Innovative non-conformal finite element methods for augmented surgery

Internship presentation

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Introduction

Presentation of the teams

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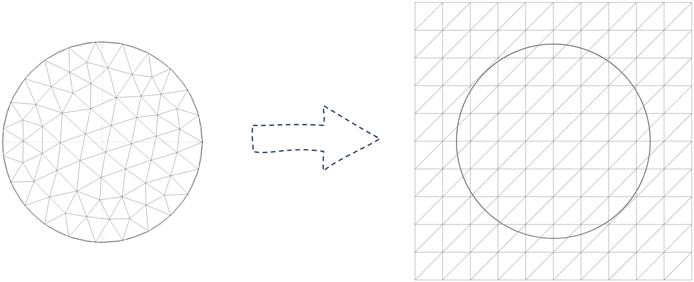
- project-team Inria affiliated with MLMS ("Machine Learning, Modélisation et Simulation") and with the Inria Nancy Center.
- **Aim :** to create real-time digital twins of an organ
- **Scientific challenges :**
  - scientific computing
  - data assimilation
  - machine learning
  - control
- **Main application domains :**
  - surgical training
  - surgical guidance during complex interventions

Scientific context

Abstract : Mimesis propose a new fictitious domain finite element method, the ϕ-FEM Method, given by a level-set function without requiring a mesh fitting the boundary.

- ➙ Fictitious domain methods allow to mesh complex geometries to ensure geometric quality (such as organs like the liver)
- ➙ Cartesian grid adapted for neural networks
- ➙ *Practical case:* Real-time simulation, form optimization



Objectives - Deliverables

Objective : Correct and certify the prediction of a Fourier Neural Operator (FNO)

Tools :

- the ϕ-FEM method (to be adapted for our purpose)
- the FNO implementation by Vincent Vigon with Tensorflow.

Deliverables :

- a weekly tracking report (in French)
- a github repository
- a report of the internship
- an online report generated with a tool called antora (made by a github CI)

Problem considered

Poisson problem with Dirichlet conditions :
Find $u : \Omega \to \mathbb{R}^d (d = 1, 2, 3)$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P})$$

with Δ the Laplace operator, Ω a smooth bounded open set and $\partial\Omega$ its boundary.
In this section, we defined by

$$||u_{ex} - u_{method}||_{0,\Omega}^{(rel)} = \frac{\int_\Omega (u_{ex} - u_{method})^2}{\int_\Omega u_{ex}^2}$$

the relative error between the exact solution u_{ex} and u_{method} a solution obtained by FEM or ϕ-FEM, a correction solver or the prediction of an neural network.

General methods and tools

Standard FEM method

ϕ-FEM method

Fourier Neural Operator (FNO)

General methods and tools

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Presentation of standard FEM method

Variational Problem :

$$\text{Find } u \in V \text{ such that } a(u,v) = l(v),\ \forall v \in V$$

where V is a Hilbert space, a is a bilinear form and l is a linear form.

Approach Problem :

$$\text{Find } u_h \in V_h \text{ such that } a(u_h,v_h) = l(v_h),\ \forall v_h \in V$$

with u_h an approximate solution in V_h, a finite-dimensional space dependent on h such that $V_h \subset V,\ dimV_h = N_h < \infty,\ (\forall h > 0)$

As $u_h = \sum_{i=1}^{N_h} u_i \varphi_i$ with $(\varphi_1, \ldots, \varphi_{N_h})$ a basis of V_h, finding an approximation of the PDE solution implies solving the following linear system:

$$AU = b$$

with

$$A = (a(\varphi_i, \varphi_j))_{1 \le i,j \le N_h}, \quad U = (u_i)_{1 \le i \le N_h} \quad \text{and} \quad b = (l(\varphi_j))_{1 \le j \le N_h}$$

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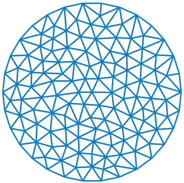
In practice

→ Construct a mesh of our Ω geometry with a family of elements (in 2D: triangle, rectangle; in 3D: tetrahedron, parallelepiped, prism) defined by

$$\mathcal{T}_h = \{K_1, \ldots, K_{N_e}\}$$

where N_e is the number of elements.

(Importance of geometric quality)



→ Construct a space of piece-wise affine continuous functions, defined by

$$V_h := P^k_{C,h} = \{v_h \in C^0(\bar{\Omega}), \forall K \in \mathcal{T}_h, v_{h|K} \in \mathbb{P}_k\}$$

where $\mathbb{P}_k$ is the vector space of polynomials of total degree less than or equal to k.

