# Enriching continuous Lagrange finite element approximation spaces using neural networks

Hélène BARUCQ[3], Michel DUPREZ[1], Florian FAUCHER[3], Emmanuel FRANCK[2], **Frédérique LECOURTIER**[1], Vanessa LLERAS[4], Victor MICHEL-DANSAC[2], and Nicolas VICTORION[3]

[1] Mimesis team, INRIA Nancy grand Est, Icube   [2] Macaron team, INRIA Nancy grand Est, IRMA   [3] Makutu team, INRIA Bordeaux, TotalEnergies   [4] University of Montpellier
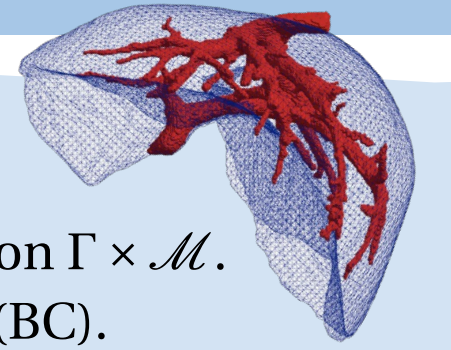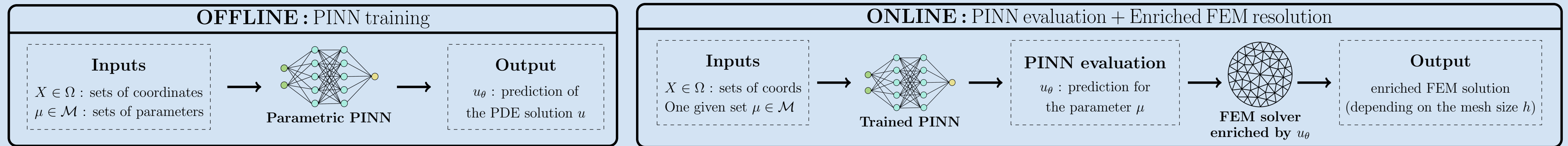
## Motivations

**Current Objective :** Develop hybrid **finite element** / **neural network** methods.
**accurate**   **quick + parameterized**

**Problem considered :**   $-\Delta u(X,\mu) = f(X,\mu)$ in $\Omega \times \mathcal{M}$,   $u(x,\mu) = 0$ on $\Gamma \times \mathcal{M}$.
Poisson problem with homogeneous Dirichlet boundary conditions (BC).

**OFFLINE :** PINN training

| Inputs | Output |
|---|---|
| $X \in \Omega$ : sets of coordinates $\mu \in \mathcal{M}$ : sets of parameters | $u_\theta$ : prediction of the PDE solution $u$ |

Parametric PINN

**ONLINE :** PINN evaluation + Enriched FEM resolution

| Inputs | PINN evaluation | Output |
|---|---|---|
| $X \in \Omega$ : sets of coords One given set $\mu \in \mathcal{M}$ | $u_\theta$ : prediction for the parameter $\mu$ | enriched FEM solution (depending on the mesh size $h$) |

Trained PINN — FEM solver enriched by $u_\theta$

**Long term objective :** Create real-time digital twins of an organ (e.g. liver).

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## How improve PINN prediction ? - Using enriched FEM

