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Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries

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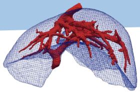
Inria MIMESIS

Introduction

Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.

finite element: accurate — neural network: quick + parameterized

Parametric linear elliptic PDE : For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{L}\big(u; \boldsymbol{x}, \boldsymbol{\mu}\big) = f(\boldsymbol{x}, \boldsymbol{\mu}),$$

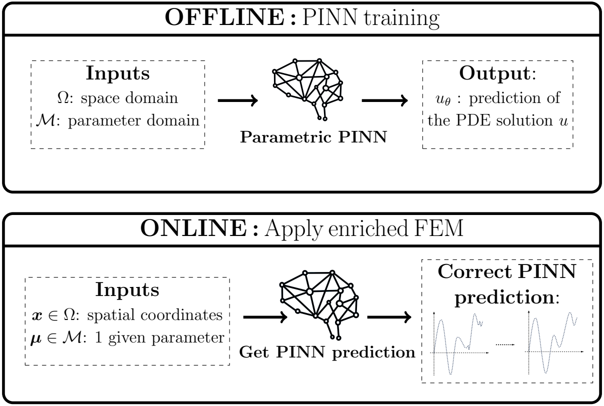
where $\mathcal{L}$ is the parametric differential operator defined by

$$\mathcal{L}(\cdot; \boldsymbol{x}, \boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x}, \boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\mathrm{Pe}} \nabla \cdot (D(\boldsymbol{x}, \boldsymbol{\mu}) \nabla u).$$

Ω	Spatial domain
d	Spatial dimension
$\boldsymbol{x} = (x_1, \ldots, x_d)$	Spatial coordinates
$\mathcal{M}$	Parameter space
p	Number of parameters
$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$	Parameter vector

f	Right-hand side
R	Reaction coefficient
C	Convection coefficient
D	Diffusion matrix
Pe	Péclet number

Pipeline of the Enriched FEM



Correction : Enriched continuous Lagrange finite element approximation spaces using the PINN prediction.

Physics-Informed Neural Networks

TODO

Finite Element Method

TODO

How improve PINN prediction with FEM ?

Additive approach

TODO

Theorerical results

TODO

Numerical results - 2D Poisson problem

2D Poisson problem

TODO

Numerical results - 2D anysotropic Elliptic problem

2D anysotropic Elliptic problem

TODO

Conclusion

Conclusion

TODO

References

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Appendix

Appendix 1 : Standard FEM

Appendix 1 : General Idea

TODO