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Standard FEM method

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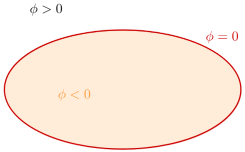
Problem

We pose $u = \phi w + g$ such that

$$\begin{cases} -\Delta u = f, \text{ on } \Omega, \\ u = g, \text{ in } \Gamma, \end{cases}$$

where ϕ is the level-set function and Ω and Γ are given by :

$$\Omega = \{\phi < 0\} \quad \text{and} \quad \Gamma = \{\phi = 0\}.$$



The level-set function ϕ is supposed to be known on $\mathbb{R}^d$ and sufficiently smooth. For instance, the signed distance to Γ is a good candidate

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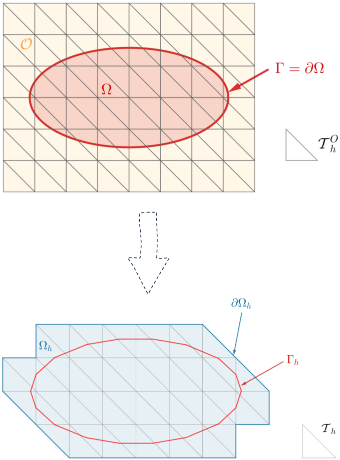
Fictitious domain



- $\mathcal{O}$: fictitious domain such that $\Omega \subset \mathcal{O}$
- $\mathcal{T}_h^{\mathcal{O}}$: simple quasi-uniform mesh on $\mathcal{O}$
- $\phi_h = I_{h,\mathcal{O}}^{(l)}(\phi) \in V_{h,\mathcal{O}}^{(l)}$: approximation of ϕ

  with $I_{h,\mathcal{O}}^{(l)}$ the standard Lagrange interpolation operator on

$$V_{h,\mathcal{O}}^{(l)} = \left\{v_h \in H^1(\mathcal{O}) : v_{h|_T} \in \mathbb{P}_l(T)\ \forall T \in \mathcal{T}_h^{\mathcal{O}}\right\}$$

- $\Gamma_h = \{\phi_h = 0\}$: approximate boundary of Γ
- $\mathcal{T}_h$: sub-mesh of $\mathcal{T}_h^{\mathcal{O}}$ defined by

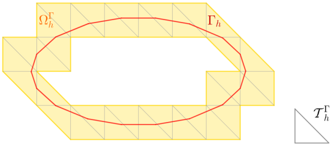
$$\mathcal{T}_h = \left\{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \{\phi_h < 0\} \neq \emptyset\right\}$$

- Ω_h : domain covered by the $\mathcal{T}_h$ mesh defined by

$$\Omega_h = \left(\cup_{T \in \mathcal{T}_h} T\right)^O$$

($\partial\Omega_h$ its boundary)

Facets and Cells sets

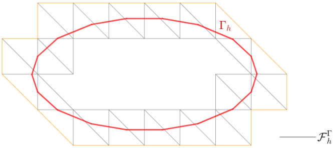


- $\mathcal{T}_h^\Gamma \subset \mathcal{T}_h$: contains the mesh elements cut by Γ_h, i.e.

$$\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma_h \neq \emptyset\},$$

- Ω_h^Γ : domain covered by the $\mathcal{T}_h^\Gamma$ mesh, i.e.

$$\Omega_h^\Gamma = \left(\cup_{T \in \mathcal{T}_h^\Gamma} T\right)^o$$



- $\mathcal{F}_h^\Gamma$: collects the interior facets of $\mathcal{T}_h$ either cut by Γ_h or belonging to a cut mesh element, i.e.

$$\mathcal{F}_h^\Gamma = \{E \text{ (an internal facet of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h : T \cap \Gamma_h \neq \emptyset \text{ and } E \in \partial T\}$$

Direct Method - Poisson problem

Approach Problem : Find $w_h \in V_h^{(k)}$ such that

$$a_h(w_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h^{(k)}$$

where

$$a_h(w, v) = \int_{\Omega_h} \nabla(\phi_h w) \cdot \nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w, v),$$

$$l_h(v) = \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v)$$

and

$$V_h^{(k)} = \left\{ v_h \in H^1(\Omega_h) : v_{h|_T} \in \mathbb{P}_k(T),\ \forall T \in \mathcal{T}_h \right\}.$$

For the non homogeneous case, we replace

$$u = \phi w \quad \rightarrow \quad u = \phi w + g$$

by supposing that g is currently given over the entire Ω_h.

