example, in the last lecture, we took g(x)=x^2-3, which has two zeros: - sqrt(3) and sqrt(3). The corresponding f has the expression f(x)=x - (x^2-3)/(2x)=x/2+3/(2x) . It can be proved that the basin of attaction of - sqrt(3) is (-infinity,0), while the basin of attaction of sqrt(3) is (0,infinity). We checked this using the stair-step diagram.
We check now for few positive initial values, trying also to see the rapid convergence.
As a novelty, we take also complex values with positive real part for the initial values and we will see that they are also in the basin of attraction of sqrt(3). It is proved that any such point
(that is, z=x+I*y with x>0) is in the basin of attraction of sqrt(3).

In the end consider another example, g(z)=z^4-1. It has 4 complex roots (1, -1, I and -I) and we will represent in the complex plane the 4 basins of attraction.
We will check that 2 and 2+I are in the basin of attraction of 1 (green zone). Also, 2+3*I and 2*I are in the basin of attraction of I (red zone).
The blue zone is the basin of attraction of -1 , while the yellow zone is the basin of attraction of - I .
We also check the initial values 1+I and 2+2*I but we do not understand anything. It is known (proved) that they are not in any basin of attraction.

```
> restart: evalf(sqrt(3));
```

1.732050808 **(1)**

```
> x:=1.7; for i from 1 to 10 do x:=evalf(x/2+3/(2*x)); print(x) od:
```

$x := 1.7$

1.732352941
1.732050834
1.732050808
1.732050808
1.732050808
1.732050808
1.732050808
1.732050808
1.732050808
1.732050808 **(2)**

```
> x:=100; for i from 1 to 10 do x:=evalf(x/2+3/(2*x)); print(x) od:
```

$x := 100$

50.01500000
25.03749100
12.57865566
6.408577457

$$3.438350033$$
$$2.155430774$$
$$1.773631961$$
$$1.732538223$$
$$1.732050876$$
$$1.732050808 \tag{3}$$

```
> x:=0.1; for i from 1 to 10 do x:=evalf(x/2+3/(2*x)); print(x) od:
```

$$x := 0.1$$
$$15.05000000$$
$$7.624667774$$
$$4.009063770$$
$$2.378684078$$
$$1.819942804$$
$$1.734173128$$
$$1.732052106$$
$$1.732050808$$
$$1.732050808$$
$$1.732050808 \tag{4}$$

```
> z:=1+I; for i from 1 to 10 do z:=evalf(z/2+3/(2*z)); print(z) od:
```

$$z := 1 + \mathrm{I}$$
$$1.250000000 - 0.2500000000\,\mathrm{I}$$
$$1.778846154 + 0.1057692307\,\mathrm{I}$$
$$1.729695598 + 0.00292246550\,\mathrm{I}$$
$$1.732049935 - 0.000003977850\,\mathrm{I}$$
$$1.732050808 + 2.004\;10^{-12}\,\mathrm{I}$$
$$1.732050808 + 0.\,\mathrm{I}$$
$$1.732050808 + 0.\,\mathrm{I}$$
$$1.732050808 + 0.\,\mathrm{I}$$
$$1.732050808 + 0.\,\mathrm{I}$$
$$1.732050808 + 0.\,\mathrm{I} \tag{5}$$

```
> z:=1.7-100*I; for i from 1 to 16 do z:=evalf(z/2+3/(2*z)); print
  (z) od:
```

$$z := 1.7 - 100.\,\mathrm{I}$$
$$0.8502549263 - 49.98500433\,\mathrm{I}$$
$$0.4256377746 - 24.96250184\,\mathrm{I}$$
$$0.2138431915 - 12.42117825\,\mathrm{I}$$
$$0.1090000114 - 6.089863416\,\mathrm{I}$$
$$0.05890721412 - 2.798699648\,\mathrm{I}$$
$$0.04072960007 - 0.8636239687\,\mathrm{I}$$
$$0.1020958639 + 1.301200509\,\mathrm{I}$$
$$0.1409449856 - 0.4951277655\,\mathrm{I}$$
$$0.8682231422 + 2.554866404\,\mathrm{I}$$
$$0.6129753549 + 0.7511018364\,\mathrm{I}$$

