present

FINC 462/662 -- Fixed Income Securities

FINC-462/662: Fixed Income Securities

Forwards

Spring 2022

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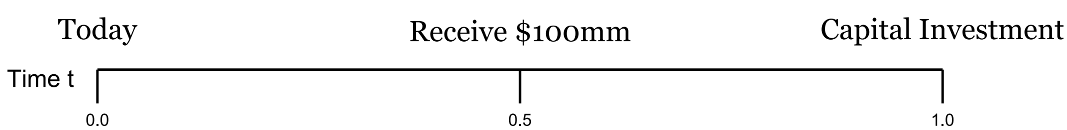
Overview

Our goals for today

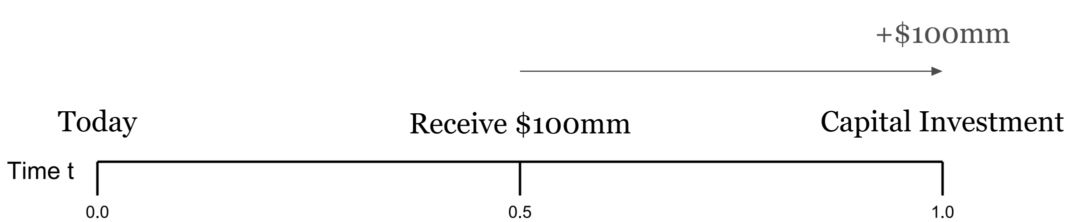
- ☐ Understand and calculate forward rates.
- ☐ Understand how forward rates and spot rates are connected.
- ☐ Know the Expectation Hypothesis and use it to interpret expectations about Fed Monetary Policy.
- ☐ Understand the relation between forward rates and forward contracts.
- ☐ Value a forward rate agreement.

Forward Rates

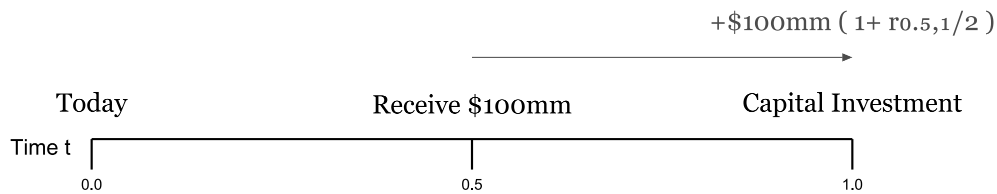
- Lets' consider the following example.
- Suppose, XYZ Widgets has made a sale on credit. The client will pay 100mm in six months. XYZ does not have cash flow needs in six months, but will be making a capital investment in 12 months.



- **What should XYZ do with the $100mm?**
  1. Keep it in a safe ⟶ will still have 100mm at t = 1



- **What should XYZ do with the $100mm?**
  1. Keep it in a safe → will still have 100mm at t = 1.
  2. Invest in a six-month Treasury note at t = 0.5 at the then-prevailing interest rate for another six months.
     - Let the six-month Treasury rate (semi-annually compounded) at time t=0.5 be $r(0.5, 1) \equiv r_{0.5,1}$. Note that we do *not* know today at t=0 what this rate will be.
     - Then the cash flow at t=1 is $100 \times \left(1 + \frac{r_{0.5 \to 1}}{2}\right)$



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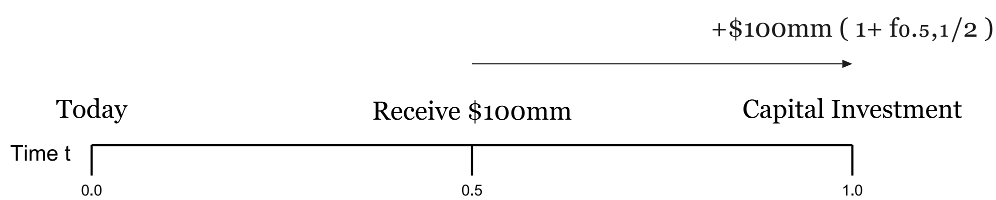
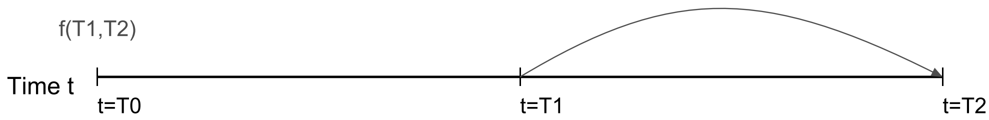
     - Then the cash flow at t=1 is $100 \times \left(1 + \frac{r_{0.5 \to 1}}{2}\right)$
  3. Enter into a contract today that let's XYZ invest the $100mm for six-months starting at t=0.5 at an interest rate of $f_{0.5,1}$.
     - Note that this interest rate is agreed upon today and thus known (whereas $r_{0.5,1}$ is not known today at t=0).
     - What should this interest rate be?

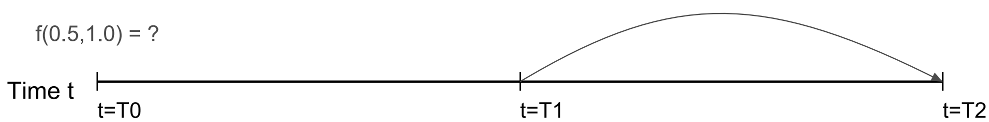


Forward Rates

A forward rate $f(T_1, T_2) \equiv f_{T_1,T_2}$ is the interest rate set *today* at time t=0 for an investment from time T_1 to T_2.



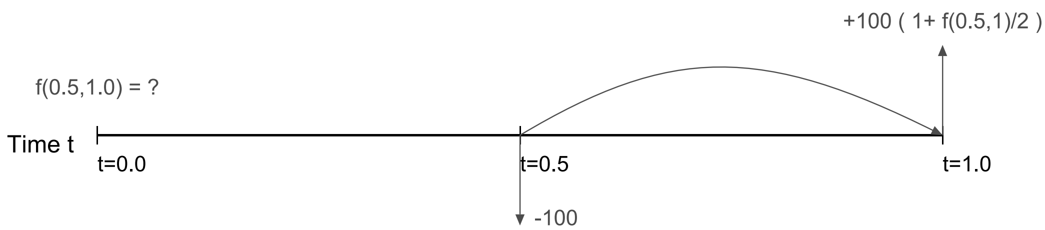
- **What must fhe forward rate $f_{0.5,1}$ be?**
- It turns out that forward rates are related to spot rates (zero-coupon yields) through the Law of One Price.
- This means that we can synthetically create a forward rate agreement using two zero-coupon bonds. Let's see how this can be done.