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Stabilization terms

Independent parameter of h

Jump on the interface E

$$G_h(w,v) = \boxed{\sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right]} + \boxed{\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v)}$$

1st order term

$$G_h^{rhs}(v) = \boxed{-\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v)}$$

$-$

2nd order term

$$\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \left(\Delta(\phi_h w) + f\right)\Delta(\phi_h v)$$

1st term : Ghost penality [Burman, 2010], ensure continuity of the solution by penalizing gradient jumps.

2nd term : require the solution to verify the strong form on Ω_h^Γ.

Purpose :

- → reduce the errors created by the "fictitious" boundary
- → ensure the correct condition number of the finite element matrix
- → permit to restore the coercivity of the bilinear scheme

Dual method - Poisson Problem

Problem : Find u on Ω_h and p on Ω_h^Γ such that

$$\int_{\Omega_h} \nabla u \nabla v - \int_{\partial\Omega_h} \frac{\partial u}{\partial n} v + \frac{\gamma}{h^2} \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \left(u - \frac{1}{h}\phi p\right)\left(v - \frac{1}{h}\phi q\right)$$

$$+ G_h(u,v) = \int_{\Omega_h} fv + G_h^{rhs}(v), \;\; \forall v \text{ on } \Omega_h, \;\; q \text{ on } \Omega_h^\Gamma$$

with γ an other positive stabilization parameter and G_h and G_h^{rhs} the stabilization terms defined previously.
For the non homogeneous case, we replace

$$\int_T \left(u - \frac{1}{h}\phi p\right)\left(v - \frac{1}{h}\phi q\right) \quad \rightarrow \quad \int_T \left(u - \frac{1}{h}\phi p - g\right)\left(v - \frac{1}{h}\phi q\right)$$

by assuming g is defined on Ω_h^Γ

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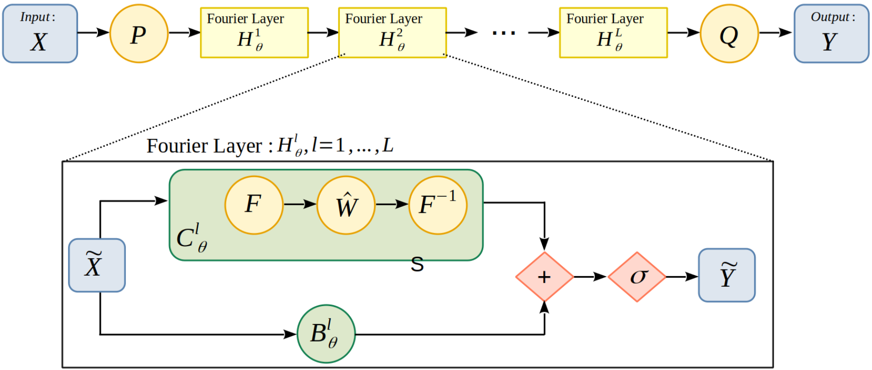
General methods and tools

Standard FEM method

ϕ-FEM method

Fourier Neural Operator (FNO)

Architecture of the FNO



Input *X* of shape (bs,ni,nj,nk) **Output** *Y* of shape (bs,ni,nj,1)
with bs the batch size, ni and nj the grid resolution and nk the number of channels.

Important points

- widely used in PDE solving and constitute an active field of research
- FNO are Neural Operator networks : Unlike standard neural networks, which learn using inputs and outputs of fixed dimensions, neural operators **learn operators, which are mappings between spaces of functions**.
- **Mesh resolution independent :** can be evaluated at any data resolution without the need for retraining

Objective

FNO Training :

Input : Train data .

$$X_{train} = \{ f_i, g_i, \phi_i \}$$
$$Y_{train} = \{ w_i \}$$
$$i = 1, \ldots, n_{data}$$

→

FNO training .

$$loss_\theta = \| \phi w_\theta - \phi w_{true} \|_{H^2(\Omega_h)}$$

→

Output .

$$Y_{pred} = \{ \phi_i w_{\theta,i} \}$$
$$i = 1, \ldots, n_{data}$$

Correction on the FNO predictions :

Input : Test data .

$$X_{test} = \{ f_i^{test}, g_i^{test}, \phi_i^{test} \}$$
$$i = 1, \ldots, n_{test}$$

→

FNO

$$Y_{test} = \{ \phi_i^{test} w_{\theta,i}^{test} \}$$

→

Correction :
- → *by adding*
- → *by multiplying*
- → *by multiplying on an elevated problem*

