

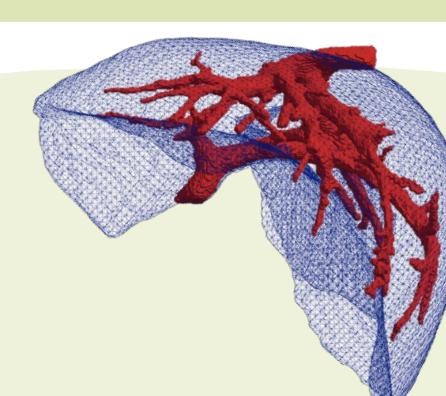
COMBINING FINITE ELEMENT METHODS AND NEURAL NETWORKS TO SOLVE ELLIPTIC PROBLEM ON COMPLEX 2D GEOMETRIES

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MIMESIS

Scientific context

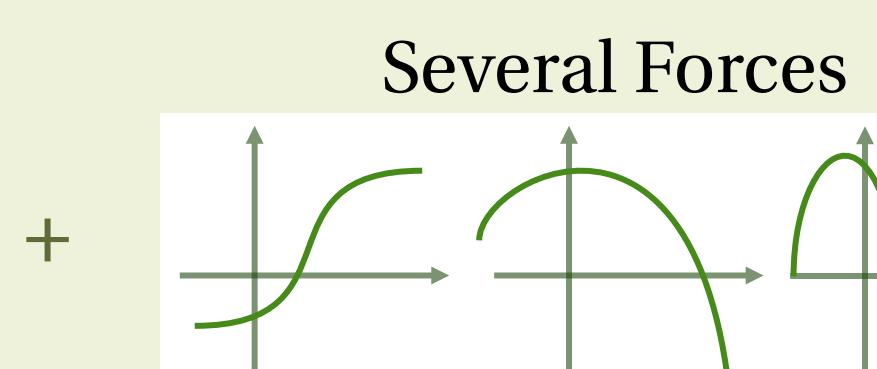
Create real-time twins of an organ (e.g. liver)



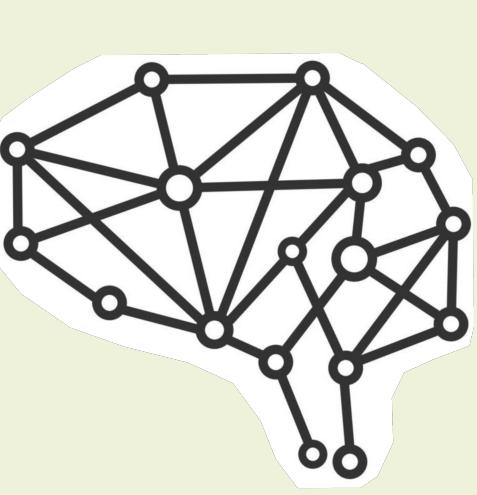
Current Objective : Develop hybrid **finite element / neural network** methods.
accurate quick + parameterized

