Chapter 11

CAPM Extensions and Multi-Factor Models

Reference: Elton, Gruber, Brown, and Goetzmann (2014) 14 and 16

11.1 Factor Investment

A number of "factors", for instance, based on firm characteristics like size and profitability, have shown good performance over long periods of time. It is therefore common to base investment strategies on those characteristics—and a large number of funds and other investment vehicles have been developed to facilitate this. This approach is called "factor investing" (and sometimes also "smart beta"). It is essentially a dynamic trading strategy (since the characteristics change over time), so trading costs must be managed.

In studies of investment fund performance, it is often found that the abnormal performance (α) can be explained by a fairly small set of factors. This suggests that fund managers have been able to invest in those characteristics that have historically paid off.

Empirical Example 11.1 *(Fama-French factors) Figure 11.1 illustrate several of the factors discussed by Fama and French (1993) and Fama and French (2015), while Table 11.1 summarises the return patterns. It is clear that several of the factors have earned substantial risk premia (average excess returns) and have fairly reasonable volatilities. As a consequence, the Sharpe ratios are good. In addition, several of these portfolios are virtually uncorrelated with the market excess return, so the α values are similar to the average excess returns. Also, the volatility and beta sorted portfolios (low minus high) have distinctly negative betas.*

Systematic abnormal performance (α values) is a rejection of CAPM, so this chapters discusses attempts to theoretically motivate and empirically test different multi-factor models.

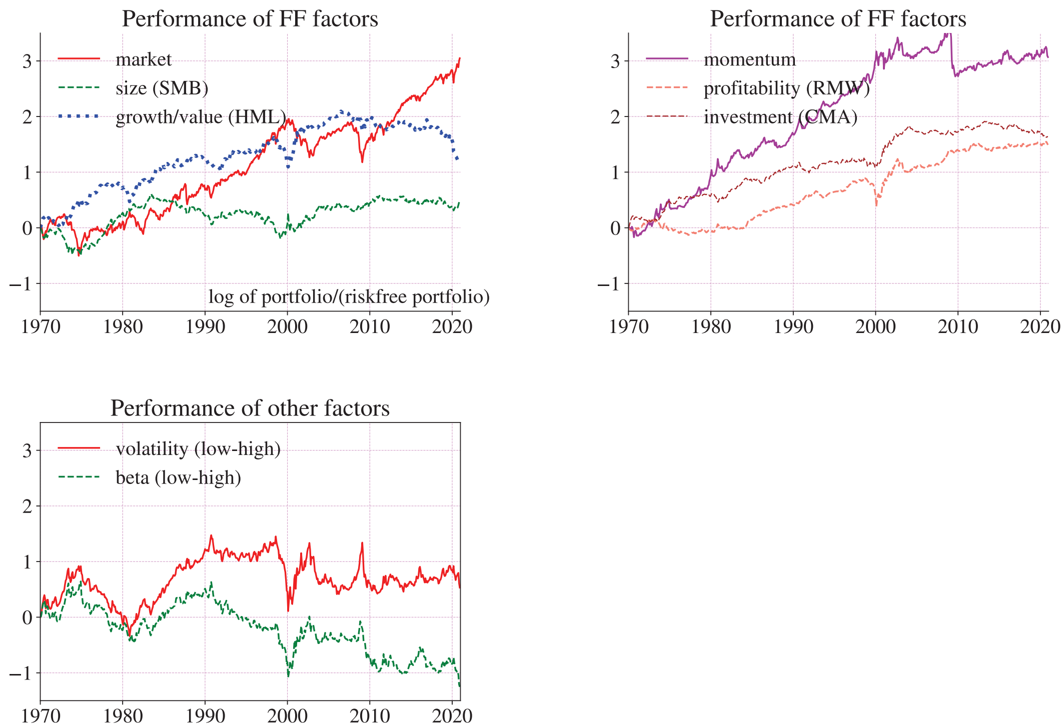


Figure 11.1: Portfolio choice with background risk

11.2 Multi-Factor Models

A multi-factor model extends the market model by allowing more factors to explain the return on an asset. In terms of excess returns it could be

