

Enriching continuous Lagrange finite element approximation spaces using neural networks

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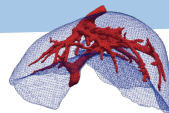
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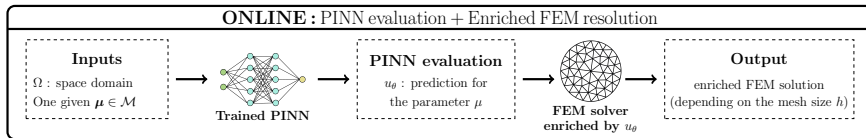
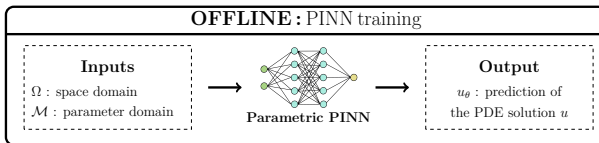
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Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.
accurate quick + parameterized



Problem considered

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :

We consider $\Omega = [-1, 1]^2$ a squared domain and $\mathbf{e}_y = (0, 1)$.

Find the velocity $\mathbf{u} = (u, v)$, the pressure p and the temperature T such that

$$\begin{cases}
 \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega & \text{(incompressibility)} \\
 (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} - g(\beta T + 1) \mathbf{e}_y = 0 & \text{in } \Omega & \text{(momentum)} \\
 \mathbf{u} \cdot \nabla T - k_f \Delta T = 0 & \text{in } \Omega & \text{(energy)}
 \end{cases} \quad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, μ the viscosity and k_f the thermal conductivity. [Coulaud et al., 2024]

Boundary Conditions:

- $\mathbf{u} = 0$ on $\partial\Omega$
- $T = 1$ on the left wall ($x = -1$) and $T = -1$ on the right wall ($x = 1$)
- $\frac{\partial T}{\partial n} = 0$ on the top and bottom walls ($y = \pm 1$)

Problem considered

Objective: Simulate the flow for a range of $\mu = (\mu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :

We consider $\mathbf{x} = (x, y) \in \Omega$ and $\mathbf{e}_y = (0, 1)$.

Find $U = (u, v, p, T)$ such that

$$\begin{cases} R_{inc}(U; \mathbf{x}, \mu) = 0 & \text{in } \Omega & \text{(incompressibility)} \\ R_{mom}(U; \mathbf{x}, \mu) = 0 & \text{in } \Omega & \text{(momentum)} \\ R_{ener}(U; \mathbf{x}, \mu) = 0 & \text{in } \Omega & \text{(energy)} \end{cases} \quad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, μ the viscosity and k_f the thermal conductivity. [Coulaud et al., 2024]

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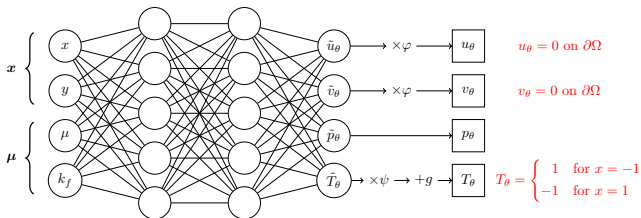
Physics-informed neural network (PINN)

Neural Network considered

We consider a parametric NN with 4 inputs and 4 outputs, defined by

$$U_{\theta}(\mathbf{x}, \boldsymbol{\mu}) = (u_{\theta}, v_{\theta}, p_{\theta}, T_{\theta})(\mathbf{x}, \boldsymbol{\mu}).$$

The Dirichlet boundary conditions are imposed on the outputs of the MLP by a **post-processing** step. [Sukumar and Srivastava, 2022]



We consider two levelsets functions φ and ψ , and the linear function g defined by

$$\varphi(x, y) = (x - 1)(x + 1)(y - 1)(y + 1),$$

$$\psi(x, y) = (x - 1)(x + 1) \quad \text{and} \quad g(x, y) = 1 - (x + 1).$$

PINN training

Approximate the solution of (\mathcal{P}) by a PINN : Find the optimal weights θ^* , such that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(J_{inc}(\theta) + J_{mom}(\theta) + J_{ener}(\theta) + J_{ad}(\theta) \right), \quad (\mathcal{P}_\theta)$$

where the different cost functions¹ are defined by

adiabatic condition

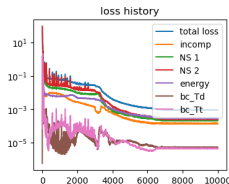
$$J_{ad}(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega|_{y=\pm 1}} \left| \frac{\partial T_\theta(\mathbf{x}, \boldsymbol{\mu})}{\partial n} \right|^2 d\mathbf{x} d\boldsymbol{\mu},$$

3 residual losses

$$J_\bullet(\theta) = \int_{\mathcal{M}} \int_{\Omega} |R_\bullet(U_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

with U_θ the parametric NN and \bullet the PDE considered (i.e. *inc*, *mom* or *ener*).

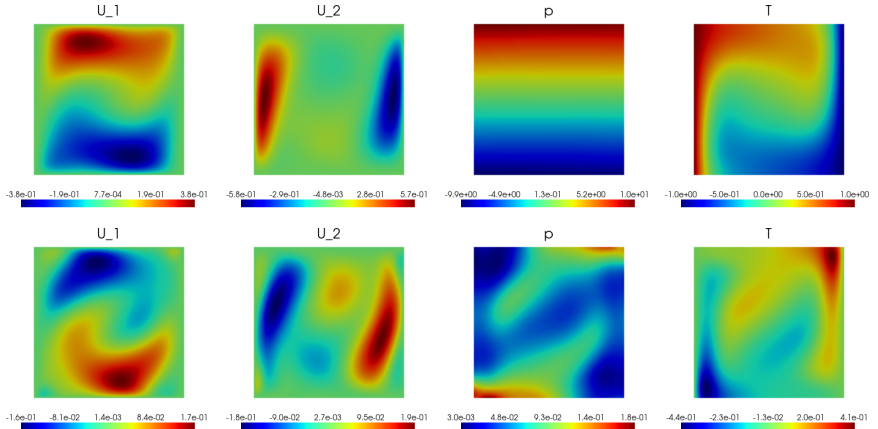
Network - MLP		Training (ADAM / LBFGS)			
<i>layers</i>	40, 60, 60, 60, 40	<i>lr</i>	7e-3	<i>N_{col}</i>	40000
<i>σ</i>	sine	<i>n_{epochs}</i>	10000	<i>N_{bc}</i>	30000



¹ Discretized by a random process using Monte-Carlo method.

PINN solution

$$\mu^{(1)} = (0.1, 0.1), \mu^{(2)} = (0.05, 0.05) \text{ and } \mu^{(3)} = (0.01, 0.01)$$



TODO : renommer figure u_θ . . . (solutions et erreurs)

Finite element method (FEM)

Newton method

TODO

Finite Element Methods¹

TODO

¹[Ern and Guermond, 2004]

Enriched finite element method using PINN

Newton method - Additive approach

TODO

Numerical results

Numerical results

TODO

Conclusion

TODO

References

Guillaume Coulaud, Maxime Le, and Régis Duvigneau. Investigations on Physics-Informed Neural Networks for Aerodynamics, 2024.

A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. 2004.

N. Sukumar and A. Srivastava. Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks. 2022.