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FINC 462/662 -- Fixed Income Securities

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Measures of Bond Price Volatility

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Overview

Our goals for today

- [ ] Understand how bond prices change with yields.
- [ ] Calculate a Macaulay Duration and understand what it is, conceptually.
- [ ] Understand what Modified Duration is and how to calculate it.
- [ ] Understand the link between Macaulay and Modified Duration.
- [ ] Use Modified Duration to approximate bond price changes.

Price-Yield Relation

- Recall that we calculate the price P of a semi-annual coupon bond with coupon rate c, semi-annual coupon cash flows C, face value $F = 100$, and time-to-maturity T (in years) by discounting all cash flows using the bond's yield to maturity y.

$$P = \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 0.5}} + \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 1.0}} + \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 1.5}} + \ldots + \frac{100+C}{\left(1+\frac{y}{2}\right)^{2\times T}}$$

- Using the annuity formula, we can calculate the bond price P as follows.

$$P = \frac{C}{y/2} \times \left(1 - \frac{1}{(1+\frac{y}{2})^{2\times T}}\right) + \frac{100}{(1+\frac{y}{2})^{2\times T}}$$

Example

- Let's consider a bond with $T = 3$ years to maturity, coupon rate $c = 2.0\%$ (paid semi-annually), and yield-to-maturity $y = 4.0$ %.
- The bond price is

$$P = \frac{1.0}{4.0\%/2} \times \left(1 - \frac{1}{(1+\frac{4.0\%}{2})^{2\times 3}}\right) + \frac{100}{(1+\frac{4.0\%}{2})^{2\times 3}} = \$94.3986$$

- Last time, we looked at the bond **Price-Yield** relation.
- Recall that we arrived at the price-yield relation by selecting values for y and calculating the bond price $P(y)$ while keeping the coupon rate c and time-to-maturity T the same.
    - The notation $P(y)$ means the bond price if the yield-to-maturity is y (keeping coupon rate c and time-to-maturity T fixed).
- Then, we plotted the different values of y on the horizontal axis and the corresponding bond prices $P(y)$ on the vertical axis.
- In short, we plotted pairs

$$(y, P(y))$$

- where y is the yield and $P(y)$ is the bond price when we use the yield y to discount the bond's cash flows.
- Let's illustrate this again below.

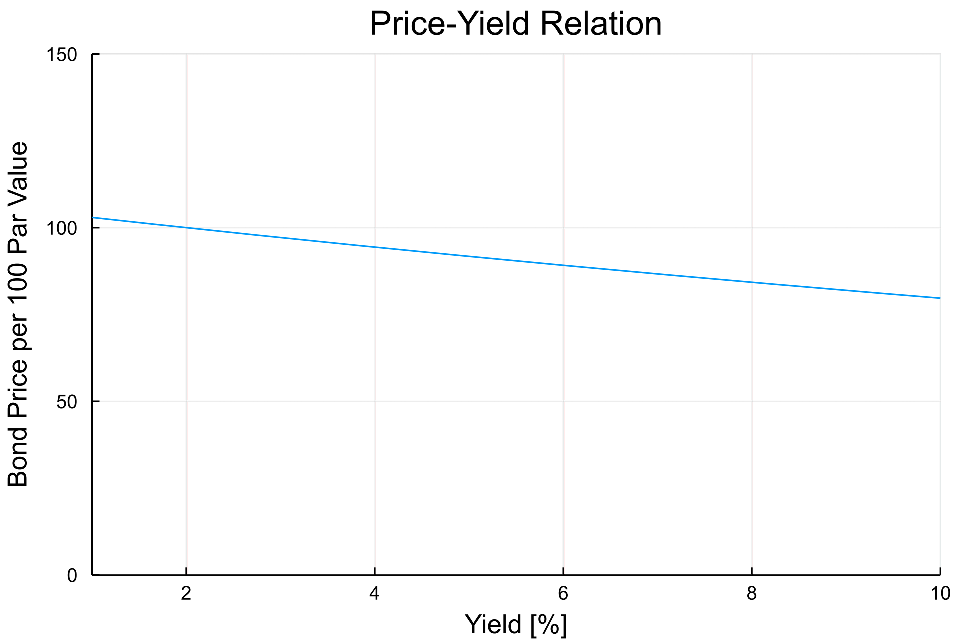
	Yield	Price	T	CouponRate
1	1.0	102.948	3.0	2.0
2	2.0	100.0	3.0	2.0
3	3.0	97.1514	3.0	2.0
4	4.0	94.3986	3.0	2.0
5	5.0	91.7378	3.0	2.0
6	6.0	89.1656	3.0	2.0
7	7.0	86.6786	3.0	2.0
8	8.0	84.2736	3.0	2.0
9	9.0	81.9474	3.0	2.0
10	10.0	79.6972	3.0	2.0

- When we plot the *Yield* column on the horizontal axis and the *Price* column on the vertical axis, we get the price-yield relation.
- The price-yield relation shows us the bond price P for specific values of the bond's yield to maturity y.

