# 主題探討

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17 May 2023

# 目錄

# Solution to Problem 1

## Hard case

When $n+1$ is a composite number $(n = 3, 5, 7, 8, 9, 11)$

- Step 1: Find prime $p = kn + 1 (k \in \mathbb{N})$(Dirichlet's Theorem). Consider a subextension of $\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}$. Taking $n = 5$ as an example, $p = 2 \times 5 + 1 = 11$, consider $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})/\mathbb{Q}$ of degree 5.

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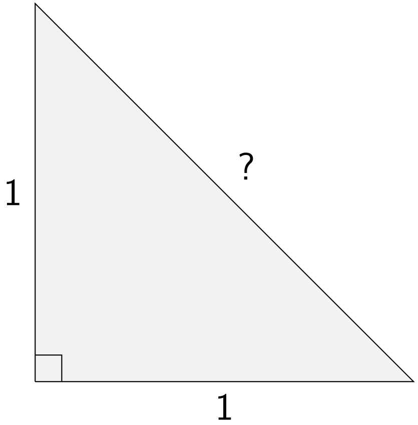
- Step 2: Consider the polynomial $P = \sum_{i=1}^{5}(x - (\zeta_{11}^{i} + \zeta_{11}^{-i}))$. It is invariant under $Aut_{\mathbb{Q}}\mathbb{Q}(\zeta_{11})$, indicating that this is the minimal polynomial of $\zeta_{11} + \zeta_{11}^{-1}$ is

# Solution to Problem 1

## Hard case

- Step 3: Consider the automorphism $f : \zeta_{11}^{i} + \zeta_{11}^{-i} \mapsto \zeta_{11}^{ia} + \zeta_{11}^{-ia}$. $Aut_{\mathbb{Q}}\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1}) \simeq \mathbb{Z}/5\mathbb{Z}$

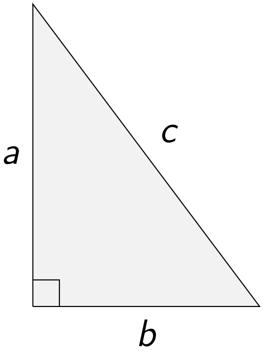
# 引入



# 第一次數學危機

## 勾股定理/畢達哥拉斯定理

$$g:\quad \begin{tikzcd} K \arrow[r] & K \\ \alpha_i \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & g(\alpha_i) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$



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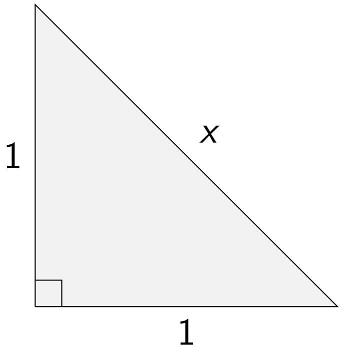
**Example**

$$1^2 + 1^2 = x^2$$

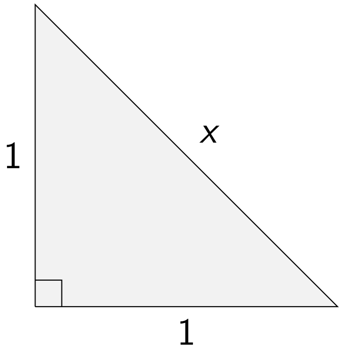


# 第一次數學危機

**Example**

$$1^2 + 1^2 = x^2$$

$$x = \sqrt{2}$$



# 第一次數學危機

- 畢達哥拉斯學派–“數即萬物” 思想

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- $\forall$ 數字x, 可以被表示為$\frac{p}{q}(p, q \in \mathbb{Z}, (p, q) = 1)$

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- $\forall$ 數字x,可以被表示為$\frac{p}{q}(p, q \in \mathbb{Z}, (p, q) = 1)$
- $\forall x \in \mathbb{Q}$
- $\sqrt{2} \notin \mathbb{Q}$

# 新聞

**Here's How Two New Orleans Teenagers Found a New Proof of the Pythagorean Theorem**

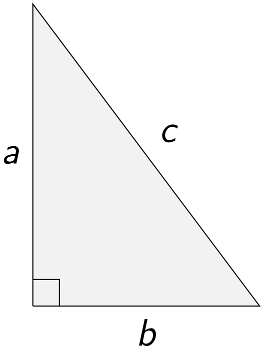
An inspirational example of how elementary math is open to everyone

