How to work with complex geometries in PINNs ?

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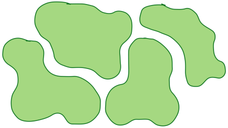
Problem considered



Poisson problem with Dirichlet conditions :
Find $u : \Omega \to \mathbb{R}^d (d = 1, 2, 3)$ such that

$$\begin{cases} -\Delta u(X) = f(X) & \text{in } \Omega, \\ u(X) = g(X) & \text{on } \partial\Omega \end{cases}$$

with Δ the Laplace operator, Ω a smooth bounded open set and Γ its boundary.
For the following examples, we will consider $f(X) = 1$ and $g(X) = 0$.

Standard PINNs : We are looking for θ_u such that

$$\theta_u = \operatorname*{argmin}_{\theta} w_r\, J_r(\theta) + w_{bc}\, J_{bc}(\theta)$$

where w_r and w_{bc} are the respective weights associated with

$$J_r = \int_\Omega (\Delta u + f)^2 \quad \text{and} \quad J_{bc} = \int_{\partial\Omega} (u - g)^2.$$

Remark : In practice, we use a Monte-Carlo method to discretize the cost function by random process.

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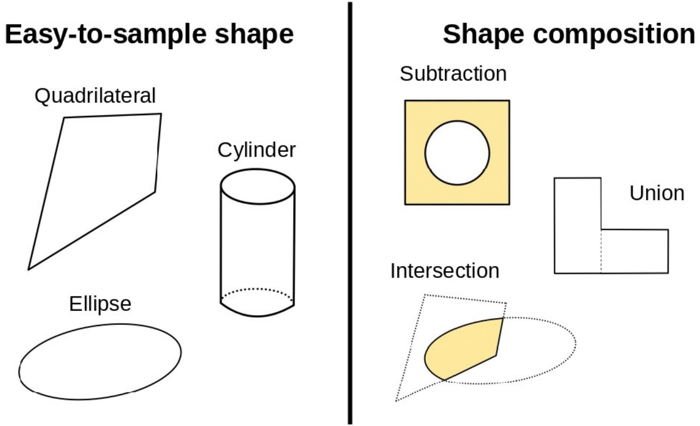
Simple geometry

Claim on PINNs : No mesh, so easy to go on complex geometry !



In practice : Not so easy ! We need to find how to sample in the geometry.

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Complex geometry

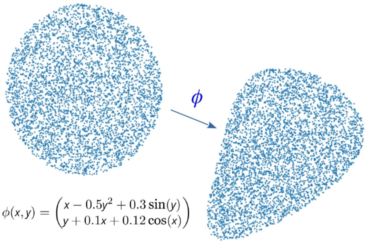
1st approach : Mapping	**2nd approach :** LevelSet function

1st approach : Mapping

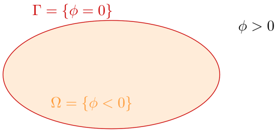
Idea :

- ➞ Ω_0 a simple domain (as circle)
- ➞ Ω a target domain
- ➞ A mapping from Ω_0 to Ω :

$$\Omega = \phi(\Omega_0)$$



2nd approach : LevelSet function



Advantages :

- ➞ Sample is easy in this case.
- ➞ Allow to impose in hard the BC :

$$u_\theta(X) = \phi(X) w_\theta(X) + g(X)$$

Natural LevelSet :

Signed Distance Function (SDF)

Problem : SDF is a $\mathcal{C}^0$ function

$\Rightarrow$ its derivatives explodes

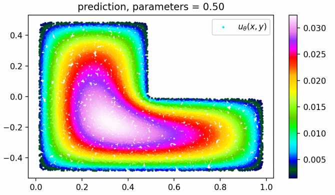
$\Rightarrow$ we need a regular levelset

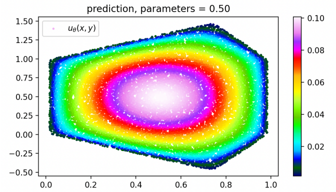
Construct smooth SDF I

1st solution : Approximation theory [4]

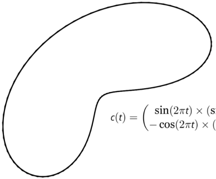
$\Delta\phi$ can be singular at the boundary. Sampling at ϵ to it solve the problem.

Polygonal domain Appendix 1	**Curved domain** Appendix 2





Minus : Use of a parametric curve $c(t)$.



$$c(t) = \begin{pmatrix} \sin(2\pi t) \times (\sin(2\pi t)^3 + \cos(2\pi t)^5) \\ -\cos(2\pi t) \times (\sin(2\pi t)^3 + \cos(2\pi t)^5) \end{pmatrix}$$

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Construct smooth SDF II

2nd solution : Learn the levelset. [1]

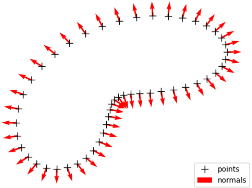
➙ How make that ? with a PINNs.

If we have a boundary domain Γ, the SDF is solution to the Eikonal equation:

$$\begin{cases} ||\nabla\phi(X)|| = 1, \ X \in \mathcal{O} \\ \phi(X) = 0, \ X \in \Gamma \\ \nabla\phi(X) = n, \ X \in \Gamma \end{cases}$$

with $\mathcal{O}$ a box which contains Ω completely and n the exterior normal to Γ.

Advantage : No need for parametric curves.