### Additive approach

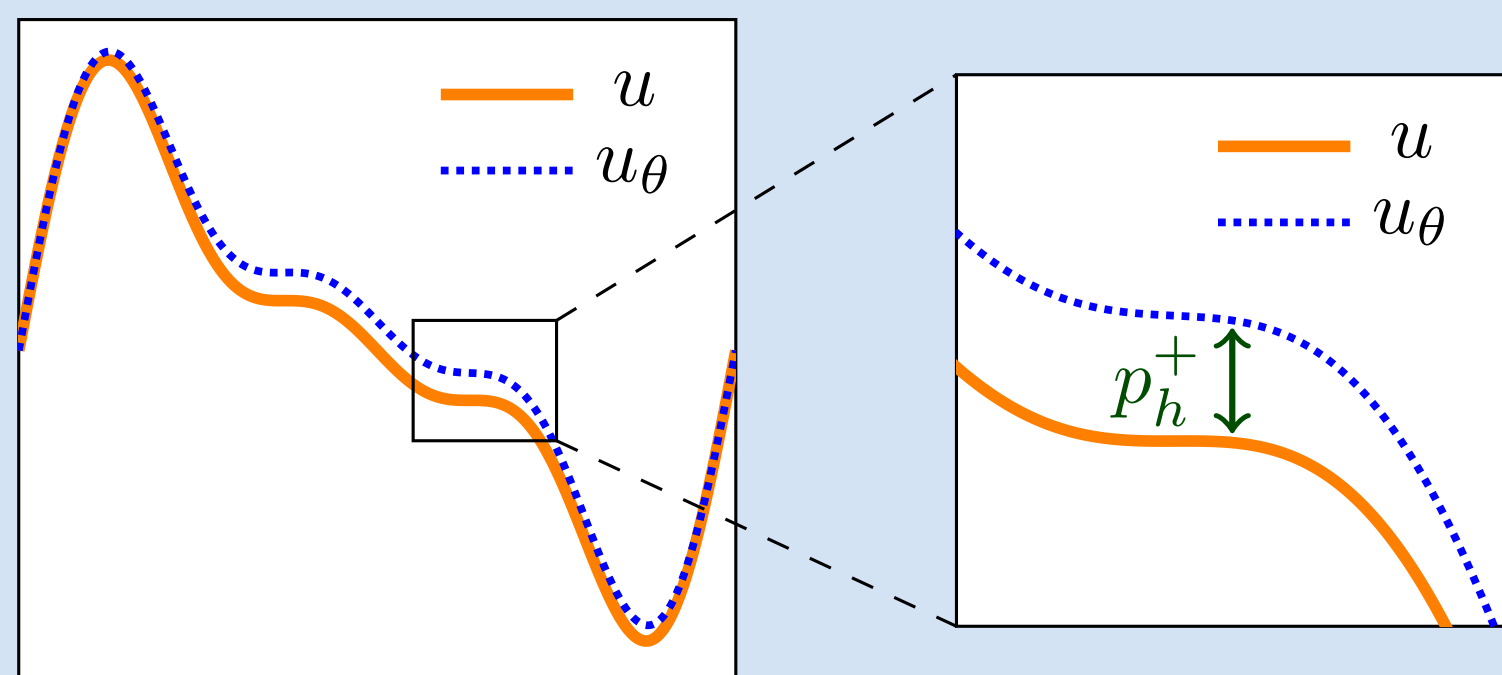The enriched approximation space is defined by
$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \ p_h^+ \in V_h^0\}$$
with $V_h^0$ the standard continuous Lagrange FE space and the weak problem becomes
$$\text{Find } p_h^+ \in V_h^0, \ \forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathscr{P}_h^+)$$
with modified boundary conditions and
$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad l(v) = \int_\Omega f\,v.$$



### Convergence analysis

$u$ : solution of the Poisson problem.   $u_\theta$ : prediction of the PINN [RPK19].

**Theorem 1: Convergence analysis of the standard FEM [EG]**

We denote $u_h \in V_h^0$ the discrete solution of standard FEM with $V_h^0$ a $\mathbb{P}_k$ Lagrange space. Thus,
$$|u - u_h|_{H^1} \le C_{H^1} h^k |u|_{H^{k+1}},$$
$$\|u - u_h\|_{L^2} \le C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

**Theorem 2: Convergence analysis of the enriched FEM [F L+25]**

We denote $u_h^+ \in V_h^+$ the discrete solution of $(\mathscr{P}_h^+)$ with $V_h^+$ a $\mathbb{P}_k$ Lagrange space. Thus
$$|u - u_h^+|_{H^1} \le \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} \left(C_{H^1} h^k |u|_{H^{k+1}}\right),$$
and
$$\|u - u_h^+\|_{L^2} \le \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} \left(C_{L^2} h^{k+1} |u|_{H^{k+1}}\right).$$