*article link:https://keith-mcnulty.medium.com/heres-how-two-new-orleans-teenagers-found-a-new-proof-of-the-pythagorean-theorem-b4f6e7e9ea2d*

# 一個新的證明

**勾股定理/畢達哥拉斯定理**

$$a^2 + b^2 = c^2$$



# 證明

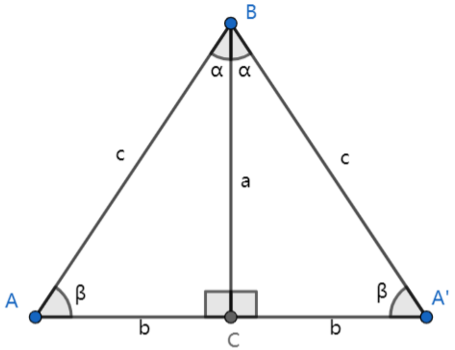
## 證明

將$\triangle ABC$沿長邊$BC(a > b)$反轉,得到$\triangle A'BC$,那麼

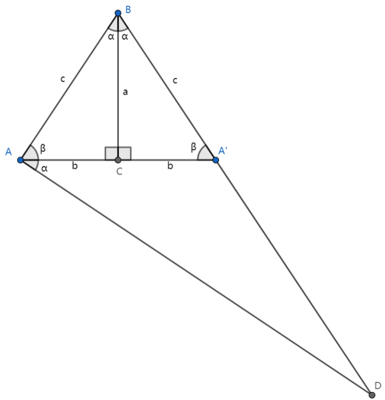
$$\alpha + \beta = 90^\circ \qquad (1)$$

$$cos\alpha = sin\beta = \frac{a}{c} \qquad (2)$$

$$tan\alpha = \frac{b}{a} \qquad (3)$$



# 證明




## 證明

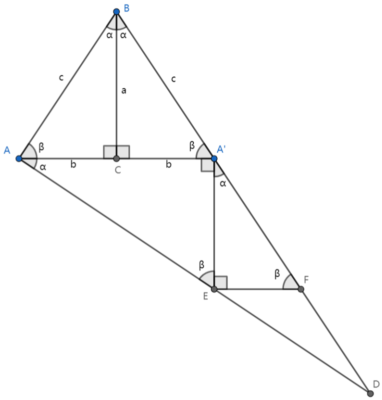
作$\angle BAD = 90°$,$AD$交$BA'$於點$D$

則$\angle A'AD = 90° - \beta = \alpha$

# 證明

## 證明

過點$A'$作$AA'$的垂線，交$AD$與點$E$
再過點$E$作$A'E$的垂線，交$A'D$與點$F$
則$\angle A'AE = \angle FA'E = \alpha$
則$\angle A'EA = \angle A'FE = \beta$



# 證明

## 證明

在$\triangle AA'E$中，

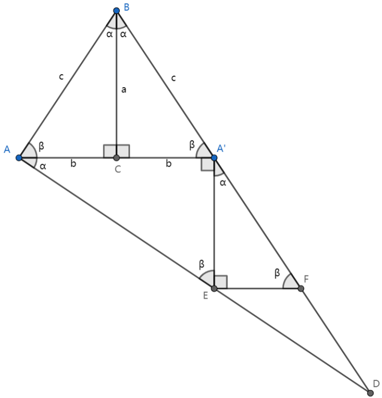
$$AE = \frac{AA'}{\cos\alpha} = \frac{2bc}{a}$$

$$A'E = AA'\tan\alpha = \frac{2b^2}{a}$$

在$\triangle A'EF$中，

$$A'F = \frac{A'E}{\cos\alpha} = \frac{2b^2c}{a^2}$$

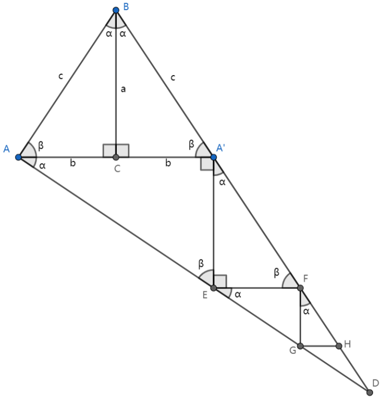
$$EF = A'E\tan\alpha = \frac{2b^3}{a^2}$$



