

2nd CSI

Development of hybrid finite element/neural network methods to help create digital surgical twins

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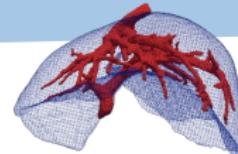
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June 12, 2025

★ - Update 2025

Scientific context

Context : Create real-time digital twins of an organ (e.g. liver).



Objective : Develop an hybrid finite element / neural network method.
accurate quick + parameterized

★ **Parametric elliptic convection/diffusion PDE :** For one or several $\mu \in \mathcal{M}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\mathcal{L}(u ; \boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}, \boldsymbol{\mu}), \quad (\mathcal{P})$$

where \mathcal{L} is the parametric differential operator defined by

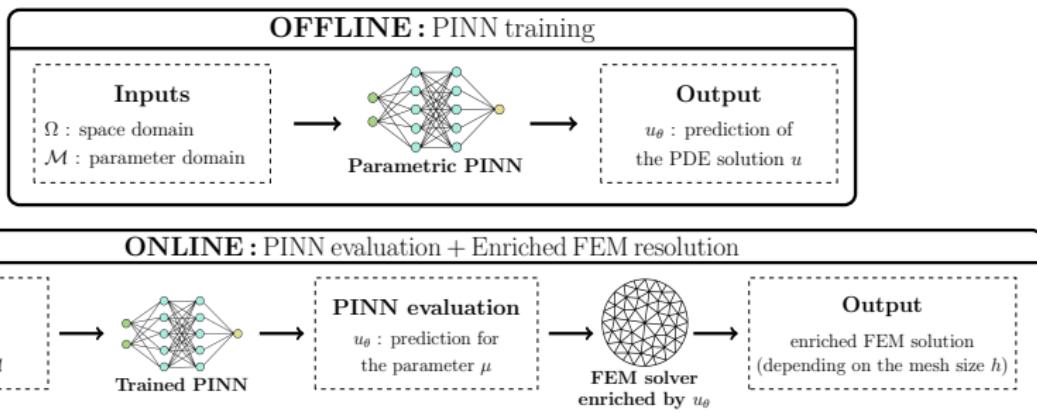
$$\mathcal{L}(\cdot ; \boldsymbol{x}, \boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x}, \boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\text{Pe}} \nabla \cdot (D(\boldsymbol{x}, \boldsymbol{\mu}) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Pipeline of the Enriched FEM

Enriched FEM = Combination of 2 standard methods

- **PINNs** : Physics Informed Neural Networks Appendix 1.1
 - **FEMs** : Finite Element Methods Appendix 1.2



Remark: The PINN prediction enriched Finite element approximation spaces.

Enriched finite element method using PINNs

Additive approach

★ Numerical results

This section is based on [F. Lecourtier et al., 2025].

Enriched finite element method using PINNs

Additive approach

★ Numerical results

Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(p_h^+, v_h) = I(v_h) - a(u_\theta, v_h), \quad (\mathcal{P}_h^+)$$

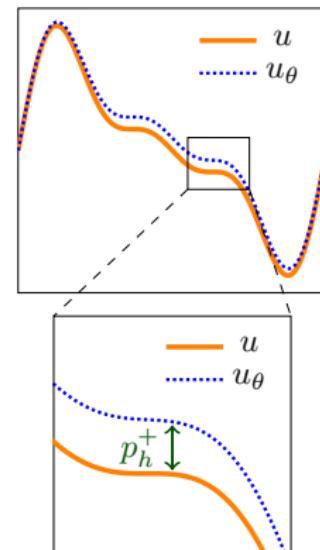
with the **enriched trial space** V_h^+ defined by

$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.



Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of (\mathcal{P}_h) with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leq C_{H^1} h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [F. Lecourtier et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of (\mathcal{P}_h^+) with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leq \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^+\|_{L^2} \leq \frac{\|u - u_\theta\|_{H^{k+1}}}{\|u\|_{H^{k+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

Gains of the additive approach.

Enriched finite element method using PINNs

Additive approach

★ Numerical results

1st problem considered

Problem statement: Considering an **Anisotropic Elliptic problem with Dirichlet BC**:

$$\begin{cases} -\operatorname{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0, 1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$).

Right-hand side :

$$f(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{0.025\sigma^2}\right).$$

Diffusion matrix : (symmetric and positive definite)

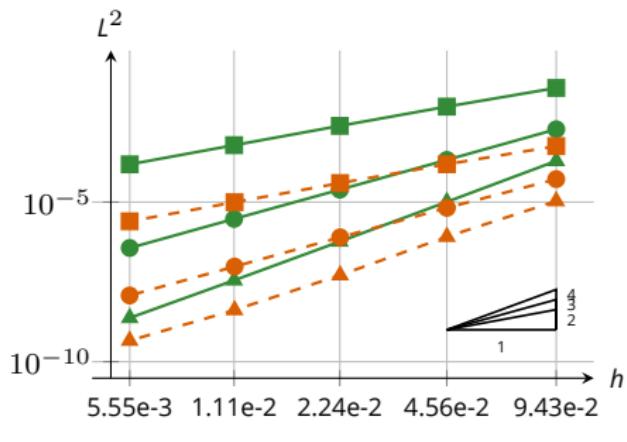
$$D(\mathbf{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

PINN training: Imposing BC exactly with a level-set function.

