Enriching continuous Lagrange finite element approximation spaces using neural networks

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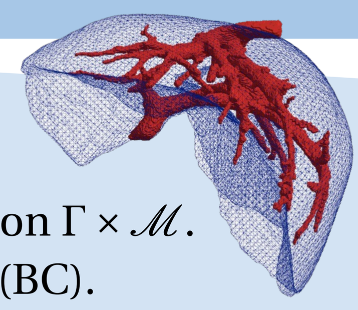
Motivations



Current Objective : Develop hybrid **finite element** / **neural network** methods.
accurate — quick + parameterized

Problem considered : $-\Delta u(X,\mu) = f(X,\mu) \quad \text{in } \Omega\times\mathcal{M}, \quad u(x,\mu)=0 \quad \text{on } \Gamma\times\mathcal{M}.$
Poisson problem with homogeneous Dirichlet boundary conditions (BC).

OFFLINE : PINN training

Inputs: $X\in\Omega$: sets of coordinates; $\mu\in\mathcal{M}$: sets of parameters → Parametric PINN → Output: u_θ : prediction of the PDE solution u

ONLINE : PINN evaluation + Enriched FEM resolution

Inputs: $X\in\Omega$: sets of coords; One given set $\mu\in\mathcal{M}$ → Trained PINN → PINN evaluation: u_θ : prediction for the parameter μ → FEM solver enriched by u_θ → Output: enriched FEM solution (depending on the mesh size h)

Long term objective : Create real-time digital twins of an organ (e.g. liver).

How improve PINN prediction ? - Using enriched FEM

Additive approach

The enriched approximation space is defined by

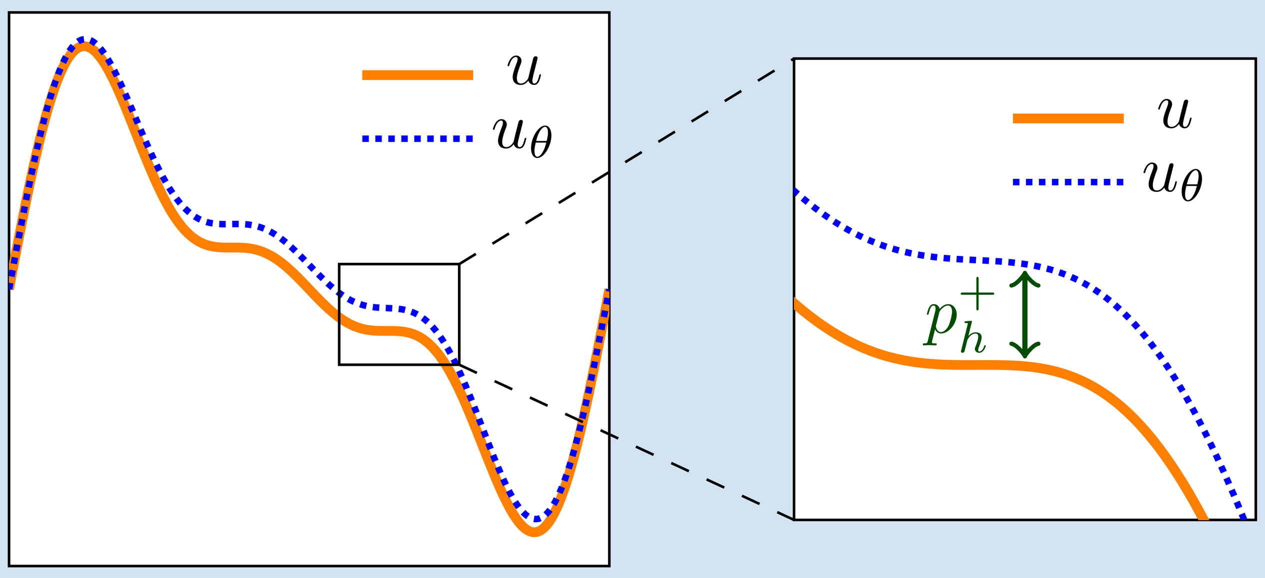
$$V_h^+ = \{u_h^+ = u_\theta + p_h^+ ,\ p_h^+\in V_h^0\}$$

with V_h^0 the standard continuous Lagrange FE space and the weak problem becomes

$$\text{Find } p_h^+\in V_h^0,\ \forall v_h\in V_h^0,\ a(p_h^+,v_h) = l(v_h) - a(u_\theta,v_h), \qquad (\mathcal{P}_h^+)$$

with modified boundary conditions and

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v, \quad l(v)=\int_\Omega f\,v.$$



Convergence analysis

u : solution of the Poisson problem. u_θ : prediction of the PINN [RPK19].