# 證明

## 證明

過點$F$作$EF$的垂線，交$AD$與點$G$
再過點$G$作$FG$的垂線，交$A'D$與點$H$
則$\angle FEG = \angle GFH = \alpha$



# 證明

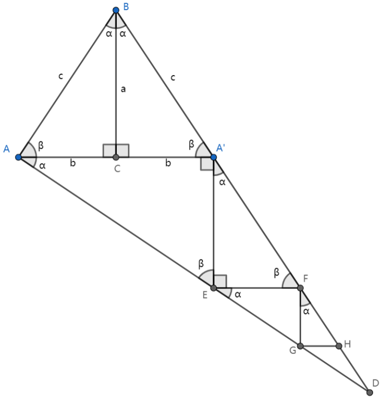
## 證明

由於$\triangle EFD \sim \triangle AA'D$

相似比$\lambda$為$\frac{EF}{AA'} = \frac{\frac{2b^3}{a^2}}{2b} = \frac{b^2}{a^2}$

則$EG = \lambda AE = \frac{b^2}{a^2}AE$

$FH = \lambda A'F = \frac{b^2}{a^2}A'F$



# 證明

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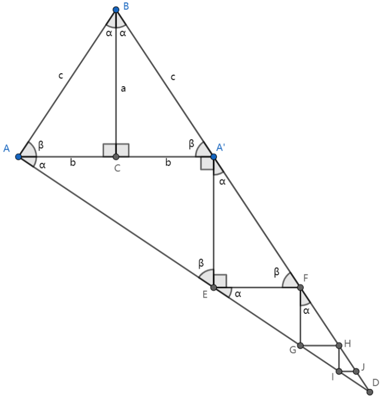
重複這種作法，則由於

$$\triangle GHD \overset{\lambda}{\sim} \triangle EFD \overset{\lambda}{\sim} \triangle AA'D$$

$$GI = \lambda EG = \lambda^2 AE$$

$$HJ = \lambda FH = \lambda^2 A'F$$

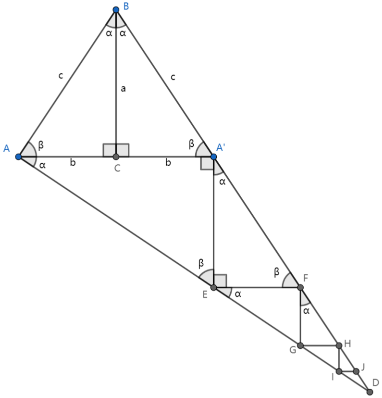


# 證明

## 證明

因此，根據等比數列求和公式

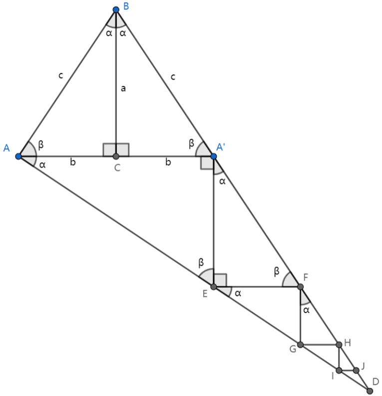
$$
\begin{aligned}
AD &= AE + EG + GI + \ldots \\
&= AE + \lambda AE + \lambda^2 AE + \ldots \\
&= AE\frac{1-\lambda^{+\infty}}{1-\lambda}
\end{aligned}
$$



# 證明

## 證明

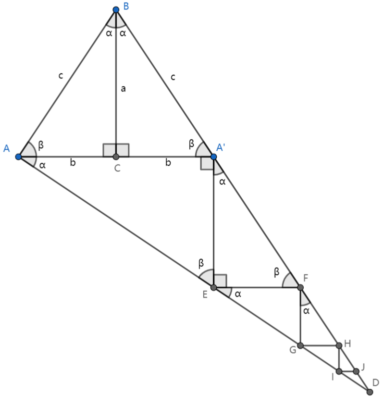
但由于$a > b$,所以
$\lambda = \frac{b^2}{a^2} \in (0,1), \lambda^{+\infty} = 0$



# 證明

## 證明

$$AD = AE\frac{1-\lambda^{+\infty}}{1-\lambda}$$
$$= \frac{2bc}{a}\left(\frac{1-0}{1-\frac{b^2}{a^2}}\right)$$
$$= \frac{2abc}{a^2-b^2}$$



