Combining Finite Element methods and Neural Networks to solve elliptic problem on complex 2D geometries

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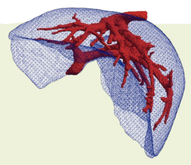
MIMESIS

Scientific context

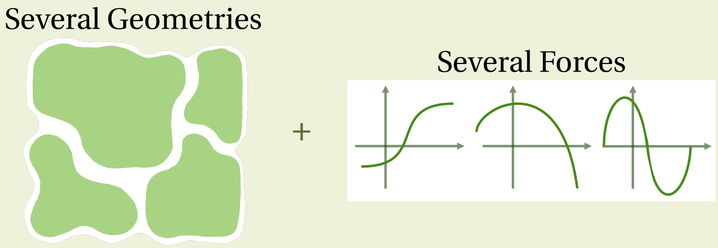
Create real-time digital twins of an organ (e.g. liver)



Current Objective : Develop hybrid **finite element** / **neural network** methods.
accurate quick + parameterized

OFFLINE

Several Geometries + Several Forces → Train a PINNs



ONLINE

1 Geometry - 1 Force → Get PINNs prediction → Correct prediction with FEM

Poisson problem with Dirichlet boundary conditions

Find $u : \Omega \to \mathbb{R}^d (d = 1,2,3)$ such that

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \Gamma \end{cases} \qquad (\mathscr{P})$$

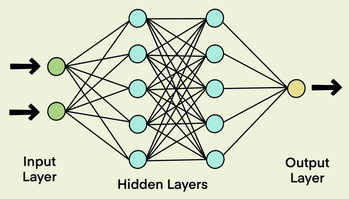
with Δ the Laplace operator, Ω a smooth bounded open set and Γ its boundary.

PINNs - Physics-Informed Neural Networks CITE

Implicit neural representation.

$$u_\theta(x) = u_{NN}(x)$$

with u_{NN} a neural network (e.g. a MLP).



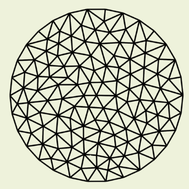
DoFs Minimization Problem.
Considering the least-square form of $(\mathscr{P})$, our discrete problem is

$$\bar{\theta} = \underset{\theta \in \mathbb{R}^m}{\operatorname{argmin}}\, \alpha J_{\text{in}}(\theta) + \beta J_{\text{bc}}(\theta)$$

with m the number of parameters of the NN and

$$J_{\text{in}}(\theta) = \frac{1}{2}\int_\Omega (\Delta u_\theta + f)^2 \quad \text{and} \quad J_{\text{bc}}(\theta) = \frac{1}{2}\int_{\partial\Omega} (u_\theta - g)^2$$

Finite Element methods



Variational Problem. Let's define V as a Hilbert space.

$$\text{Find } u \in V \text{ such that, } \forall v \in V,\ a(u,v) = l(v)$$

with a a bilinear form and l a linear form defined as

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \quad \text{and} \quad l(v) = \int_\Omega f v.$$

Approach Problem. Let's define V_h as a finite-dimensional subspace of V ($V_h \subset V$).

$$\text{Find } u_h \in V \text{ such that, } \forall v_h \in V,\ a(u_h, v_h) = l(v_h)$$

with $u_h \in V_h$ an approximate solution of u and $N_h = \dim(V_h)$.

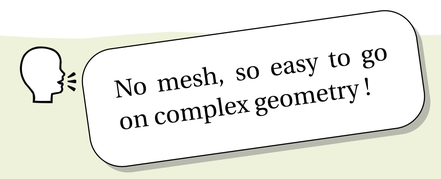
Linear System. Let's define $\{\varphi_1, \dots, \varphi_{N_h}\}$ a basis of V_h.

$$AU = b$$

with

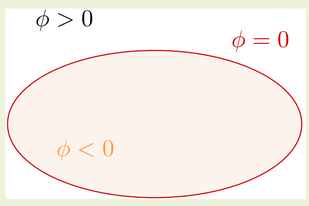
$$A = \left(a(\varphi_i, \varphi_j)\right)_{1 \le i,j \le N_h}, \quad U = \left(u_i\right)_{1 \le i \le N_h} \quad \text{and} \quad b = \left(l(\varphi_j)\right)_{1 \le j \le N_h}$$

How to deal with complex geometry in PINNs ?



⚠ **In practice :** Not so easy ! We need to find **how to sample in the geometry**.

Approach by levelset. CITE



Advantages :

➙ Sample is easy in this case.

➙ Allow to impose in hard the BC (no more J_{bc}) :

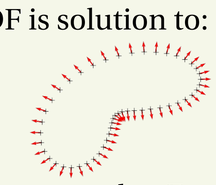
$$u_\theta(X) = \phi(X)\, w_\theta(X) + g(X)$$

with ϕ a levelset function and w_θ a NN.

Levelset considered. A regularized Signed Distance Function (SDF).

Eikonal equation. If we have a boundary domain Γ, the SDF is solution to:

$$\begin{cases} ||\nabla \phi(X)|| = 1,\ X \in \mathscr{O} \\ \phi(X) = 0,\ X \in \Gamma \\ \nabla \phi(X) = n,\ X \in \Gamma \end{cases}$$

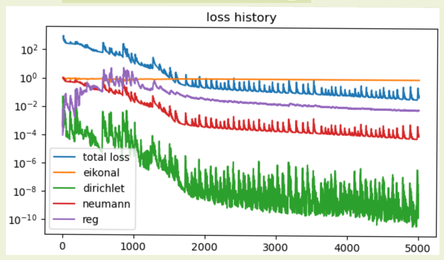


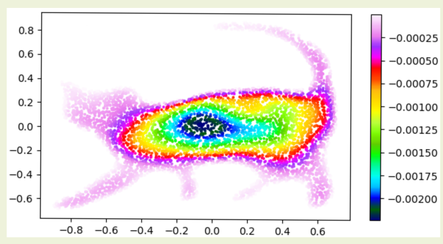
with $\mathscr{O}$ a box which contains Ω completely and n the exterior normal to Γ.

How to do that ? with a PINNs CITE, by adding the following regularization term

$$J_{\text{reg}} = \int_{\mathscr{O}} |\Delta \phi|^2.$$

Result : Levelset learning.





Result : Poisson on Cat.

➙ Solving $(\mathscr{P})$ with $f = 1$ (non parametric) and homogeneous Dirichlet BC ($g = 0$).

➙ Looking for $u_\theta = \phi w_\theta$ with ϕ the levelset learned.

How can we improve PINNs prediction ? - Using FEM-type methods

TODO