Correction

Problems considered

Methods considered

Numerical results

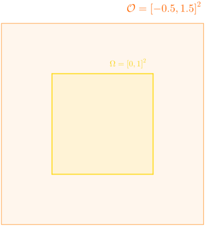
Correction

Problems considered

Methods considered

Numerical results

Trigonometric solution on a Square



- Level-set function :

$$\phi(x, y) = x(1 - x)y(1 - y)$$

- Analytical solution : (Homogeneous if $\varphi = 0$)

$$u_{ex}(x, y) = S \times sin\left(2\pi fx + \varphi\right) \times sin\left(2\pi fy + \varphi\right)$$

- Source term :

$$f(x, y) = 8\pi^2 Sf^2 sin\left(2\pi fx + \varphi\right) sin\left(2\pi fy + \varphi\right)$$

  - $S = 0.5 \in [0, 1]$: amplitude of the signal
  - $f \in \mathbb{N}$: "frequency" of the signal
  - $\varphi \in [0, 1]$: phase at the origin

		f = 1	f = 2	f = 3	f = 4
Homogeneous	**fem**	2.61e-03	1.04e-02	2.34e-02	4.12e-02
	phifem	1.84e-03	1.44e-02	3.69e-02	6.84e-02
Non-homogeneous	**fem**	2.30e-03	9.53e-03	2.19e-02	3.88e-02
	phifem	1.20e-04	5.74e-04	1.34e-03	2.39e-03

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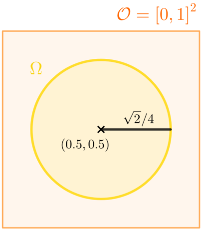
Unknown solution on a Circle



→ Level-set function :

$$\phi(x,y) = -1/8 + (x-1/2)^2 + (y-1/2)^2$$

→ Source term :

$$f(x,y) = exp\left(-\frac{(x-\mu_0)^2 + (y-\mu_1)^2}{2\sigma^2}\right)$$

- $\sigma \sim \mathcal{U}([0.1, 0.6])$
- $\mu_0, \mu_1 \sim \mathcal{U}([0.5 - \sqrt{2}/4, 0.5 + \sqrt{2}/4])$

with the condition $\phi(\mu_0, \mu_1) < -0.05$

→ a reference solution u_{ref} : over-refined $\mathbb{P}^1$ solution obtained with FEM

Correction

Problems considered

Methods considered

Numerical results

Correction by adding

We are given $\tilde{\phi}$ an "initial" solution to the problem under consideration.
We will consider

$$\tilde{u} = \tilde{\phi} + \tilde{C}$$

We want to find $\tilde{C} : \Omega \to \mathbb{R}^d$ solution to the problem

$$\begin{cases} -\Delta \tilde{u} = f, & \text{on } \Omega, \\ \tilde{u} = g, & \text{in } \Gamma. \end{cases}$$

with $\tilde{C} = \phi C$ for the ϕ-FEM method.
Rewriting the problem, we seek to find $\tilde{C} : \Omega \to \mathbb{R}^d$ solution to the problem

$$\begin{cases} -\Delta \tilde{C} = \tilde{f}, & \text{on } \Omega, \\ \tilde{C} = 0, & \text{in } \Gamma. \end{cases} \qquad (\mathcal{C}_+)$$

with $\tilde{f} = f + \Delta\tilde{\phi}$.
In practice, it may be useful to integrate by parts the term containing $\Delta\tilde{\phi}$.

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Correction by multiplying

We will considering

$$\tilde{u} = \tilde{\phi} C$$

We want to find $C : \Omega \to \mathbb{R}^d$ solution to the problem

$$\begin{cases} -\Delta(\tilde{\phi} C) = f, & \text{on } \Omega, \\ C = 1, & \text{on } \Gamma. \end{cases} \qquad (\mathcal{C}_\times)$$

In the non-homogeneous case, it is important to impose the boundary conditions either by the direct method or by the dual method.