Example

- Face Value F [$]: 100
- Coupon Rate c [% p.a.]: 2.0
- Yield y [% p.a.]: 4.0
- Time to maturity T [years]:

Reset



- We see that as the yield increases, the bond price decreases.
- This is referred to as **inverse** relation between prices and yields.
    - It means that prices and yields move in opposite directions.
- We also see that the relation between prices and yields is not a straight line, but the relation has curvature. It is **convex**.

- Let us now consider two bonds, *Bond A* and *Bond B*.

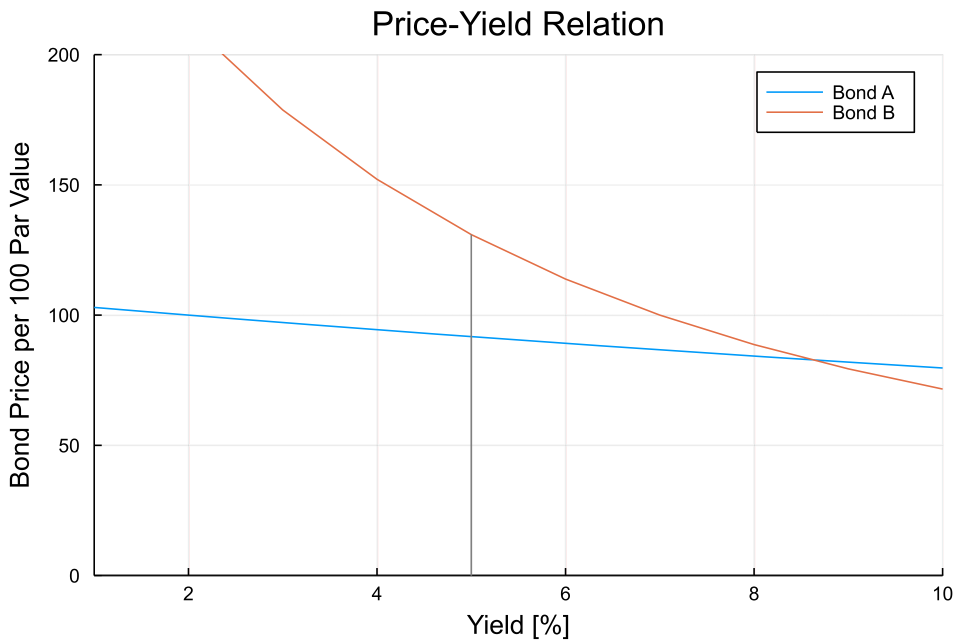
Bond A

- Face Value F [$]: 100
- Cpn Rate c [%]: 2.0
- Maturity T [yr]: 3

Bond B

- Face Value F [$]: 100
- Cpn Rate c [%]: 7.0
- Maturity T [yr]: 30

Reset



- Notice that the price of Bond A and Bond B change very differently when yields change.
- Let's show this by looking at the prices of Bond A and Bond B for different values of the bond yield y in a table.

Yield	Price	T	CouponRate
1.0	102.948	3.0	2.0
2.0	100.0	3.0	2.0
3.0	97.1514	3.0	2.0
4.0	94.3986	3.0	2.0
5.0	91.7378	3.0	2.0
6.0	89.1656	3.0	2.0
7.0	86.6786	3.0	2.0
8.0	84.2736	3.0	2.0
9.0	81.9474	3.0	2.0
10.0	79.6972	3.0	2.0

Yield	Price	T	CouponRate
1.0	255.177	30.0	7.0
2.0	212.388	30.0	7.0
3.0	178.761	30.0	7.0
4.0	152.141	30.0	7.0
5.0	130.909	30.0	7.0
6.0	113.838	30.0	7.0
7.0	100.0	30.0	7.0
8.0	88.6883	30.0	7.0
9.0	79.362	30.0	7.0
10.0	71.6061	30.0	7.0

- We will use the following notation.
- ΔP is the dollar price change of a bond.
    - For example, when the bond price is $100 and it increases to $102, then $\Delta P = 2$.
- $\frac{\Delta P}{P}$ is the percent change in the price of a bond.
    - For example, when the bond price is $100 and it increases to $102, then $\frac{\Delta P}{P} = 2\%$
- Δy is the change in the yield of the bond in decimals.
    - For example, when the bond yield is 4% and it increases to 5%, then the yield change is 1% and we write $\Delta y = 0.01$.
- $P(y)$ is the bond price when the yield-to-maturity is y (keeping time-to-maturity T and coupon rate c fixed)

Bond Duration

- **Duration** is a measure used to quantify how sensitive bond prices are to changes in interest rates.
- We will define duration as the **percent change of the bond price $\frac{\Delta P}{P}$ to a change in interest rates** Δy.
    - This **duration** is referred to as the **Modified Duration (MD)**.
    - More specifically (note the minus sign)

$$\text{Modified Duration} = -\frac{\Delta P/P}{\Delta y}$$

- There is a second definition of a bond's duration, the so-called **Macaulay Duration (D)**.
    - This duration measure can be interpreted as the average time-to-payment for a bond.