- Suppose we observe the following zero-coupon bonds

Bond	Time-to-maturity T	Price	Spot Rate
X	0.5	99.0099	2 %
Z	1.0	97.0662	3 %

- Let's consider a portfolio of Bond X and Bond Z.
- Suppose we invest in x units of Bond X and z units in Bond Z.
    - For instance, $x = 1$ means that we buy one Bond X (with 100 par amount).
- Recall that when we agree to the forward rate $f_{0.5,1}$ we invest an amount, say 100, at t=0.5. This is a cash-outflow.
- In turn, we have a cash inflow of principal plus interest at time t=1.
- Thus, in order to replicate the forward rate agreement, we need to have a cash outflow at t=0.5 and a cash inflow at t=1.0.
- Let's replicate this pattern using zero-coupon bonds.



- To get a cash outflow at t=0.5, we short 1 unit of Bond X.
    - The short requires us to pay back the par amount of 100 at t=0.5, thus creating a cash outflow.
- By taking the short position in Bond X, we create a cash inflow today in the amount of the market price of Bond X.
- However, in the forward rate agreement, we have no cash flow today at t=0.
- Thus, we buy z units of Bond Z such that we pay as much for the position in Bond Z as we receive from the short position in Bond X. In doing this, we have a zero cash flow today.
- Since we short $x = 1$ units of Bond X, we receive the market price of Bond X in the amount of P= $99.0099 today.
- Since the price of 1 unit of Bond Z is P= $97.0662, we need to buy more than one unit. Specifically, we buy

$$z = \frac{99.0099}{97.0662} = 1.02 \text{ units}$$

Bond	Units	Cash Flow t=0	Cash Flow t=0.5	Cash Flow t=1.0
X	$x = -1$	+99.0099	-100	0
Z	$z = 1.02$	-99.0099	0	102.0024
Portfolio		0	-100	102.0024

- This means that we can lock in a six-month interest rate of 2.0024 percent to invest 100 starting in six months from now (at t=0.5) for another six months until time t=1.
- The corresponding annualized rate is 4.0049 percent.
- Thus, by no arbitrage, the forward rate $f_{0.5,1.0}$ must be equal to 4.0049 percent.

$$f_{0.5,1.0} \stackrel{!}{=} 0.040049$$

- Suppose that someone agrees to a forward rate $f_{0.5,1.0} = 0.03$.
- Can we lock in a risk-free profit?
- The answer is yes. Since the forward rate of 3% is too low relative to what it should be, we **borrow** at the forward rate and invest in the two bonds (Bond X and Bond z) that we used to replicate the forward rate agreement.
- Let's illustrate this in the next table.

Bond	Units	Cash Flow t=0	Cash Flow t=0.5	Cash Flow t=1.0
Forward	1	0	+100	$-100 \times (1 + \frac{f(0.5,1)}{2}) = -101.5$
X	$x = -1$	+99.0099	-100	0
Z	$z = 1.02$	-99.0099	0	102.0024
Portfolio		0	-100	0.5024

- Thus, if someone agrees a forward rate of $f_{0.5,1.0} = 3\%$, we can earn a riskfree profit of $ 0.5024 for each 100 invested.

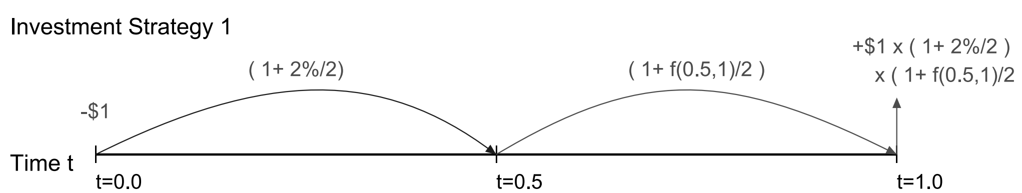
- In the previous example, we have shown how to use zero-coupon bonds to calculate forward rates.
- Alternatively, we can use spot rates. Let's illustrate this next by using the same two bonds as in the previous example.

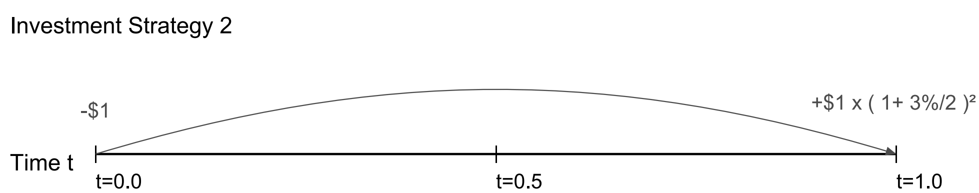
- Suppose we observe the following zero-coupon bonds

Bond	Time-to-maturity T	Spot Rate
X	0.5	2 %
Z	1.0	3 %

- We want to determine the forward rate $f_{0.5,1}$ for investing \$1 starting in six months for another six months.
- The basic idea is that we consider two investment strategies at t=0, where one strategy uses the forward rate and the other uses spot rates only. The maturity of these two investment strategies is equal to the maturity of the forward rates.
    - In this example the forward rate ends at t=1.0, so the two investment strategies will invest 1 from t=0 to t=1.0.

- **Consider two different investment strategies**
    1. Invest \$1 at the six-month spot rate $r_{0.5}$ today and agree to a forward rate agreement at $f_{0.5,1}$ for the second six-month period until t=1.0.
    2. Invest \$1 today at the one-year spot rate $r_{1.0}$ for one year.





- Both strategies involve an initial investment at t = 0 of $ 1.
- The cash flows from both stragies are risk-free and known as of today.
- By no arbitrage, Strategy 1 and Strategy 2 must have the same cash flow at t=1.

$$\left(1+\frac{2\%}{2}\right) \times \left(1+\frac{f_{0.5,1}}{2}\right) \stackrel{!}{=} \left(1+\frac{3\%}{2}\right)^2$$

- Thus, the forward rate $f_{0.5,1}$ is

$$f_{0.5,1} = \frac{\left(1+\frac{3\%}{2}\right)^2}{\left(1+\frac{2\%}{2}\right)} - 1 = 0.04005 = 4.005\%$$

Calculating Forward Rates

- To illustrate how the approach we took in the previous example can be applied to calculate forward rates for other periods, let's consider the following spot rates.
- The goal is to calculate the six-months forward rates starting in six-months from now, starting at 12-months from now, etc.
    - Assume that all spot rates are semi-annually compounded.