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Correction by multiplying (elevated problem)

We introduced an initial modified problem : Find $\hat{u} : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} -\Delta \hat{u} = f, & \text{in } \Omega, \\ \hat{u} = g + m, & \text{on } \Gamma, \end{cases} \tag{$\mathcal{P}^{\mathcal{M}}$}$$

with $\hat{u} = u + m$ and m a constant.
We then apply the multiplication correction on the elevated problem by considering

$$\hat{\phi} = \tilde{\phi} + m$$

and so we look for $C : \Omega \to \mathbb{R}^d$ solution to the problem

$$\begin{cases} -\Delta(\hat{\phi} C) = f, & \text{in } \Omega, \\ C = 1, & \text{on } \Gamma. \end{cases} \tag{$\mathcal{C}_{\times}^{\mathcal{M}}$}$$

In the case of this correction, it is important to impose the boundary conditions either by the direct method, or by the dual method.

Theoretical results

We consider $\hat{u_{ex}} = u_{ex} + m$ the exact solution of the elevated problem $(\mathcal{P}^{\mathcal{M}})$ with u_{ex} the exact solution of the initial problem $(\mathcal{P})$.
Here, we are interested by $(\mathcal{C}_{\times}^{\mathcal{M}})$ with standard FEM on a disturbed solution defined by

$$\hat{\phi} = \tilde{\phi} + m = \hat{u_{ex}} + \epsilon P$$

with $\tilde{\phi} = u_{ex} + \epsilon P$.

- ➙ We have $||\hat{u_{ex}} - \hat{u_h}||_0 \leq ch^{k+1}||\hat{\phi}||_\infty \, |C|_{k+1}$
- ➙ With m sufficiently large, the error no longer depends on the solution but only on the perturbation P :

$$\left|\left|\frac{\hat{u_{ex}}}{\hat{\phi}} - C_h\right|\right|_{0,\Omega} \leq ch^{k+1}\epsilon\,||P''||_{0,\Omega}$$

- ➙ When m tends to infinity, the solution obtained with multiplication correction on an elevated problem $(\mathcal{C}_{\times}^{\mathcal{M}})$ converges to the solution obtained with correction by adding $(\mathcal{C}_{+})$.

Correction

Problems considered

Methods considered

Numerical results

Correction on exact solution

We consider the trigonometric solution on the square ($n_{vert} = 100$) and

$$\tilde{\phi} = u_{ex} \in \mathbb{P}_{10}$$

Results with FEM :

Homogeneous case :

	FEM	Corr_add	Corr_add_IPP	Corr_mult
f = 1	2.61e-03	5.16e-11	1.28e-13	2.48e-13
f = 2	1.04e-02	1.43e-11	1.27e-13	2.49e-13
f = 3	2.34e-02	8.07e-12	1.26e-13	2.49e-13
f = 4	4.12e-02	1.28e-11	1.24e-13	2.49e-13

Non homogeneous case :

	FEM	Corr_add	Corr_add_IPP	Corr_mult
f = 1	2.30e-03	4.74e-11	1.10e-13	1.72e-13
f = 2	9.53e-03	1.79e-11	1.15e-13	2.16e-13
f = 3	2.19e-02	1.19e-11	1.17e-13	2.29e-13
f = 4	3.88e-02	8.32e-12	1.18e-13	2.35e-13

Results with ϕ-FEM :

Homogeneous case :

	PhiFEM	Corr_add	Corr_add_IPP	Corr_mult
f = 1	1.84e-03	2.21e-11	2.21e-13	3.08e-13
f = 2	1.44e-02	8.80e-12	4.77e-12	4.80e-12
f = 3	3.69e-02	1.30e-10	1.29e-10	1.73e-10
f = 4	6.84e-02	1.27e-09	1.27e-09	2.02e-09

Non homogeneous case :

	PhiFEM	Corr_add	Corr_add_IPP	Corr_mult
f = 1	1.20e-04	1.42e-11	2.96e-13	2.02e-11
f = 2	5.74e-04	9.83e-12	4.45e-12	2.90e-11
f = 3	1.34e-03	1.16e-10	1.15e-10	1.78e-10
f = 4	2.39e-03	1.13e-09	1.14e-09	1.46e-09

Remark : For the addition, we hope $C = 0$ and for the multiplication $C = 1$ (on Ω).

Correction on disturbed solution I

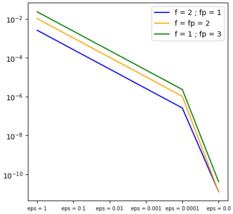
We consider the trigonometric solution on the square ($n_{vert} = 100$ and $\varphi = 0$) and

$$\tilde{\phi} = u_{ex} + \epsilon P \in \mathbb{P}_{10}$$

where ϵ a real number and P the perturbation (same expression as u_{ex} with (S_p, f_p, φ_p) parameters). Consider $\varphi_p = 0$ so that $P = 0$ on Γ and therefore $\tilde{\phi} = 0$ on Γ.

