present

FINC 462/662 - Fixed Income Securities

FINC-462/662: Fixed Income Securities

Measures of Bond Price Volatility

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Overview

Our goals for today

- [ ] Understand why we use Convexity and how to calculate it.
- [ ] Calculate the Convexity of a portfolio.
- [ ] Use Modified Duration to hedge interest rate risk.
- [ ] Use Modified Duration and Convexity to hedge interest rate risk.

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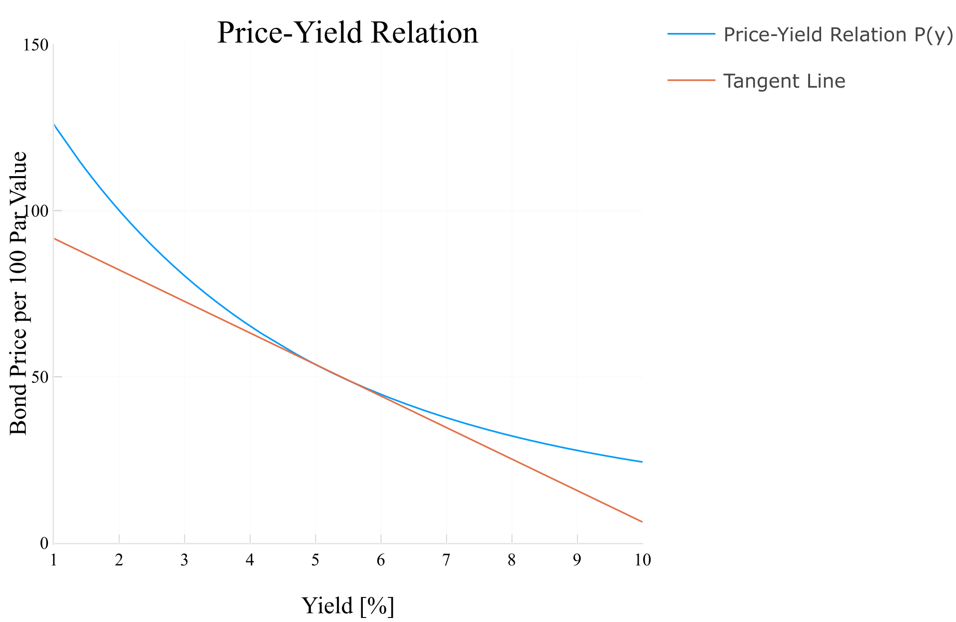
Modified Duration and the Price-Yield Relation

- Face Value F [$]: 100
- Coupon Rate c [% p.a.]: 2.0
- Time to maturity T [years]: 30

Reset



- The tangent line approximates the price-yield relation closely near the tangency point.
- The modified duration can be interpreted as giving us the percent price change of the bond when we assume that the price-yield relation is represented by the tangent line.

- Recall that we compute modified duration $MD(y)$ using

$$MD(y) \approx -\frac{P(y+\Delta y) - P(y-\Delta y)}{2 \times \Delta y} \times \frac{1}{P(y)}$$

- And recall that we approximate the percent price change of the bond using

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y$$

- We noted, however, that the previous equation becomes inaccurate as yield changes increase.
- We now add a "Convexity" term to this equation that takes into account the convex shape of the price-yield relation.
- This will improve the accuracy of our approximation formula.

- Let's first define the **convexity** CX of a standard semi-annual coupon bond with price P, time-to-maturity T, coupon rate c (paid-semiannually), semi-annual coupon cash flows of C, face value F and yield-to-maturity of y (semi-annually compounded).

- To shorten the notation, we will
  - use $n = 1, \ldots, N$ to denote the coupon period.
    - n=1 corresponds to t=0.5 (the first coupon period).
    - The last coupon period N corresponding to $2 \times T$.
  - use Y to denote the per-period yield, i.e. $Y = y/2$.
- The **per-period** convexity is given by

$$\mathrm{CX} = \frac{1}{P \times (1+Y)^2} \times \sum_{n=1}^{N} \left[\frac{C}{(1+Y)^n} \left(n^2 + n\right) \right]$$

- To get the **annual** convexity we divide CX by the square of the number of periods in a year.
  - For instance, for semi-annual coupon bonds, there are two periods per year.
  - Thus, we divide CX by 4 (since $2^2 = 4$).