Theoretical gain of the additive approach.
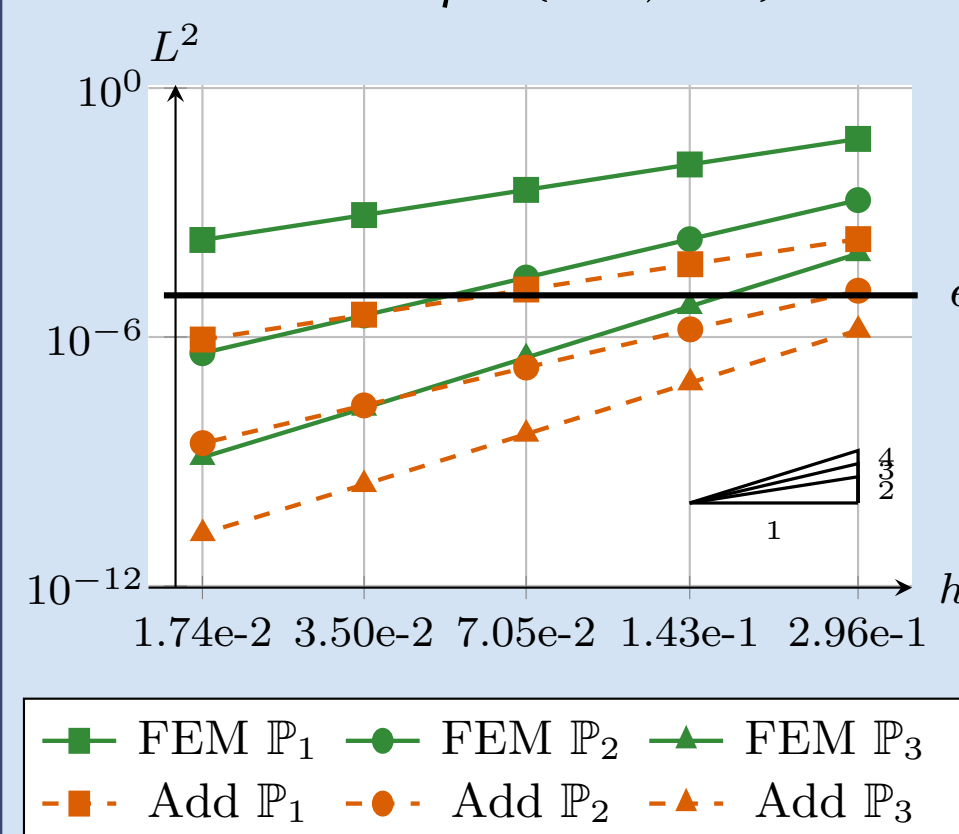
### Numerical results - Considered problem

→ Spatial domain : $\Omega = [-0.5\pi, 0.5\pi]^2$
→ Parametric domain : $\mathcal{M} = [-0.5, 0.5]^2$
→ Analytical solution :
$$u_{ex}\big((x,y),\mu\big) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right)\sin(2x)\sin(2y)$$
with $\mu = (\mu_1, \mu_2) \in \mathcal{M}$ (**parametric**) and the associated source term $f$.

### Numerical results - Improve errors

**Error estimates :** $\mu = (0.05, 0.22)$.



**Gains achieved :** 50 sets of parameters.
$$S = \{\mu^{(1)}, \ldots, \mu^{(50)}\}$$

| | Gains in $L^2$ rel error of our method w.r.t. FEM | | |
|---|---|---|---|
| k | min | max | mean |
| 1 | 134.32 | 377.36 | 269.39 |
| 2 | 67.02 | 164.65 | 134.85 |
| 3 | 39.52 | 72.65 | 61.55 |

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$
Cartesian mesh : $20^2$ nodes.

### Numerical results - Improve numerical costs

$N_{\text{dofs}}$ required to reach the same error $e$ : $\mu = (0.05, 0.22)$.

| k | e | $N_{\text{dofs}}$ FEM | $N_{\text{dofs}}$ Add |
|---|---|---|---|
| 1 | $1 \cdot 10^{-3}$ | 14,161 | 64 |
| | $1 \cdot 10^{-4}$ | 143,641 | 576 |
| 2 | $1 \cdot 10^{-4}$ | 6,889 | 225 |
| | $1 \cdot 10^{-5}$ | 31,329 | 1,089 |
| 3 | $1 \cdot 10^{-5}$ | 6,724 | 784 |
| | $1 \cdot 10^{-6}$ | 20,164 | 2,704 |

FEM Matrix ($N_{\text{dofs}} \times N_{\text{dofs}}$)
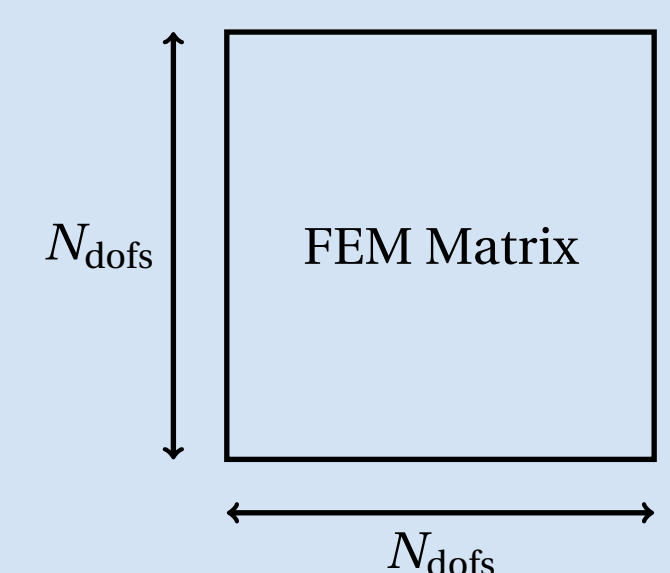
Less degrees of freedom $\Rightarrow$ { Lower numerical cost / Faster simulation }