- 0.5-year spot rate $r_{0.5}$ [%]: 2.0
- 1.0-year spot rate $r_{1.0}$ [%]: 3.0
- 1.5-year spot rate $r_{1.5}$ [%]: 3.5
- 2.0-year spot rate $r_{2.0}$ [%]: 3.0
- 2.5-year spot rate $r_{2.5}$ [%]: 4.0
- 3.0-year spot rate $r_{3.0}$ [%]: 4.5
- 3.5-year spot rate $r_{3.5}$ [%]: 4.75
- 4.0-year spot rate $r_{4.0}$ [%]: 5.0
- 4.5-year spot rate $r_{4.5}$ [%]: 5.1
- 5.0-year spot rate $r_{5.0}$ [%]: 5.25

Reset

- Calculate the forward rate $f_{1,1.5}$ starting in 1-year from now to invest for another six months.

1. Strategy 1: Invest at the 1.0-year spot rate and invest at the forward rate $f_{1,1.5}$ for another six months starting at t=1.

$$\left(1+\frac{r_{1.0}}{2}\right)^2 \times \left(1+\frac{f_{1,1.5}}{2}\right)$$

2. Strategy 2: Invest at 1.5-year spot rate.

$$\left(1+\frac{r_{1.5}}{2}\right)^3$$

3. Strategy 1 and 2 must have the same cash flow at t=1.5.

$$\left(1+\frac{r_{1.0}}{2}\right)^2 \times \left(1+\frac{f_{1,1.5}}{2}\right) \stackrel{!}{=} \left(1+\frac{r_{1.5}}{2}\right)^3$$

- Plugging in the values for the spot rates.

$$\left(1+\frac{3.0\%}{2}\right)^2 \times \left(1+\frac{f_{1,1.5}}{2}\right) \stackrel{!}{=} \left(1+\frac{3.5\%}{2}\right)^3$$

- Solve for $f_{1.0,1.5}$

$$\left(1+\frac{f_{1,1.5}}{2}\right) = \frac{\left(1+\frac{3.5\%}{2}\right)^3}{\left(1+\frac{3.0\%}{2}\right)^2}$$

$$f_{1,1.5} = 2 \times \left(\frac{\left(1+\frac{3.5\%}{2}\right)^3}{\left(1+\frac{3.0\%}{2}\right)^2} - 1\right)$$

$$f_{1,1.5} = 0.045037 = 4.5037\%$$

- Calculate the forward rate $f_{1.5,2.0}$ starting in 1.5-year from now to invest for another six months.

1. Strategy 1: Invest at the 1.5-year spot rate and invest at the forward rate $f_{1.5,2.0}$ for another six months starting at t=1.5.

$$\left(1+\frac{r_{1.5}}{2}\right)^{2\times 1.5} \times \left(1+\frac{f_{1.5,2.0}}{2}\right)^{2\times 0.5}$$

2. Strategy 2: Invest at 2.0-year spot rate.

$$\left(1+\frac{r_{2.0}}{2}\right)^{2\times 2}$$

3. Strategy 1 and 2 must have the same cash flow at t=2.0.

$$\left(1+\frac{r_{1.5}}{2}\right)^3 \times \left(1+\frac{f_{1.5,2.0}}{2}\right) \stackrel{!}{=} \left(1+\frac{r_{2.0}}{2}\right)^4$$

- Plugging in the values for the spot rates.

$$\left(1+\frac{3.5\%}{2}\right)^3 \times \left(1+\frac{f_{1.5,2.0}}{2}\right) \stackrel{!}{=} \left(1+\frac{3.0\%}{2}\right)^4$$

- Solve for $f_{1.5,2.0}$

$$\left(1+\frac{f_{1.5,2.0}}{2}\right) = \frac{\left(1+\frac{3.0\%}{2}\right)^4}{\left(1+\frac{3.5\%}{2}\right)^3}$$

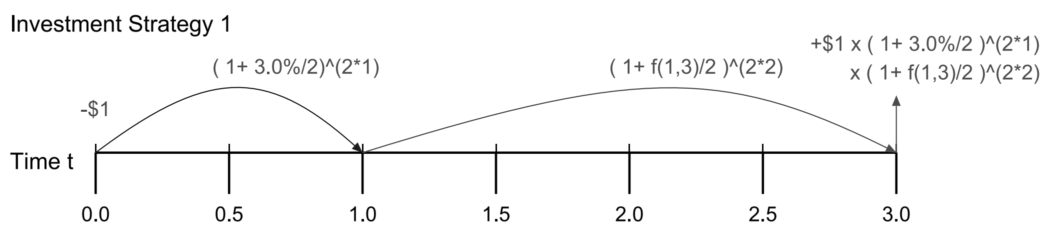
$$f_{1.5,2.0} = 2 \times \left(\frac{\left(1+\frac{3.0\%}{2}\right)^4}{\left(1+\frac{3.5\%}{2}\right)^3} - 1\right)$$

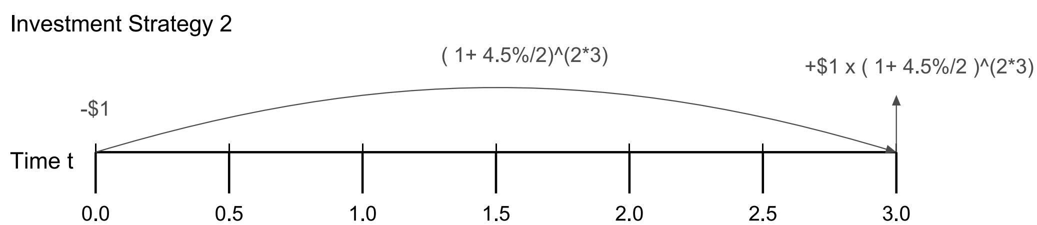
$$f_{1.5,2.0} = 0.015074 = 1.5074\%$$

- Repeating this process to $f_{4.5,5.0}$ gives the following six-month forward rates