- For a zero-coupon bond, the formula simplifies.
- Specifically, for a zero-coupon bond with time-to-maturity T and yield-to-maturity y (**annually compounded**)

$$\mathrm{CX} = \frac{T^2 + T}{(1+y)^2}$$

- Instead of using the previous formula, we can use the following to calculate the convexity CX.

$$\mathrm{CX} = \frac{P(y + \Delta y) + P(y - \Delta y) - 2 \times P(y)}{(\Delta y)^2} \times \frac{1}{P(y)}$$

- Recall the notation:
  - ΔP is the dollar price change of a bond.
  - $\frac{\Delta P}{P}$ is the percent change in the price of a bond.
  - Δy is the change in the yield of the bond in decimals.
  - $P(y)$ is the bond price when the yield-to-maturity is y (keeping time-to-maturity T and coupon rate c fixed).

- We can now approximate bond price changes more precisely by using

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y + \frac{1}{2} \times \mathrm{CX} \times (\Delta y)^2$$

Example

- Face Value F [$]: 100
- Coupon Rate c [% p.a.]: 8.0
- Yield y [% p.a.]: 6.0
- Time to maturity T [years]: 10

Reset

- Consider a semi-annual bond with time-to-maturity $T = 10$ years, face value $F = 100$, coupon rate $c = 8.0\%$, semi-annual coupon cash flows of $C = 4.0$ and yield-to-maturity $y = 6.0\%$.
- Calculate the convexity CX of this bond.

$$\mathrm{CX} = \frac{P(y + \Delta y) + P(y - \Delta y) - 2 \times P(y)}{(\Delta y)^2} \times \frac{1}{P(y)}$$

- We start by selecting the yield change $\Delta y = 0.2\%$
- First, we calculate the bond price $P(y)$

$$P(y) = \frac{C}{y/2} \times \left(1 - \frac{1}{(1+\frac{y}{2})^{2\times T}}\right) + \frac{F}{(1+\frac{y}{2})^{2\times T}} = \frac{4.0}{6.0\%/2} \times \left(1 - \frac{1}{\left(1+\frac{6.0\%}{2}\right)^{2\times 10}}\right) +$$

- Next, using $\Delta y = 0.2\%$

$$P(y+\Delta y) = \frac{C}{(y+\Delta y)/2} \times \left(1 - \frac{1}{\left(1+\frac{y+\Delta y}{2}\right)^{2\times T}}\right) + \frac{F}{\left(1+\frac{y+\Delta y}{2}\right)^{2\times T}} = \frac{4.0}{6.2\%/2} \times \left(1 -\right.$$

- Similarly,

$$P(y-\Delta y) = \frac{C}{(y-\Delta y)/2} \times \left(1 - \frac{1}{\left(1+\frac{y-\Delta y}{2}\right)^{2\times T}}\right) + \frac{F}{\left(1+\frac{y-\Delta y}{2}\right)^{2\times T}} = \frac{4.0}{5.8\%/2} \times \left(1 -\right.$$

- Thus,

$$\mathrm{CX} = \frac{113.2668 + 116.5176 - 2 \times 114.8775}{(0.2\%)^2} \times \frac{1}{114.8775} = 63.925643$$

- Recall that the modified duration of the bond is

$$MD(y) = -\frac{P(y+\Delta y) - P(y-\Delta y)}{2 \times \Delta y} \times \frac{1}{P(y)}$$

