First order linear differential equations.

-recap-

1.3.1. a) $x' + \frac{1}{t} \cdot x = 0$. Here $I = (0, \infty)$

This is a first order linear homogenous differential equations with variable coefficient $a(t) = \frac{1}{t}$.

Let $A(t) = \int_0^t \frac{1}{s} ds = \ln|t|$

Method 1: (integrating factor method)

Here $\mu(t) = e^{\ln|t|}$ is an integrating factor of the equation

$\Rightarrow x' + \frac{1}{t} \cdot x = 0 \; | \cdot t \quad \Rightarrow x' \cdot t + x = 0$

Notice that $(x \cdot t)' = x' \cdot t + x$

$\Rightarrow (x \cdot t)' = 0$

Integrateing with respect to $t \Rightarrow x \cdot t = c$, $c \in \mathbb{R}$

Thus, the general solution is $x = \frac{c}{t}$, $c \in \mathbb{R}$

Method 2: (separation of variables method)

We look for solutions (non-null). We write the eq as:

$$x' = -\frac{1}{t} x$$

We separate the dependent variable x from the independent variable t:

$$\frac{x'(t)}{x(t)} = -\frac{1}{t}$$

We integrate the above equation (we look for primitives for each side of the equation.

$$\ln|x(t)| = -\ln|t| + \underset{\to \ln c}{C}$$

$$\ln|x(t)| = \ln\left|\frac{c}{t}\right| \Rightarrow \underbrace{x(t) = \frac{c}{t}}_{\text{general solution}}, \quad c \in \mathbb{R}$$

1.3.2. c) $x' + \frac{2t}{1+t^2} \cdot x = 3.$

Notice that the equation is a first order linear nonhomogenous differential equation with variable coefficient $a(t) = \frac{2t}{1+t^2}$.

The nonhomogenous part is $f(t) = 3$.

Here $I := \mathbb{R}$

Let $A(t) = \int_0^t a(s)\,ds = \int_0^t \frac{2s}{1+s^2}\,ds = \ln(1+t^2)$

Method 1: (integrating factor method)

The integrating factor of the given equation is:

$$\mu(t) = e^{A(t)} = e^{\ln(1+t^2)} = 1+t^2$$

Let the equation:

$$x' + \frac{2t}{1+t^2} \cdot x = 3 \quad \Big| \cdot (1+t^2)$$

$$\left.\begin{array}{l} x' \cdot (1+t^2) + 2t \cdot x = 3(1+t^2) \\ \text{Notice that: } \left[x \cdot (1+t^2)\right]' = x' \cdot (1+t^2) + x \cdot 2t \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left[x \cdot (1+t^2)\right]' = 3(1+t^2)$$

Now we integrate with respect to t, the above eq.

$$\Rightarrow x \cdot (1+t^2) = 3t + t^3 + C \Rightarrow x = \frac{3t+t^3}{1+t^2} + \frac{C}{1+t^2}$$

Thus, the general solution of the given equation:

$$x(t) = \frac{C}{1+t^2} + \frac{3t+t^3}{1+t^2}, \quad C \in \mathbb{R}.$$

Method 2 : (separation of variables method & Lagrange method)

st1: We write the linear homog eq associated :

$$x' + \frac{2t}{1+t^2} \cdot x = 0. \Rightarrow \frac{dx}{dt} = -\frac{2t}{1+t^2} \cdot x \qquad \left(x' = \frac{dx}{dt} \right)$$

We separate the variables : $\frac{dx}{x} = -\frac{2t}{1+t^2} \cdot dt$

We integrate : $\int \frac{dx}{x} = -\int \frac{2t}{1+t^2} \cdot dt$

$\Rightarrow \ln|x| = -\ln|1+t^2| + \ln C$

$x_h = \frac{C}{1+t^2}$ the general solution of the homogenous equation.

$(c \in \mathbb{R})$

st2: We now apply the Lagrange method to find a particular solution, denoted x_p of the nonhomog. eq.