Macaulay Duration

- The Macaulay Duration D of a bond with price P can be interpreted as the weighted-average time-to-payment.
- The reason becomes clear when we look at how it is calculated.
- For a semi-annual coupon bond with coupon rate c% (paid semi-annually), semi-annual coupon cash flows C, time to maturity T and yield-to-maturity y, the Macaulay Duration (D) is

$$D = 0.5 \times w_{0.5} + 1.0 \times w_{1.0} + 1.5 \times w_{1.5} + \ldots + T \times w_T = \sum_{t=0.5}^{T} t \times w_t$$

$$w_t = \frac{\text{PV of time-t coupon cash flow}}{P}$$

- Note that the weights w_t sum to one.
- Recall that the present value (PV) of the time-t coupon cash flow is

$$\text{PV of time-t coupon cash flow} = \frac{C}{(1+\frac{y}{2})^{2\times t}}$$

and the bond price P is

$$P = \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 0.5}} + \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 1.0}} + \frac{C}{\left(1+\frac{y}{2}\right)^{2\times 1.5}} + \ldots + \frac{100+C}{\left(1+\frac{y}{2}\right)^{2\times T}}$$

Example

- Consider a Treasury note with time-to-maturity in $T = 10$ years, coupon rate $c = 8.0\%$, and yield to maturity $y = 6.0\%$.
  - The semi-annual coupon cash flows are $C = \frac{c}{2} \times F = \frac{8.0\%}{2} \times 100 = 4.0$
  - The bond price is

$$P = \frac{4.0}{6.0\%/2} \times \left(1 - \frac{1}{(1+\frac{6.0\%}{2})^{2\times 10}}\right) + \frac{100}{(1+\frac{6.0\%}{2})^{2\times 10}} = \$114.8775$$

- Face Value F [$]: 100
- Coupon Rate c [% p.a.]: 8.0
- Yield y [% p.a.]: 6.0
- Time to maturity T [years]: 10

Reset

	Time	Coupon	PVCoupon	P	Wt	t_times_Wt
1	0.5	4.0	3.8835	114.877	0.0338055	0.0169028
2	1.0	4.0	3.77038	114.877	0.0328209	0.0328209
3	1.5	4.0	3.66057	114.877	0.031865	0.0477974
4	2.0	4.0	3.55395	114.877	0.0309369	0.0618737
5	2.5	4.0	3.45044	114.877	0.0300358	0.0750895
6	3.0	4.0	3.34994	114.877	0.029161	0.0874829
7	3.5	4.0	3.25237	114.877	0.0283116	0.0990906
8	4.0	4.0	3.15764	114.877	0.027487	0.109948
9	4.5	4.0	3.06567	114.877	0.0266864	0.120089
10	5.0	4.0	2.97638	114.877	0.0259091	0.129546
more						
20	10.0	104.0	57.5823	114.877	0.50125	5.0125

- Using the Macaulay Duration formula, we calculate D as follows

$$D = 0.5 \times \frac{\frac{C}{(1+\frac{y}{2})^{2\times 0.5}}}{P} + 1.0 \times \frac{\frac{C}{(1+\frac{y}{2})^{2\times 1.0}}}{P} + 1.5 \times \frac{\frac{C}{(1+\frac{y}{2})^{2\times 1.5}}}{P} + 2.0 \times \frac{\frac{C}{(1+\frac{y}{2})^{2\times 2.0}}}{P} + 2.5 \times \frac{\frac{C}{(1+\frac{y}{2})^{2}}}{P}$$

- Plugging in the values

$$D = 0.5 \times \frac{\frac{4.0}{(1+\frac{6.0\%}{2})^{2\times 0.5}}}{114.8775} + 1.0 \times \frac{\frac{4.0}{(1+\frac{6.0\%}{2})^{2\times 1.0}}}{114.8775} + 1.5 \times \frac{\frac{4.0}{(1+\frac{6.0\%}{2})^{2\times 1.5}}}{114.8775} + 2.0 \times \frac{\frac{4.0}{(1+\frac{6.0\%}{2})^{2\times 2.0}}}{114.8775} + 2.5 \times$$

- Finally, the Macaulay Duration D is

$$D = 7.2863 \text{ years}$$

- Note that the unit of the Macaulay Duration is **years**. Thus, the Macaulay Duration is oftentimes interpreted is a weighted-average time-to-payment.