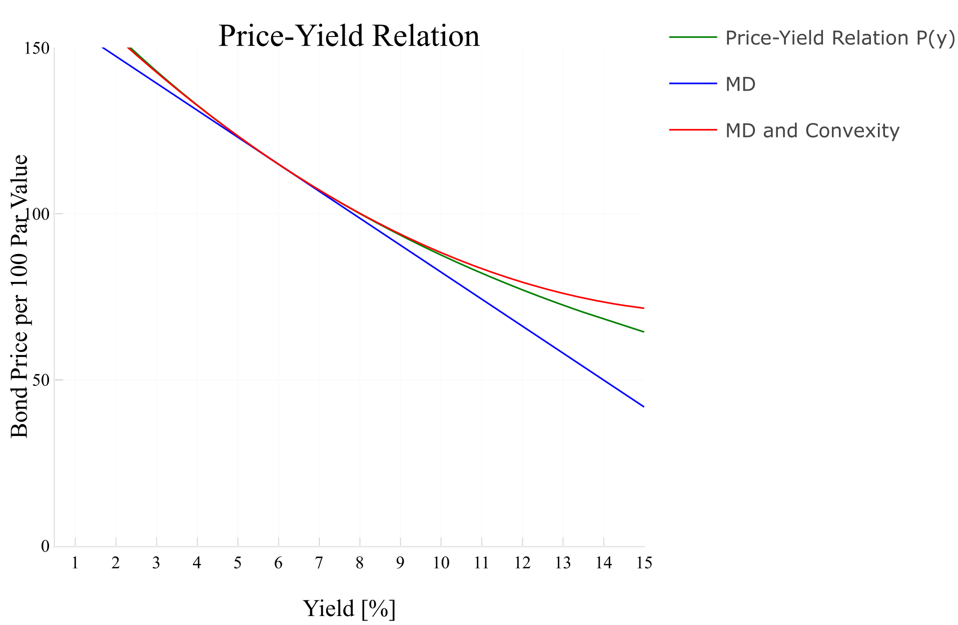
$$MD(y) = -\frac{113.2668 - 116.5176}{2 \times 0.2\%} \times \frac{1}{114.8775} = 7.074474$$

- Thus, when yield increase from $y = 6.0\%$ to $y = 6.5\%$, the approximate percent change in the bond price is

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y + \frac{1}{2} \times \text{CX} \times (\Delta y)^2$$

$$\frac{\Delta P}{P} = -7.074474 \times 0.005 + \frac{1}{2} \times 63.925643 \times (0.005)^2 = -0.034573 = -3.4573\%$$

	CurrentYield	NewYield	YieldChange	ActualPrice	MDPrice	CXPrice	MD_PriceChange
1	6.0	0.5	-5.5	173.067	159.576	170.683	44.6984
2	6.0	1.0	-5.0	166.456	155.512	164.692	40.6349
3	6.0	1.5	-4.5	160.151	151.449	158.884	36.5714
4	6.0	2.0	-4.0	154.137	147.385	153.26	32.5079
5	6.0	2.5	-3.5	148.398	143.322	147.82	28.4444
6	6.0	3.0	-3.0	142.922	139.258	142.563	24.3809
7	6.0	3.5	-2.5	137.694	135.195	137.49	20.3174
8	6.0	4.0	-2.0	132.703	131.131	132.6	16.254
9	6.0	4.5	-1.5	127.936	127.068	127.894	12.1905
10	6.0	5.0	-1.0	123.384	123.004	123.372	8.12698
	more						
30	6.0	15.0	9.0	64.3193	41.7347	71.4763	-73.1428



Convexity of a Bond Portfolio

- Thus far, we have considered the case of a single bond and have calculated its convexity.
- When we have a portfolio of bonds, we calculate the convexity of the bond portfolio using the convexities of the individual bonds in the portfolio.

- Specifically, suppose the bond portfolio consists of B bonds. We denote the individual bonds by $b = 1, \ldots, B$.
- The portfolio is assumed to consist of N_b units of each bond b.
- Each bond is assumed to have a price of P_b per $100 par value.
- We write the fraction of the position in bond b to the total portfolio value $P_{\text{Portfolio}}$ as

$$w_b = \frac{n_b \times P_b}{P_{\text{Portfolio}}}$$

- Note that the total value of the bond portfolio is

$$P_{\text{Portfolio}} = n_1 \times P_1 + \ldots + n_B \times P_B$$

- Then, we calculate the convexity $CX_{\text{Portfolio}}$ of the bond portfolio as the weighted average of the convexities of the individual bonds (CX_i).

$$CX_{\text{Portfolio}} = w_1 \times CX_1 + w_2 \times CX_2 + \ldots + w_B \times CX_B$$

Example

- Suppose that you own a portfolio of zero-coupon bonds. All yields are annually compounded.
- Calculate the convexity of the portfolio.