Numerical results

Error estimates : 1 set of parameters.

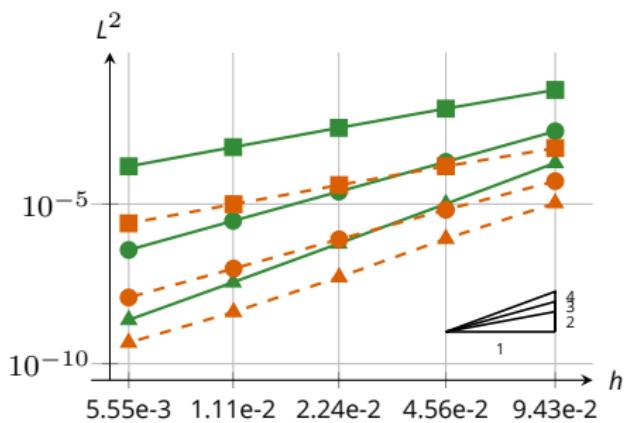
$$\mu = (0.51, 0.54, 0.52, 0.55)$$



Numerical results

Error estimates : 1 set of parameters.

$$\mu = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

Gains in L^2 rel error of our method w.r.t. FEM			
k	min	max	mean
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$$N = 20$$

$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Cartesian mesh : N^2 nodes.

2nd problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \frac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x, y) \in \mathbb{R}^2, 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($\rho = 1$).

Analytical solution :

$$u(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1 \sqrt{x^2 + y^2})}{\ln(4)},$$

Boundary conditions :

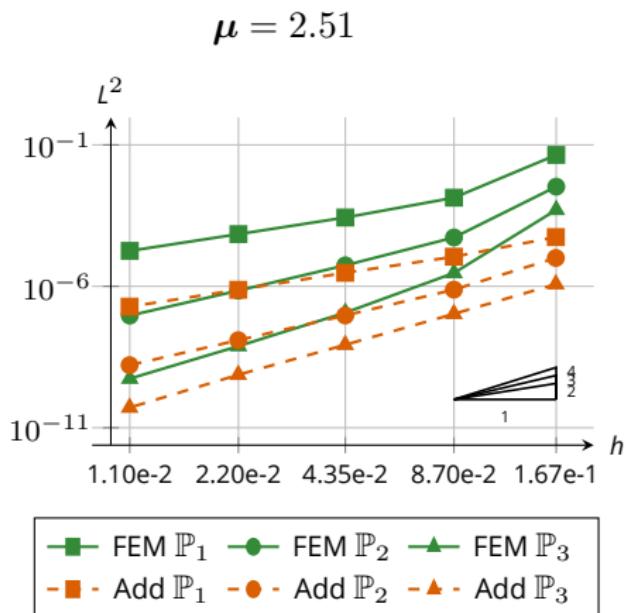
$$g(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\mathbf{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

PINN training: Imposing mixed BC exactly in the PINN¹.

¹[Sukumar and Srivastava, 2022]

Numerical results

Error estimates : 1 set of parameters.



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

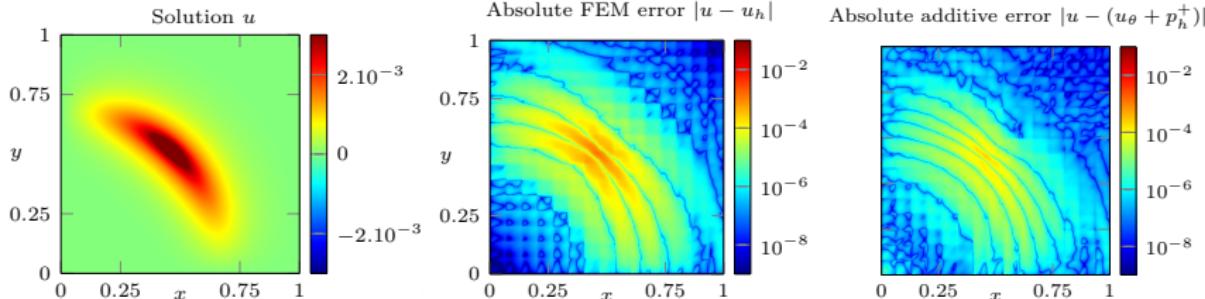
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