Modified Duration

- Recall that we define duration as the **percent change of the bond price $\frac{\Delta P}{P}$ to a change in interest rates Δy**.
  - This **duration** is referred to as the **Modified Duration (MD)**
  - More specifically, let the current yield-to-maturity be y and let the current bond price be $P(y)$.
  - Then, the modified duration $MD(y)$ is (note the minus sign)

$$MD = -\frac{\%\text{ change in bond price}}{\text{change in interest rates}} = -\frac{\frac{\Delta P}{P(y)}}{\Delta y}$$

- Thus, the modified duration has the following interpretation:

> If interest rates *increase* by 1 percentage point, then the *percent decrease* in the bond price is approximately equal to the modified duration.

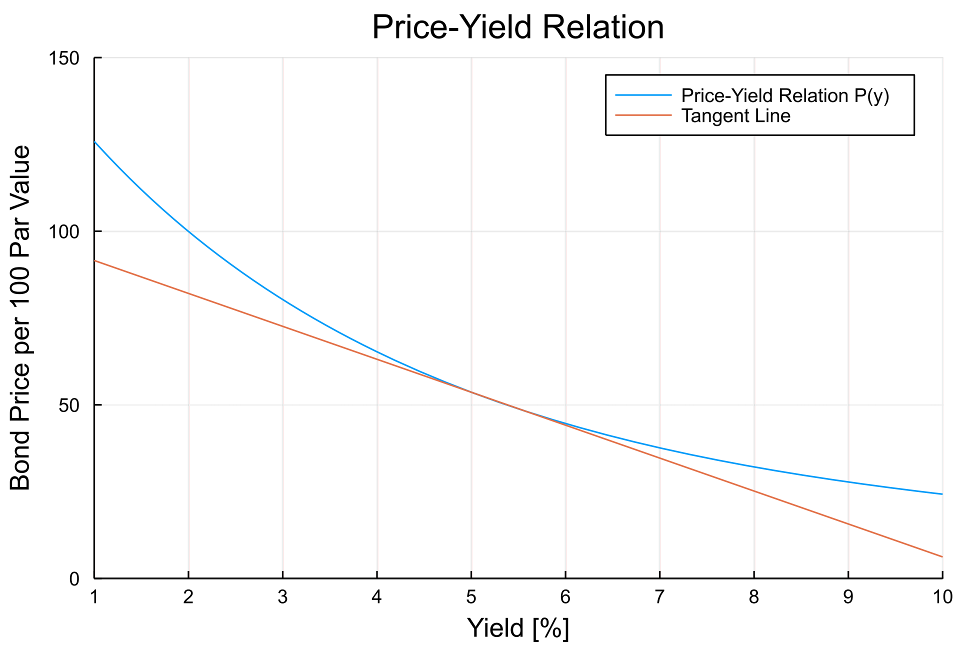
> Thus, modified duration can be interpreted as the approximate percentage change in price for a 100-basis-point change in yield.

- **Example**: Suppose modified duration is 7. An increase in the yield from 5% to 6% is a price drop of about 7%.

Modified Duration and the Price-Yield Relation

- Face Value F [$]: 100
- Coupon Rate c [% p.a.]: 2.0
- Time to maturity T [years]: 30

Reset



- The tangent line approximates the price-yield relation closely near the tangency point.
- The modified duration can be interpreted as giving us the percent price change of the bond when we assume that the price-yield relation is represented by the tangent line.
- Specifically, the modified duration MD is

$$MD \approx -\text{slope of the tangency line} \times \frac{1}{P}$$

- Since we know from basic calculus what the slope of the tangency line is, we have a short-cut to calculating the modified duration.
- Specifically, recall that the slope of the tangency at the point y is

$$m = \frac{P(y+\Delta y) - P(y-\Delta y)}{2 \times \Delta y}$$

- This is a first-order linear approximation of the price-yield relation at today's yield y.
  - It means that if the yield goes up by one percentage point, the price changes by approximately this slope (times 0.01).

- We now use this insight about m to get a formula for the modified duration $MD(y)$.

$$MD(y) \approx -\frac{P(y+\Delta y) - P(y-\Delta y)}{2 \times \Delta y} \times \frac{1}{P(y)}$$

- This means that in order to calculate the modified duration given today's bond price $P(y)$ and yield y we do two calculations:
  1. Increase the yield-to-maturity from y to $y + \Delta y$ and calculate the bond price $P(y + \Delta y)$ (pick a small value for Δy, e.g. $\Delta y = 0.001$).
  2. Decrease the yield-to-maturity from y to $y - \Delta y$ and calculate the bond price $P(y - \Delta y)$ (pick a small value for Δy, e.g. $\Delta y = 0.001$).
  3. Plug the values for $P(y + \Delta y)$ and $P(y - \Delta y)$ into the modified duration formula and calculate $MD(y)$.