Theorem 1: Convergence analysis of the standard FEM [EG]

We denote $u_h\in V_h^0$ the discrete solution of standard FEM with V_h^0 a $\mathbb{P}_k$ Lagrange space. Thus,

$$|u-u_h|_{H^1} \le C_{H^1}\, h^k |u|_{H^{k+1}},$$
$$\|u-u_h\|_{L^2} \le C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [F L+25]

We denote $u_h^+\in V_h^+$ the discrete solution of $(\mathcal{P}_h^+)$ with V_h^+ a $\mathbb{P}_k$ Lagrange space. Thus

$$|u-u_h^+|_{H^1} \le \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}}\right),$$

and

$$\|u-u_h^+\|_{L^2} \le \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

Theoretical gain of the additive approach.

Numerical results - Considered problem

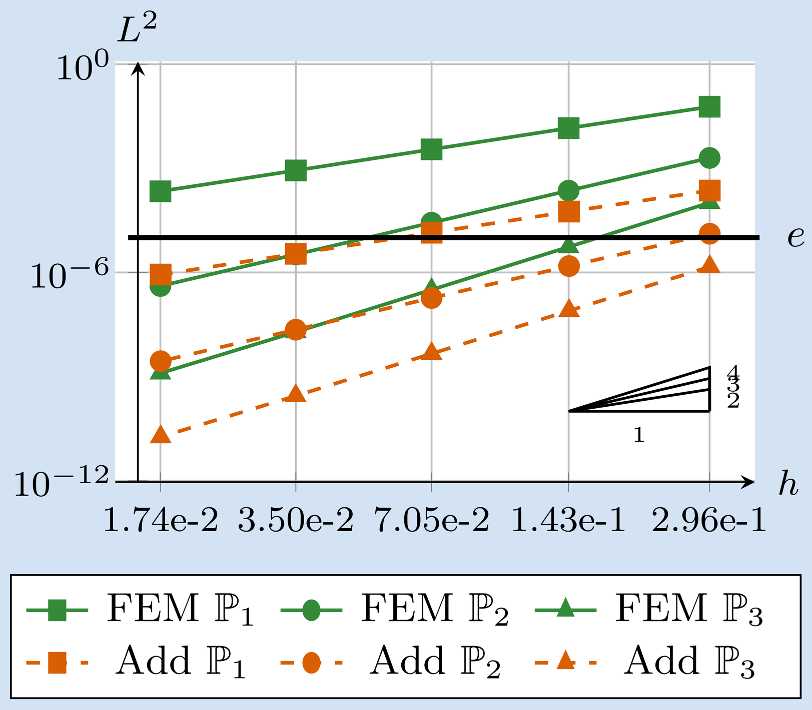
→ Spatial domain : $\Omega = [-0.5\pi, 0.5\pi]^2$
→ Parametric domain : $\mathcal{M} = [-0.5, 0.5]^2$
→ Analytical solution :

$$u_{ex}((x,y),\mu) = \exp\left(-\frac{(x-\mu_1)^2+(y-\mu_2)^2}{2}\right)\sin(2x)\sin(2y)$$

with $\mu=(\mu_1,\mu_2)\in\mathcal{M}$ (**parametric**) and the associated source term f.

Numerical results - Improve errors

Error estimates : $\mu = (0.05, 0.22)$.