$$h = 1.33 \cdot 10^{-1}$$

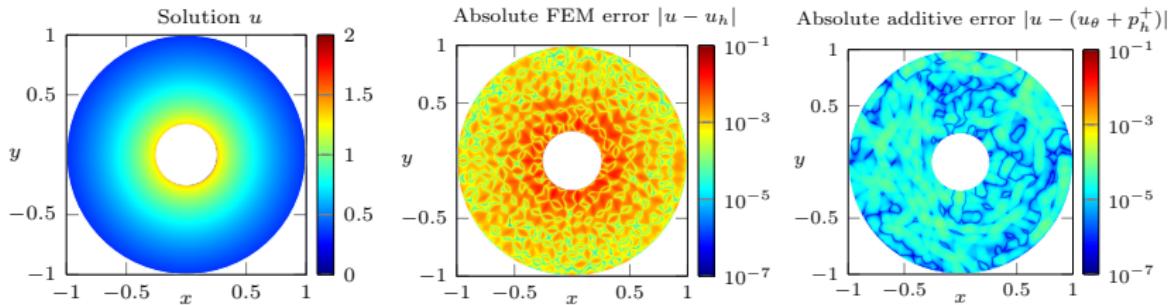
$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Numerical solutions

1st problem : $\mu = (0.46, 0.52, 0.05, 0.12)$



2nd problem : $\mu = 2.51$



New lines of research

Complex geometries

★ Posteriori error estimates

★ Non linear PDEs

New lines of research

Complex geometries

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★ Non linear PDEs

Learn a regular levelset

Theorem 3: [Clémot and Digne, 2023]

If we have a boundary domain Γ , the SDF is solution to the Eikonal equation:

$$\begin{cases} \|\nabla\phi(x)\| = 1, & x \in \mathcal{O} \\ \phi(x) = 0, & x \in \Gamma \\ \nabla\phi(x) = n, & x \in \Gamma \end{cases}$$



with \mathcal{O} a box which contains Ω completely and n the exterior normal to Γ .

Objective: Move on to complex geometries by using a levelset function to

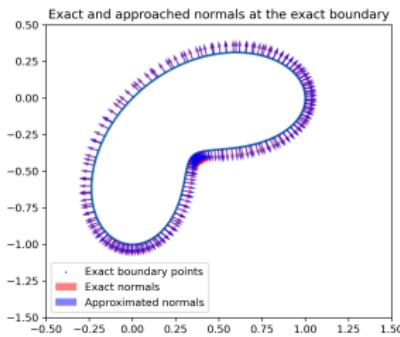
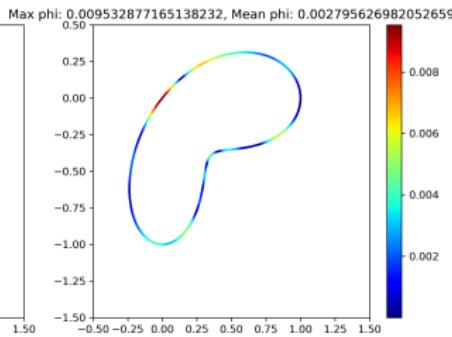
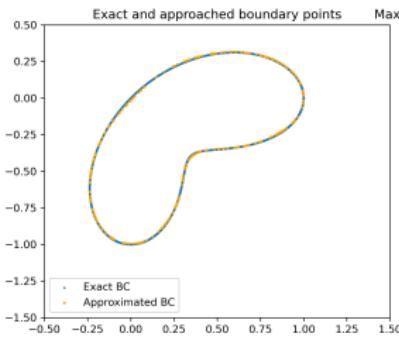
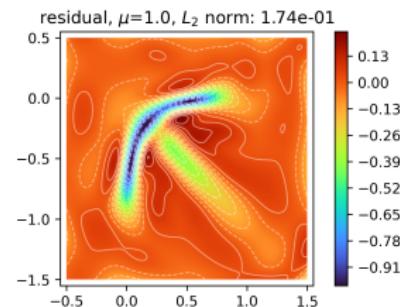
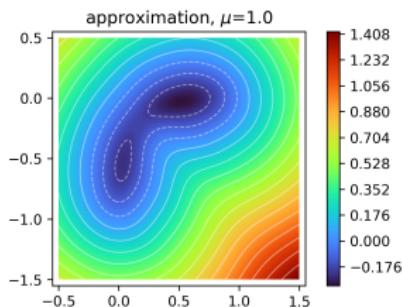
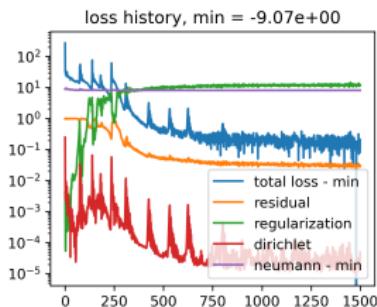
- Sample points in the domain Ω for the PINN training.
- Impose exactly the boundary condition in PINN [Sukumar and Srivastava, 2022].