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## Perspectives

### Mesh adaptation

**Construction of the mesh:** $N_{\text{dofs}} = 598$.



$|\Delta u_\theta + f|$

**Apply enriched FEM:**



— Add $\mathbb{P}_1$ (cartesian mesh)
— Add $\mathbb{P}_1$ (adapted mesh)
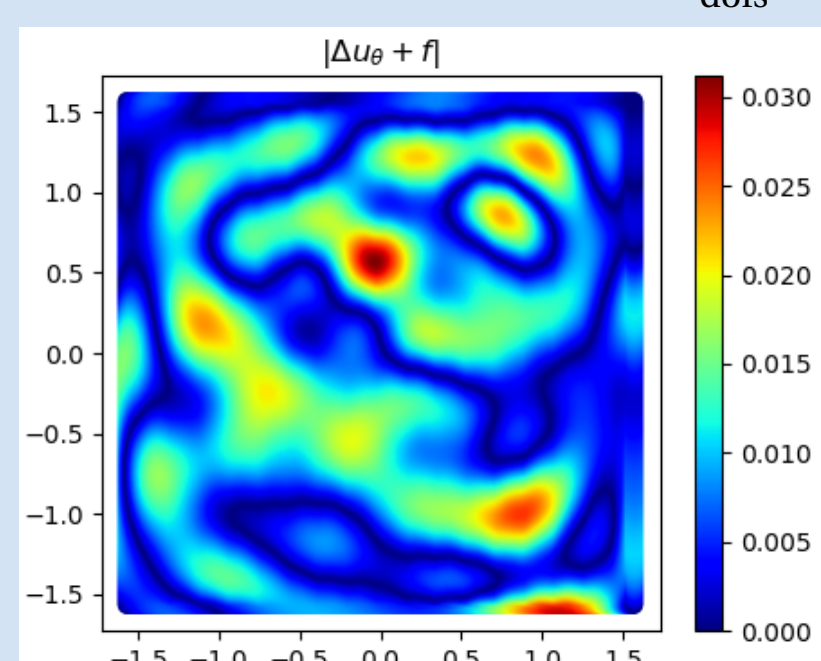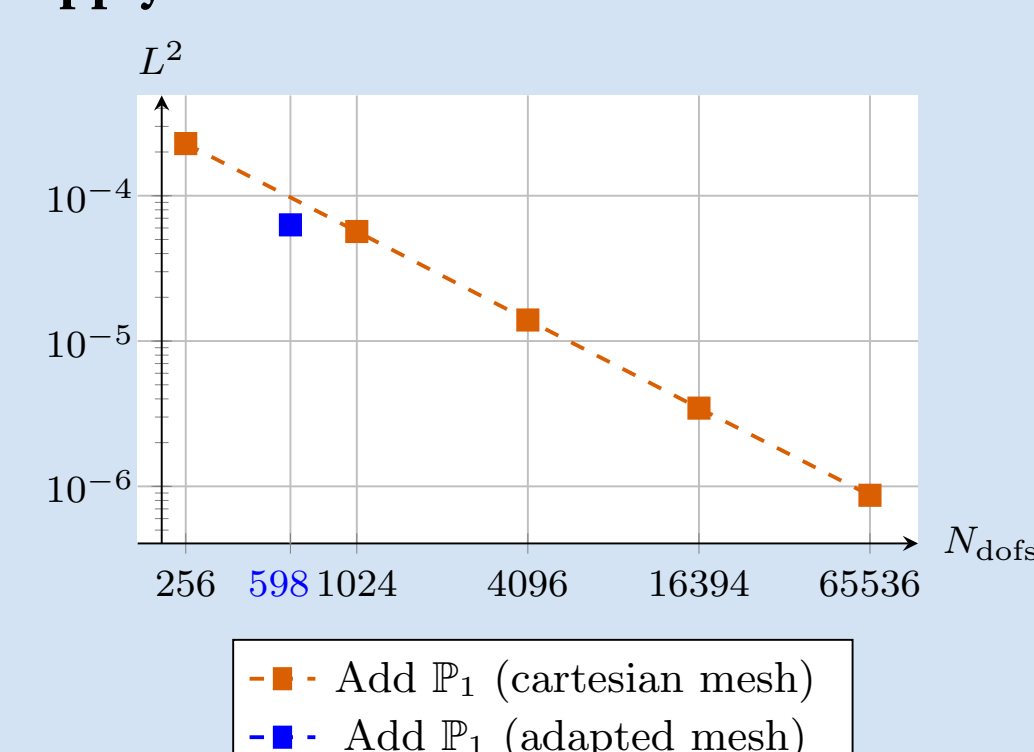
### More complex geometries
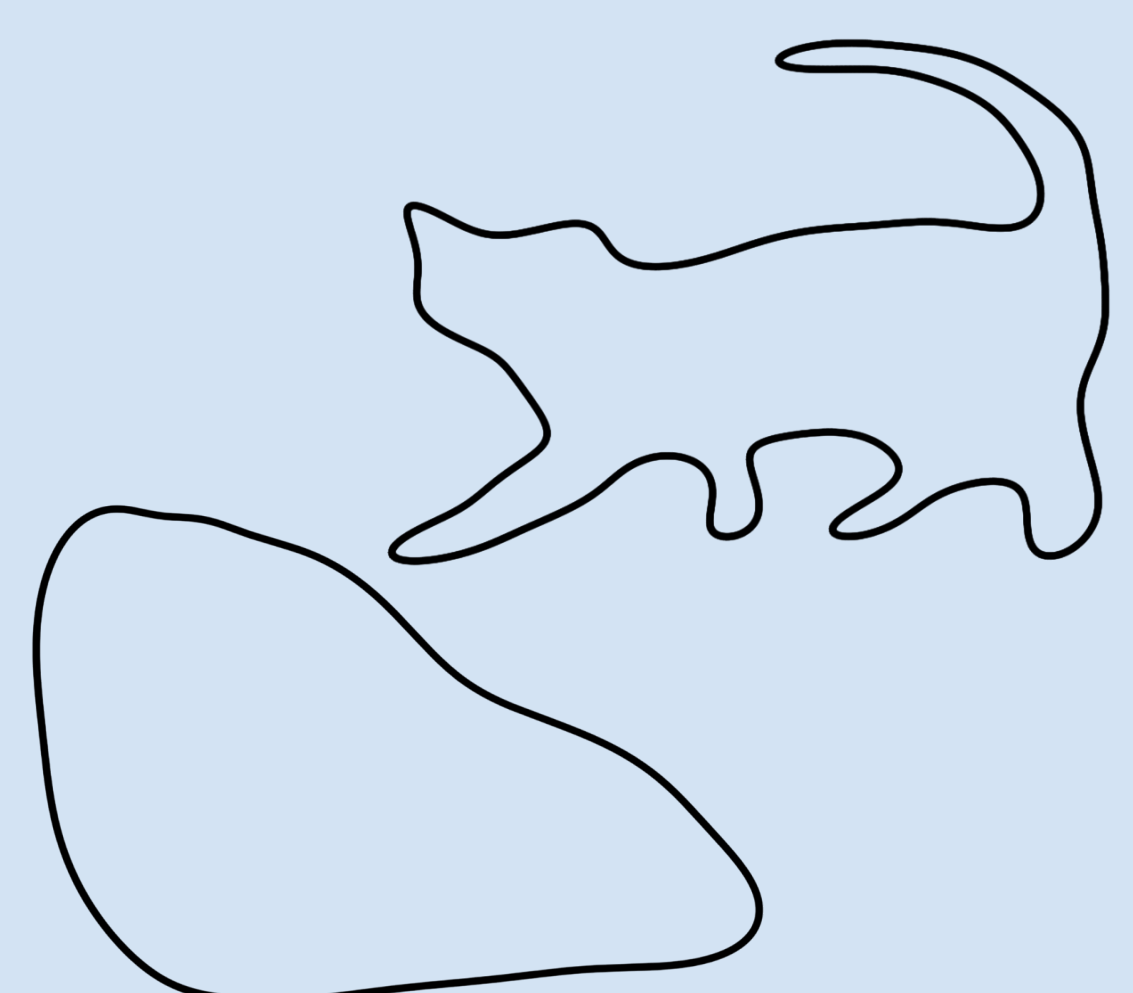
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[RPK19] M. Raissi, P. Perdikaris, and G. E. Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *J. Comput. Phys.* 378 (2019), pp. 686–707.