# 證明

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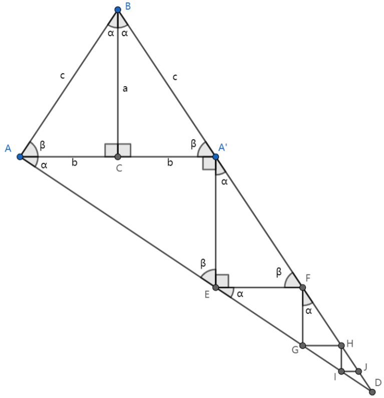
同地，根據等比數列求和公式

$$\begin{aligned} A'D &= A'F + FH + HJ + \dots \\ &= A'F + \lambda A'F + \lambda^2 A'F + \dots \\ &= A'F \frac{1-\lambda^{+\infty}}{1-\lambda} \end{aligned}$$



# 證明

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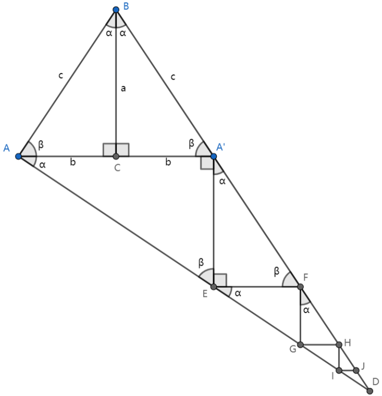
$$
\begin{aligned}
A'D &= A'F\frac{1-\lambda^{+\infty}}{1-\lambda} \\
&= \frac{2b^2c}{a^2}\left(\frac{1-0}{1-\frac{b^2}{a^2}}\right) \\
&= \frac{2b^2c}{a^2-b^2}
\end{aligned}
$$

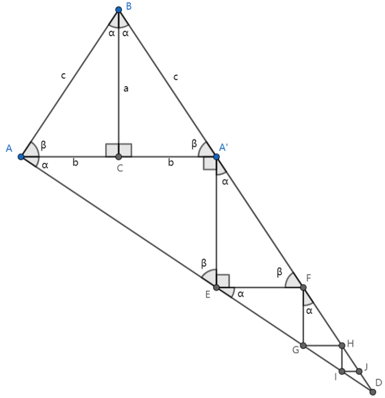


# 證明

## 證明

由于$BD = BA' + A'D$,所以

$$
\begin{aligned}
BD &= BA' + A'D \\
&= c + \frac{2b^2c}{a^2 - b^2} \\
&= \frac{a^2c + b^2c}{a^2 - b^2}
\end{aligned}
$$



# 證明

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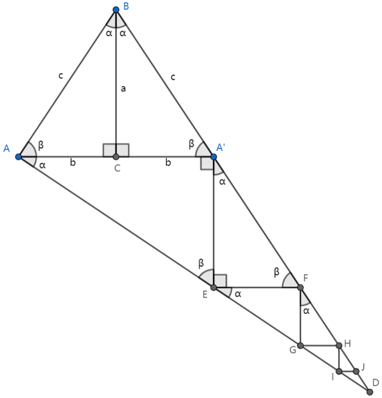
在$\triangle ABD$,

$$sin\angle ABD = sin2\alpha = \frac{AD}{BD} = \frac{\frac{2abc}{a^2-b^2}}{\frac{a^2c+b^2c}{a^2-b^2}} = \frac{2ab}{a^2+b^2}$$

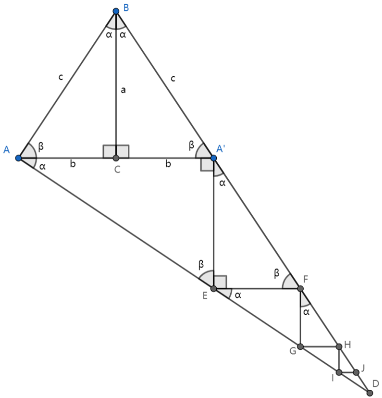


# 證明

## 證明

在$\triangle ABA'$,由正弦定理

$$\frac{AA'}{\sin\angle ABD} = \frac{A'B}{\sin\angle BAA'}$$
$$\frac{2b}{\frac{2ab}{a^2+b^2}} = \frac{c}{\frac{a}{c}}$$
$$\frac{a^2+b^2}{a} = \frac{c^2}{a}$$



# 思考

證明完畢Q.E.D.?

