present

FINC 462/662 - Fixed Income Securities

FINC-462/662: Fixed Income Securities

Bond Pricing Fundamentals

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Overview

Goals for today

- [ ] Calculate the present values of future cash flows, including bonds, annuities, perpetuities, and other arbitrary cash flows..
- [ ] Price securities using the observed prices of other securities and the Law of One Price.
- [ ] Construct an arbitrage trade if the Law of One Price is violated.
- [ ] Calculate the price of a coupon bond.

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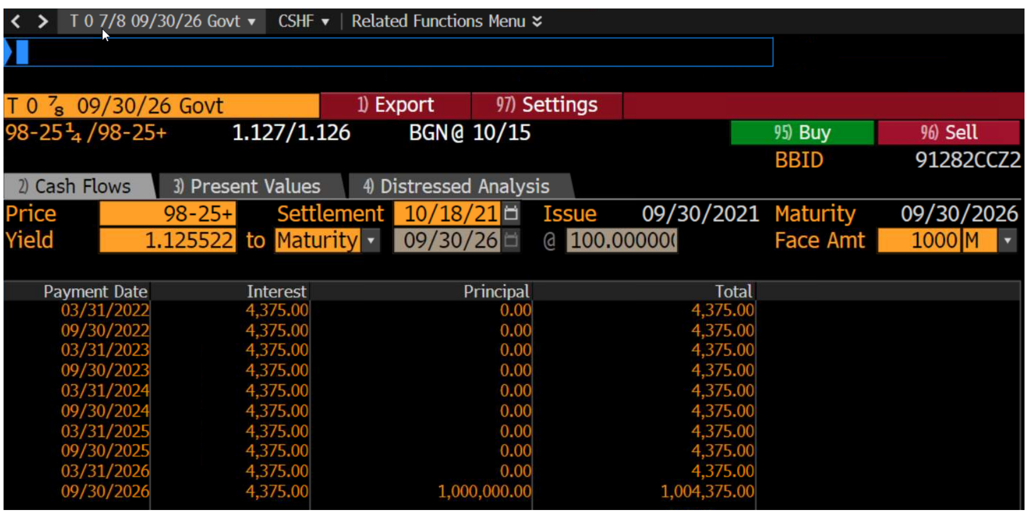
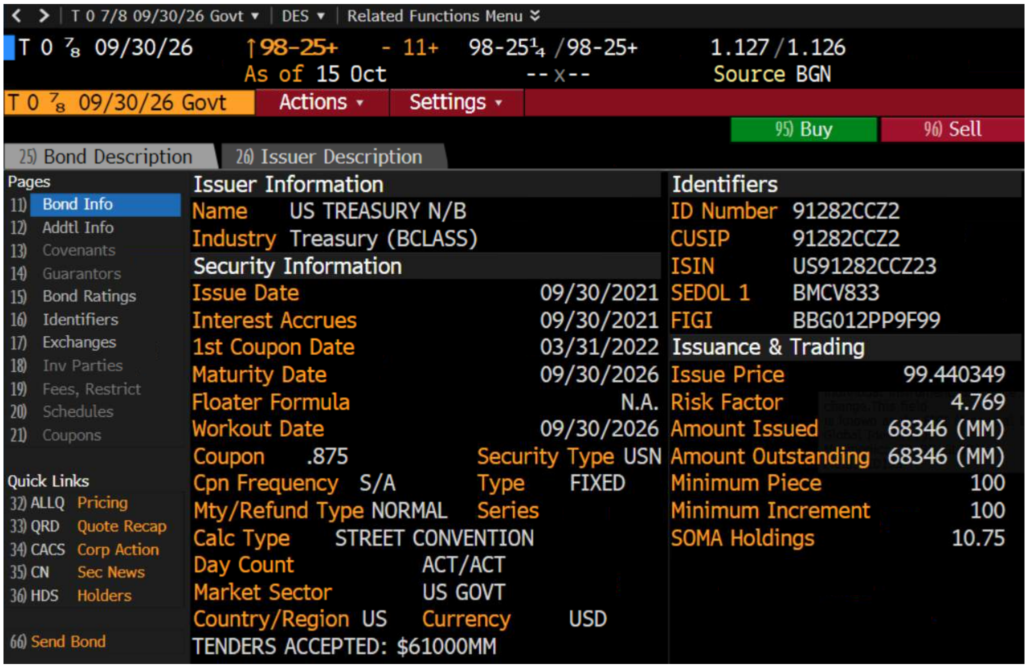
Coupon Bond Cash Flows