- Let's compute the modified duration of a Treasury note with time-to-maturity in $T = 10$ years, coupon rate $c = 8.0$ %, coupon cash flow $C = 4.0$ and yield to maturity $y = 6.0$ %.
- We pick $\Delta y = 0.2\%$ (20 basis points).

1. We calculate $P(y + \Delta y)$

$$P(y+\Delta y) = \frac{C}{(y+\Delta y)/2} \times \left(1 - \frac{1}{\left(1+\frac{(y+\Delta y)}{2}\right)^{2\times T}}\right) + \frac{100}{\left(1+\frac{(y+\Delta y)}{2}\right)^{2\times T}}$$

$$P(y+\Delta y) = \frac{4.0}{6.2\%/2} \times \left(1 - \frac{1}{\left(1+\frac{6.2\%}{2}\right)^{2\times 10}}\right) + \frac{100}{\left(1+\frac{6.2\%}{2}\right)^{2\times 10}} = 113.266767$$

2. We calculate $P(y - \Delta y)$

$$P(y-\Delta y) = \frac{C}{(y-\Delta y)/2} \times \left(1 - \frac{1}{\left(1+\frac{(y-\Delta y)}{2}\right)^{2\times T}}\right) + \frac{100}{\left(1+\frac{(y-\Delta y)}{2}\right)^{2\times T}}$$

$$P(y+\Delta y) = \frac{4.0}{5.8\%/2} \times \left(1 - \frac{1}{\left(1+\frac{5.8\%}{2}\right)^{2\times 10}}\right) + \frac{100}{\left(1+\frac{5.8\%}{2}\right)^{2\times 10}} = 116.517557$$

3. We calculate the modified duration $MD(y)$

$$MD(y) = -\frac{P(y + \Delta y) - P(y - \Delta y)}{2 \times \Delta y} \times \frac{1}{P(y)}$$

$$MD(6.0\%) = -\frac{113.2668 - 116.5176}{2 \times 0.002} \times \frac{1}{114.8775} = 7.074474$$

- This means that when interest rates increase by 1 percentage point, the price of the bond declines by 7.07 percent.

Relation between Macaulay Duration and Modified Duration

- For a semi-annual coupon bond with yield y (semi-annually compounded), the Macaulay Duration D and Modified Duration MD are related as follows:

$$MD(y) = \frac{D}{1 + \frac{y}{2}}$$

- For a zero-coupon bond, the Macaulay Duration is equal to the maturity of the bond.
- Thus, for a zero coupon bond with maturity in T years (and semi-annual yield y), we have

$$MD(y) = \frac{T}{1 + \frac{y}{2}}$$

Using Duration to estimate Bond Price Changes

- To estimate percent-changes in the price of a bond $(\Delta P/P)$ when the yield changes by Δy, we use

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y$$

- Intuitively, this means

$$\% \text{ Change in Bond Price} = -\text{Modified Duration} \times \text{Change in Yield}$$

- However, this equation is an approximation, and it gives a result that is different from the *actual* price change as yield changes become larger.
  - We see this in the price-yield relation. The tangent line approximates the price-yield relation close to the tangent point at the current yield y, but gets more and more inaccurate as we move away from the tangent point. The reason why the approximation gets worse is because the price-yield relation is convex.

- To illustrate this, let's consider a Treasury note with time-to-maturity $T = 10$ years, coupon rate $c = 8.0$ %, coupon cash flows of $C = 4.0$, face value $F = 100$, and yield to maturity $y = 6.0$ %.
- The bond's modified duration is $MD(6.0\%) = 7.074474$
- Suppose interest rates increase and the bond's yield changes by 0.5% ($\Delta y = 0.0050$).
- We calculate the percentage change in the bond price as

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y = -7.074474 \times 0.005 = -0.035372 = -3.54\%$$

- Thus, we estimate that the bond price declines by -3.54% when the bond's yield increases by 0.5% (50 basis points).

- What is the *actual* percent-change in the bond price?

$$\frac{\Delta P}{P} = \frac{P(y + \Delta y) - P(y - \Delta y)}{P} = \frac{P(6.0\% + 0.5\%) - P(y = 6.0\% - 0.5\%)}{P}$$

$$\frac{\Delta P}{P} = \frac{110.9045 - 114.8775}{114.8775} = -0.034584 = -3.46\%$$

- We see that the actual percent price change is -3.46% and the approximated percent price change is -3.54%.
- Thus, there is an approximation error.
- How large this error is depends on how large the yield change Δy is.
- Let's illustrate this in a table.