Forward Rate	Value	Calculation
$f(0.5, 1.0)$	4.005%	$2 * ((1+0.03/2)^{2*1.0}/(1+0.02/2)^{2*0.5} - 1)$
$f(1.0, 1.5)$	4.5037%	$2 * ((1+0.035/2)^{2*1.5}/(1+0.03/2)^{2*1.0} - 1)$
$f(1.5, 2.0)$	1.5074%	$2 * ((1+0.03/2)^{2*2.0}/(1+0.035/2)^{2*1.5} - 1)$
$f(2.0, 2.5)$	8.0495%	$2 * ((1+0.04/2)^{2*2.5}/(1+0.03/2)^{2*2.0} - 1)$
$f(2.5, 3.0)$	7.0184%	$2 * ((1+0.045/2)^{2*3.0}/(1+0.04/2)^{2*2.5} - 1)$
$f(3.0, 3.5)$	8.0258%	$2 * ((1+0.05/2)^{2*3.5}/(1+0.045/2)^{2*3.0} - 1)$
$f(3.5, 4.0)$	5.0%	$2 * ((1+0.05/2)^{2*4.0}/(1+0.05/2)^{2*3.5} - 1)$
$f(4.0, 4.5)$	5.9018%	$2 * ((1+0.051/2)^{2*4.5}/(1+0.05/2)^{2*4.0} - 1)$
$f(4.5, 5.0)$	4.1022%	$2 * ((1+0.05/2)^{2*5.0}/(1+0.051/2)^{2*4.5} - 1)$

- In the previous example, we have calculated six-month forward rates.
- Let's consider next, how we calculate forward rates starting at some time t in the future for investing for another n months.
- Let's start by calculating the forward rate $f_{1.0,3.0}$ starting in one year from now for investing for another two years
- We take a similar approach as in the previous examples.
- Strategy 1:
    - Invest for 1-year at the one-year spot rate starting today and then invest at the 2-year forward rate starting in year 1.
- Strategy 2:
    - Invest for 3-years at the three-year spot rate starting today.





- Strategy 1:
    - Invest for 1-year at the one-year spot rate starting today and then invest at the 2-year forward rate starting in year 1.

$$\left(1+\frac{r_1}{2}\right)^{2\times1} \times \left(1+\frac{f_{1,3}}{2}\right)^{2\times2}$$

- Strategy 2:
  - Invest for 3-years at the three-year spot rate starting today.

$$\left(1+\frac{r_3}{2}\right)^{2\times 3}$$

- Strategy 1 and 2 must have the same cash flow at t=3.

$$\left(1+\frac{r_1}{2}\right)^{2\times 1}\times\left(1+\frac{f_{1,3}}{2}\right)^{2\times 2}\stackrel{!}{=}\left(1+\frac{r_3}{2}\right)^{2\times 3}$$

- Plugging in the values for the spot rates.

$$\left(1+\frac{3.0\%}{2}\right)^{2}\times\left(1+\frac{f_{1,3}}{2}\right)^{4}\stackrel{!}{=}\left(1+\frac{4.5\%}{2}\right)^{6}$$

$$\left(1+\frac{f_{1,3}}{2}\right)^{4}=\frac{\left(1+\frac{4.5\%}{2}\right)^{6}}{\left(1+\frac{3.0\%}{2}\right)^{2}}$$

$$f_{1,3}=2\times\left(\frac{\left(1+\frac{4.5\%}{2}\right)^{6/4}}{\left(1+\frac{3.0\%}{2}\right)^{2/4}}-1\right)$$

$$f_{1,3}=0.052542=5.2542\%$$

- This approach to calculating forward rates works in general.
- Specifically, when thinking about how to calculate forward rates, think in terms of how we can invest at different horizons.
- In particular, suppose that we are considering f_{T_1,T_2}.
  - Buy a zero-coupon bond with maturity T_2.
  - Buy a zero-coupon bond with maturity T_1 and agree to a forward contract at a rate of f_{T_1,T_2}.
  - Since both strategies are risk-free, have the same investment, can be entered into today, and pay out at the same time, we can equate their final cash flows.

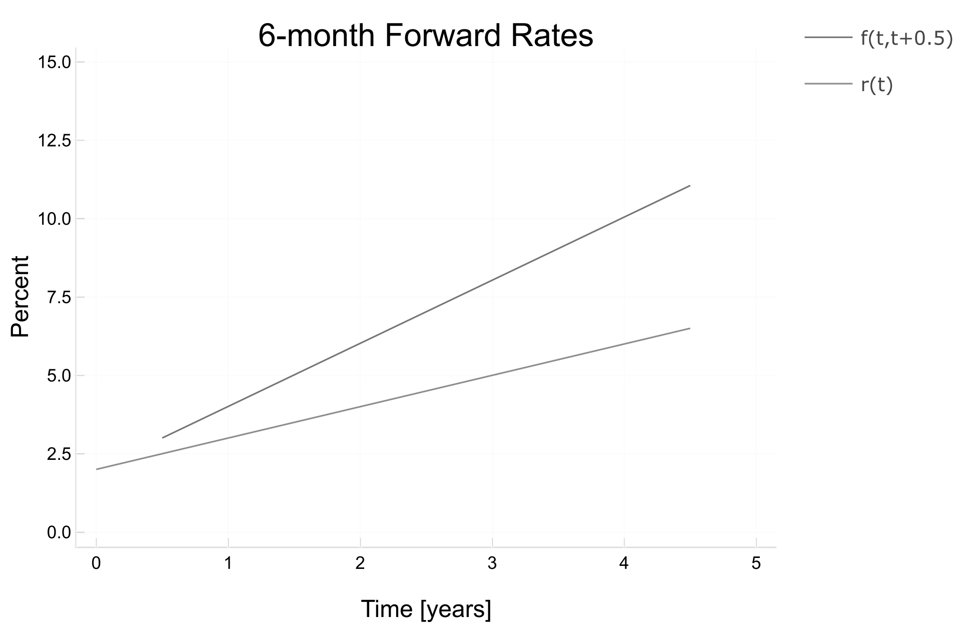
Term Structure and Expectations Hypothesis

- **Upward-Sloping Yield Curve**

- 0.5-year spot rate $r_{0.5}$ [%]: 2.0
- 1.0-year spot rate $r_{1.0}$ [%]: 2.5
- 1.5-year spot rate $r_{1.5}$ [%]: 3.0
- 2.0-year spot rate $r_{2.0}$ [%]: 3.5
- 2.5-year spot rate $r_{2.5}$ [%]: 4.0
- 3.0-year spot rate $r_{3.0}$ [%]: 4.5
- 3.5-year spot rate $r_{3.5}$ [%]: 5.0
- 4.0-year spot rate $r_{4.0}$ [%]: 5.5
- 4.5-year spot rate $r_{4.5}$ [%]: 6.0
- 5.0-year spot rate $r_{5.0}$ [%]: 6.5