Payment Date	Interest	Principal	Total
03/31/2022	4,375.00	0.00	4,375.00
09/30/2022	4,375.00	0.00	4,375.00
03/31/2023	4,375.00	0.00	4,375.00
09/30/2023	4,375.00	0.00	4,375.00
03/31/2024	4,375.00	0.00	4,375.00
09/30/2024	4,375.00	0.00	4,375.00
03/31/2025	4,375.00	0.00	4,375.00
09/30/2025	4,375.00	0.00	4,375.00
03/31/2026	4,375.00	0.00	4,375.00
09/30/2026	4,375.00	1,000,000.00	1,004,375.00



Example

Set Coupon Rate

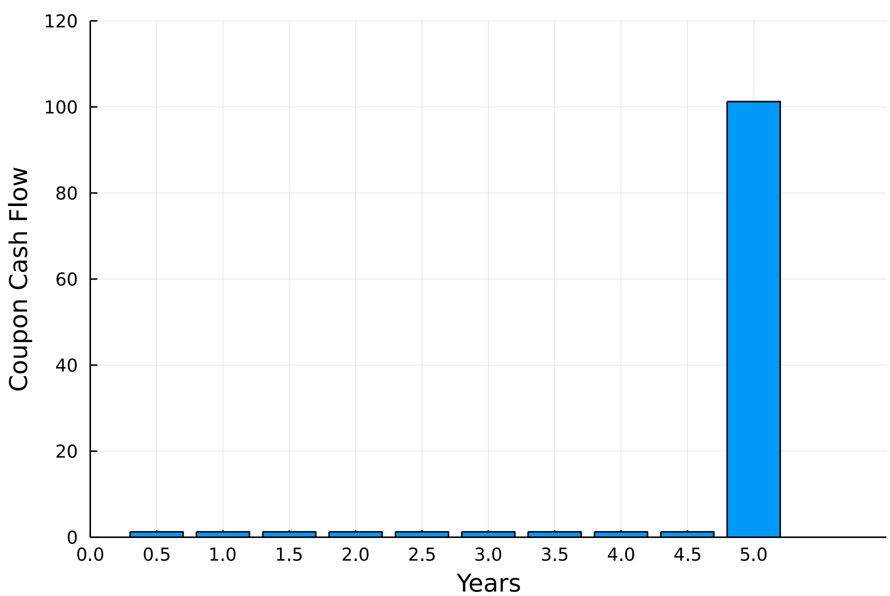
2.5

Coupon Rate: 2.5%

Set Time to Maturity

5.0

Time to Maturity: 5.0 years



Bond Pricing Building Blocks

- Time Value of Money
- Present Value
- Future Value
- Perpetuity
- Annuity
- Law of One Price
- Short-Selling
- Pricing Treasury Bonds
- Continuous Compounding

Time Value of Money and Interest Rates

- Suppose you won the lottery and you can choose to receive your prize of $1000 today or one year from today.
- Clearly, you prefer to get the $1,000 today instead of waiting for another year.
- However, suppose you were offered $1,100 one year from today for waiting another year.
- Let's say this sounds like a fair deal to you, i.e. you are indifferent between having $1,000 today or $1,100 one year from today.
- How is your choice related to interest rates?
- Your choice reveals that each dollar today is worth 10% more one year from today.

$$\$1,000 \times (1 + 10\%) \stackrel{!}{=} \$1,100$$

$$\$1 \times (1 + 10\%) \stackrel{!}{=} \$1.10$$

- In other words, you require to earn interest at an annual rate of r=10%

$$\$1 \times (1 + r) \stackrel{!}{=} \$1.10$$

$$r = \frac{\$1.10}{\$1.00} - 1 = 0.010 = 10\%$$

- The interest rate r in the example reflects you individual choice.
- When we observe an interest rate r in financial markets, we can think of this interest rate as an aggregate of all the individual choice investors make.
- How can we use the interest rate r that we observe in financial markets to tell us how "the market" decides in the lottery example.
- Suppose, we observe r=5%.
- This tells us that a value today of 1,000 is worth

$$\$1,000 \times (1 + r) = \$1,000 \times (1 + 5\%) = \$1,000 \times (1 + 0.05) = \$1,050$$