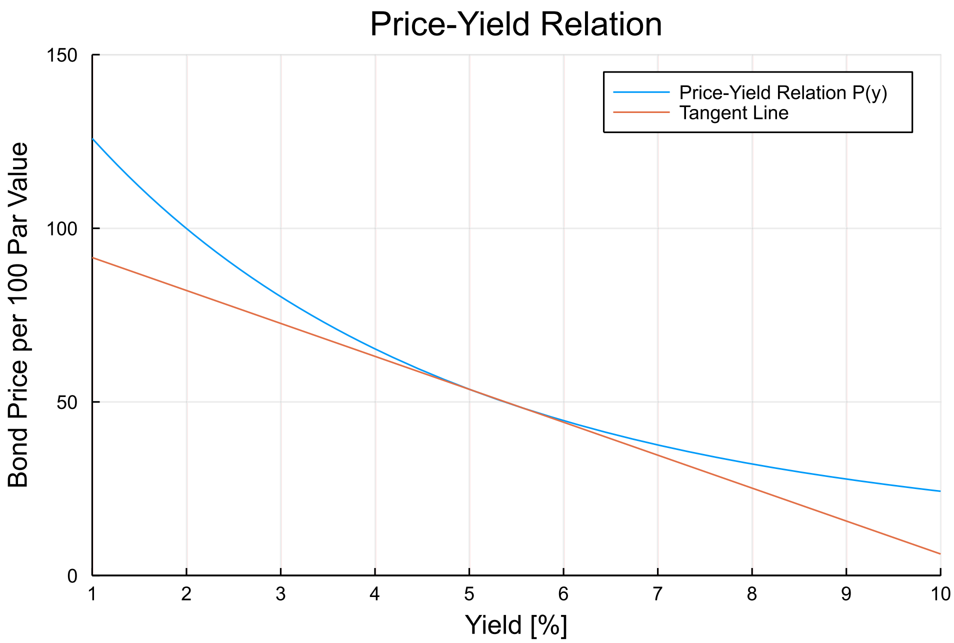
	CurrentYield	NewYield	YieldChange	ActualPrice	MDPrice	MD_PriceChan
1	6.0	0.5	-5.5	173.067	159.576	44.6984
2	6.0	1.0	-5.0	166.456	155.512	40.6349
3	6.0	1.5	-4.5	160.151	151.449	36.5714
4	6.0	2.0	-4.0	154.137	147.385	32.5079
5	6.0	2.5	-3.5	148.398	143.322	28.4444
6	6.0	3.0	-3.0	142.922	139.258	24.3809
7	6.0	3.5	-2.5	137.694	135.195	20.3174
8	6.0	4.0	-2.0	132.703	131.131	16.254
9	6.0	4.5	-1.5	127.936	127.068	12.1905
10	6.0	5.0	-1.0	123.384	123.004	8.12698
more						
30	6.0	15.0	9.0	64.3193	41.7347	-73.1428

- The take-away is that using Modified Duration to calculate price changes of a bond works well when yield changes are small.
- The approximation error becomes severe when yield changes become large.
- As shown below, this is because the actual price-yield relation is convex (not linear as on the straight line).
- We can improve the approximation by taking the curvature of the price-yield relation into account.
- To achieve this, we will add a **convexity** term to the price change formula.
- Specifically,

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y$$

- becomes

$$\frac{\Delta P}{P} = -MD(y) \times \Delta y + \frac{1}{2}\text{ Convexity }(\Delta y)^2$$



Practice Problem

- A portfolio manager is considering the purchase of a bond with a 5.5% coupon rate that pays interest annually and matures in three years. If the required rate of return on the bond is 5%, the price of the bond per 100 of par value is closest to:
  - (A) 98.65
  - (B) 101.36
  - (C) 106.43

Hint

Practice Problem

Which bond will most likely experience the smallest percent change in price if the market discount rates for all three bonds increase by 100 bps?

Bond	Price	Coupon Rate	Time-to-Maturity T
A	101.886	5%	2 years
B	100.000	6%	2 years
C	97.327	5%	3 years

Hint

Duration with Term Structure of Interest Rates

- Thus far, we have assumed that the interest rate that changes is the yield of the bond y. Implicit in this assumption is that we assumed that the term structure of interest rates is flat and all interest rates change by the same small amount Δy.
- Next, we consider how we compute modified duration, when we are given a term structure of interest rates.

- When we are given a term structure of interest rates, we can still use the modified duration formula

$$MD = -\frac{P^{+} - P^{-}}{2 \times \Delta r} \times \frac{1}{P}$$

- and calculate percent price changes using

$$\frac{\Delta P}{P} = -MD \times \Delta r$$

- In the formula for MD,
  - the term P^{+} is the price of the bond when the entire term structure is shifted upward by the same amount Δr (e.g. $\Delta r = 0.001$).
  - the term P^{-} is the price of the bond when the entire term structure is shifted down by the same amount Δr.
  - As before, P is the current bond price.