Bond	Maturity	Yield	Face value
H	1	2%	40
I	2	3%	40
J	3	5%	40
K	4	6%	40
L	5	8%	1040

- Let's first calculate the the prices of the zero coupon bonds per $100 face value.
- Recall, that the price of a T-year maturity zero-coupon bond with yield y_T (annually compounded) is given by

$$P_T = \frac{100}{(1+y_T)^T}$$

- Next, let's calculate the number of units n_b for each bond b in the portfolio.
- The number of bonds is simply the actual face value divided by 100 face value (which we used to calculate the bond price).
  - For instance for bond H, it is $40/$100=0.4

- Next, we calculate the convexities of the zero-coupon bonds.
- Recall that for a zero-coupon bond with time-to-maturity T and yield-to-maturity y (**annually compounded**), the convexity is

$$\mathrm{CX} = \frac{T^2+T}{(1+y)^2}$$

- For instance, for bond L it is $MD_5 = \frac{(5^2+5)}{(1+8\%)^2} = 25.720165$

- Next, we calculate the total value of the bond portfolio.
- The value of the bond portfolio $P_{\text{Portfolio}}$ is the sum of the values of the positions in the individual bonds. The position in bond b is worth the number of units times the bond price, i.e. $n_b \times P_b$.

- Now we can calculate the portfolio weights

$$w_b = \frac{n_b \times P_b}{P_{\text{Portfolio}}}$$

- As the last step, we compute the convexity of the portfolio $CX_{\text{Portfolio}}$

$$CX_{\text{Portfolio}} = w_1 \times CX_1 + w_2 \times CX_2 + \ldots + w_B \times CX_B$$

$$CX_{\text{Portfolio}} = 0.0886 + 0.2506 + 0.442 + 0.6627 + 21.3933$$

$$CX_{\text{Portfolio}} = 22.837159$$

Hedging Interest Rate Risk

- Consider a bond **portfolio** with modified duration MD and convexity CX. Suppose that the portfolio has a current value (price) of P.
- We would like to protect the **value** of our portfolio against changes in interest rates.
  - An increase in interest rates typically leads to a drop in value of a long bond portfolio.

- To illustrate how we can achieve this, recall how we calculate the percentage price change in the value of a bond portfolio, given the portfolio's duration and convexity.

$$\frac{\Delta P}{P} = -MD \times \Delta y + \frac{1}{2} \times CX \times (\Delta y)^2$$

- In the equation, we want the percentage price change to be zero, because thenour portfolio does not change in value when interest rates change by Δy.

$$\frac{\Delta P}{P} \stackrel{!}{=} 0$$

- How can we achieve this?

- Looking at the right-hand side of the equation for $\frac{\Delta P}{P}$, it must be the case that

$$-MD \times \Delta y + \frac{1}{2} \times CX \times (\Delta y)^2 \stackrel{!}{=} 0$$

$$\text{for } \textbf{all } \Delta y.$$

- A straight-forward way to do this is to construct the portfolio in such a way that the modified duration of the portfolio is zero and the convexity of the portfolio is zero.

$$MD \stackrel{!}{=} 0$$

$$CX \stackrel{!}{=} 0$$

- To simplify the calculations, let's start by requiring that **only** the modified duration of the bond portfolio be zero.
    - We will consider modified duration and convexity *jointly* later.

Hedging Interest Rate Risk using Duration

- Suppose that we are a large firm and that we have issued a bond with $ 1000 par value. The bond is a zero-coupon bond with maturity in 10 years.
- Suppose that all interest rates are 4.0%.

- Let's first determine what the value of our liability is today.
- Recall that the price of a 10-year zero coupon bond with $ 1000 par value when interest rate are 4.0 % is

$$P_{10} = \frac{F}{(1+r)^{10}} = \frac{1000}{(1+4.0\%)^{10}} = 675.5642$$

- Thus, the present value of what we owe is $ 675.5642.