$$\begin{aligned} R_i^e &= \beta_{im} R_m^e + \beta_{iF} R_F^e + \varepsilon_i, \text{ where} \\ \mathrm{E}\,\varepsilon_i &= 0, \mathrm{Cov}(R_m^e, \varepsilon_i) = 0, \mathrm{Cov}(R_F^e, \varepsilon_i) = 0. \end{aligned} \tag{11.1}$$

The pricing implication is a multi-beta model

$$\mu_i^e = \beta_{im} \mu_m^e + \beta_{iF} \mu_F^e. \tag{11.2}$$

Remark 11.2 *(When factors are not excess returns*) Equation 11.2 assumes that the factor can be expressed as an excess return—but that is not always the case. For instance, it could be that the second factor is a macro variable like inflation surprises. Then there are two possible ways to proceed. First, find that portfolio which mimics the movements in the inflation surprises best and use the excess return of that (factor mimicking) portfolio in*

	$\mu, \%$	$\sigma, \%$	SR	β	$\alpha, \%$
market	7.11	15.92	0.45	1.00	−0.00
size	1.61	10.64	0.15	0.19	0.25
growth/value	3.08	10.28	0.30	−0.15	4.14
momentum	7.26	15.12	0.48	−0.17	8.48
profitability	3.20	7.73	0.41	−0.11	3.95
investment	3.54	6.87	0.52	−0.16	4.71
volatility (low-high)	3.66	22.27	0.16	−0.89	9.96
beta (low-high)	−0.48	19.41	−0.02	−0.81	5.31

Table 11.1: Descriptive statistics of excess returns of different US equity portfolios (including the Fama-French factors and more), annualised. Monthly data 1970:01-2020:12.

(11.1) and (11.2). Second, we could instead reformulate the model by adding an intercept in (11.2) and let R^e_F denote whatever the factor is (not necessarily an excess return) and then estimate the factor risk premium, corresponding to μ^e_F in (11.2), by using a cross-section of different assets ($i = 1, 2, \ldots$).

We will consider several *theoretical* multi-factor models: the "CAPM with background risk" as well as a consumption-based model.

There are also several *empirically motivated* multi-factor models, that is, empirical models that have been found to work well (even if the theoretical foundation might be a bit weak). For instance, Fama and French (1993) estimate a three-factor model (capturing the market, the difference between small and large firms and the difference between value firms and growth firms) and show that it performs much better than CAPM. Also, the multi-factor model by MSCIBarra is widely used in the financial industry. It uses a set of firm characteristics (rather than macro variables) as factors, for instance, size, volatility, price momentum, and industry/country (see Stefek (2002)). This model is often used to value firms without a price history (for instance, before an IPO) or to find mispriced assets.

11.6 Testing Multi-Factors Models

Provided all factors are *excess returns*, we can test a multi-factor model by testing whether $\alpha = 0$ in the regression

$$R_{it}^e = \alpha + b_{io} R_{ot}^e + b_{ip} R_{pt}^e + ... + \varepsilon_{it}. \tag{11.45}$$

The t-test of the null hypothesis that $\alpha_i = 0$ uses the fact that, under fairly mild conditions, the t-statistic has an asymptotically normal distribution, that is

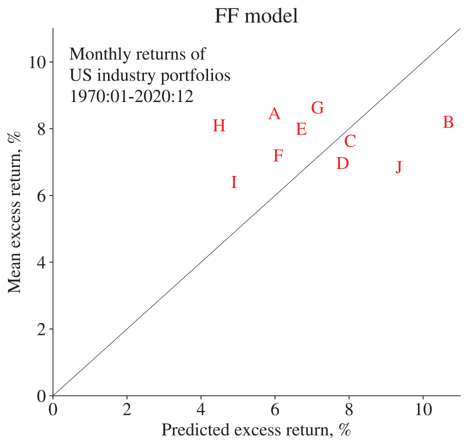
$$\frac{\hat{\alpha}_i}{\text{Std}(\hat{\alpha}_i)} \xrightarrow{d} N(0,1) \text{ under } H_0 : \alpha_i = 0. \tag{11.46}$$