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證明完畢Q.E.D.?

NO!

# 證明

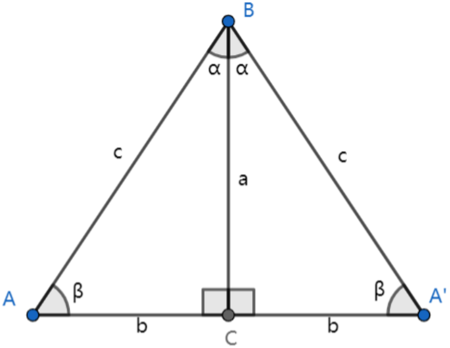
## 證明

$a = b$時

$\angle ABA' = 90^\circ$

由射影定理，$AB^2 = AC \cdot AA'$

$c^2 = b \cdot 2b = 2b^2 = a^2 + b^2$

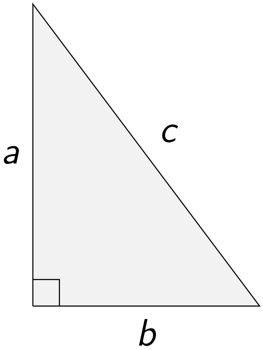


# 證明

**Proof.**

$$a^2 + b^2 = c^2$$

$\square$



# 延伸1

滿足$a^2+b^2=c^2$的整數對的全部解為

$a=\pm k(r^2-s^2), b=\pm 2ksr, c=\pm k(r^2+s^2)$

及

$a=\pm 2ksr, b=\pm k(r^2-s^2), c=\pm k(r^2+s^2)$

其中,$r, s, k$滿足$r>s>0, (s, r)=1, (r+s, 2)=1, k\in\mathbb{Z}^*$

# 延伸2

費馬大定理
當整數$n > 2$時,不定方程$x^n + y^n = z^n$沒有正整數解

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- Pierre de Fermat

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- "業餘"數學家
- 律師工作枯燥而孤獨,所以喜好研究數學,數學成就不遜於數學家

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- 17世紀提出

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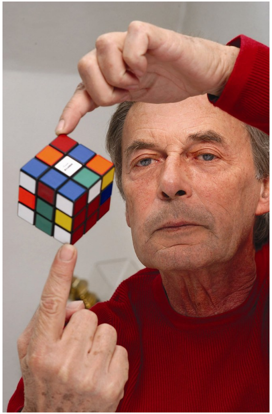
- $\iff$ 在代數曲線$\xi^n + \eta^n = 1 (n \geq 3)$ 上無顯然的有理點

# 魔方



- Ernő Rubik,Magyarország

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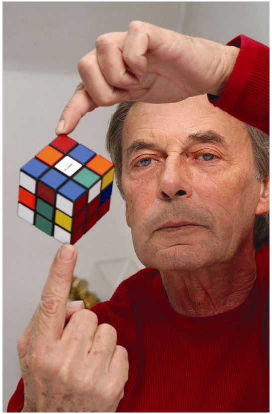

- Ernő Rubik,Magyarország
- Rubik’s cube, 1974

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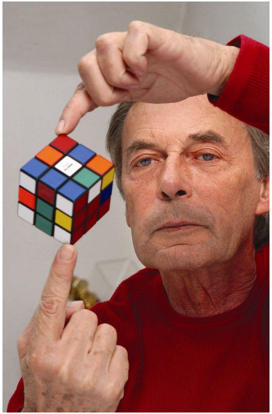

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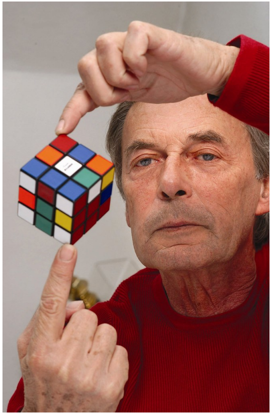

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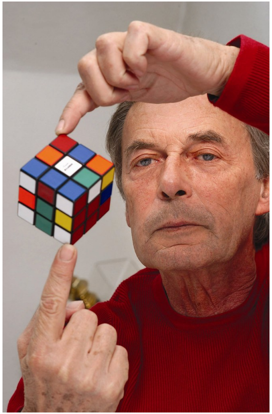

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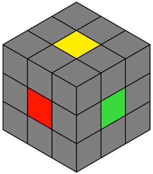
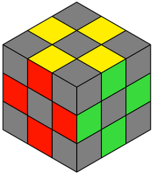
- Combinations?