Gains achieved : 50 sets of parameters.

$$S=\{\mu^{(1)},\dots,\mu^{(50)}\}$$

Gains in L^2 rel error of our method w.r.t. FEM

k	min	max	mean
1	134.32	377.36	269.39
2	67.02	164.65	134.85
3	39.52	72.65	61.55

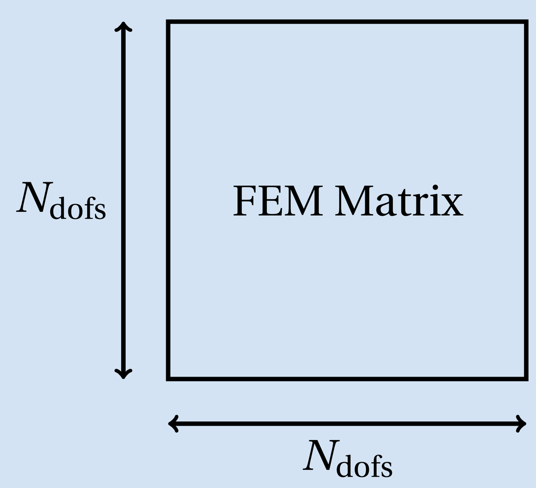
Gain : $\|u-u_h\|_{L^2} / \|u-u_h^+\|_{L^2}$

Cartesian mesh : 20^2 nodes.

Numerical results - Improve numerical costs

N_{dofs} **required to reach the same error** e **:** $\mu = (0.05, 0.22)$.

k	e	N_{dofs} FEM	N_{dofs} Add
1	$1\cdot10^{-3}$	14,161	64
	$1\cdot10^{-4}$	143,641	576
2	$1\cdot10^{-4}$	6,889	225
	$1\cdot10^{-5}$	31,329	1,089
3	$1\cdot10^{-5}$	6,724	784
	$1\cdot10^{-6}$	20,164	2,704

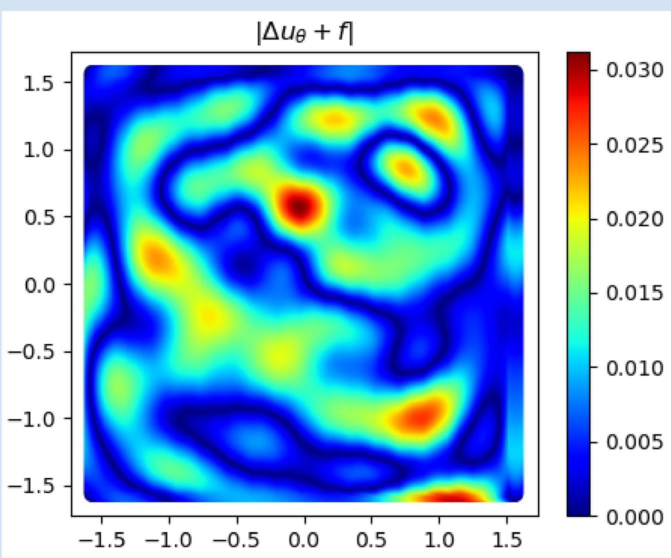


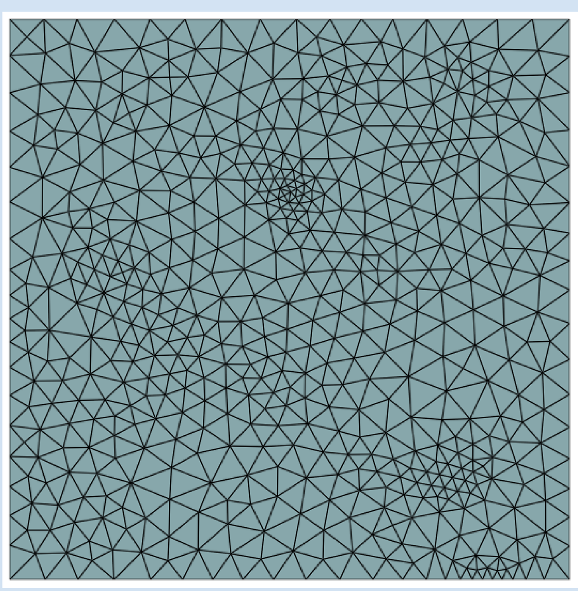
Less degrees of freedom $\Rightarrow$ Lower numerical cost; Faster simulation