Fama and French (1993) try a multi-factor model. They find that a three-factor model fits the 25 stock portfolios fairly well (two more factors are needed to also fit the seven bond portfolios that they use). This three-factor model is rejected at traditional significance levels, but it can still capture a fair amount of the variation of expected returns.

Remark 11.20 *(Fama-French factors) Fama and French (1993) use three factors: the market excess return, the return on a portfolio of small stocks minus the return on a portfolio of big stocks (SMB), and the return on a portfolio with a high ratio of book value to market value minus the return on a portfolio with a low ratio (HML). All three are excess returns (although only the first is in excess of a riskfree return), since they are long-short portfolios. He and Ng (1994) try to relate these factors to macroeconomic series.*

Remark 11.21 *(Returns on long-short portfolios*) Suppose you invest x into asset i, but finance that by short-selling asset j. (You sell enough of asset j to raise x.) The net investment is then zero, so there is no point in trying to calculate an overall return like "value today/investment yesterday - 1." Instead, the convention is to calculate an excess return of your portfolio as $R_i - R_j$ (or equivalently, $R_i^e - R_j^e$). This excess return essentially says: if your exposure (how much you invested) is x, then you have earned $x(R_i - R_j)$. To make this excess return comparable with other returns, you add the riskfree rate: $R_i - R_j + R_f$, implicitly assuming that your portfolio includes a riskfree investment of the same size as your long-short exposure (x).*

Chen, Roll, and Ross (1986) use a number of macro variables as factors—along with traditional market indices. They find that industrial production and inflation surprises



	α	t-stat	σ
A (NoDur)	2.43	2.05	8.34
B (Durbl)	-2.54	-1.32	13.56
C (Manuf)	-0.45	-0.53	5.98
D (Enrgy)	-0.91	-0.42	15.34
E (HiTec)	1.24	0.90	9.67
F (Telcm)	1.06	0.72	10.38
G (Shops)	1.43	1.11	9.07
H (Hlth)	3.56	2.43	10.32
I (Utils)	1.46	0.94	10.92
J (Other)	-2.55	-3.25	5.54

Fama-French model
Factors: US market, SMB (size), and HML (book-to-market)
Predicted excess return: $\beta_m R_m^e + \beta_{SMB} R_{SMB} + \beta_{HML} R_{HML}$

α and σ (std of residual) are in annualized %

p-val for testing if all $\alpha_i = 0$: 0.009

Figure 11.5: Fama-French regressions on US industry indices

are priced factors, while the market index might not be. For such (non-return) factors it is common to use factor mimicking portfolios: the excess return on portfolios strongly correlated with the factors.

Figure 11.5 shows some results for the Fama-French model on US industry portfolios and Figures 11.6–11.7 on the 25 Fama-French portfolios.

11.7 Appendix: Extra Material

11.7.1 The Arbitrage Pricing Model*

Reference: Ross (1976)

The *first assumption* of the Arbitrage Pricing Theory (APT) is that the return of asset i can be described as

$$R_{it} = a_i + \beta_i f_t + \varepsilon_{i,t}, \text{ where} \tag{11.47}$$
$$\mathrm{E}\,\varepsilon_{it} = 0, \mathrm{Cov}(\varepsilon_{it}, f_t) = \mathrm{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0.$$

In this particular formulation there is only one factor, f_t, but the APT allows for more