1st result : ϵ convergence.

Correction by adding with FEM :



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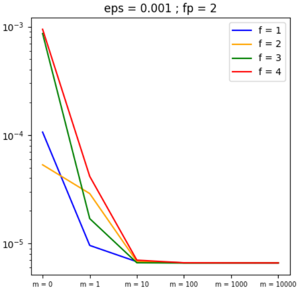
Correction on disturbed solution II

2nd result : Importance of perturbation form.

We fix $f_p = 2$ and $\epsilon = 0.001$.

Correction by multiplying on an elevated problem with FEM :

	m = 0	m = 1	m = 10	m = 100	m = 1000	m = 10000
f = 1	1.06e-04	9.60e-06	6.79e-06	6.64e-06	6.62e-06	6.62e-06
f = 2	5.32e-05	2.89e-05	6.82e-06	6.63e-06	6.62e-06	6.62e-06
f = 3	8.65e-04	1.70e-05	6.64e-06	6.60e-06	6.62e-06	6.62e-06
f = 4	9.46e-04	4.16e-05	7.02e-06	6.64e-06	6.62e-06	6.62e-06



Correction on disturbed solution III

3rd result : Convergence of the correction by multiplying on an elevated problem.

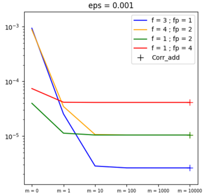
Correction by multiplying on the elevated problem with FEM and ϕ-FEM :

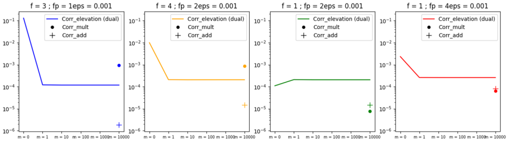
Standard FEM method :

		m = 0	m = 1	m = 10	m = 100	m = 1000	m = 10000	Corr_add
f > fp	f = 3 ; fp = 1	9.40e-04	2.54e-05	2.82e-06	2.61e-06	2.61e-06	2.61e-06	2.61e-06
	f = 4 ; fp = 2	8.82e-04	3.47e-05	1.06e-05	1.04e-05	1.04e-05	1.04e-05	1.04e-05
f < fp	f = 1 ; fp = 2	3.98e-05	1.13e-05	1.04e-05	1.04e-05	1.04e-05	1.04e-05	1.04e-05
	f = 1 ; fp = 4	7.39e-05	4.15e-05	4.12e-05	4.12e-05	4.12e-05	4.12e-05	4.12e-05

Φ–FEM method :

		Corr_mult	m = 0	m = 1	m = 10	m = 100	m = 1000	m = 10000	Corr_add
f > fp	f = 3 ; fp = 1	9.41e-04	1.30e-01	1.21e-04	1.19e-04	1.19e-04	1.19e-04	1.19e-04	1.86e-06
	f = 4 ; fp = 2	8.86e-04	9.99e-03	2.09e-04	2.07e-04	2.07e-04	2.07e-04	2.07e-04	1.50e-05
f < fp	f = 1 ; fp = 2	7.67e-06	1.10e-04	2.10e-04	2.07e-04	2.07e-04	2.07e-04	2.07e-04	1.51e-05
	f = 1 ; fp = 4	6.38e-05	2.33e-03	2.61e-04	2.60e-04	2.60e-04	2.60e-04	2.60e-04	8.11e-05





Remark : For ϕ-FEM, we do not find the same type of results.

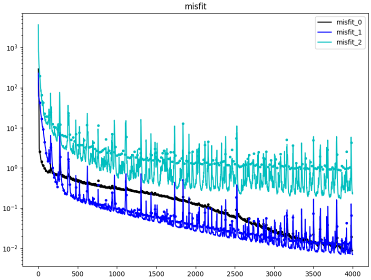
Correction on a FNO prediction I

We consider the unknown solution on the circle (with f Gaussian) and

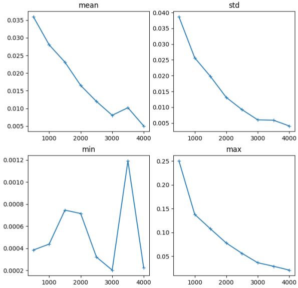
$$\tilde{\phi} = u_{FNO} \in \mathbb{P}_2$$

where u_{FNO} is the FNO prediction ($n_{vert} = 32$).