Reset

Forward Rate	Value
$f(0.5, 1.0)$	3.0012%
$f(1.0, 1.5)$	4.0037%
$f(1.5, 2.0)$	5.0074%
$f(2.0, 2.5)$	6.0123%
$f(2.5, 3.0)$	7.0184%
$f(3.0, 3.5)$	8.0258%
$f(3.5, 4.0)$	9.0343%
$f(4.0, 4.5)$	10.044%
$f(4.5, 5.0)$	11.055%

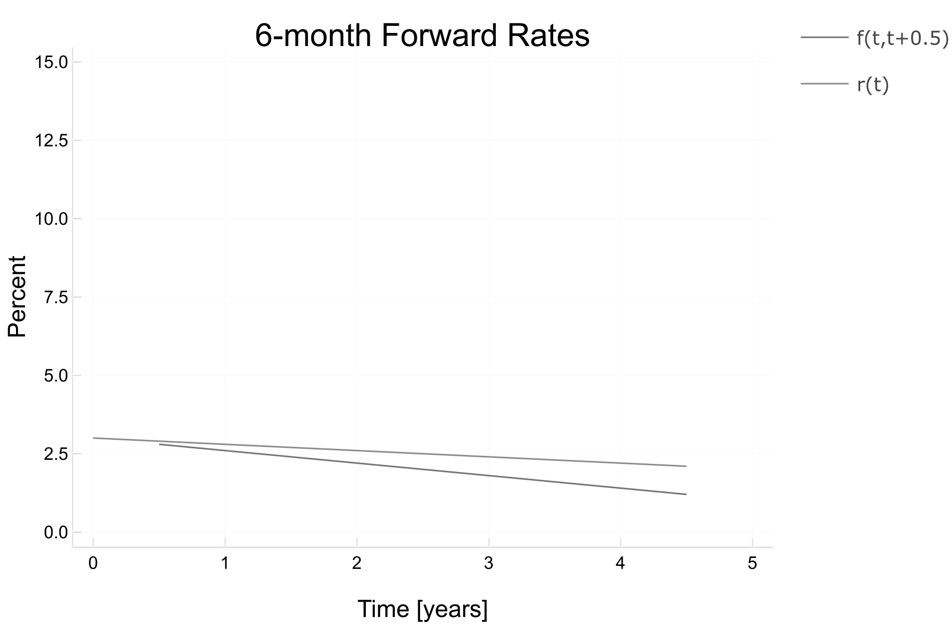


- **Downward-Sloping Yield Curve**

- 0.5-year spot rate $r_{0.5}$ [%]: 3.0
- 1.0-year spot rate $r_{1.0}$ [%]: 2.9
- 1.5-year spot rate $r_{1.5}$ [%]: 2.8
- 2.0-year spot rate $r_{2.0}$ [%]: 2.7
- 2.5-year spot rate $r_{2.5}$ [%]: 2.6
- 3.0-year spot rate $r_{3.0}$ [%]: 2.5
- 3.5-year spot rate $r_{3.5}$ [%]: 2.4
- 4.0-year spot rate $r_{4.0}$ [%]: 2.3
- 4.5-year spot rate $r_{4.5}$ [%]: 2.2
- 5.0-year spot rate $r_{5.0}$ [%]: 2.1

Reset

Forward Rate	Value
$f(0.5, 1.0)$	2.8%
$f(1.0, 1.5)$	2.6001%
$f(1.5, 2.0)$	2.4003%
$f(2.0, 2.5)$	2.2005%
$f(2.5, 3.0)$	2.0007%
$f(3.0, 3.5)$	1.801%
$f(3.5, 4.0)$	1.6014%
$f(4.0, 4.5)$	1.4018%
$f(4.5, 5.0)$	1.2022%

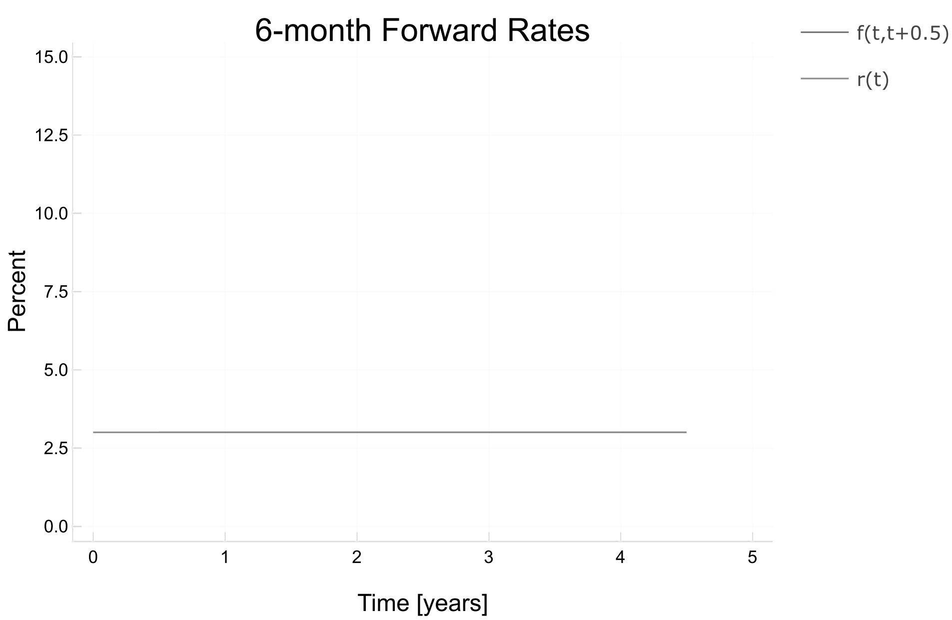


- **Flat Yield Curve**

Reset

- 0.5-year spot rate $r_{0.5}$ [%]: 3.0
- 1.0-year spot rate $r_{1.0}$ [%]: 3.0
- 1.5-year spot rate $r_{1.5}$ [%]: 3.0
- 2.0-year spot rate $r_{2.0}$ [%]: 3.0
- 2.5-year spot rate $r_{2.5}$ [%]: 3.0
- 3.0-year spot rate $r_{3.0}$ [%]: 3.0
- 3.5-year spot rate $r_{3.5}$ [%]: 3.0
- 4.0-year spot rate $r_{4.0}$ [%]: 3.0
- 4.5-year spot rate $r_{4.5}$ [%]: 3.0
- 5.0-year spot rate $r_{5.0}$ [%]: 3.0