➙ set of boundary points

➙ exterior normals at Γ

(evaluated at this points)

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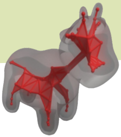
Learn LevelSet I



Objective of the paper :

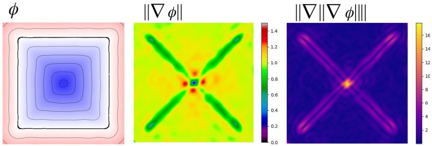
Learn topological Skeleton (by learning SDF) Appendix 3

→ Skeleton correspond exactly to the gradient singularity

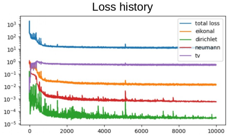
→ Adding the following term in the loss

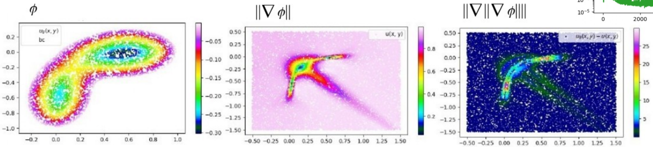
$$\int_{\mathcal{O}} ||\nabla||\nabla\phi||(p)||dp$$

(Total Variation Regularization)



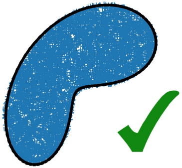
1st test : Eikonal equation with TV Regularization [1]



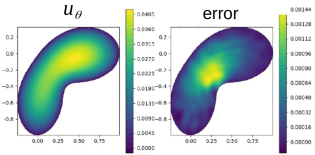


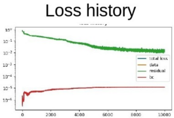
Learn LevelSet I

Sampling :



Classical PINNs : Looking for u_θ.

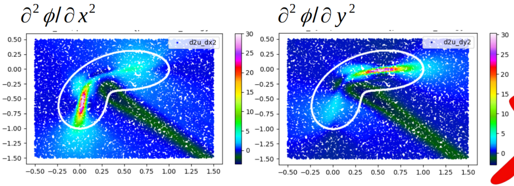




$\|u_{ref} - u_\theta\|_{L^2} = 2.20\text{e-}2$

Minus : Costly boundary points generation.

PINNs - Impose BC in hard : Looking for $u_\theta = \phi w_\theta$.



Levelset derivatives explodes.

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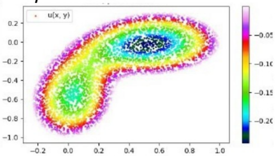
Learn LevelSet II

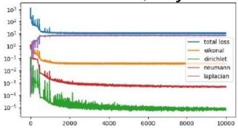
2nd test : We replace the TV term by a penalization on the laplacian of the levelset

$$J_{reg} = \int_{\mathcal{O}} |\Delta \phi|^2$$

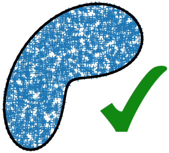
ϕ



Loss history

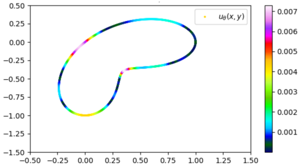


Sampling :



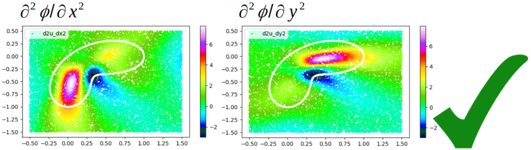
Dirichlet error on the boundary : Looking for $u_\theta = \phi w_\theta$.

Max : 7.29e-3 ; Mean : 1.88e-3



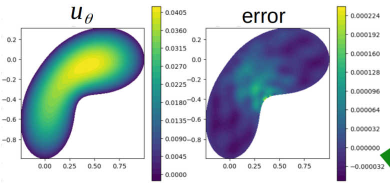
Learn LevelSet II

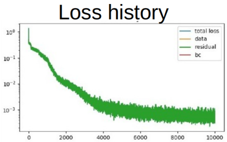
Derivatives :



$\Rightarrow$ We can impose in hard boundary conditions

PINNs - Impose BC in hard :





$\|u_{ref} - u_\theta\|_{L^2} = 1.43\text{e-}3$

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Conclusion

2 main questions :

➙ How to sample in complex domains?

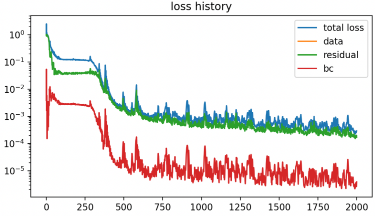
- Using mapping
- Using Levelset (Approximation theory/Learning)

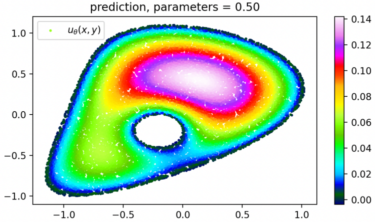
➙ How can we obtain a levelset that usable for imposing boundary conditions in hard ?

By learning the Eikonal equation with penalisation of the levelset Laplacian

To go further : We can combine the option.

(Mapping for the big domain. Level set for the hole.)





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Thank you !

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Bibliography

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[2] Pierre-Alain Fayolle. Signed Distance Function Computation from an Implicit Surface.

[3] M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*.

[4] N. Sukumar and Ankit Srivastava. Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks. *Computer Methods in Applied Mechanics and Engineering*.

[5] Sifan Wang, Shyam Sankaran, Hanwen Wang, and Paris Perdikaris. An Expert's Guide to Training Physics-informed Neural Networks.

Appendix 1 : Polygonal domain

TO COMPLETE !

Appendix 2 : Curved domain

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Appendix 3 : Neural Skeleton

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