Conservation of Energy in Elliptical Orbits
Group Quiz

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Introduction

Most comets originate in the Oort Cloud located 50,000. A.U.'s (Astronomical Units) from the Sun. An A.U. is the distance from the sun to the Earth or 1.496×10^8 km. They orbit the sun with highly eccentric elliptical orbits. The comets start at their aphelion in the Oort cloud with nearly zero velocity while their velocity at the perihelion when they pass closest to the sun at around 0.0100 A.U. is quite large.

Questions and Responses

1a. What is the functional form of the Gravitational Potential energy (U) for the Sun? (Hint: Integrate (anti-derivative) the Gravitational force with respect to r)

Response: The force of gravity was first found, such that

$$F = \frac{GM_cM_s}{r^2} \tag{1}$$

where

F = gravitational force
$G = 6.67 \times 10^{-11} \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}$
M_c = mass of the comet
M_s = mass of the sun
r = distance from the comet to the sun

This equation was then simplified to

$$F = GM_cM_sr^{-2} \tag{2}$$

By integrating the gravitational force with respect to r, the gravitational potential energy was found

$$\int F dr \tag{3}$$

$$W = -GM_Cm_sr^{-1} \tag{4}$$

$$U = \frac{-GM_cM_s}{r} \tag{5}$$

1b. Create a relatively accurate sketch of the comet's orbit. Identify the perihelion, aphelion, elliptical foci, Oort cloud, Sun, and Earth.

Response: Using these values, a rough sketch was drawn

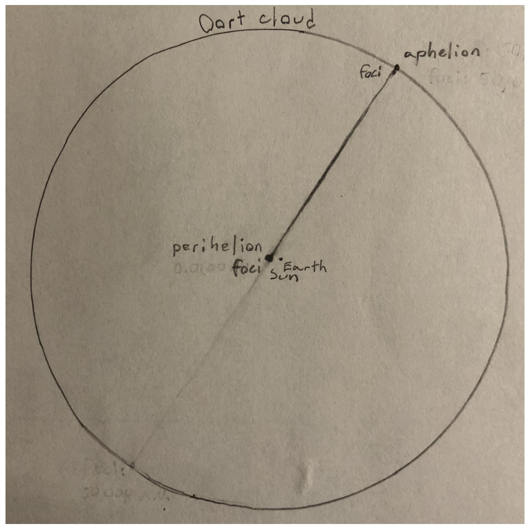

Figure 1: An orbital overview.

2. Determine the velocity of the comet at its perihelion.

Response: By using the definition of kinetic energy as

$$KE = \frac{1}{2}mv^2 = PE_1 - PE_2 \tag{6}$$

where

KE = kinetic energy
m = mass (kg)
v = velocity (m/s)
PE = potential energy

and substituting PE_1 and PE_2 with the gravitational potential energy equation found in equation 5, the velocity can be isolated and solved for:

$$\frac{1}{2}M_c v^2 = \frac{-GM_cM_s}{r_p} - \frac{-GM_cM_s}{r_a} \tag{7}$$

where

M_s = mass of the sun (1.99×10^{30} kg)
r_p = distance from the comet to the sun at perihelion
r_a = distance from the comet to the sun at aphelion

$$\frac{1}{2}v^2 = \frac{-GM_s}{r_p} - \frac{-GM_s}{r_a} \tag{8}$$

$$v = \sqrt{2\left(\left(\frac{-GM_s}{r_p}\right) - \left(\frac{-GM_s}{r_a}\right)\right)} \tag{9}$$

$$v = \sqrt{2\left(\left(\frac{-6.67\times10^{-11}\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}\cdot1.99\times10^{30}\ \mathrm{kg}}{(0.0100\ \mathrm{A.U.})\left(1.50\times10^{8}\frac{\mathrm{km}}{\mathrm{A.U.}}\right)\left(1000\frac{\mathrm{m}}{\mathrm{km}}\right)}\right) - \left(\frac{-6.67\times10^{-11}\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}\cdot1.99\times10^{30}\ \mathrm{kg}}{(50000\ \mathrm{A.U.})\left(1.50\times10^{8}\frac{\mathrm{km}}{\mathrm{A.U.}}\right)\left(1000\frac{\mathrm{m}}{\mathrm{km}}\right)}\right)\right)}$$

$$v = 348000\mathrm{m/s}$$

3. Using Kepler's second law, determine the velocity of the comet at aphelion.

Response: By using an expression of Kepler's second law:

$$v_a = v_p\left(\frac{h_p}{h_a}\right) \tag{10}$$

where

v_a = velocity of the comet at perihelion
v_p = velocity of the comet at aphelion
h_p = distance from the comet to the sun at perihelion
h_a = distance from the comet to the sun at aphelion

and filling in known values as well as the perihelion velocity found from equation 5, the velocity of the comet at aphelion can be found

$$v_a = 348000\ \mathrm{m/s}\ \left(\frac{0.0100\ \mathrm{A.U.}}{50000\ \mathrm{A.U.}}\right)$$

$$v_a = 0.0696\ \mathrm{m/s}$$

4. Write an equation for the comet's elliptical orbit.

Response: With the equation for an ellipse as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{11}$$

where

$x =$ distance from the comet to the center of the ellipse in the horizontal axis
$y =$ distance from the comet to the center of the ellipse in the vertical axis
$a =$ distance from the center of the ellipse to either perihelion or aphelion
$b =$ distance from the horizontally widest point of the ellipse to its center