Forward Rate	Value
$f(0.5, 1.0)$	3.0%
$f(1.0, 1.5)$	3.0%
$f(1.5, 2.0)$	3.0%
$f(2.0, 2.5)$	3.0%
$f(2.5, 3.0)$	3.0%
$f(3.0, 3.5)$	3.0%
$f(3.5, 4.0)$	3.0%
$f(4.0, 4.5)$	3.0%
$f(4.5, 5.0)$	3.0%



Expectations Hypothesis

- The *Expectations Hypothesis* says that our best estimates of future spot rates come from forward rates.
- For example, the expected six-month spot rate in six-months is equal to today's six-month forward rate

$$E[r(0.5, 1)] = f(0.5, 1)$$

- If $f(0.5, 1) > r(0, 0.5)$, then we expect the six-month interest rates to go up.
- While the theory is intuitive, it is only a partial explanation of the yield curve.

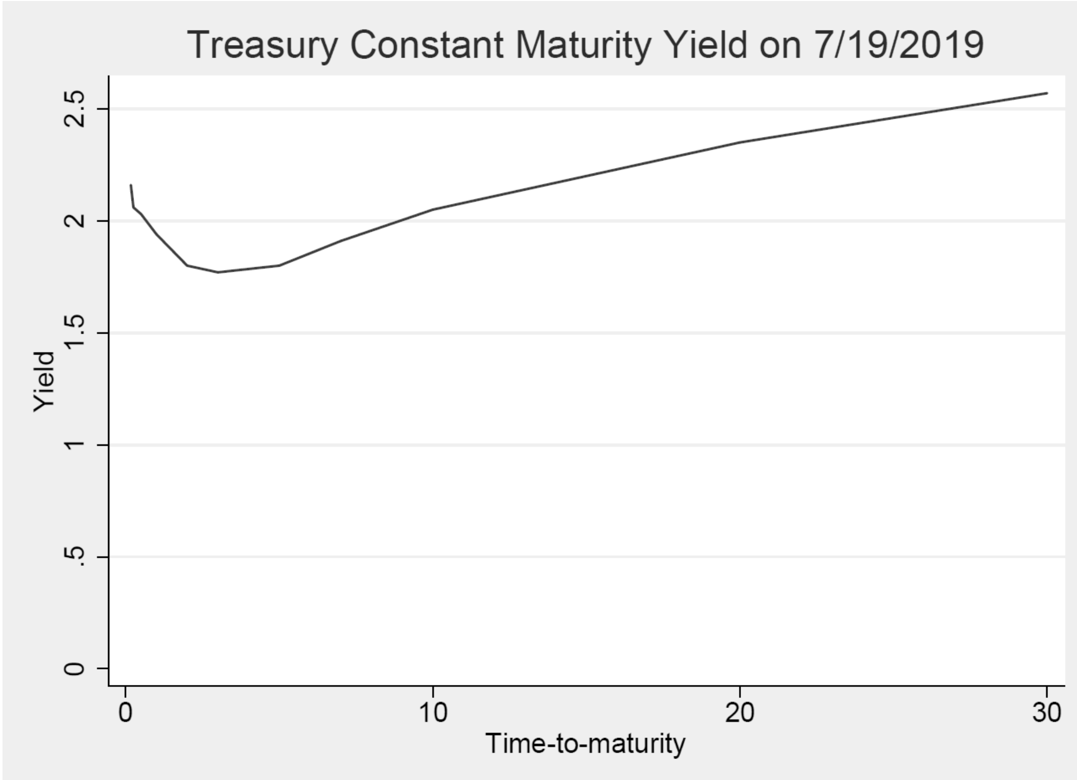
ECONOMY | U.S. ECONOMY

Fed Officials Signal Quarter-Point Rate Cut Likely at July Meeting

Central bank's move viewed as insurance against sharper slowdown

Bank of New York President John Williams's comments on Thursday touched off market speculation that the Fed was preparing for a a half-point rate cut, prompting a rare clarification from the New York Fed. PHOTO: LUCAS JACKSON/REUTERS

By Nick Timiraos
Updated July 19, 2019 6:32 pm ET



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Fed Cuts Rates by a Quarter Point, Ends Portfolio Runoff

Central bankers cite inflation, global developments as reasons for cut

Federal Reserve Chairman Jerome Powell **PHOTO:** MANDEL NGAN/AGENCE FRANCE-PRESSE/GETTY IMAGES

By Nick Timiraos
July 31, 2019 2:03 pm ET

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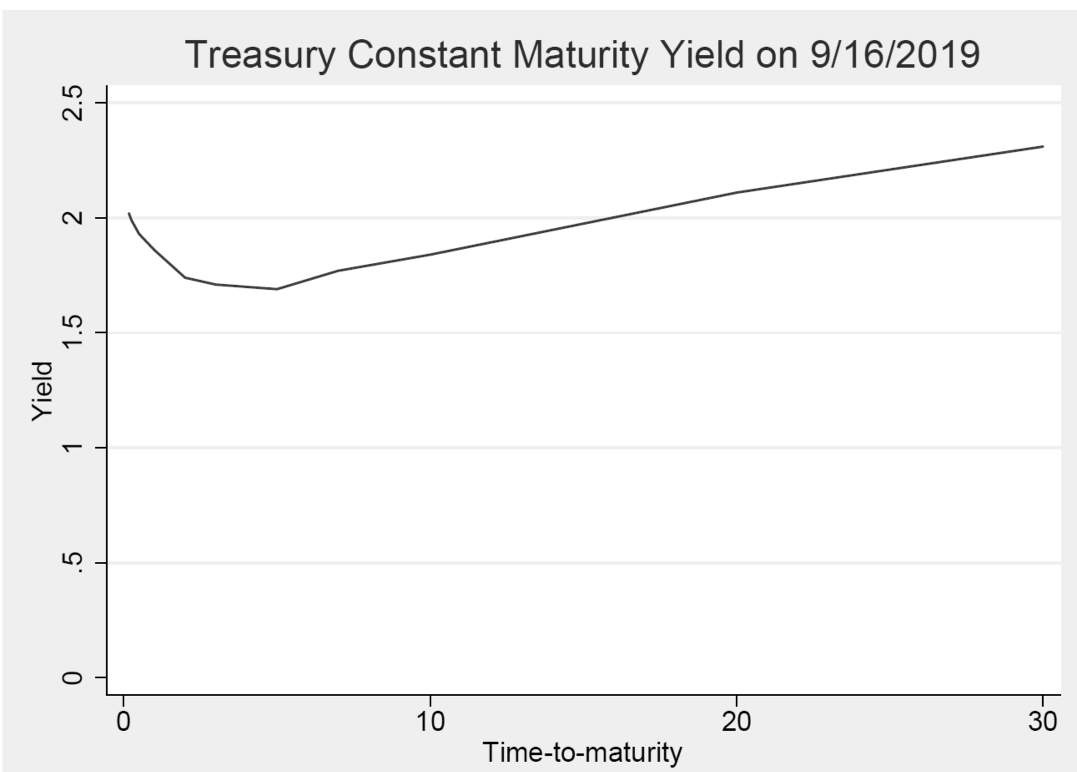
Fed Lines Up Another Quarter-Point Rate Cut