$$1.284751451 - 0.8231526038\,I$$
$$1.470120295 + 0.1187675999\,I$$
$$1.748768694 - 0.02251135625\,I$$
$$1.731988607 - 0.00021600484\,I$$
$$1.732050795 + 7.7558\ 10^{-9}\,I$$
$$1.732050808 - 5.6\ 10^{-17}\,I \tag{6}$$

```
> restart:
> g:=unapply(z^4-1,z);
```

$$g := z \rightarrow z^4 - 1 \tag{7}$$

```
> solve(z^4-1);
```

$$1,\ -1,\ I,\ -I \tag{8}$$

```
> f:=unapply(expand(z-g(z)/D(g)(z)),z);
```

$$f := z \rightarrow \frac{3}{4}\,z + \frac{1}{4\,z^3} \tag{9}$$

```
> solve(f(z)=z);
```

$$1,\ -1,\ I,\ -I \tag{10}$$

```
> factor(D(f)(z));
```

$$\frac{3}{4}\,\frac{(z-1)\,(z+1)\,(z^2+1)}{z^4} \tag{11}$$

```
> z:=2; for i from 1 to 10 do z:=evalf(3/4*z+1/(4*z^3)); print(z);
  od:
```

$$z := 2$$
$$1.531250000$$
$$1.218068351$$
$$1.051884020$$
$$1.003714083$$
$$1.000020564$$
$$1.000000001$$
$$1.000000000$$
$$1.000000000$$
$$1.000000000$$
$$1.000000000 \tag{12}$$

```
> z:=2+I; for i from 1 to 10 do z:=evalf(3/4*z+1/(4*z^3)); print(z)
  ; od:
```

$$z := 2 + I$$
$$1.504000000 + 0.7280000000\,I$$
$$1.139610535 + 0.4936849880\,I$$
$$0.8987703616 + 0.2474216182\,I$$
$$0.8877755873 - 0.0370807737\,I$$
$$1.019405798 + 0.01670106264\,I$$
$$1.000167328 + 0.00093726411\,I$$
$$0.9999987254 + 4.723507\ 10^{-7}\,I$$

$$1.000000000 - 1.8062\ 10^{-12}\ I$$
$$1.000000000 + 0.\ I$$
$$1.000000000 + 0.\ I \tag{13}$$

```
> z:=2+3*I; for i from 1 to 12 do z:=evalf(3/4*z+1/(4*z^3)); print
  (z); od:
```

$$z := 2 + 3\ I$$
$$1.494765589 + 2.248975876\ I$$
$$1.108604751 + 1.684346661\ I$$
$$0.8014301380 + 1.257935555\ I$$
$$0.5263761251 + 0.9336129437\ I$$
$$0.1918161448 + 0.7064334628\ I$$
$$-0.3113072824 + 0.9759747239\ I$$
$$-0.0475836356 + 0.8716909858\ I$$
$$0.02551355122 + 1.024513785\ I$$
$$0.00180257546 + 1.000002154\ I$$
$$2.6291\ 10^{-8} + 0.9999951262\ I$$
$$-3.8442\ 10^{-13} + 1.000000000\ I$$
$$0. + 1.000000000\ I \tag{14}$$

```
> z:=2*I; for i from 1 to 10 do z:=evalf(3/4*z+1/(4*z^3)); print(z)
  ; od:
```