OFFLINE

Several Geometries

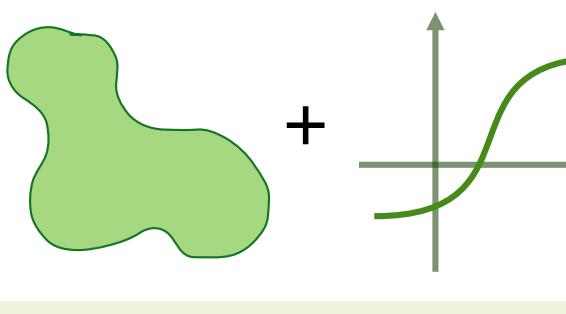


Train a PINNs

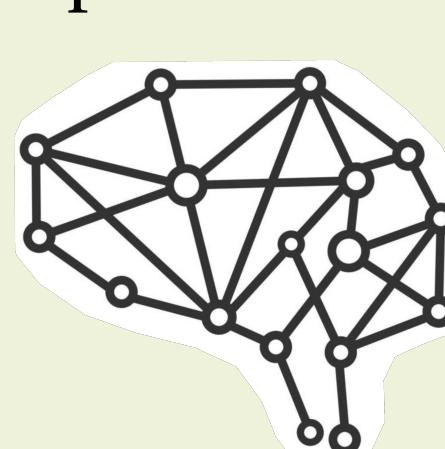


ONLINE

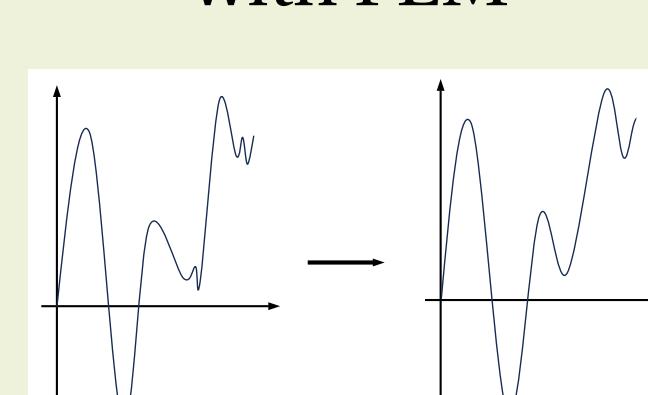
1 Geometry - 1 Force



Get PINNs prediction



Correct prediction with FEM



Poisson problem with Dirichlet boundary conditions

Find $u : \Omega \rightarrow \mathbb{R}^d$ ($d = 1, 2, 3$) such that

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \Gamma \end{cases} \quad (\mathcal{P})$$

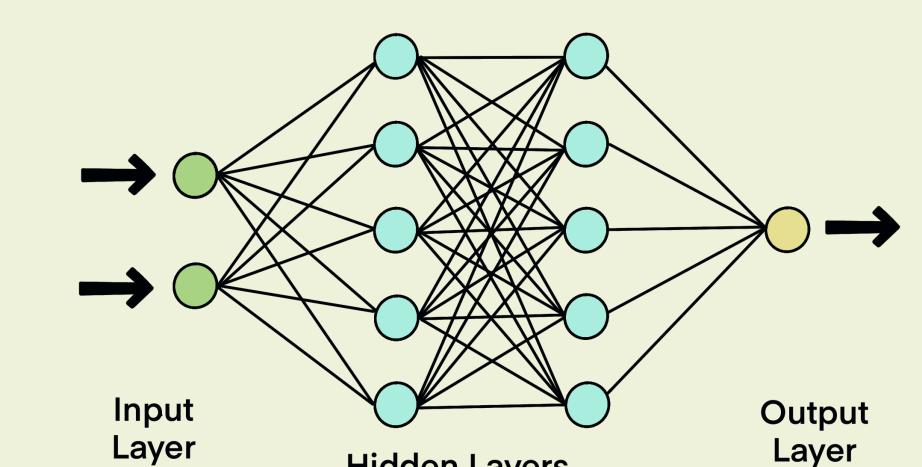
with Δ the Laplace operator, Ω a smooth bounded open set and Γ its boundary.

PINNs - Physics-Informed Neural Networks [RPK19]

Implicit neural representation.

$$u_\theta(x) = u_{NN}(x)$$

with u_{NN} a neural network (e.g. a MLP).



DoFs Minimization Problem.