- By issuing the bond we have created a liability that fluctuates in value as interest rates change.
    - Note that issuing a bond is similar to taking a short position in the bond.

- Specifically, we know that the bond has a modified duration MD of

$$MD_{10} = \frac{T}{1+y} = \frac{10}{1+4.0\%} = 9.6154$$

- Recall that this means that when interest rates decrease by 100 basis points, the value of our liability increases by around 9.62 percent.

$$\frac{\Delta P_{10}}{P_{10}} = -MD_{10} \times \Delta y = -9.62 \times \Delta y$$

- We want to hedge our exposure to this liability.
- To hedge our exposure, we can buy/sell a 2-year zero-coupon bond in the financial market.

- Recall that the modified duration of this 2-year zero-coupon bond is

$$MD_2 = \frac{T}{1+y} = \frac{2}{1+4.0\%} = 1.9231$$

- This means that when interest rates decrease by 100 basis points, the value the bond *increases* by around 1.92 percent.

$$\frac{\Delta P_2}{P_2} = -MD_2 \times \Delta y = -1.92 \times \Delta y$$

- The idea is that we owe more on the liability, when interest rates decrease, and the value of the 2-year bond increases.
- Our bond portfolio will then consist of the 10-year liability and the 2-year bond.
- Since we consider modified duration only, the percentage price change in the value of our portfolio is

$$\frac{\Delta P}{P} = -MD \times \Delta y$$

- We want $\frac{\Delta P}{P}$ to be zero.

- We can visualize our portfolio by thinking about it as a balance sheet where the liability side consists of the bond we have just issued.
- The asset side will consist of the 2-year bond that we will use to hedge the interest rate risk of the bond we have issued. Suppose the market value of our position in the 2-year bond is \$ x.

Assets	**Liabilities**
`2-year bond: x`	`10-year Bond: 675.5642`

- To quantify the interest rate sensitivity of assets and liabilities, let's add the modified durations of the 2-year bond and the 10-year bond.
- Recall that the modified duration of a zero-coupon bond with time-to-maturity T is $MD = T/1 + y$.

Assets	Liabilities
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2`: 1.9231`	MD_{10}`: 9.6154`

- Suppose that yields increase by Δy. What is the percentage change in the value of assets/liabilities?

Assets	Liabilities
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2`: 1.9231`	MD_{10}`: 9.6154`

Assets	Liabilities
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2`: 1.9231`	MD_{10}`: 9.6154`
$\frac{\Delta B_2}{B_2} = -MD_2 \times \Delta y$	$\frac{\Delta B_{10}}{B_{10}} = -MD_{10} \times \Delta y$

- Suppose that yields increase by Δy. What is the change in *dollar* terms of the value of assets/liabilities?
- The (approximate) change in dollar terms of the value of a bond with T-years to maturity and modified duration MD_T is

$$\Delta B_T = B_T \times (-MD_T) \times \Delta y$$

- Using this insight, the balance sheet can be written as

Assets	**Liabilities**
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2`: 1.9231`	MD_{10}`: 9.6154`
$\Delta B_2 = B_2 \times (-MD_2) \times \Delta y$	$\Delta B_{10} = B_{10} \times (-MD_{10}) \times \Delta y$

- Plugging in the values:
  - B_2 is the value in the 2-year bond, i.e. `x` .
  - B_{10} is the value in the 10-year bond, i.e. `675.5642` .
  - MD_2 `= 9.6154`
  - MD_{10} `= 1.9231`

Assets	**Liabilities**
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2`: 1.9231`	MD_{10}`: 9.6154`
$\Delta B_2 = x \times (-1.9231) \times \Delta y$	$\Delta B_{10} = 675.5642 \times (-9.6154) \times \Delta y$

- Hedging interest rate risk means that the total change in the value of assets and liabilities should be zero
  - The change in the value of the liability is offset by the change in value of the asset.

- Thus, it must be the case that

$$x \times (-1.9231) \times \Delta y \stackrel{!}{=} 675.5642 \times (-9.6154) \times \Delta y$$

- This must be true for all Δy which means that our position on the 2-year zero coupon bond x must be

$$x = 675.5642 \times \frac{(-9.6154)}{(-1.9231)} = 3377.8208$$

- Thus, we buy \$ 3377.8208 of the 2-year zero-coupon bond.