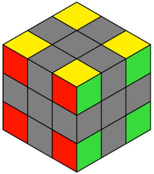
# 結構與限制——3x3



**Centre Pieces**

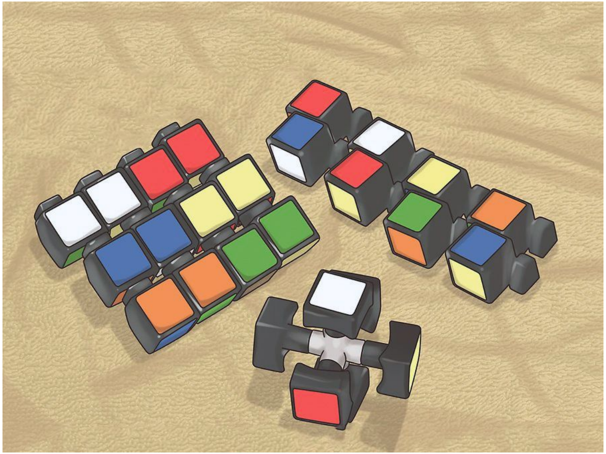


**Edge Pieces**

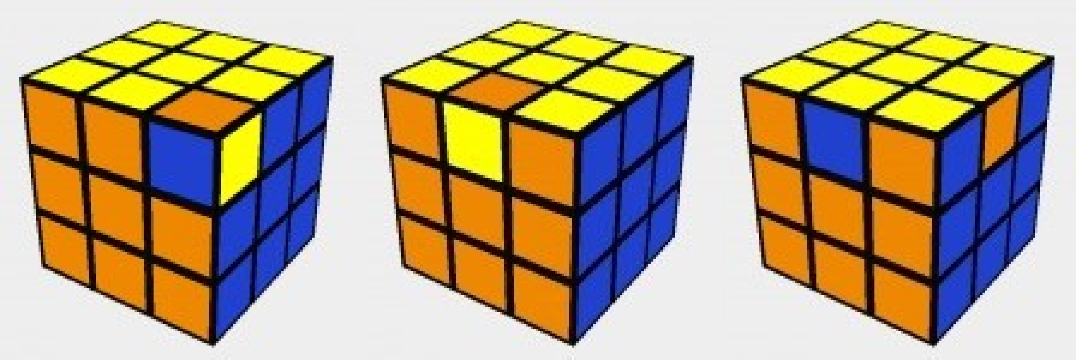


**Corner Pieces**

# 結構與限制——3x3



# 結構與限制——3x3



# 3x3

- 中心:$C_{center} = 1$

## 3x3

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- 角塊(總共8個):$C_{corner} = P_8^8 \cdot 3^8 \cdot \underbrace{\frac{1}{3}}_{\text{限制}}$

# 3x3

- 中心:$C_{center} = 1$
- 角塊(總共8個):$C_{corner} = P_8^8 \cdot 3^8 \cdot \underbrace{\frac{1}{3}}_{限制}$
- 棱塊(總共12個）:$C_{edge} = P_{12}^{12} \cdot 2^{12} \cdot \underbrace{\frac{1}{2}}_{限制} \cdot \underbrace{\frac{1}{2}}_{限制}$

$$T_{3x3} = \underbrace{C_{center} \cdot C_{corner} \cdot C_{edge}}_{乘法原理} = 43252003274489860000$$

# God's number

- 三階魔方任意狀態最少復原步數

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- 三階魔方任意狀態最少復原步數
- God's number
- 20

# NxN cube combinations

$$C_{NxN} = \underbrace{\underbrace{7! \cdot 3^6}_{\text{角塊}} \cdot \underbrace{\left(24 \cdot 2^{10} \cdot 12\right)^{N mod 2}}_{\text{中心棱塊和正中心}} \cdot \underbrace{24!^{\lfloor \frac{N-2}{2} \rfloor}}_{\text{非中心棱塊}} \cdot \underbrace{\left(\frac{24!}{4!^6}\right)^{\lfloor \left(\frac{N-2}{2}\right)^2 \rfloor}}_{\text{非正中心中心塊}}}_{\text{乘法原理}}$$

find $\{f\}$ that satisfies: $\forall$x,y$\in \mathbb{R}$,
$f(x) - f(y) < |x - y|^2$

# THE END