Considering the least-square form of (\mathcal{P}) , our discrete problem is

$$\bar{\theta} = \underset{\theta \in \mathbb{R}^m}{\operatorname{argmin}} \alpha J_{\text{in}}(\theta) + \beta J_{\text{bc}}(\theta)$$

with m the number of parameters of the NN and

$$J_{\text{in}}(\theta) = \frac{1}{2} \int_{\Omega} (\Delta u_\theta + f)^2 \quad \text{and} \quad J_{\text{bc}}(\theta) = \frac{1}{2} \int_{\partial\Omega} (u_\theta - g)^2$$

Finite Element methods

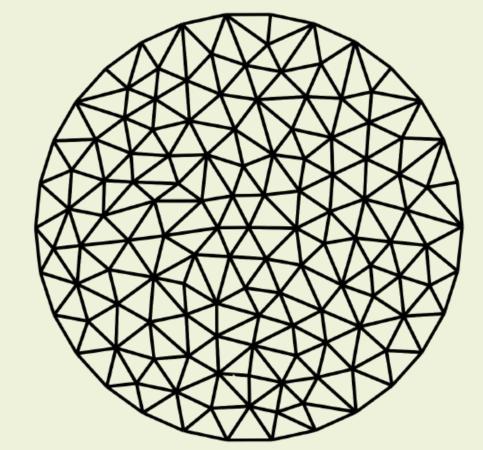
Variational Problem.

Let's define V as a Hilbert space.

$$\text{Find } u \in V \text{ such that, } \forall v \in V, a(u, v) = l(v)$$

with a a bilinear form and l a linear form defined as

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{and} \quad l(v) = \int_{\Omega} f v.$$



Approach Problem.

Let's define V_h as a finite-dimensional subspace of V ($V_h \subset V$).

$$\text{Find } u_h \in V_h \text{ such that, } \forall v_h \in V_h, a(u_h, v_h) = l(v_h)$$

with $u_h \in V_h$ an approximate solution of u and $N_h = \dim(V_h)$.

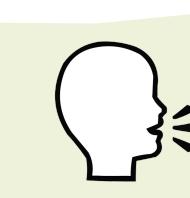
Linear System.

Let's define $\{\varphi_1, \dots, \varphi_{N_h}\}$ a basis of V_h .

$$AU = b$$

with

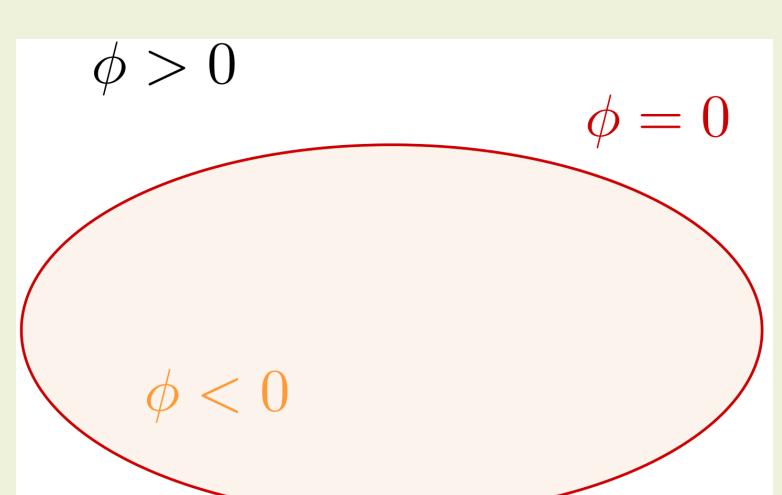
$$A = (a(\varphi_i, \varphi_j))_{1 \leq i, j \leq N_h}, \quad U = (u_i)_{1 \leq i \leq N_h} \quad \text{and} \quad b = (l(\varphi_j))_{1 \leq j \leq N_h}$$



No mesh, so easy to go on complex geometry!

How to deal with complex geometry in PINNs ?

Approach by levelset. [SS22]



Advantages :

- Sample is easy in this case.
 - Allow to impose hard BC (no more J_{bc}):
- $$u_\theta(X) = \phi(X) w_\theta(X) + g(X)$$
- with ϕ a levelset function and w_θ a NN.