- To illustrate this, suppose we want to calculate the modified duration MD of a bond when the term structure of interest rates is upward sloping.
- Specifically, suppose we are given a 5-year bond with face value $F = 100$, coupon rate $c = 4\%$, and annual coupon cash flows $C = 4$.

- Assume the following zero-coupon yield curve with **annual** compounding.

t	1 year	2 year	3 year	4 year	5 year
r_t	r_1=2%	r_2=3%	r_3=5%	r_4=6%	r_5=8%

- Recall that under **annual** compounding, the price of a T-year bond is caluclated as

$$P = \frac{C}{(1+r_1)^1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C+F}{(1+r_T)^T}$$

- To calculate the modified duration MD, let's first calculate the current bond price.

$$P = \frac{C}{(1+r_1)^1} + \frac{C}{(1+r_2)^2} + \frac{C}{(1+r_3)^3} + \frac{C}{(1+r_4)^4} + \frac{F+C}{(1+r_5)^5}$$

$$P = \frac{4.0}{(1+2\%)^1} + \frac{4.0}{(1+3\%)^2} + \frac{4.0}{(1+5\%)^3} + \frac{4.0}{(1+6\%)^4} + \frac{104.0}{(1+8\%)^5} = 85.09633$$

- Next, we shift the term structure of interest rates up and down by $\Delta r = 0.1\%$.
- Shifting the term structure up by $+\Delta r$ gives us

t	1 year	2 year	3 year	4 year	5 year
r_t	$r_1 = 2.1$	$r_2 = 3.1$	$r_3 = 5.1$	$r_4 = 6.1$%	$r_5 = 8.1$

$$P^{+} = \frac{4.0}{(1+2.1\%)^1} + \frac{4.0}{(1+3.1\%)^2} + \frac{4.0}{(1+5.1\%)^3} + \frac{4.0}{(1+6.1\%)^4} + \frac{104.0}{(1+8.1\%)^5} = 84.736$$

- Shifting the term structure down by $-\Delta r = -0.1\%$ gives us

t	1 year	2 year	3 year	4 year	5 year
r_t	$r_1 = 1.9$	$r_2 = 2.9$	$r_3 = 4.9$	$r_4 = 5.9\%$	$r_5 = 7.9$

$$P^- = \frac{4.0}{(1+1.9\%)^1} + \frac{4.0}{(1+2.9\%)^2} + \frac{4.0}{(1+4.9\%)^3} + \frac{4.0}{(1+5.9\%)^4} + \frac{104.0}{(1+7.9\%)^5} = 85.4579$$

- Thus, the modified duration MD is

$$MD = -\frac{P^+ - P^-}{2 \times \Delta r} \times \frac{1}{P}$$

$$MD = -\frac{84.736617 - 85.457986}{2 \times 0.01\%} \times \frac{1}{85.09633} = 4.238545$$

- We can now compute bond price changes when the term structure of interest rates shifts in parallel, i.e. all zero-coupon yields r_t increase or decrease by the same amount Δr.
- To illustrate this, suppose the term structure of interest rates shifts up by 0.2%.
- Then, the approximate dollar price change of the bond is

$$\Delta P = -MD \times \Delta r \times P = -4.238545 \times 0.2\% \times 85.0963 = -0.721369$$

- For comparison, the actual price change is -0.717495.

Duration of Bond Portfolios

- Thus far, we have considered the case of a single bond and have calculated the modified duration.
- When we have a portfolio of bonds, we calculate the modified duration of the bond portfolio using the modified durations of the individual bonds in the portfolio.
- Specifically, suppose the bond portfolio consists of B bonds. We denote the individual bonds by $b = 1, \ldots, B$.
- The portfolio is assumed to consist of N_b units of each bond b.
- Each bond is assumed to have a price of P_b per 100 par value.

- We write the fraction of the position in bond b to the total portfolio value $P_{\text{Portfolio}}$ as

$$w_b = \frac{n_b \times P_b}{P_{\text{Portfolio}}}$$

- Note that the total value of the bond portfolio is

$$P_{\text{Portfolio}} = n_1 \times P_1 + \ldots + n_B \times P_B$$

- Then, we calculate the modified duration $MD_{\text{Portfolio}}$ of the bond portfolio as the weighted average of the modified durations of the individual bonds.

$$MD_{\text{Portfolio}} = w_1 \times MD_1 + w_2 \times MD_2 + \ldots + w_B \times MD_B$$

Example

- Suppose that you own a portfolio of zero-coupon bonds. All yields are annually compounded.
- Calculate the modified duration of the portfolio.