- Let's call the $1,000 today **Present Value (PV)** and the $1,050 to be received in one year the **Future Value (FV)**.
- Thus, in the example

$$\mathrm{PV} \times (1 + r) = \mathrm{FV}$$

- Putting the PV on the left-hand side, we have the fundamental present-value relationship.

$$\mathrm{PV} = \frac{\mathrm{FV}}{(1 + r)}$$

- We just looked at a one year period.
- However, it is simple to to write down the same relation when the future cash flow occurs two years from today. Then,

$$\mathrm{PV} = \frac{\mathrm{FV}_2}{(1 + r)^2}$$

- In general, for t years

$$\mathrm{PV} = \frac{\mathrm{FV}_t}{(1 + r)^t}$$

- where FV_t means the future value (FV) in t years.

Present Value

Important

Annual Compounding

The present value of a cash flow FV_t to be received in t years given the interest rate r (also called discount rate) is

$$\mathrm{PV} = \frac{\mathrm{FV}_t}{(1 + r)^t}$$

Future Value

Important

Annual Compounding

The future value FV_t in t years of a cash flow with present value (PV) given the interest rate r is

$$\mathrm{FV}_t = \mathrm{PV} \times (1+r)^t$$

Present Value Example

- Instead of

- Future Value (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Time t [years]: 2

Reset

$$\mathrm{PV} = \frac{\mathrm{FV}_t}{(1+r)^t} = \frac{\$100.0}{(1+0.02)^2} = \$96.116878$$

Future Value Example

- Present Value (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Time t [years]: 2

Reset

$$\mathrm{FV}_t = \mathrm{PV} \times (1+r)^t = \$100.0 \times (1+0.02)^2 = \$104.04$$

Present value of multiple cash flows

- If there are multiple cash flows in the future in t=1, 2, 3, ... T years from today, then we calculate the present value of these cash flows as follows.
  1. calculate the individual present values of each future cash flow: PV_t for $t = 1, \ldots, T$
  2. sum up the individual present values: $PV_1 + PV_2 + \ldots + PV_T$

Example

- Future Values (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Time t [years]: 5

Reset

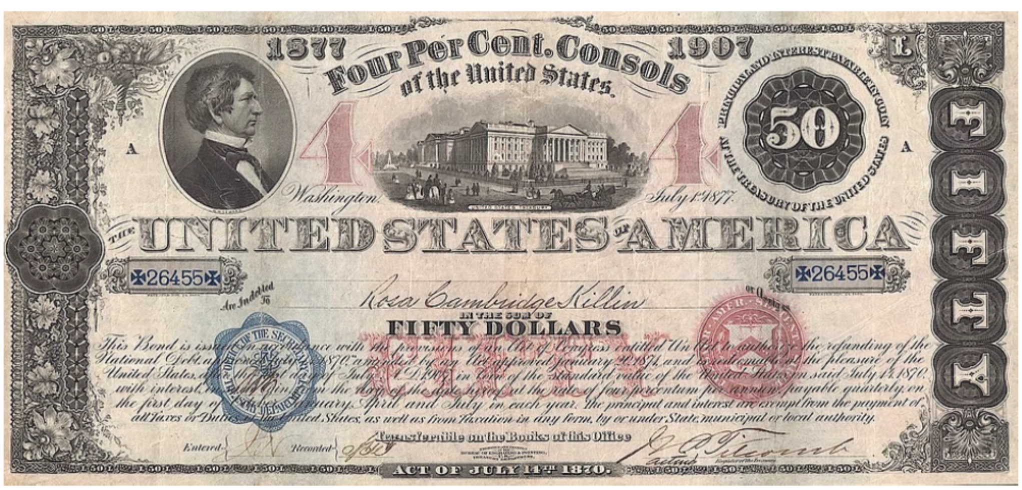
	Time	FutureValue	PresentValue	Calculation
1	1	100.0	98.0392	"100.0 * 1/(1+2.0%)^1=98.0392"
2	2	100.0	96.1169	"100.0 * 1/(1+2.0%)^2=96.1169"
3	3	100.0	94.2322	"100.0 * 1/(1+2.0%)^3=94.2322"
4	4	100.0	92.3845	"100.0 * 1/(1+2.0%)^4=92.3845"
5	5	100.0	90.5731	"100.0 * 1/(1+2.0%)^5=90.5731"