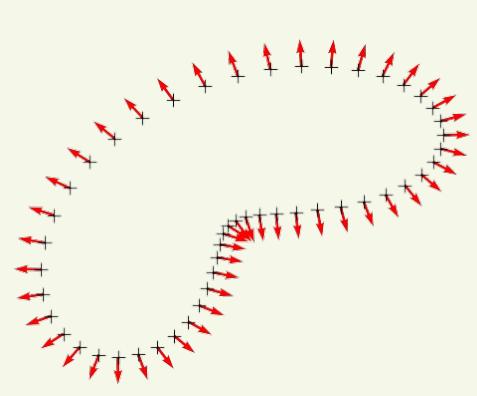
Levelset considered.

A regularized Signed Distance Function (SDF).

Theorem 1: Eikonal equation.

If we have a boundary domain Γ , the SDF is solution to:

$$\begin{cases} \|\nabla \phi(X)\| = 1, X \in \mathcal{O} \\ \phi(X) = 0, X \in \Gamma \\ \nabla \phi(X) = n, X \in \Gamma \end{cases}$$

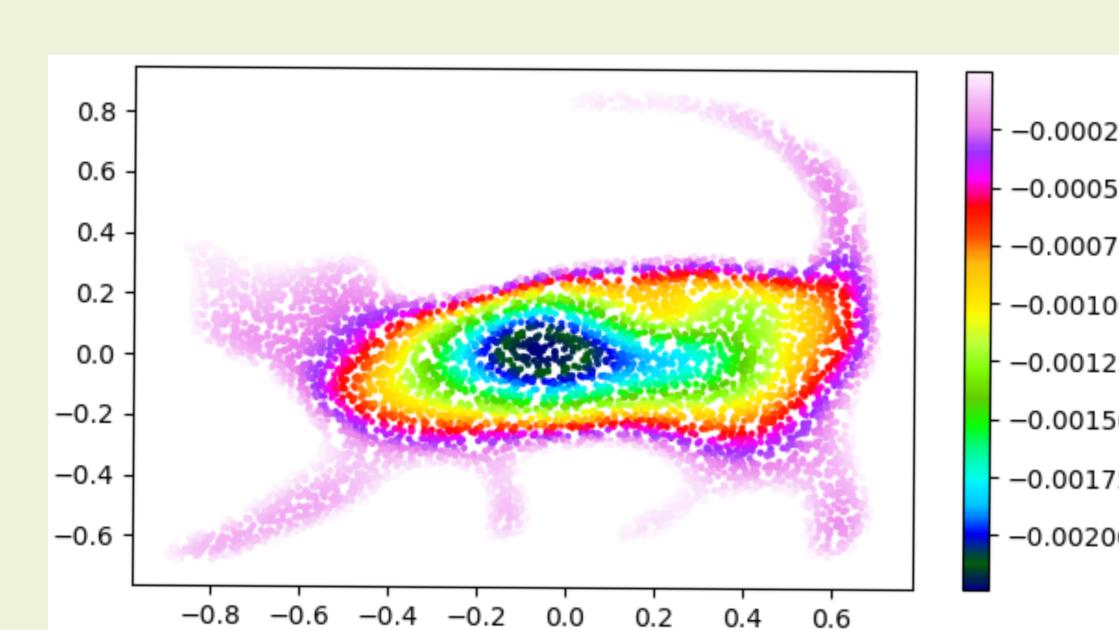
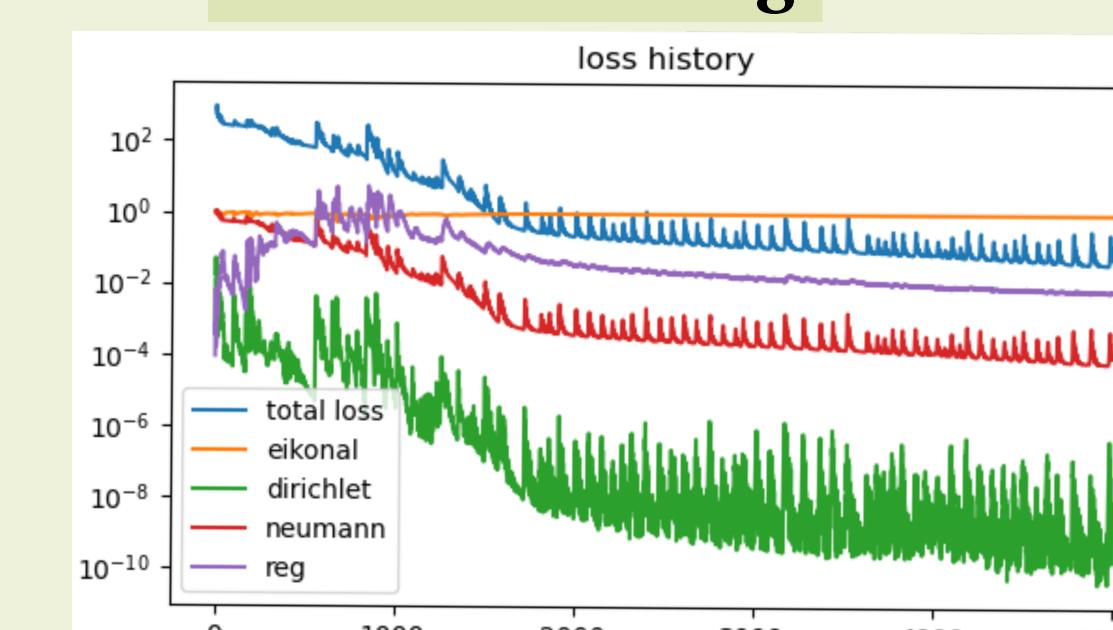