Bond	Maturity	Yield	Face value
H	1	2%	40
I	2	3%	40
J	3	5%	40
K	4	6%	40
L	5	8%	1040

- Let's first calculate the the prices of the zero coupon bonds per $100 face value.
- Recall, that the price of a T-year maturity zero-coupon bond with yield y_T (annually compounded) is given by

$$P_T = \frac{100}{(1+y_T)^T}$$

	Bond	Maturity	Yield	FaceValue	PricePer100
1	"H"	1	2	40	98.0392
2	"I"	2	3	40	94.2596
3	"J"	3	5	40	86.3838
4	"K"	4	6	40	79.2094
5	"L"	5	8	1040	68.0583

- Next, let's calculate the number of units n_b for each bond b in the portfolio.
- The number of bonds is simply the actual face value divided by 100 face value (which we used to calculate the bond price).
    - For instance for bond H, it is $40/$100=0.4

	Bond	Maturity	Yield	FaceValue	PricePer100	nB
1	"H"	1	2	40	98.0392	0.4
2	"I"	2	3	40	94.2596	0.4
3	"J"	3	5	40	86.3838	0.4
4	"K"	4	6	40	79.2094	0.4
5	"L"	5	8	1040	68.0583	10.4

- Next, we calculate the modified durations of the zero-coupon bonds.
- Recall that the modified duration MD of a zero-coupon bond with T-years to maturity and yield y_T (annually compounded) is

$$MD = \frac{T}{1+y}$$

- For instance, for bond L it is $MD_5 = 5/(1+8\%) = 4.6296$

	Bond	Maturity	Yield	FaceValue	PricePer100	nB	MD
1	"H"	1	2	40	98.0392	0.4	0.980392
2	"I"	2	3	40	94.2596	0.4	1.94175
3	"J"	3	5	40	86.3838	0.4	2.85714
4	"K"	4	6	40	79.2094	0.4	3.77358
5	"L"	5	8	1040	68.0583	10.4	4.62963

- Next, we calculate the total value of the bond portfolio.
- The value of the bond portfolio P_b is the sum of the values of the positions in the individual bonds. The position in bond b is worth the number of units times the bond price, i.e. $n_b \times P_b$.

	Bond	Maturity	Yield	FaceValue	PricePer100	nB	MD	Pb
1	"H"	1	2	40	98.0392	0.4	0.980392	850.963
2	"I"	2	3	40	94.2596	0.4	1.94175	850.963
3	"J"	3	5	40	86.3838	0.4	2.85714	850.963
4	"K"	4	6	40	79.2094	0.4	3.77358	850.963
5	"L"	5	8	1040	68.0583	10.4	4.62963	850.963

- Now we can calculate the portfolio weights

$$w_b = \frac{n_b \times P_b}{P_{\text{Portfolio}}}$$

	Bond	Maturity	Yield	FaceValue	PricePer100	nB	MD	Pb	
1	"H"	1	2	40	98.0392	0.4	0.980392	850.963	0.0
2	"I"	2	3	40	94.2596	0.4	1.94175	850.963	0.0
3	"J"	3	5	40	86.3838	0.4	2.85714	850.963	0.0
4	"K"	4	6	40	79.2094	0.4	3.77358	850.963	0.0
5	"L"	5	8	1040	68.0583	10.4	4.62963	850.963	0.8

- As the last step, we compute the modified duration of the portfolio $MD_{\text{Portfolio}}$

$$MD_{\text{Portfolio}} = w_1 \times MD_1 + w_2 \times MD_2 + \ldots + w_B \times MD_B$$

	Bond	Maturity	Yield	FaceValue	PricePer100	nB	MD	Pb	
1	"H"	1	2	40	98.0392	0.4	0.980392	850.963	0.0
2	"I"	2	3	40	94.2596	0.4	1.94175	850.963	0.0
3	"J"	3	5	40	86.3838	0.4	2.85714	850.963	0.0
4	"K"	4	6	40	79.2094	0.4	3.77358	850.963	0.0
5	"L"	5	8	1040	68.0583	10.4	4.62963	850.963	0.8

$$MD_{\text{Portfolio}} = 0.0452 + 0.086 + 0.116 + 0.1405 + 3.8508$$

$$MD_{\text{Portfolio}} = 4.238521$$

Wrap-Up

Our goals for today

- [x] Understand how bond prices change with yields.
- [x] Calculate a Macaulay Duration and understand what it is, conceptually.
- [x] Understand what Modified Duration is and how to calculate it.
- [x] Understand the link between Macaulay and Modified Duration.
- [x] Use Modified Duration to approximate bond price changes.
- [x] Know how to calculate the Modified Duration of a bond portfolio.

Reading

Fabozzi, Fabozzi, 2021, Bond Markets, Analysis, and Strategies, 10th Edition
Chapter 4