Though neither a nor b are known, a is quickly solved for as follows

$$a = \frac{1}{2}(h_a + h_p) \tag{12}$$

$$a = \frac{1}{2}(50000.\text{ A.U.} + 0.0100\text{ A.U.})$$

$$a = 25000.005\text{ A.U.}$$

Though b still cannot be found, c, the distance from the foci to the center of the ellipse, is solved for through

$$c = a - h_p \tag{13}$$

$$c = 25000.005\text{ A.U.} - 0.0100\text{ A.U.}$$

$$c = 24999.995\text{ A.U.}$$

With a and c now found, b is isolated from the equation

$$c^2 = a^2 - b^2 \tag{14}$$

$$b = \sqrt{a^2 - c^2} \tag{15}$$

and is solved for by inputting the known values of a and c

$$b = \sqrt{(25000.005\text{ A.U.})^2 - (24999.995\text{ A.U.})^2}$$

$$b = 22.4\text{ A.U.}$$

Thus, the equation for the ellipse of the comet's orbit is

$$\frac{x^2}{(2.50 \times 10^4\text{ A.U.})^2} + \frac{y^2}{(22.4\text{ A.U.})^2} = 1 \tag{16}$$

5. What is the minimal velocity a comet must have at the Oort cloud to overcome the gravitational pull or the sun and escape the solar system?

Response: With the equation for escape velocity as

$$v = \sqrt{\frac{2GM_s}{r}} \tag{17}$$

the escape velocity of the comet at aphelion is found:

$$v = \sqrt{\frac{2\left(-6.67 \times 10^{-11} \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right) \cdot (1.99 \times 10^{30}\ \mathrm{kg})}{(50000\ \mathrm{A.U.})\left(1.50 \times 10^{8} \frac{\mathrm{km}}{\mathrm{A.U.}}\right)\left(1000 \frac{\mathrm{m}}{\mathrm{km}}\right)}}$$

$$v = 188\ \mathrm{m/s}$$

6. Is the direction of the velocity in question 5 important? Explain.

Response: The direction of the velocity in question 5 is important. To escape the Sun's orbit the velocity must be tangent to the orbit at aphelion, as other directions may change the orbit but will not result in an orbital escape.

7. Use Kepler's third law to estimate the orbital period of the comet.

Response: Given a general form of Kepler's third law

$$\frac{T_c^2}{R_c^3} = \frac{T_e^2}{R_e^3} \tag{18}$$

$$T_c = \sqrt{R_c^3 \times \frac{T_e^2}{R_e^3}} \tag{19}$$

where

T_c = period of the comet's orbit (seconds)
R_c = average radius of the comet's orbit (meters)
T_e = period of the earth's orbit (3.154×10^7 seconds)
R_e = average radius of the earth's orbit (1.496×10^{11} meters)

and a calculated average orbit size of the comet

$$R_c = \frac{h_a + h_p}{2} \tag{20}$$

$$R_c = \frac{50000 \text{ A.U.} + 0.0100 \text{ A.U.}}{2}$$
$$R_c = 25000.005 \text{ A.U.}$$

the period can be found such that

$$T_c = \sqrt{\left(25000.005 \text{ A.U.} * 1.496 \times 10^{11} \frac{\text{m}}{\text{A.U.}}\right)^3 \times \frac{(3.154 \times 10^7 \text{ s})^2}{(1.496 \times 10^{11} \text{ m})^3}}$$

$$T_c = 1.24673 \times 10^{14} \text{ s}$$
$$T_c(\text{years}) = (1.24673 \times 10^{14} \text{ s}) * (3.17098 \times 10^{-8} \frac{\text{yr}}{\text{s}})$$
$$T_c(\text{years}) = 3.95 \times 10^6 \text{ yr}$$

8. If the Sun had the same radius as the earth (6.38×10^6 m), what velocity would the comet have if it passed within 100. km of the surface?

Response: The new perihelion was found by summing the radius of the earth and the comet's distance above the surface:

$$h_p = 6.38 \times 10^6 \text{ m} + 1.00 \times 10^5 \text{m}$$

$$h_p = 6.48 \times 10^6 \text{ m}$$

With the new perihelion, equation 5 was used to find the comet's velocity at perihelion

$$v = \sqrt{2\left(\left(\frac{-6.67\times10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\cdot1.99\times10^{30}\text{ kg}}{6.48\times10^6\text{ m}}\right) - \left(\frac{-6.67\times10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\cdot1.99\times10^{30}\text{ kg}}{(50000\text{ A.U.})\left(1.50\times10^8\frac{\text{km}}{\text{A.U.}}\right)\left(1000\frac{\text{m}}{\text{km}}\right)}\right)\right)}$$

$$v = 6.40 \times 10^6 \text{ m/s}$$