- What is the **face value** of the position in the 2-year zero-coupon bond that has a market value of $ 3377.8208?
- Recall that the market value of a zero-coupon bond with face value F and time-to-maturity T when the discount rate is y (annually-compounded) is

$$P = \frac{F}{(1+y)^T}$$

- Pluggin in the market value of the 2-year bond and solving for the face value F

$$3377.8208 = \frac{F}{(1+4.0\%)^2}$$

$$F = \$3653.45$$

- With the hedge, the market value of our assets and liabilities is

Assets	Liabilities
2-year bond: $ 3377.8208	10-year Bond: $ 675.5642
Face value: $ 3653.45	Face value: $ 1000.0

- Let's verify that the hedge works.
- The market value of our portfolio (assets minus liabilities) is

$$\$3377.8208 - \$675.5642 = \$2702.26$$

- To check the hedge, we calculate the value of the portfolio for different changes in yield Δy.

	Delta_y	PortfolioValue
1	-3	2676.18
2	-2	2691.23
3	-1	2699.63
4	0	2702.26
5	1	2699.87
6	2	2693.16
7	3	2682.72

Hedging Interest Rate Risk using Duration and Convexity

- In the previous example, the duration hedge worked well for small changes in interest rates.
- Can we improve the hedge for larger changes in interest rates by hedging convexity as well (i.e. heding both duration and convexity)?
- Let's consider the same setup as in the previous example.

- Suppose that we are a large firm and that we have issued a bond with $ 1000 par value. The bond is a zero-coupon bond with maturity in 10 years.
- Suppose that all interest rates are 4.0%.
- Suppose that we have an additional bond to invest in to hedge convexity. This bond is a 30-year zero coupon bond.

- Let's first calculate the duration, convexity, the percentage price change and the dollar price in response to a yield change Δy for each of the three bonds.
- 10-year Zero-coupon bond (liability)
  - $MD_{10} = \frac{T}{1+y} = \frac{10}{1+4.0\%} = 9.6154$
  - $\text{CX}_{10} = \frac{T^2+T}{(1+y)^2} = \frac{110}{(1+4.0\%)^2} = 101.7012$
  - $\frac{\Delta P_{10}}{P_{10}} = -MD_{10} \times \Delta y + \frac{1}{2} \times CX_{10} \times (\Delta y)^2$
  - $\Delta P_{10} = P_{10} \times (-MD_{10}) \times \Delta y + P_{10} \times \frac{1}{2} \times CX_{10} \times (\Delta y)^2$

- 2-year Zero-coupon bond (liability)
  - $MD_2 = \frac{T}{1+y} = \frac{2}{1+4.0\%} = 1.9231$
  - $\text{CX}_2 = \frac{T^2+T}{(1+y)^2} = \frac{6}{(1+4.0\%)^2} = 5.5473$
  - $\frac{\Delta P_2}{P_2} = -MD_2 \times \Delta y + \frac{1}{2} \times CX_2 \times (\Delta y)^2$
  - $\Delta P_2 = P_2 \times (-MD_2) \times \Delta y + P_2 \times \frac{1}{2} \times CX_2 \times (\Delta y)^2$

- 30-year Zero-coupon bond (liability)
  - $MD_{30} = \frac{T}{1+y} = \frac{30}{1+4.0\%} = 28.8462$
  - $\mathrm{CX}_{30} = \frac{T^2+T}{(1+y)^2} = \frac{930}{(1+4.0\%)^2} = 859.8373$
  - $\frac{\Delta P_{30}}{P_{30}} = -MD_{30} \times \Delta y + \frac{1}{2} \times CX_{30} \times (\Delta y)^2$
  - $\Delta P_{30} = P_{30} \times (-MD_{30}) \times \Delta y + P_{30} \times \frac{1}{2} \times CX_{30} \times (\Delta y)^2$

- Next, let's write down the balance sheet as in the previous example.
- The asset side of the balance sheet now has the 2-year zero-coupon bond and the 30-year zero coupon bond.