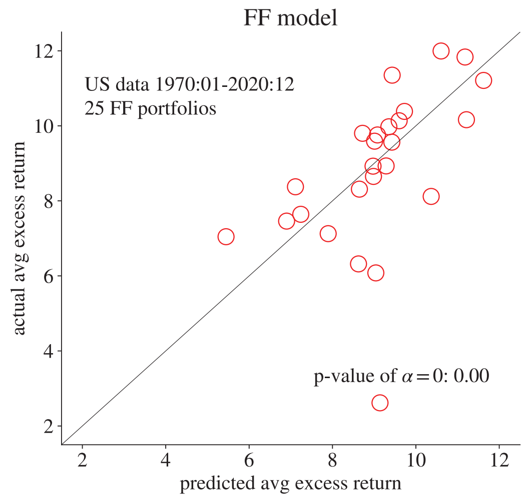


Figure 11.6: FF, FF portfolios

factors. Notice that (11.47) assumes that any correlation of two assets (i and j) is due to movements in f_t—the residuals are assumed to be uncorrelated.

The *second assumption* of APT is that financial markets are very well developed—so well developed that it is possible to form portfolios that "insure" against almost all possible outcomes. To be precise, the assumption is that it is possible to form a zero cost portfolio (buy some, sell some) that has a zero exposure to the factor and also (almost) no idiosyncratic risk. In essence, this assumes that we can form a (non-trivial) zero-cost portfolio of the risky assets that is riskfree. In formal terms, the assumption is that there is a non-trivial portfolio (with the value v_j of the position in asset j) such that $\Sigma_{i=1}^N v_i = 0$ (zero cost), $\Sigma_{i=1}^N v_i \beta_i = 0$ (zero exposure to the factor) and $\Sigma_{i=1}^N v_i^2 \operatorname{Var}(\varepsilon_{i,t}) \approx 0$ (well diversified). The requirement that the portfolio is non-trivial means that at least some $v_j \neq 0$. This riskfree portfolio has a zero cost, so it must have a return of zero (and thus also an expected return of zero) or otherwise there are arbitrage opportunities.

Together, these assumptions imply that for (every asset) we have

$$\mathrm{E}\, R_{it} = R_f + \beta_i \lambda, \tag{11.48}$$

where λ is (typically) an unknown constant. The important feature is that there is a linear relation between the risk premium (expected excess return) of an asset and its beta. This expression generalizes to the multi-factor case.

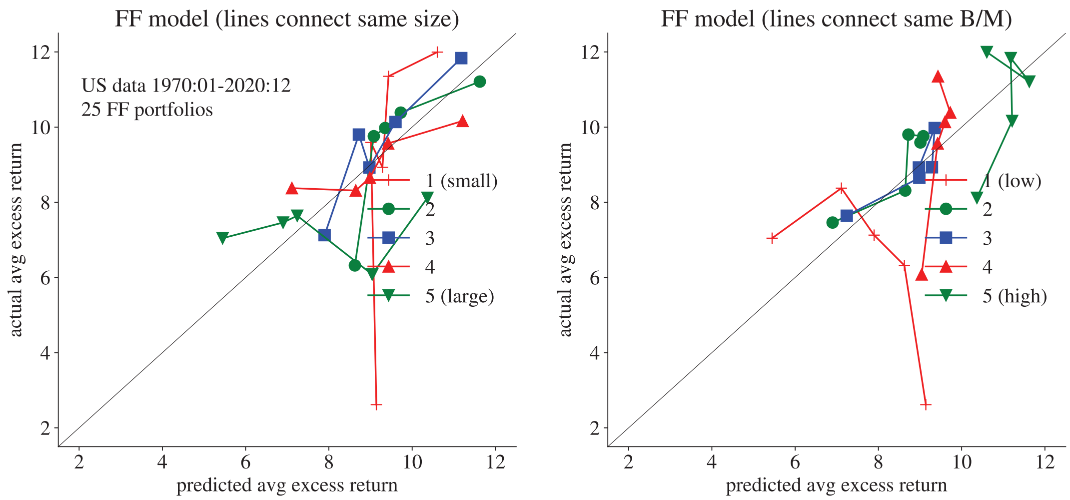


Figure 11.7: FF, FF portfolios

Example 11.22 *(APT with three assets) Suppose there are three well-diversified portfolios (that is, with no residual) with the following factor models*

$$\begin{aligned} R_{1,t} &= 0.01 + 1 f_t \\ R_{2,t} &= 0.01 + 0.25 f_t, \text{ and} \\ R_{3,t} &= 0.01 + 2 f_t. \end{aligned}$$