We look for $\varphi \in C^1(\mathbb{R})$ s.t. $x_p = \varphi(t) \cdot \frac{1}{1+t^2}$

$$x_p' = \varphi'(t) \cdot \frac{1}{1+t^2} + \varphi(t) \cdot \left(-\frac{2t}{(1+t^2)^2} \right)$$

Replace x_p, x_p' in the equation:

$$\varphi'(t) \cdot \frac{1}{1+t^2} \underbrace{- \varphi(t) \cdot \frac{2t}{(1+t^2)^2} + \frac{2t}{1+t^2} \cdot \varphi(t) \cdot \frac{1}{1+t^2}}_{\text{this terms always cancel out}} = 3$$

$$\Rightarrow \varphi'(t) \cdot \frac{1}{1+t^2} = 3 \Rightarrow \varphi'(t) = 3(1+t^2) \quad \Big| \int dt$$

$$\Rightarrow \varphi(t) = 3t + t^3 \Rightarrow x_p = (3t + t^3) \cdot \frac{1}{1+t^2}$$

st3: $x = x_h + x_p$

$$x = \frac{c}{1+t^2} + \frac{3t+t^3}{1+t^2}, \quad c \in \mathbb{R}$$

- the general solution of the given equation

Method 3: (direct application of Prop 2.16).

[Prop 2.16. The general solution of the first order linear nonhomogenous differential equation (4) is:

$$x(t) = c \cdot e^{-A(t)} + \int_{t_0}^{t} e^{-A(t)+A(s)} f(s) \cdot ds \quad , \quad c \in \mathbb{R} .$$]

Here $A(t) = \int_0^t \frac{2s}{1+s^2} ds = \ln(1+t^2)$

$f(t) = 3 \quad , \quad t_0 = 0.$

$$\Rightarrow x(t) = c \cdot e^{-\ln(1+t^2)} + \int_0^t e^{-\ln(1+t^2)+\ln(1+s^2)} \cdot 3\, ds$$

$$= c \cdot \frac{1}{1+t^2} + \frac{1}{1+t^2} \cdot \int_0^t (1+s^2) \cdot 3\, ds$$

$$= \frac{c}{1+t^2} + \frac{1}{1+t^2} \cdot (3t + t^3)$$

$$= \frac{c}{1+t^2} + \frac{3t+t^3}{1+t^2} \quad , \quad c \in \mathbb{R}$$

– the general solution of the given nonhomogenous equation.

1.3.2. d) $x' - \frac{2}{t}x = t^2 \sin(2t) - 4t^3 \quad , t \in (0,\infty)$

- first order linear nonhomogenous equation
- the variable coefficient : $a(t) = -\frac{2}{t}$
- the nonhomogenous part : $f(t) = t^2 \cdot \sin(2t) - 4t^3$.
- $I = (0,\infty)$
- $A(t) = \int_0^t a(s)ds = -2\ln t$

Method 1: (integrating factor method)

- the integrating factor : $\mu(t) = e^{A(t)} = e^{-2\ln t} = \frac{1}{t^2}$
- multiply the equation with the integrating factor:

$$x' - \frac{2}{t}\cdot x = t^2 \cdot \sin(2t) - 4t^3 \quad \Big| \cdot \frac{1}{t^2}$$

$$x' \cdot \frac{1}{t^2} - \frac{2}{t^3}\cdot x = \sin(2t) - 4t \qquad (1)$$

- notice that the left hand side of the above equation can be written:

$$\left(x \cdot \frac{1}{t^2}\right)' = x' \cdot \frac{1}{t^2} - x \cdot \frac{2}{t^3} \qquad (2)$$

$$\overset{(1)}{\underset{(2)}{\Rightarrow}} \quad \left(x \cdot \frac{1}{t^2}\right)' = \sin(2t) - 4t \qquad \Big| \int dt$$

$$x \cdot \frac{1}{t^2} = \int (\sin 2t - 4t)\, dt$$

$$x \cdot \frac{1}{t^2} = -\frac{1}{2}\cos 2t - \cancel{4}^{2}\frac{t^2}{\cancel{2}} + c$$

$$x = \left(-\frac{1}{2}\cos 2t - 2t^2\right)\cdot t^2 + c \cdot t^2$$

$$x = c \cdot t^2 - \frac{t^2}{2}\cdot \cos 2t - 2t^4 \qquad , c \in \mathbb{R}$$

Method 2 (separation of variables method & Lagrange method)

st1: • first we solve the linear homogenous eq. associated

$x' - \frac{2}{t} \cdot x = 0 \quad \Rightarrow \frac{dx}{dt} = \frac{2}{t} \cdot x$

We separate the variables and integrate:

$\frac{dx}{x} = \frac{2}{t} dt \quad \Rightarrow \ln|x| = 2 \ln|t| + \ln C \; , \; C \in \mathbb{R}$

$\Rightarrow x_h = C \cdot t^2$ - the general solution of the homogenous equation.

st2: • apply the Lagrange method.