Training on 4000 epochs (bs=64,lr=0.01) :

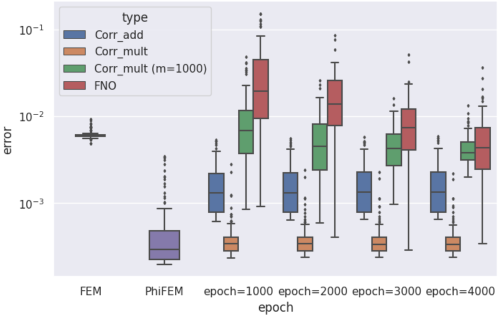


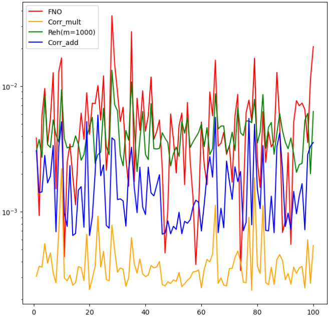
Test sample ($n_{test} = 100$, $n_{vert} = 32$) :



Correction on a FNO prediction II

We will apply the different correction methods on the FNO prediction on the test sample.





Remark : We should try to reduce the resolution for correction, maybe we will gain in the time-to-error ratio.

Correction with other networks I

Idea : using a neural network to predict a single solution at any point in the domain

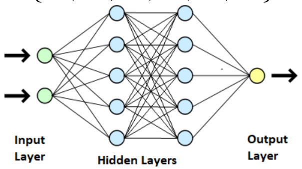
input data = a collection of n_{pts} 2D points : $\{(x_i, y_i)\}_{i=1,\dots,n_{pts}}$

output = solution at each of these points : $\{u_i\}_{i=1,\dots,n_{dots}}$

with $u_i = \phi(x_i, y_i)w_i$ and $w_i = w_\theta(x_i, y_i)$.

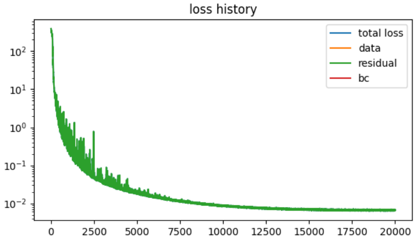
We try with a Physics-Informed Neural Networks (PINNs).

model = MLP with 6-layer network ($\{10, 20, 60, 60, 20, 10\}$)



$$loss = mse(\Delta(\phi(x_i, y_i)w_{\theta,i}) + f_i)$$

Training over 20000 epochs ($n_{test} = 20000$) :



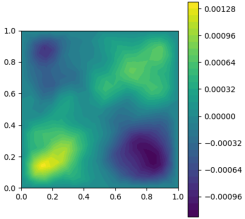
Correction with other networks II

We consider the trigonometric solution on the square and $\tilde{\phi} = u_{PINNs} \in \mathbb{P}_{10}$ where u_{PINNs} is the PINNs prediction ($n_{vert} = 32$) and we have

$$||u_{ex} - \tilde{\phi}||_{0,\Omega}^{(rel)} = 1.93e-3.$$

We will correct this solution with the correction by adding (with FEM and ϕ-FEM).

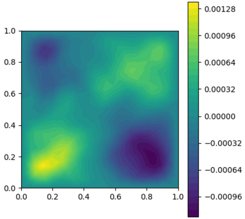
Standard FEM method :



$$||u_{ex} - \tilde{\phi}C||_{0,\Omega}^{(rel)} = 1.13e-4$$

divided by 246.60 (FEM : 2.80e-2)

ϕ-FEM method :



$$||u_{ex} - \tilde{\phi}C||_{0,\Omega}^{(rel)} = 1.27e-4$$

divided by 151.34 (ϕ-FEM : 1.92e-2)

MIMESIS

Conclusion

Conclusion

- obtain numerical results on analytical solutions $\Rightarrow$ correction methods considered functional and theoretical results confirmed
- correction methods on the FNO predictions not satisfactory
- try to increase the degree of the solution :
  - Legendre and MLP not satisfactory
  - PINNs : reduction of the error made by conventional methods (FEM and ϕ-FEM) by a factor of around 100 (correction by addition).