APT then holds if there is a portfolio with v_i invested in asset i, so that the cost of the portfolio is zero (which implies that the weights must be of the form v_1, v_2, and $-v_1 - v_2$ respectively) such that the portfolio has zero sensitivity to f_t, that is

$$\begin{aligned} 0 &= v_1 \times 1 + v_2 \times 0.25 + (-v_1 - v_2) \times 2 \\ &= v_1 \times (1-2) + v_2 \times (0.25 - 2) \\ &= -v_1 - v_2 \times 1.75. \end{aligned}$$

There is clearly an infinite number of such weights but they all obey the relation $v_1 = -v_2 \times 1.75$. Notice the requirement that there is no idiosyncratic volatility is (here) satisfied by assuming that none of the three portfolios have any idiosyncratic noise.

Example 11.23 *(APT with two assets) Example 11.22 would not work if we only had the first two assets. To see that, the portfolio would then have to be of the form $(v_1, -v_1)$ and* *it is clear that* $v_1 \times 1 - v_1 \times 0.25 = v_1(1 - 0.25) \neq 0$ *for any non-trivial portfolio (that is, with* $v_1 \neq 0$*).*

One of the main drawbacks with APT is that it is silent about both the number of factors and their definition. In many empirical implications, the factors—or the factor mimicking portfolios—are found by some kind of statistical method.

11.7.2 CAPM without a Riskfree Rate*

This section states the main result for CAPM when there is no riskfree asset. It uses two basic ingredients.

First, suppose investors behave as if they had mean-variance preferences, so they choose portfolios on the mean-variance frontier (of risky assets only). Different investors may have different portfolios, but they are all on the mean-variance frontier. The market portfolio is a weighted average of these individual portfolios, and therefore itself on the mean-variance frontier. (Linear combinations of efficient portfolios are also efficient.)

Second, consider the market portfolio. We know that we can find some other efficient portfolio (denote it R_z) that has a zero covariance (beta) with the market portfolio, $\text{Cov}(R_m, R_z) = 0$. (Such a portfolio can actually be found for any efficient portfolio, not just the market portfolio.) Let v_m be the portfolio weights of the market portfolio, and Σ the variance-covariance matrix of all assets. Then, the portfolio weights v_z that generate R_z must satisfy $v_m' \Sigma v_z = 0$ and $v_z' \mathbf{1} = 1$ (sum to unity). The intuition for how the portfolio weights of the R_z assets is that some of the weights have the same sign as in the market portfolio (contributing to a positive covariance) and some other have the opposite sign compared to the market portfolio (contributing to a negative covariance). Together, this gives a zero covariance.

See Figure 11.8 for an illustration.

The main result is then the "zero-beta" CAPM

$$\text{E}(R_i - R_z) = \beta_i \, \text{E}(R_m - R_z). \tag{11.49}$$

Proof. (*of (11.49)) An investor (with initial wealth equal to unity) chooses the portfolio weights (v_i) to maximize

$$\begin{aligned} \text{E}\, U(R_p) &= \text{E}\, R_p - \frac{k}{2} \text{Var}(R_p), \text{ where} \\ R_p &= v_1 R_1 + v_2 R_2 \text{ and } v_1 + v_2 = 1, \end{aligned}$$