With trade uncertainty escalating, the central bank chief is weighing diverse opinions from inside his own institution

Fed Chairman Jerome Powell will update the public on his outlook in a discussion Friday with the head of the Swiss National Bank. **PHOTO:** SARAH SILBIGER/REUTERS

By Nick Timiraos
Sept. 5, 2019 5:30 am ET

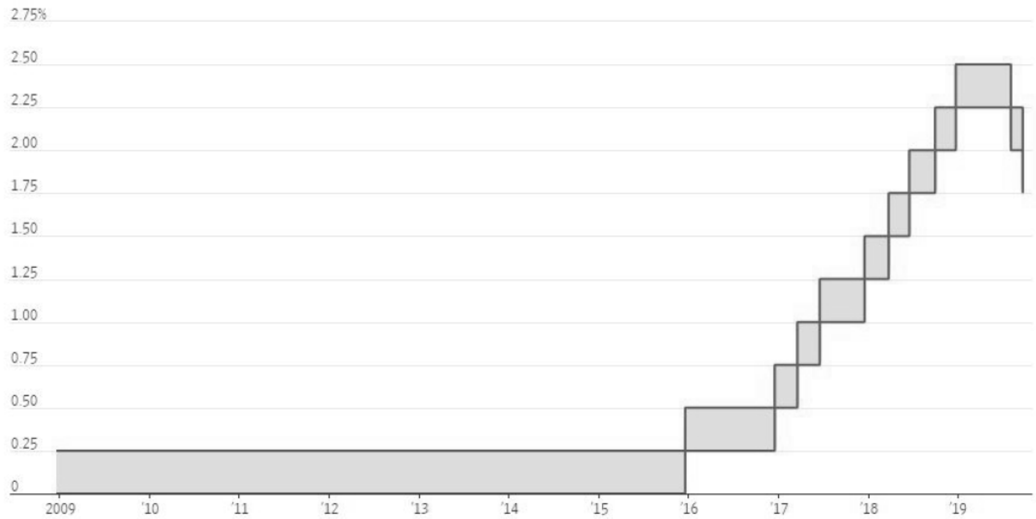


ECONOMY

Fed Cuts Rates by Quarter Point but Faces Growing Split

Central bankers divided over Wednesday's decision and the outlook for further reductions.

Federal Reserve rate target



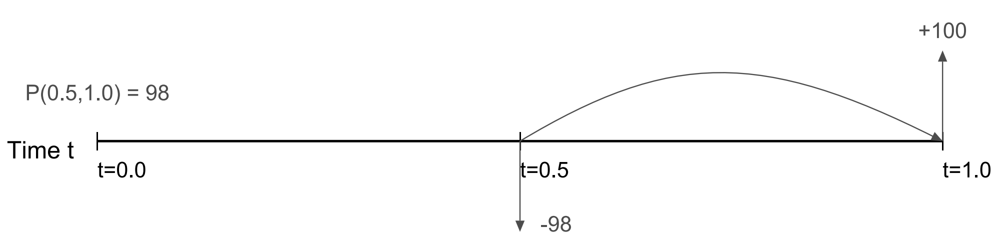
Source: Federal Reserve

By Nick Timiraos

Updated Sept. 18, 2019 6:12 pm ET

Forward Contracts

- So far, we have talked about what are known as *forward rate agreements*.
- In reality, many contracts are **forward contracts**.
- Rather than agreeing on an interest rate, two parties agree on a *future purchase price* that a bond will be purchased at.
- Let $P(T_1, T_2)$ be the price agreed upon today to purchase a bond at time T_1. The bond matures at time T_2.
- Suppose that $P(0.5, 1) = 98$.
- This means that we agree today (time 0) to buy a bond at time $T_1 = 0.5$. The bond matures at time $T_2 = 1$. The purchase price will be \$98 at time $T_1 = 0.5$.



- Suppose we observe the following zero-coupon bonds

Bond	Time-to-maturity T	Price	Spot Rate
X	0.5	99.0099	2.0 %
Z	1.0	97.0662	3.0 %

- Suppose we enter into a position of x units in Bond X and z units in Bond Z.

Bond	Units	$t = 0$	$t = 0.5$	$t = 1.0$
X	x	$-99.0099 \times x$	$100 \times x$	0
Z	z	$-97.0662 \times z$	0	$100 \times z$
Portfolio		0	?	100

- Let's select $z = 1$, i.e. we buy 1 unit of Bond Z.
- To get a zero cash flow at $t = 0$ it must be the case that

$$-99.0099 \times x - 97.0662 \times 1 = 0$$

$$x = -\frac{97.0662}{99.0099} = -0.9804$$

- Thus, we take a short position in 0.9804 units of Bond X.
- This creates a cash flow today of

$$+P_X \times 0.9804 = +97.0662$$

- The resulting cash flows are

- Suppose we enter into a position of x units in Bond X and z units in Bond Z.