with \mathcal{O} a box which contains Ω completely and n the exterior normal to Γ .

How to do that? with a PINNs [CD23], by adding the following regularization term

$$J_{\text{reg}} = \int_{\mathcal{O}} |\Delta \phi|^2.$$

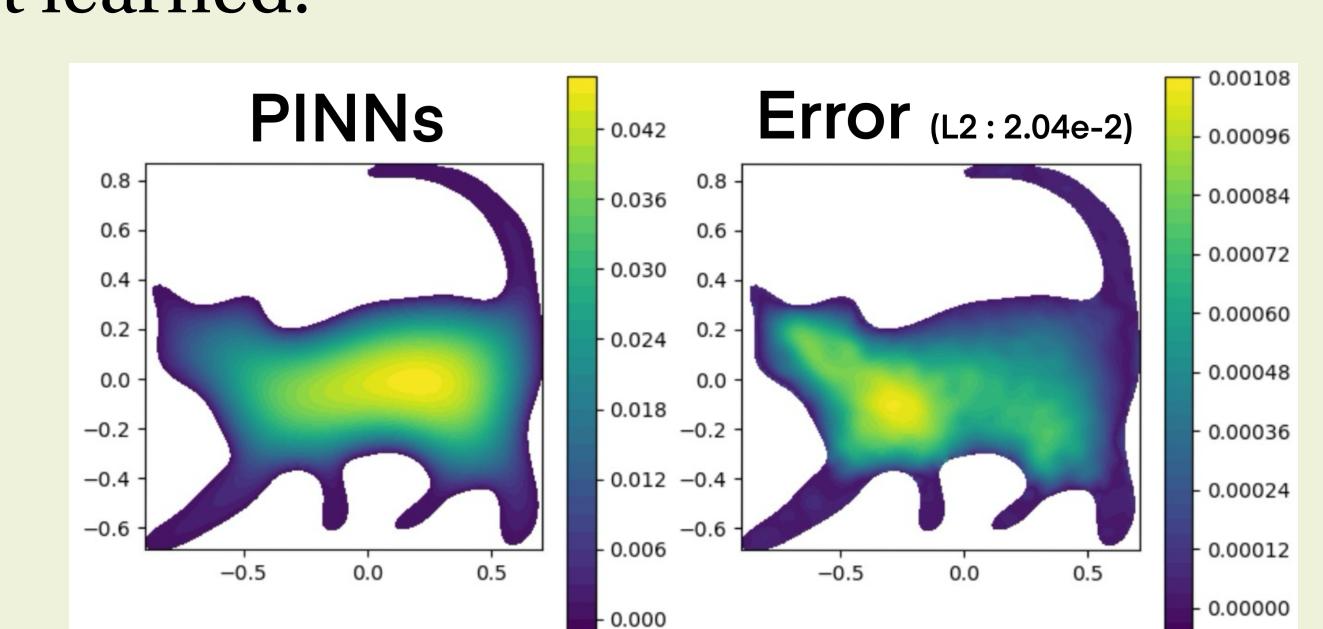
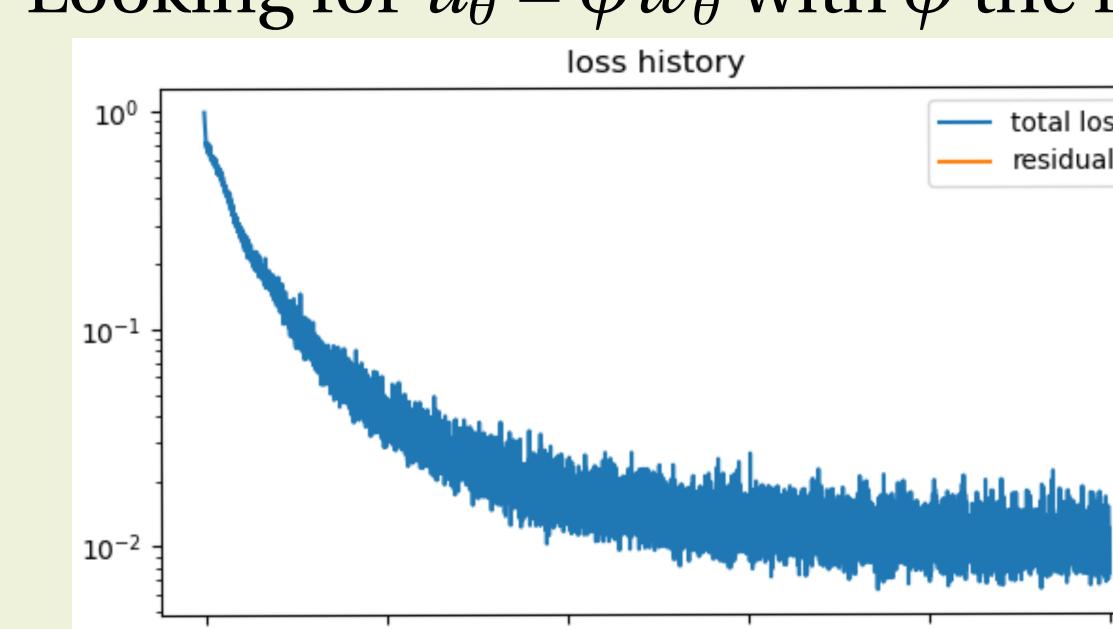
Result : Levelset learning.



Result : Poisson on Cat.

→ Solving (\mathcal{P}) with $f = 1$ (non parametric) and homogeneous Dirichlet BC ($g = 0$).

→ Looking for $u_\theta = \phi w_\theta$ with ϕ the levelset learned.



How can we improve PINNs prediction ? - Using FEM-type methods

Additive approach. Considering u_{NN} as the prediction of our PINNs for (\mathcal{P}) , the correction problem consists in writing the solution as

$$\tilde{u} = u_\theta + \tilde{C}$$

and searching $\tilde{C} : \Omega \rightarrow \mathbb{R}^d$ such that

$$\begin{cases} -\Delta \tilde{C} = \tilde{f}, & \text{in } \Omega, \\ \tilde{C} = 0, & \text{on } \Gamma, \end{cases} \quad (\mathcal{P}^+)$$

with $\tilde{f} = f + \Delta u_\theta$.