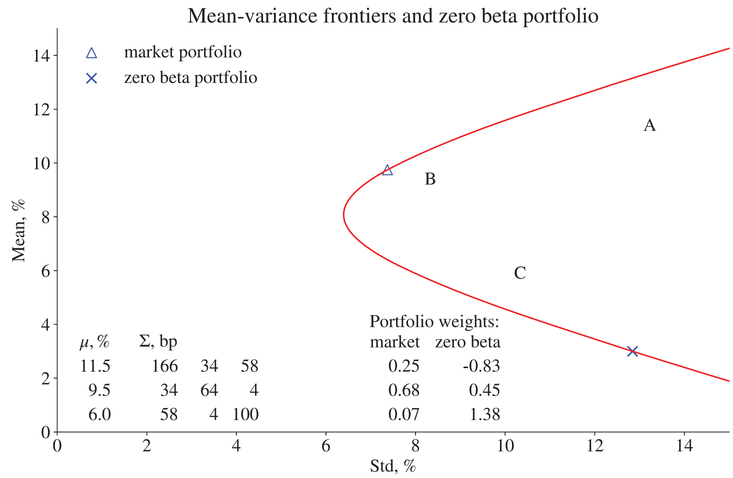


Figure 11.8: Zero-beta model

where we assume two risky assets. Combining gives the Lagrangian

$$L = v_1\mu_1 + v_2\mu_2 - \frac{k}{2}\left(v_1^2\sigma_{11} + v_2^2\sigma_{22} + 2v_1v_2\sigma_{12}\right) + \lambda(1 - v_1 - v_2).$$

The first order conditions (for v_1 and v_2) are that the partial derivatives equal zero

$$\begin{aligned}
0 &= \partial L/\partial v_1 = \mu_1 - k\,(v_1\sigma_{11} + v_2\sigma_{12}) - \lambda \\
0 &= \partial L/\partial v_2 = \mu_2 - k\,(v_2\sigma_{22} + v_1\sigma_{12}) - \lambda \\
0 &= \partial L/\partial \lambda = 1 - v_1 - v_2
\end{aligned}$$

Notice that

$$\sigma_{1m} = \mathrm{Cov}(R_1, \underbrace{v_1R_1 + v_2R_2}_{R_m}) = v_1\sigma_{11} + v_2\sigma_{12},$$

and similarly for σ_{2m}. We can then rewrite the first order conditions as

$$\begin{aligned}
0 &= \mu_1 - k\sigma_{1m} - \lambda \qquad \text{(a)} \\
0 &= \mu_2 - k\sigma_{2m} - \lambda \\
0 &= 1 - v_1 - v_2
\end{aligned}$$

Take a weighted average of the first two equations with the weights v_1 and v_2 respectively

$$\begin{aligned} v_1\mu_1 + v_2\mu_2 - \lambda &= k\,(v_1\sigma_{1m} + v_2\sigma_{2m}) \\ \mu_m - \lambda &= k\sigma_{mm}, \end{aligned} \tag{b}$$

which follows from the fact that

$$\begin{aligned} v_1\sigma_{1m} + v_2\sigma_{2m} &= v_1 \operatorname{Cov}(R_1, v_1R_1 + v_2R_2) + v_2 \operatorname{Cov}(R_2, v_1R_1 + v_2R_2) \\ &= \operatorname{Cov}(v_1R_1 + v_2R_2, v_1R_1 + v_2R_2) \\ &= \operatorname{Var}(R_m). \end{aligned}$$

Divide (a) by (b)

$$\begin{aligned} \frac{\mu_1 - \lambda}{\mu_m - \lambda} &= \frac{k\sigma_{1m}}{k\sigma_{mm}} \text{ or} \\ \mu_1 - \lambda &= \beta_1(\mu_m - \lambda) \end{aligned}$$

Applying this equation on a return R_z with a zero beta (against the market) gives.

$$\mu_z - \lambda = 0(\mu_m - \lambda), \text{ so we notice that } \lambda = \mu_z.$$

Combining the last two equations gives (11.49). ■