Perspectives :

- try adding PINNs to the output of the FNO (add the PINNs as a layer output that would replace the decomposition into a series of polynomials) $\Rightarrow$ solution at any point in the domain
- carry out some documentation work to find more suitable models than the FNO
- consider more complex and time-varying geometries (such as 3D organ geometries)

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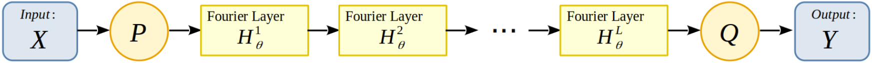
Description of the FNO architecture



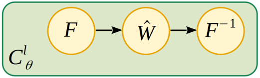
- perform a P transformation, to move to a space with more channels (to build a sufficiently rich representation of the data)
- apply L Fourier layers defined by

$$\mathcal{H}^l_\theta(\tilde{X}) = \sigma\left(\mathcal{C}^l_\theta(\tilde{X}) + \mathcal{B}^l_\theta(\tilde{X})\right), \; l = 1, \ldots, L$$

  with $\tilde{X}$ the input of the current layer and
  - σ an activation function (ReLU or GELU)
  - $\mathcal{C}^l_\theta$: convolution sublayer (convolution performed by Fast Fourier Transform)
  - $\mathcal{B}^l_\theta$: "bias-sublayer"
- return to the target dimension by performing a Q transformation (in our case, the number of output channels is 1)

Fourier Layer Structure

Convolution sublayer : $\mathcal{C}^l_\theta(X) = \mathcal{F}^{-1}(\mathcal{F}(X) \cdot \hat{W})$



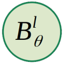
- $\hat{W}$: a trainable kernel
- $\mathcal{F}$: 2D Discrete Fourier Transform (DFT) defined by

$$\mathcal{F}(X)_{ijk} = \frac{1}{ni}\frac{1}{nj}\sum_{i'=0}^{ni-1}\sum_{j'=0}^{nj-1} X_{i'j'k} e^{-2\sqrt{-1}\pi\left(\frac{ii'}{ni}+\frac{jj'}{nj}\right)}$$

$\mathcal{F}^{-1}$: its inverse.

- $(Y \cdot \hat{W})_{ijk} = \sum_{k'} Y_{ijk'}\hat{W}_{ijk'} \quad \Rightarrow \quad$ applied channel by channel

Bias-sublayer : $\mathcal{B}^l_\theta(X)_{ijk} = \sum_{k'} X_{ijk}W_{k'k} + B_k$



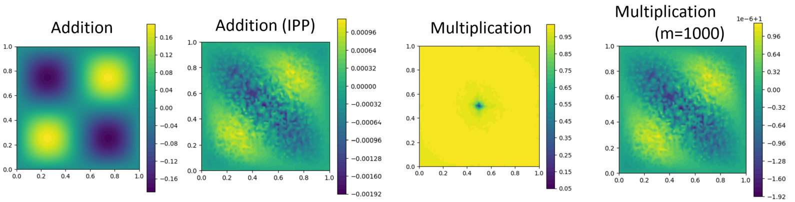
- 2D convolution with a kernel of size 1
- allowing channels to be mixed via a kernel without allowing interaction between pixels.

Correction on a ϕ-FEM solution

We consider the trigonometric solution on the square and

$$\tilde{\phi} = u_{\phi-FEM} \in \mathbb{P}_2$$

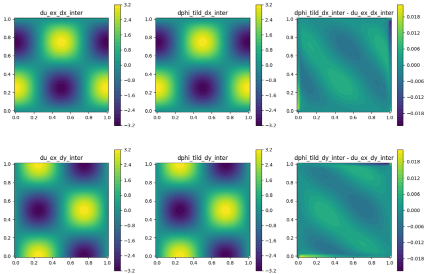
where $u_{\phi-FEM}$ is the solution obtained with ϕ-FEM ($n_{vert} = 32$).
We will correct this solution with the different correction methods by using FEM.
We obtain the following correction terms ($\tilde{C}$ for addition and C for multiplication) :



	Add	Add (IPP)	Mult	Mult (m=1000)
L2_rel	3.64e-01	1.03e-03	1.91e-03	1.03e-03

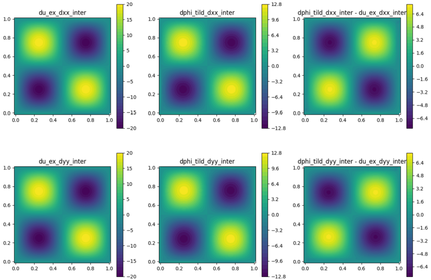
Correction on a ϕ-FEM solution - Derivatives

We compute the first and second derivatives of $\tilde{\phi}$ according to *x* and *y*.



1st derivatives

2nd derivatives



We can see that the second derivatives are quite far from the true derivatives.