Bond	Units	$t = 0$	$t = 0.5$	$t = 1.0$
X	$x = -0.9804$	$+97.0662$	-98.0368	0
Z	$z = 1$	-97.0662	0	100
Portfolio		0	-98.0368	100

- Thus, the price agreed upon *today* to purchase a bond at time $T_1 = 0.5$ that has maturity date at $T_2 = 1$ must be

$$P(0.5, 1) = \$98.0368$$

Pricing Forward Contracts

- Thus far, we have calculated the forward price as of today (at time $t = 0$).
- Recall that at inception ($t = 0$), the value of a forward contract is "fair", i.e. it has a value of zero.
  - Today's cash flow is zero.
- What is the forward price at some time $t > 0$ after we entered into the contract?
- As time passes, the value of a forward may become non-zero.
  - Whether a forward rate agreement has a positive or negative value depends on how interest rates have changed since the contract was entered into.

- Suppose that today ($t = 0$) is December 31, 2021 and the term structure is as shown below.

Maturity T	Maturity Date	Spot Rate
0.5	June 30, 2022	2%
1	Dec 31, 2022	3%
1.5	June 30, 2023	3.5%
2	Dec 31, 2023	3%
2.5	June 30, 2024	4%
3	Dec 31, 2024	4.5%

- Suppose, we enter into a forward rate agreement f(June 2023, Dec 2023) where we agree to invest 100 at the forward rate for six months starting on June 30, 2023.
- Recall that the forward rate is

$$f(\text{June 2023}, \text{Dec 2023}) = 2 \times \left(\frac{\left(1 + \frac{r_{2.0}}{2}\right)^{2\times 2.0}}{\left(1 + \frac{r_{1.5}}{2}\right)^{2\times 1.5}} - 1 \right) = 0.015074 = 1.5074\%$$

- As a concept check, let's verify that the initival value today of the forward rate agreement is indeed zero.

Date	Dec 2021	June 2022	Dec 2022	June 2023	Dec 2023
	$t = 0$	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2$
Forward	0	0	0	-100	$100 \times (1 + f_{1.5,2}/2) = 100.7537$

- The present value of the forward rate agreement is

$$\frac{-100}{(1+\frac{3.5\%}{2})^{2\times 1.5}}+\frac{100.7537}{(1+\frac{3.0\%}{2})^{2\times 2.0}}=0$$

- Thus, the initial value today of the forward rate agreement is indeed zero.

- Suppose now that six months have passed, so that it is now June 30, 2022.
- Suppose that the term structure is now

Maturity T	Maturity Date	Spot Rate
0.5	Dec 31, 2022	2.0%
1.0	June 30, 2023	2.5%
1.5	Dec 31, 2023	3.0%
2.0	June 30, 2024	4.0%
2.5	Dec 31, 2024	5.0%

- What is the forward rate for investing for six months starting in June 2023 now? Let's calculate $f(\text{June 2023}, \text{Dec 2023})$.

$$f(\text{June 2023}, \text{Dec 2023}) = 2\times\left(\frac{\left(1+\frac{r_{1.5}}{2}\right)^{2\times 1.5}}{\left(1+\frac{r_{1.0}}{2}\right)^{2\times 1.0}}-1\right)=0.040037=4.0037\%$$

- Now, in June 2022, the fair forward rate for investing for six months starting in June 2023 is 4.0037%.
- The fair forward rate that we agreed to when we entered the contract in December 2021 was 1.5074%.
- Since the forward rates are different, what is the contract now worth that we entered into in December 2021?
- To answer this question, we calculate the present value of the cash flows from the forward rate agreement that we entered into in December 2021.

Date	June 2022	Dec 2022	June 2023	Dec 2023
	$t=0.0$	$t=1.5$	$t=2.0$	$t=2.5$
Forward	0	0	-100	100.7537

$$\text{Value of Forward} = \frac{-100}{\left(1+\frac{r_{1.0}}{2}\right)^{\times 1.0}}+\frac{100.7537}{\left(1+\frac{r_{1.5}}{2}\right)^{2\times 1.5}}=-1.1936$$

Eurodollar Futures

- In practice, most trades are made in the Eurodollar Futures market rather than in forwards.
- Conceptually, there are many similarities between forwards and futures, but there are important differences.

Future	Forwards
- Traded on exchanges with a clearinghouse	- Traded over-the-counter
- Standardized agreements	- Agreements customized between buyer and seller
- Mark-to-market, collateral	- Intended to go to final settlement

- Eurodollar Futures were introduced by the CME Group in December 1981:

> "… they have often been characterized as the 'Swiss Army knife' of the futures industry…"
> *Labuszewski and Co.*

> "Eurodollar contracts do not appear to have much of a future… Five months after their noisy launch on the Chicago Mercantile Exchange, Eurodollar contracts still haven't caught on."
> *Institutional Investor, 1982*

- Eurodollar Futures Contract
    - Notional size: in units of 1mm
    - March, June, September, and December contracts out 10 years, plus next four months from today
    - Last trading day: Second London bank business day prior to third Wednesday of contract month
    - Final settlement: Cash settlement relative to 3-month LIBOR rather than physical delivery.

Prices as of May 28, 2010 (90-day LIBOR deposits)

Ticker	Expiration	Price	Rate (%)
EDM0	6/14/10	99.400	0.600
EDU0	9/13/10	99.155	0.845
EDZ0	12/13/10	99.005	0.995
EDH1	3/14/11	98.875	1.125
EDM1	6/13/11	98.705	1.295
EDU1	9/19/11	98.495	1.505
EDZ1	12/19/11	98.245	1.755
EDH2	3/19/12	98.010	1.990

Source: Tuckman and Serrat

For current futures prices, see CME Group.

- Let's consider the futures contract expiring on 3/19/12.
    - Basically, locking in an interest rate of 1.99% (on an annual basis) for a 90-day investment starting on March 19, 2012.
    - $1,000,000 invested on March 19 , 2012
    - $1,000,000 × (1 + 0.0199 × 90/360) = 1,004,975 on June 17, 2012.
    - Absent the institutional differences, on May 28, 2010, we basically have

$$f(\text{March 19, 2012, June 17, 2012}) = 1.99\%$$

Wrap-Up

Our goals for today

- [x] Understand and calculate forward rates.
- [x] Understand how forward rates and spot rates are connected.
- [x] Know the Expectation Hypothesis and use it to interpret expectations about Fed Monetary Policy.
- [x] Understand the relation between forward rates and forward contracts.
- [x] Value a forward rate agreement.

Reading

Fabozzi, Fabozzi, 2021, Bond Markets, Analysis, and Strategies, 10th Edition
Chapter 29