Problem considered.

Poisson on Square

- Solving (\mathcal{P}) with homogeneous Dirichlet BC ($g = 0$) and $\Omega = [-0.5\pi, 0.5\pi]^2$.
- Analytical levelset function: $\phi(x, y) = (x - 0.5\pi)(x + 0.5\pi)(y - 0.5\pi)(y + 0.5\pi)$

→ Analytical solution:

$$u_{\text{ex}}(x, y) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y)$$

with $\mu_1, \mu_2 \in [-0.5, 0.5]$ (parametric).

Theoretical results.

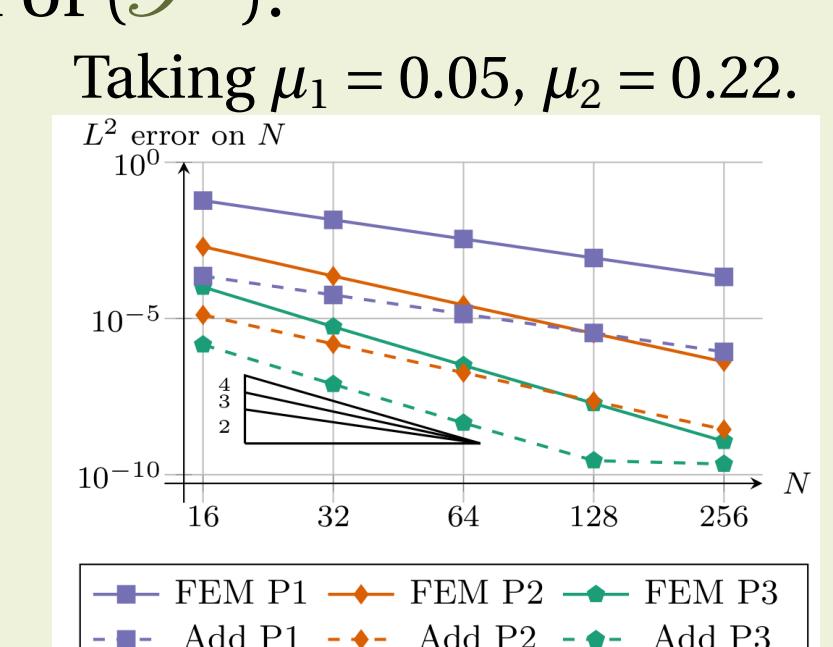
We denote u the solution of (\mathcal{P}) and u_h the discrete solution of (\mathcal{P}^+) .

Theorem 2: [Lec+ss]

Considering V_h as a \mathbb{P}_k Lagrange space, we have

$$\|u - u_h\|_0 \lesssim \frac{\|u - u_\theta\|_{H^{k+1}}}{\|u\|_{H^{k+1}}} h^{k+1} \|u\|_{H^{k+1}}$$

Remark: We note N the number of nodes in each direction of the square.



Result : Gains using additive approach.

Considering a set of $n_p = 50$ parameters: $\{(\mu_1^{(1)}, \mu_2^{(1)}), \dots, (\mu_1^{(n_p)}, \mu_2^{(n_p)})\}$.

Solution \mathbb{P}_1	Gains on PINNs				Gains on FEM					
	N	min	max	mean	std	N	min	max	mean	std
	20	15.7	48.35	33.64	5.57	134.31	377.36	269.4	43.67	
	40	61.47	195.75	135.41	23.21	131.18	362.09	262.12	41.67	

[CD23] Clémot and Digne. "Neural skeleton: Implicit neural representation away from the surface". In: *Computers and Graphics* (2023).

[Lec+ss] Lecourtier et al. "Enhanced finite element methods using neural networks". In: (in progress).

[RPK19] Raissi, Perdikaris, and Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *Journal of Computational Physics* (2019).

[SS22] Sukumar and Srivastava. "Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks". In: *Computer Methods in Applied Mechanics and Engineering* (2022).