Present Value = 98.0392 + 96.1169 + 94.2322 + 92.3845 + 90.5731 = 471.345951

Perpetuities

- In the previous example, we calculated the present value of multiple future cash flows that were all equal to $100.0 by calculating the present value of each individual future cash flow.
- Suppose now that we are paid $100.0 each year forever.
- Calculating all individual cash flows is not feasible, of course.

Types of perpetuities exist in reality



Example

- Future Values (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Time t [years]: 5

Reset

	Time	FutureValue	PresentValue
1	1	100.0	98.0392
2	2	100.0	96.1169
3	3	100.0	94.2322
4	4	100.0	92.3845
5	5	100.0	90.5731

Present Value = $ 471.346

- Compare the present value to

$$\frac{FV_5}{r} = \frac{100.0}{0.02} = 5000.0$$

Present Value of Perpetuity

Important

The present value today (time $t = 0$) of a perpetuity paying a dollar cash flow of C forever is

$$\text{PV} = \frac{\text{C}}{r}$$

Time t	**0**	**1**	**2**	**3**	**...**
Cash Flow	0	C	C	C	C

Time t	**0**	**1**	**2**	**3**	**...**
Cash Flow	0	C	C	C	C

Growing Perpetuity

Example

- In the case of a perpetuity the cash flows are always the same
- In a "growing perpetuity" the cash flow grow at a constant percentage rate g **after** the first cash flow.

- Future Values (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Growth rate g [% p.a.]: 1.0
- Time t [years]: 5

Reset

	Time	FutureValue	PresentValue
1	1	100.0	98.0392
2	2	101.0	97.078
3	3	102.01	96.1263
4	4	103.03	95.1839
5	5	104.06	94.2507

Present Value = $ 480.6782

- Compare the present value to

$$\frac{FV_5}{r-g} = \frac{100.0}{0.02-0.01} = 10000.0$$

Present Value of Growing Perpetuity

Important

The present value today (time $t=0$) of a perpetuity paying a dollar cash flow of C forever that grows at a constant percentage rate g each period **after** the first cash flow is

$$\text{PV} = \frac{\text{FV}}{r-g}$$

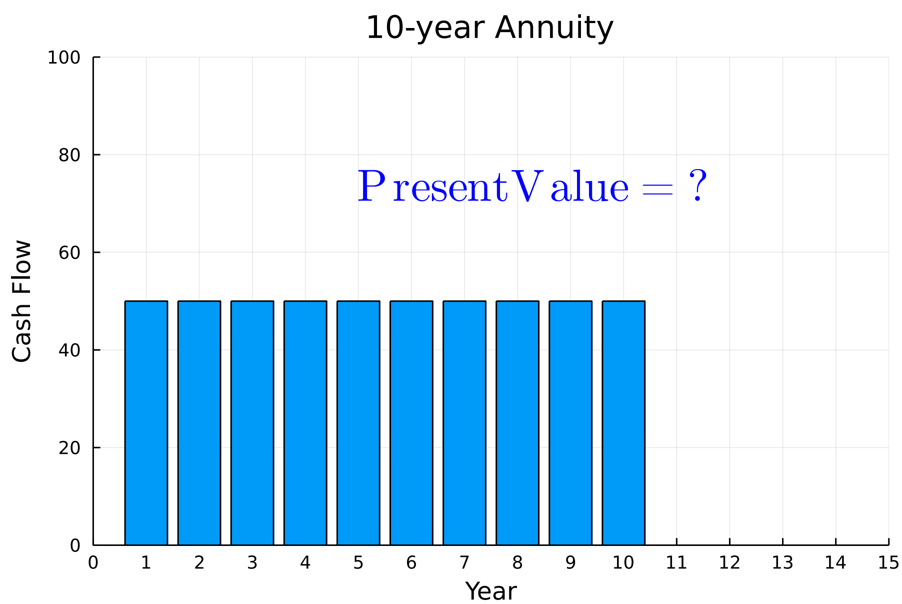
Time t	**0**	**1**	**2**	**3**	**4**	**...**
`Cash Flow`	`0`	C	$C \times (1+g)$	$C \times (1+g)^2$	$C \times (1+g)^3$	$\dots$