- We assume that we enter into a position with market value `z` in the 30-year zero coupon bond.

Assets	**Liabilities**
`2-year bond: x`	`10-year Bond: 675.5642`
`30-year bond: z`	

- Next, let's write down the balance sheet as in the previous example.

Assets	**Liabilities**
`2-year bond: x`	`10-year Bond: 675.5642`
MD_2: 1.9231	MD_{10}: 9.6154
CX_2: 5.5473	CX_{10}: 101.7012
$\Delta B_2 = x \times (-1.9231) \times \Delta y + x \times \frac{1}{2}(5.5473) \times (\Delta y)^2$	$\Delta B_{10} = 675.5642 \times (-9.6154) \times \Delta y + 6$
`30-year bond: z`	
MD_{30}: 28.8462	
CX_{30} 859.8373	
$\Delta B_{30} = z \times (-28.8462) \times \Delta y + z \times \frac{1}{2}(859.8373) \times (\Delta y)^2$	

- Similar to the previous example, we want the total change in the value of assets and liabilities to be zero (when interest rates change by Δy). This means that the change in value of assets must be equal to the change in value of liabilities.

- This means, we need to have

$$\Delta B_2 + \Delta B_{30} = \Delta B_{10}$$

$$x \times (-1.9231) \times \Delta y + x \times \frac{1}{2}(5.5473) \times (\Delta y)^2 + z \times (-28.8462) \times \Delta y + z \times \frac{1}{2}(859.8373) \times$$

- Since this equation must hold for all Δy and for all $(\Delta y)^2$, we can look at all terms in Δy and in $(\Delta y)^2$ separately.
- Terms in Δy: **Modified Duration Equation**
  - $x \times (-1.9231) \times \Delta y + z \times (-28.8462) \times \Delta y = 675.5642 \times (-9.6154) \times \Delta y$
- Terms in $(\Delta y)^2$: **Convexity Equation**
  - $x \times \frac{1}{2}(5.5473) \times (\Delta y)^2 + z \times \frac{1}{2}(859.8373) \times (\Delta y)^2 = 675.5642 \times \frac{1}{2}(101.7012) \times$

- How do we solve these two equations for x and z ?
- Let's first rewrite the equations by collecting all terms in x and y on the left-hand side and the constant terms on the right-hand side and by dropping the Δy and $(\Delta y)^2$ terms.
- $$1.9231 \times x + 28.8462 \times z = 6495.8093$$

$$2.7737 \times x + 429.9186 \times z = 34352.8377$$

▶ How to solve the system of equations using Excel

- The solution to this system of 2 equations in 2 unknowns is

$$x = 2412.7292, z = 64.3394$$

- Thus, we enter a position with market value of \$ 2412.7292 in the 2-year bond, and a position with market value of \$ 64.3394 in the 30-year bond.
- The corresponding face values in the 2-year bond and the 30-year bonds are

$$F_2 = 2609.61$$

$$F_{30} = 208.68$$

- The balance sheet is now
- Next, let's write down the balance sheet as in the previous example.

Assets	Liabilities
2-year bond: 2412.7292 Face value F_2: 2609.61	10-year Bond: 675.5642 Face value F_{10}: 1000.0
30-year bond: 64.3394 Face value F_{30}: 208.68	

- Let's verify that the hedge works.
- The market value of our portfolio (assets minus liabilities) is

$$\$2412.7292 + 64.3394 - \$675.5642 = \$1801.5$$

- To check the hedge, we calculate the value of the portfolio for different changes in yield Δy.

	Delta_y	PortfolioValue
1	-3	1807.72
2	-2	1803.13
3	-1	1801.68
4	0	1801.5
5	1	1801.36
6	2	1800.48
7	3	1798.4

Wrap-Up

Our goals for today

- [x] Understand why we use Convexity and how to calculate it.
- [x] Calculate the Convexity Convexity of a portfolio.
- [x] Use Modified Duration to hedge interest rate risk.
- [x] Use Modified Duration and Convexity to hedge interest rate risk.

Reading

Fabozzi, Fabozzi, 2021, Bond Markets, Analysis, and Strategies, 10th Edition
Chapter 4