$$z := 2\ I$$
$$1.531250000\ I$$
$$1.218068351\ I$$
$$1.051884020\ I$$
$$1.003714083\ I$$
$$1.000020564\ I$$
$$1.000000001\ I$$
$$1.000000000\ I$$
$$1.000000000\ I$$
$$1.000000000\ I$$
$$1.000000000\ I \tag{15}$$

```
> z:=1+I; for i from 1 to 50 do z:=evalf(3/4*z+1/(4*z^3)); print(z)
  ; od:
```

$$z := 1 + I$$
$$0.6875000000 + 0.6875000000\ I$$
$$0.3232884110 + 0.3232884110\ I$$
$$-1.607269089 - 1.607269089\ I$$
$$-1.190399123 - 1.190399123\ I$$
$$-0.8557481470 - 0.8557481470\ I$$
$$-0.5420773679 - 0.5420773679\ I$$
$$-0.0141876679 - 0.0141876679\ I$$
$$21885.01073 + 21885.01073\ I$$
$$16413.75805 + 16413.75805\ I$$

$$12310.31854 + 12310.31854\,I$$
$$9232.738905 + 9232.738905\,I$$
$$6924.554179 + 6924.554179\,I$$
$$5193.415634 + 5193.415634\,I$$
$$3895.061726 + 3895.061726\,I$$
$$2921.296294 + 2921.296294\,I$$
$$2190.972220 + 2190.972220\,I$$
$$1643.229165 + 1643.229165\,I$$
$$1232.421874 + 1232.421874\,I$$
$$924.3164055 + 924.3164055\,I$$
$$693.2373041 + 693.2373041\,I$$
$$519.9279781 + 519.9279781\,I$$
$$389.9459836 + 389.9459836\,I$$
$$292.4594877 + 292.4594877\,I$$
$$219.3446158 + 219.3446158\,I$$
$$164.5084618 + 164.5084618\,I$$
$$123.3813464 + 123.3813464\,I$$
$$92.53600977 + 92.53600977\,I$$
$$69.40200725 + 69.40200725\,I$$
$$52.05150525 + 52.05150525\,I$$
$$39.03862850 + 39.03862850\,I$$
$$29.27897033 + 29.27897033\,I$$
$$21.95922526 + 21.95922526\,I$$
$$16.46941304 + 16.46941304\,I$$
$$12.35204579 + 12.35204579\,I$$
$$9.264001178 + 9.264001178\,I$$
$$6.947922273 + 6.947922273\,I$$
$$5.210755361 + 5.210755361\,I$$
$$3.907624770 + 3.907624770\,I$$
$$2.929671108 + 2.929671108\,I$$
$$2.194767776 + 2.194767776\,I$$
$$1.640164106 + 1.640164106\,I$$
$$1.215958030 + 1.215958030\,I$$
$$0.8772049611 + 0.8772049611\,I$$
$$0.5653110141 + 0.5653110141\,I$$
$$0.0780298024 + 0.0780298024\,I$$
$$-131.4937929 - 131.4937929\,I$$
$$-98.62034465 - 98.62034465\,I$$
$$-73.96525842 - 73.96525842\,I$$
$$-55.47394367 - 55.47394367\,I$$
$$-41.60545738 - 41.60545738\,I \qquad \textbf{(16)}$$

```
> z:=2+2*I; for i from 1 to 20 do z:=evalf(3/4*z+1/(4*z^3)); print
  (z); od:
```