We look for a particular solution : $x_p = \varphi(t) \cdot t^2$

$\Rightarrow x_p' = \varphi'(t) \cdot t^2 + 2t \cdot \varphi(t)$.

Replace x_p, x_p' in the nonhomogenous equation.

$$\varphi'(t) \cdot t^2 + \underbrace{2t \cdot \varphi(t) - \frac{2}{t} \cdot \varphi(t) \cdot t^2}_{\text{canceled out}} = t^2 \sin(2t) - 4t^3$$

$\varphi'(t) = (t^2 \sin(2t) - 4t^3) \cdot t^{-2}$

$\varphi'(t) = \sin 2t - 4t$

$\varphi(t) = \int (\sin 2t - 4t)\, dt = -\frac{1}{2} \cos 2t - 2t^2$

$\Rightarrow x_p = -\frac{t^2}{2} \cdot \cos 2t - 2t^4$ (particular solution)

st3: • General solution of the nonhomogenous equation

$x = x_h + x_p$

$x = c \cdot t^2 - \frac{t^2}{2} \cdot \cos 2t - 2t^4 \; , \; c \in \mathbb{R}$

Method 3 (direct application of Prop 2.16)

- general solution of the nonhomogenous equation:

$$x(t) = C\cdot e^{-A(t)} + \int_{t_0}^{t} e^{-A(t)+A(s)}\cdot f(s)ds \quad , C\in\mathbb{R}$$

- here $A(t) = -2\ln t$

$$f(t) = t^2\cdot\sin(2t) - 4t^3 \quad , t_0 = 0$$

$$\Rightarrow x(t) = C\cdot e^{2\ln t} + \int_0^t e^{2\ln t - 2\ln s}\cdot(s^2\cdot\sin(2s) - 4s^3)ds$$

$$x(t) = C\cdot t^2 + t^2\cdot\int_0^t \frac{1}{s^2}(s^2\cdot\sin(2s) - 4s^3)ds$$

$$x(t) = C\cdot t^2 + t^2\cdot\int_0^t(\sin 2s - 4s)ds$$

$$x(t) = C\cdot t^2 - \frac{t^2}{2}\cos 2t - 2t^4 \quad , C\in\mathbb{R}.$$

- general solution of the given nonhomog. eq.

1.3.4. Find the general solution of $x' - x = e^{t-1}$. Justify the result in two ways.

Solution:

- here $a(t) = -1$, $f(t) = e^{t-1}$, $I := \mathbb{R}$

$$A(t) = \int_0^t -ds = -t$$

Method 1: (integrating factor method):

- the integrating factor: $\mu(t) = e^{A(t)} = e^{-t}$

$$x' - x = e^{t-1} \quad |\cdot e^{-t}$$

$$x'\cdot e^{-t} - x\cdot e^{-t} = e^{-1}$$

$$(x\cdot e^{-t})' = e^{-1} \quad |\int dt$$

$$x\cdot e^{-t} = e^{-1}\cdot t + C \quad \Rightarrow \underline{x = C\cdot e^{t} + t\cdot e^{t-1}} ,$$

$C\in\mathbb{R}$

Method 2: (separation of variables method & Lagrange method)

st1: $x' - x = 0 \quad , x_h = ?$

$$\frac{dx}{dt} = x \quad \Rightarrow \frac{dx}{x} = dt \quad \Rightarrow \ln|x| = t + \ln C$$

$$\Rightarrow x_h = C \cdot e^t \quad , C \in \mathbb{R}$$

st2: $x_p = ? \quad , x_p = \varphi(t) \cdot e^t$

$$x_p' = \varphi'(t) \cdot e^t + \varphi(t) \cdot e^t$$

$$\overset{\text{nonh. eq}}{\Longrightarrow} \varphi'(t) \cdot e^t + \cancel{\varphi(t) \cdot e^t} - \cancel{\varphi(t) \cdot e^t} = e^{t-1}$$

$$\varphi'(t) = e^{-1} \quad \Rightarrow \varphi(t) = e^{-1} \cdot t$$

$$\Rightarrow x_p = t \cdot e^{t-1}$$

st3: $x = x_h + x_p$

$$\underline{x = C \cdot e^t + t \cdot e^{t-1}} \quad , C \in \mathbb{R}$$

general solution of the equation.