Perspectives

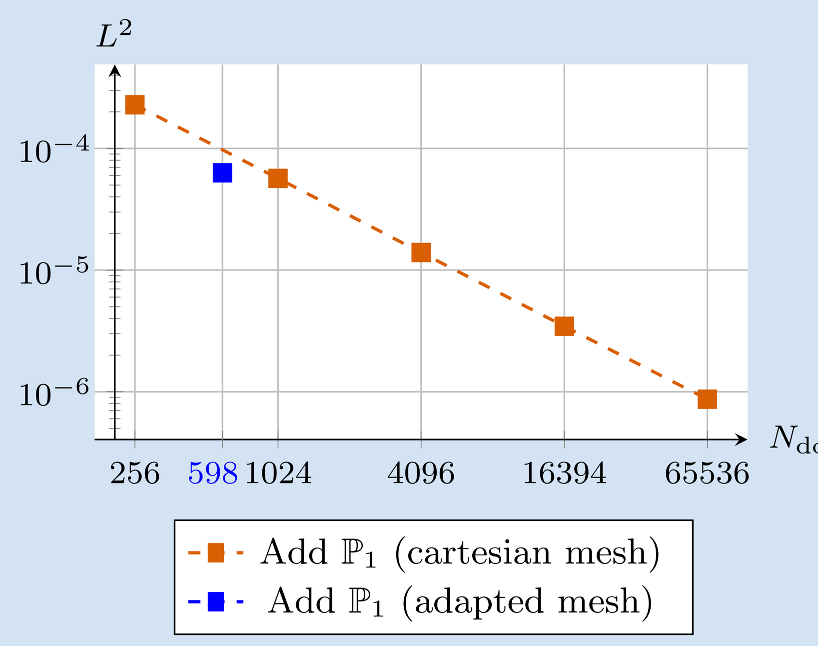
Mesh adaptation

Construction of the mesh: $N_{\text{dofs}} = 598$.

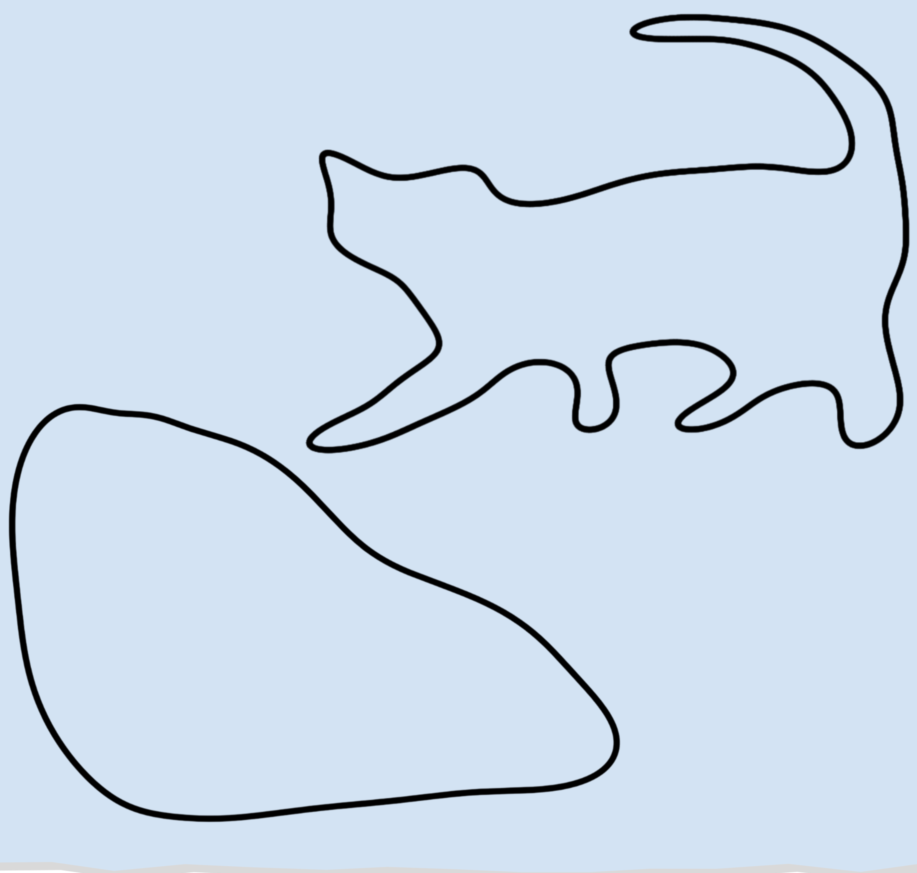




Apply enriched FEM:



More complex geometries



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