$$z := 2 + 2\,\mathrm{I}$$
$$1.492187500 + 1.492187500\,\mathrm{I}$$
$$1.100329714 + 1.100329714\,\mathrm{I}$$
$$0.7783323100 + 0.7783323100\,\mathrm{I}$$
$$0.4511976736 + 0.4511976736\,\mathrm{I}$$
$$-0.3420254918 - 0.3420254918\,\mathrm{I}$$
$$1.305565619 + 1.305565619\,\mathrm{I}$$
$$0.9510886015 + 0.9510886015\,\mathrm{I}$$
$$0.6406695511 + 0.6406695511\,\mathrm{I}$$
$$0.2428303029 + 0.2428303029\,\mathrm{I}$$
$$-4.182747868 - 4.182747868\,\mathrm{I}$$
$$-3.136206828 - 3.136206828\,\mathrm{I}$$
$$-2.350128997 - 2.350128997\,\mathrm{I}$$
$$-1.757781652 - 1.757781652\,\mathrm{I}$$
$$-1.306828626 - 1.306828626\,\mathrm{I}$$
$$-0.9521172096 - 0.9521172096\,\mathrm{I}$$
$$-0.6416762025 - 0.6416762025\,\mathrm{I}$$
$$-0.2447021051 - 0.2447021051\,\mathrm{I}$$
$$4.081943511 + 4.081943511\,\mathrm{I}$$
$$3.060538710 + 3.060538710\,\mathrm{I}$$
$$2.293223882 + 2.293223882\,\mathrm{I} \qquad (17)$$

```
> restart:
> newton4_IE := proc(x,y)
    local z,i,p,q,r,s,notclosetoroot;
    z:=x+I*y;
    notclosetoroot := true;

    for i from 1 to 50 while notclosetoroot do z:= 3/4*z+1/(4*z^3)
  ;
       # Are we close to the root -1?
       p:=evalb(abs(z+1)^2<0.0002);

       # Are we close to the root -i?
       q:=evalb(abs(z+I)^2<0.0002);

       # Are we close to the root i?
       r := evalb(abs(z-I)^2<0.0002);

       # Are we close to the root 1?
       s := evalb(abs(z-1)^2<0.0002);

       # Set flag to end loop if sufficiently close to a root.
       notclosetoroot := not(p or q or r or s);
    end do;

    # Determine the value to return. The value .6666666667
  indicates the Blue  colour, the value .1666666667 indicates the
   # Yellow colour, the value 0 indicates the Red colour, the value
```

```
.3333333333 indicates the Green colour, while the value    #
.5416666667 indicates no colour.
   if p then .6666666667 elif q then .1666666667
   elif r then 0 elif s then .3333333333
   else .5416666667 end if;
end proc:
> newton4_IE(-2,0);  newton4_IE(1,-2); newton4_IE(0,2); newton4_IE
(2,3); newton4_IE(3,2);newton4_IE(1,1);
```