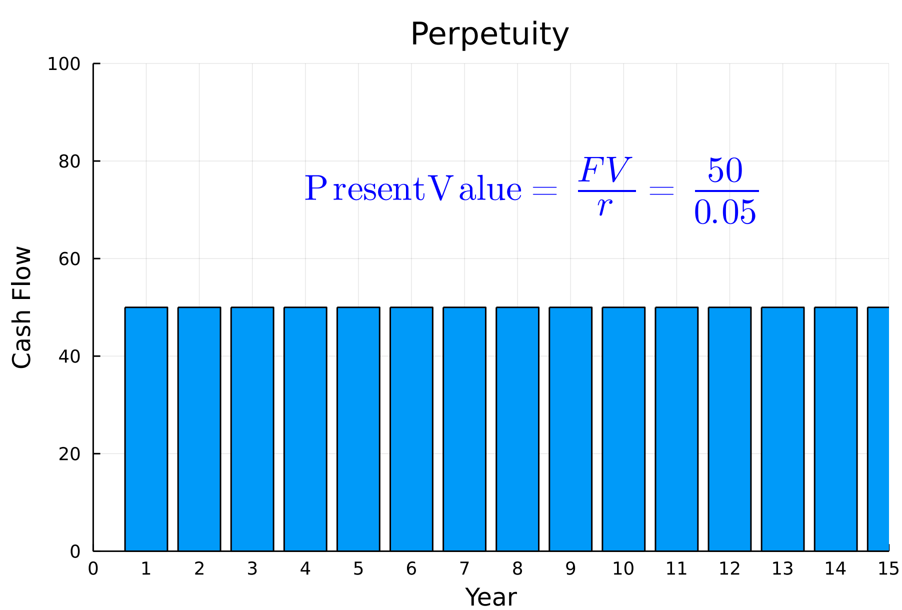
- *Note: We only consider cases where g is less than r*

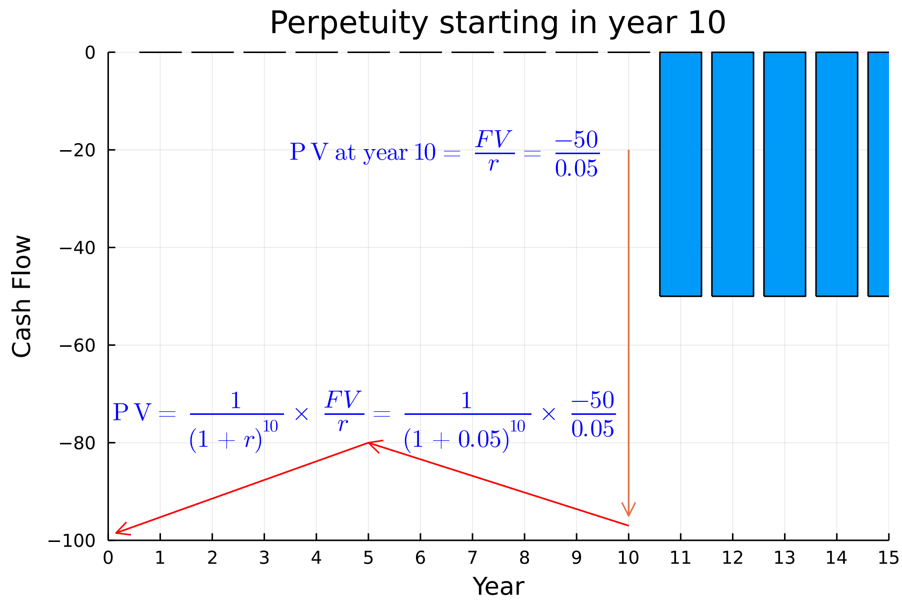
Annuity

- An annuity pays a constant cash flow of FV at the end of each period for a specific number of periods.
- It is similar to a perpetuity, except that the cash flows stop after a certain number of periods.
- Assume that the interest rate is $r = 5\%$ and we want to calculate the present value of a 30-year annuity with annual cash flows of $1.
    - A thiry-year annuity paying $1, has the first cash flow at the end of the first year $t = 1$, the next at the end of the second year $t = 2$, ..., and on final cash flow at the end of year 30 (t =30).
- An annuity is the differentce between two perpetuities. Why?

