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Lab 8: Proofs by Numerical Induction

Note to reader: these aren't exactly the most **wordy** proofs - you should make yours wordier :).

1. (some points) For all natural numbers $n \geq 2$, $3^n > n^2$.

Solution:
Claim. For all natural numbers $n \geq 2, 3^n > n^2$

Proof. (Induct on n.)
Base Case ($n = 2$):

$3^2 = 9 > 4 = 2^2$

Induction Step:

Choose a $k \in \mathbb{N}$ with $k \geq 2$, and assume $3^k > k^2$

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&> 3 \cdot k^2 && \text{(by IH)} \\
&= k^2 + k^2 + k^2 \\
&= k^2 + k \cdot k + k \cdot k \\
&\geq k^2 + 2k + 2 \cdot 2 && (\text{since } k \geq 2) \\
&= k^2 + 2k + 4 \\
&> k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

Therefore $3^{k+1} > (k+1)^2$ $\square$

2. (some points) For all natural numbers $n \geq 1$, $\sum_{i-1}^{n} 2^{i-1} = 2^n - 1$

Solution:
Claim. For all natural numbers $n \geq 1$, $\sum_{i-1}^{n} 2^{i-1} = 2^n - 1$

Proof. (Induct on n.)
Base Case ($n = 1$):

$\sum_{i=1}^{1} 2^{1-1} = 1$, and $2^1 - 1 = 1$.

So therefore $\sum_{i=1}^{1} 2^{i-1} = 2^1 - 1$

Inductive Step:

Assume that for some positive integer k, $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^k \\
&= 2^k - 1 + 2^k && \text{(by IH)} \\
&= 2(2^k) - 1 \\
&= 2^{k+1} - 1
\end{aligned}$$

So $\sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1$ $\square$

3. (some points) For all natural numbers, $n \geq 7, 3^n < n!$.

Solution: proof goes here

4. (some points) For all natural numbers $n \geq 1, \sum_{i=1}^{n} 2i = n(n+1)$.

Solution: proof goes here

5. (some points) For all $n \in \mathbb{N}, n^2 - 3n$ is even.

Solution: this is a proof

3: Claim: For all natural numbers, $n \geq 7, 3^n < n!$.

Proof: (Induct on n)

Base case: $n = 7$

$3^7 = 2187 < 5040 = n!$

Induction Step:

Choose a $k \geq 7$ and assume that $3^k < k!$

$$3^{(k+1)} = 3 \cdot 3^k$$
$$< 3 \cdot k! \quad \text{(by IH)}$$
$$= k+1 \cdot k! \quad (\text{since } k \geq 7 > 3)$$
$$= (k+1)!$$

Therefore $3^{(k+1)} < (k+1)!$ $\square$

4.- Claim: For all natural numbers $n \geq 1, \sum_{i=1}^{n} 2i = n(n+1)$.

Proof: (Induct on n)

Base case: $n = 1$

$$\sum_{i=1}^{1} 2 \cdot 1 = 2 = 1(1+1)$$

Induction Step:

Choose a $k \geq 1$ and assume that $\sum_{i=1}^{k} 2i = k(k+1)$

$$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1)$$

$$= k(k+1) + 2(k+1) \quad (\text{by IH})$$

$$= k^2 + 3k + 2$$

$$= (k+1)(k+1+1)$$

Therefore $\sum_{i=1}^{k+1} 2i = (k+1)(k+1+1)$ $\square$

5: Claim: For all $n \in \mathbb{N}, n^2 - 3n$ is even.

Proof: (Induction on n)

Base case: $n = 0$

$$0^2 - 3 \cdot 0 = 0 = 0 \cdot 2$$

Induction step

Choose a $k \in \mathbb{N}$ and assume that $k^2 - 3k$ is even.

$$\begin{aligned}(k+1)^2 - 3(k+1) &= k^2 + 1 - 3k + 3 \\ &= k^2 - 3k + 4 \\ &= k^2 - 3k + 2 \cdot 2\end{aligned}$$

Since $k^2 - 3k$ is even (by IH) and 4 is even, $k^2 - 3k + 4$ must be even, since even numbers are closed under addition.

Therefore, $(k+1)^2 - 3(k+1)$ is even. $\square$