0.6666666667

0.1666666667

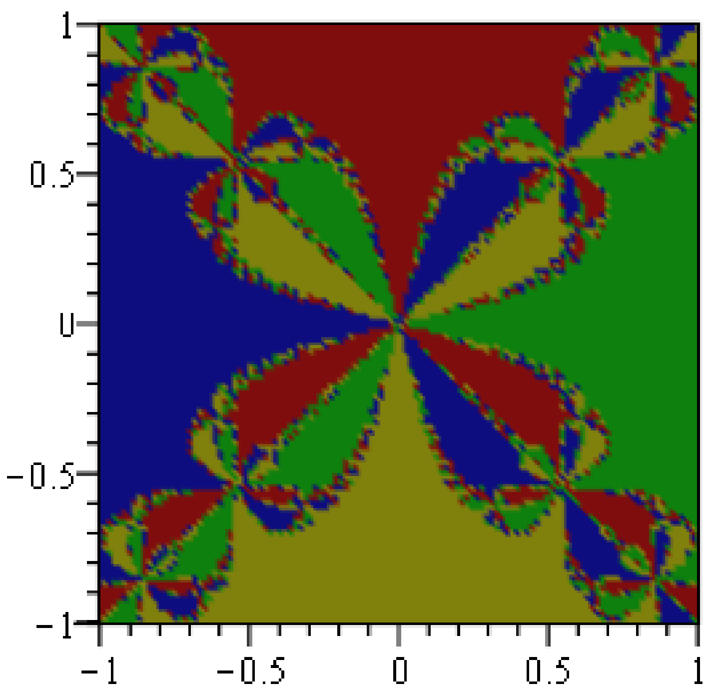
0

0

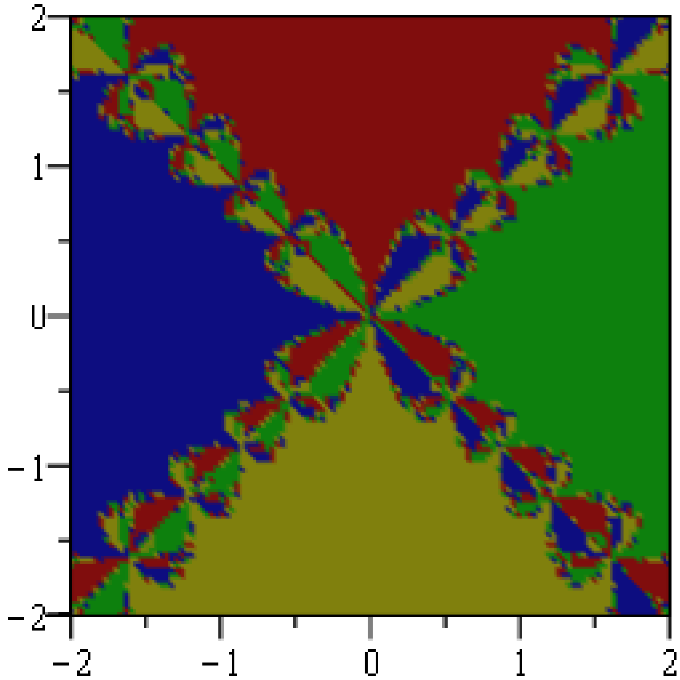
0.3333333333

Warning, computation interrupted

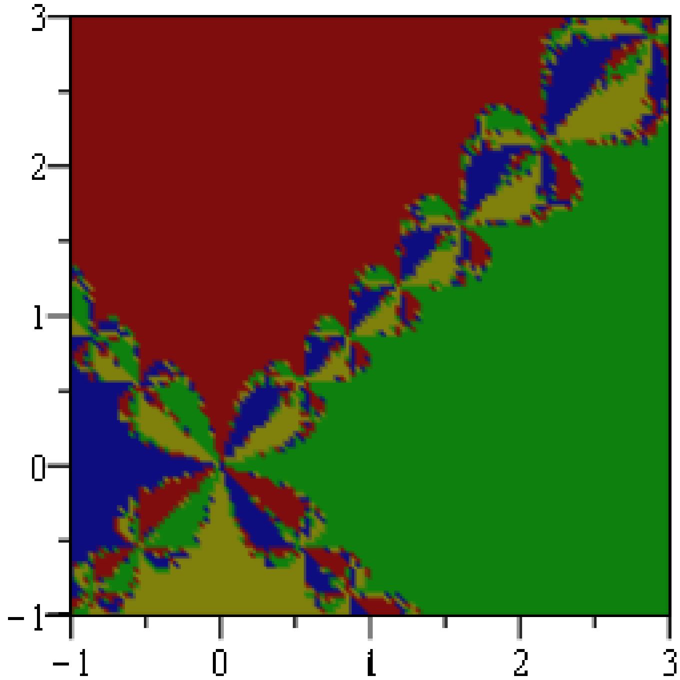
```
> plot3d(0,-1..1,-1..1,orientation=[-90,0],grid=[120,120],
   style=patchnogrid,scaling=constrained,color=newton4_IE);
```




```
> plot3d(0,-2..2,-2..2,orientation=[-90,0],grid=[120,120],
   style=patchnogrid,scaling=constrained,color=newton4_IE);
```




```
> plot3d(0,-1..3,-1..3,orientation=[-90,0],grid=[120,120],
    style=patchnogrid,scaling=constrained,color=newton4_IE);
```


3
2
1
0
−1
−1
0
1
2
3

>