Method 3: (with proposition 2.16 from the Lecture)

$$x(t) = C \cdot e^{-A(t)} + \int_0^t e^{-A(t) + A(s)} \cdot f(s)\, ds \quad , C \in \mathbb{R}$$

$$= C \cdot e^t + \int_0^t e^t \cdot e^{-s} \cdot e^{s-1}\, ds = C \cdot e^t + t \cdot e^{t-1} \quad , C \in \mathbb{R}$$

R: Method 4*: (with the characteristic equation method)

Notice that the equation can be seen as linear differential equation with constant coefficients:

st1: We solve the homogenous eq: $x' - x = 0$ with the characteristic method: $r - 1 = 0 \Rightarrow r = 1 \Rightarrow x_h = e^t \cdot C$

st2: Hint: we look for $x_p = a \cdot e^t \cdot t$ (since $f(t) = e^{-1} \cdot e^t$)

$$\Rightarrow x_p' = a \cdot e^t + a \cdot t \cdot e^t \overset{\text{eq}}{\Longrightarrow} a \cdot e^t + a \cdot \cancel{t \cdot e^t} - a \cancel{t e^t} = e^t \cdot e^{-1}$$

$$x_p = t e^{t-1}$$

st3: $x = x_h + x_p$

1.3.5. a) $x''' - x'' = 0.$

Here we notice we have only the derivative of x, (x'', x''')

We make the change of variable :

(1) $y = x'' \Rightarrow y' = x'''$ $(x = x(t),\ y = y(t))$

We obtain a first order equation:

$y' - y = 0.$ (first order linear homogenous differential equation)

We use here the separation of variables method:

$\frac{dy}{dt} = y$ | separate the variables $\Rightarrow$

$\frac{dy}{y} = dt$ | integrate $\Rightarrow$

$\ln|y| = t + \ln C$

$y = C \cdot e^t$

We replace y in (1) : $x'' = C \cdot e^t$

We find x by two successive integrations out

$\Rightarrow x' = C_1 e^t + C_2$ | $\int dt$

$\underline{x = C_1 \cdot e^t + C_2 \cdot t + C_3}$, $C_1, C_2, C_3 \in \mathbb{R}$

- general solution of the equation

Remark:** Notice that the above equation is a linear differential equation with constant coefficients. We solve this equation with the characteristic eq. method. $\Rightarrow r^3 - r^2 = 0$ (the characteristic equation)

$\Rightarrow r_1 = r_2 = 0,\ r_3 = 1 \Rightarrow x_1 = e^{0t} = 1,\ x_2 = te^{0t} = t,$ and $x_3 = e^{1 \cdot t} = e^t$

$\Rightarrow x = C_1 \cdot x_1 + C_2 x_2 + C_3 x_3 = C_1 + C_2 t + C_3 \cdot e^t,$ $C_1, C_2, C_3 \in \mathbb{R}.$

1.3.5. b) $x'' = \frac{2}{t} x'$

- here the change of variable is : $x' = y \Rightarrow x'' = y'$

$\Rightarrow y' = \frac{2}{t} \cdot y$ - differential eq of first order

$\frac{dy}{dt} = \frac{2}{t} \cdot y$ - (separate the variables)

$\frac{dy}{y} = \frac{2}{t} \cdot dt$ - (integrate the equation)

$\ln|y| = 2\ln|t| + \ln C$

$y = t^2 \cdot C$

- $x' = t^2 \cdot C \qquad | \int dt$

$x = C \cdot \frac{t^3}{3} + c_2$

$x(t) = c_1 \cdot t^3 + c_2 \quad , c_1, c_2 \in \mathbb{R}.$