Example







- Thus, the value of the 10-year annuity is the difference between the present values of the perpetuity starting today and the perpetuity starting in year 10.

$$\text{PV Annuity} = \text{PV of Perpetuity starting today} - \text{Perpetuity starting in year 10}$$

$$\left(\frac{50}{r}\right) - \left(\frac{50}{(1+r)^{10}} \times \frac{1}{r}\right)$$

$$\rightarrow \text{PV} = \left(\frac{50}{r}\right)\left(1 - \frac{1}{(1+r)^{10}}\right)$$

Present Value of Annuity

Important

The present value today (time $t = 0$) of an annuity paying a dollar cash flow of C for T years is

$$\text{PV} = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^{T}}\right)$$

Time t	0	1	2	3	4	...	T	T+1	...
Cash Flow	0	C	C	C	C	...	C	0	0

Compounding Frequencies

- Consider again the Future Value formula and suppose $t = 1$ year and assume that we compute the future value of $100 after one year. In this example, we receive interest on the $100 once after 1 year.

$$\mathrm{FV}_1 = \$100 \times (1 + r)^1$$

- Suppose now that we earn interest once after six months and again after another six months have passed.
- First, the future value after six months is

$$\mathrm{FV}_{0.5} = \$100 \times \left(1 + \frac{r}{2}\right)$$

- Next, the future value after another six months have passed is

$$\mathrm{FV}_1 = \mathrm{FV}_{0.5} \times \left(1 + \frac{r}{2}\right) = \$100 \times \left(1 + \frac{r}{2}\right)^2$$

- When interest is computed twice per year, this is referred to semi-annual compounding.

- Next, compute the future value of $100 after 2 years with semi-annual compounding.

$$\mathrm{FV}_2 = \$100 \times \left(1 + \frac{r}{2}\right) \times \left(1 + \frac{r}{2}\right) \times \left(1 + \frac{r}{2}\right) \times \left(1 + \frac{r}{2}\right)$$

- In general after T years and with semi-annual compounding, the future value of a dollar investment PV is

$$FV_T = PV \times \left(1 + \frac{r}{2}\right)^{2 \times T}$$

- Consider now the **present** value of $100 to be received in two years from now with semi-annual compounding
- Since we now know the Future value after $T = 2$ years (FV_2), we rearrange the previous equation and solve for PV

$$PV = \frac{FV_2}{\left(1 + \frac{r}{2}\right)^{2 \times T}}$$

- What if interest is compounded quarterly? Monthly? Daily?
- We can apply the same reasoning.

Present and Future Values with different compounding frequencies

Important

- Let r be the **annual** interest rate and let T be the number of years.
- Let PV be the the value today and FV_T be the future value after T years.
- Let m be the compounding frequency
  - m=1 : Annual compounding
  - m=2 : Semi-Annual compounding
  - m=4 : Quarterly compounding
  - m=12: Monthly compounding

Future Value

$$FV_T = PV \times \left(1 + \frac{r}{m}\right)^{m \times T}$$

Present Value

$$PV = FV_T \times \frac{1}{\left(1 + \frac{r}{m}\right)^{m \times T}}$$

Future Value Example

- Present Value (PV): 100.0
- Interest rate r [% p.a.]: 2.0
- Compounding frequency m: 2
- Time T [years]: 5

Reset

$$\mathrm{FV}_5 = \mathrm{PV} \times \left(1 + \frac{r}{m}\right)^{m \times T} = \$100.0 \times \left(1 + \frac{2.0\%}{2}\right)^{2 \times 5} = \$110.462213$$

Present Value Example

- Future Value (FV): 100.0
- Interest rate r [% p.a.]: 2.0
- Compounding frequency m: 2
- Time T [years]: 5

Reset

$$\mathrm{PV} = \frac{\mathrm{FV}_5}{\left(1 + \frac{r}{m}\right)^{m \times T}} = \frac{\$100.0}{\left(1 + \frac{2.0\%}{2}\right)^{2 \times 5}} = \$90.528695$$

Annuity formula with difference compounding frequencies

- The annuity formula with different compounding frequencies becomes

Present Value of Annuity

The present value today (time $t = 0$) of an annuity paying a dollar cash flow of C for T years when interest is compounded m times per year is

$$\text{PV} = \left(\frac{C}{r/m}\right)\left(1 - \frac{1}{(1+\frac{r}{m})^{m\times T}}\right)$$

Example

- Cash Flow (C): 50.0
- Interest rate r [% p.a.]: 2.0
- Compounding frequency m: 2
- Time T [years]: 5

Reset

$$\text{PV} = \left(\frac{C}{r/m}\right)\left(1 - \frac{1}{(1+\frac{r}{m})^{m\times T}}\right) = \left(\frac{\$50.0}{0.02/2}\right)\left(1 - \frac{1}{(1+\frac{0.02}{2})^{2\times 5}}\right) = 473.565227$$

Continuous Compounding

Future Value and Present Value with continuous compounding

- With continuous compounding, interest is compounded every instant.
- Mathematically, with continuous compounding the number of times that interest is compounded goes to infinity.
- Many of the models in Finance such as the Black-Scholes model use continuous compounding. This is done for tractability of the models.

Present and Future Values with continuous compounding

Important

- Let r be the **annual** interest rate (continuously compounded) and let T be the number of years.
- Let PV be the the value today and FV_T be the future value after T years.

Future Value

$$FV_T = PV \times \exp(r \times T)$$

Present Value

$$PV = FV_T \times \exp(-r \times T)$$

Example

- Future Value (FV): 50.0
- Interest rate r [% p.a.]: 2.0
- Time T [years]: 5.0

Reset

$$\text{PV} = FV_T \times \exp(-r \times T) = \$50.0 \times \exp(-0.02 \times 5.0) = \$45.241871$$

Annuity formula with continuous compounding

Present Value of Annuity with continuous compounding

The present value today (time $t = 0$) of an annuity paying a (continuous) dollar cash flow of C for T years when interest is continuously compounded is

$$\text{PV} = \frac{C}{\exp(r) - 1} \times (1 - \exp(-r \times T))$$

Example

- Cash Flow (C): 50.0
- Interest rate r [% p.a.]: 2.0
- Time T [years]: 5.0

Reset

$$\text{PV} = \frac{C}{\exp(-rT)} \times (1 - \exp(-rT)) = \frac{50.0}{\exp(-0.02 \times 5.0)} \times (1 - \exp(-0.02 \times 5.0))) = \$235$$

Converting Compounding Frequencies

- Suppose we are given an interest rate r that is compounded m times per year.
- We want to know what the equivalent interest rate is when interest is compounded n times per year.
- To do this, we first find what an investment of $1 is worth after one year given that the interest rate is r and interest is compounded m times per year.
- Then, to find the equivalent rate when interest rate is compounded n times per year, we set the amount from the previous step equal to the amount we would have when interest is compounded n times per year.

Example

- Suppose, the semi-annually compounded interest rate is 4%.
- We want to find the equivalent continuously-compounded interest rate.
- Step 1:
  - A one dollar investment after one year has grown to: $FV_1 = \$1 \times (1+\frac{r}{2})^{2\times 1} = \$1 \times (1+\frac{4\%}{2})^2 = 1.0816$
- Step 2:
  - After one year, a one dollar investment with continuous-compounding at the interest rate r_c has grown to: $FV_1 = \$1 \times \exp(r_c \times 1) = \exp(r_c)$
- Step 3:
  - Setting both equal, we can find r_c:

$$\exp(r_c) = 1.0816 \rightarrow r_c = \ln(1.0816) \rightarrow r_c = 7.8441\%$$

Wrap-Up

Our goals for today

- [x] Calculate the present values of future cash flows, including bonds, annuities, perpetuities, and other arbitrary cash flows..
- [x] Price securities using the observed prices of other securities and the Law of One Price.
- [x] Construct an arbitrage trade if the Law of One Price is violated.
- [x] Calculate the price of a coupon bond.

Reading

Fabozzi, Fabozzi, 2021, Bond Markets, Analysis, and Strategies, 10th Edition
Chapter 2