How to learn a regular levelset ? with a PINN by adding a regularization term,

$$J_{reg} = \int_{\mathcal{O}} |\Delta\phi|^2,$$

and a sample of boundary points that considers the curvature of Γ . ★

Numerical results



TODO : Ajouter résultats "Poisson on Bean" + Mettre au propre les images.

New lines of research

Complex geometries

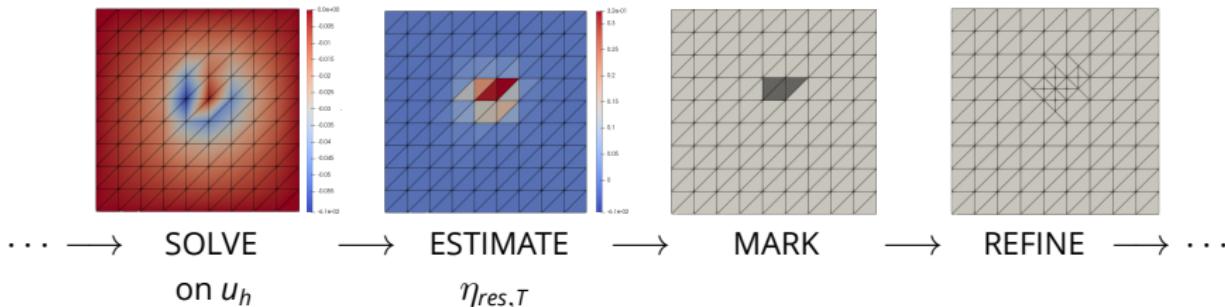
★ Posteriori error estimates

★ Non linear PDEs

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy. Appendix 4.1

Standard FEM



Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

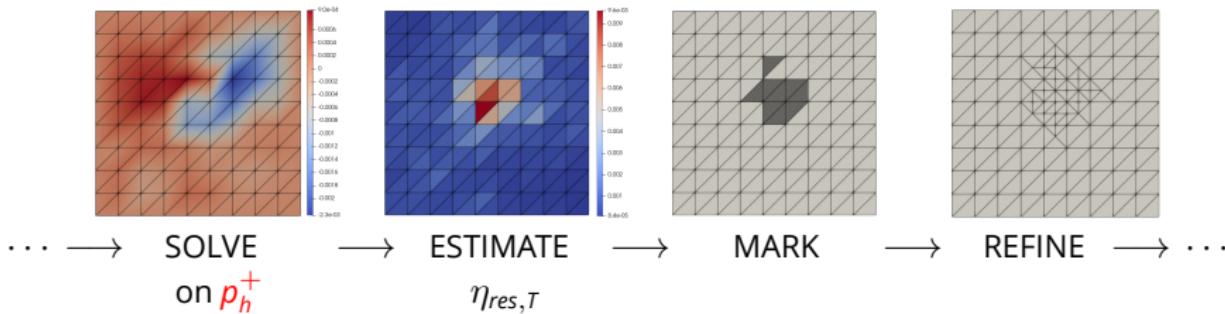
$$\eta_{res,T}^2 = h_T^4 \|\Delta u_h + f_h\|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E^2 \|[\nabla u_h \cdot n]\|_{L^2(E)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach



Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

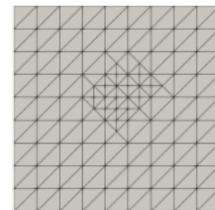
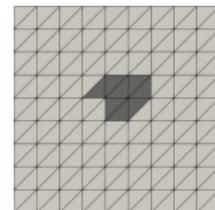
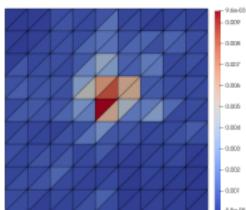
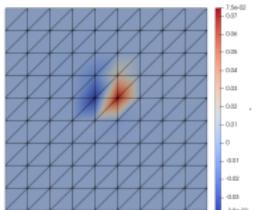
$$\eta_{res,T}^2 = h_T^4 \| (\Delta u_\theta)_h + \Delta p_h^+ + f_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E^2 \| [\nabla p_h^+ \cdot n] \|_{L^2(E)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach - No resolution

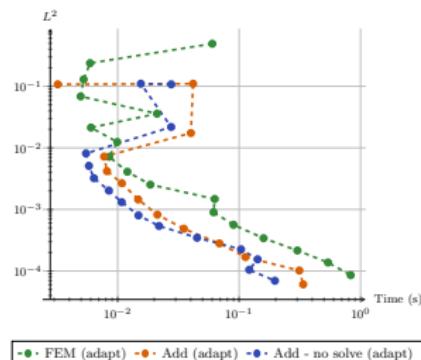
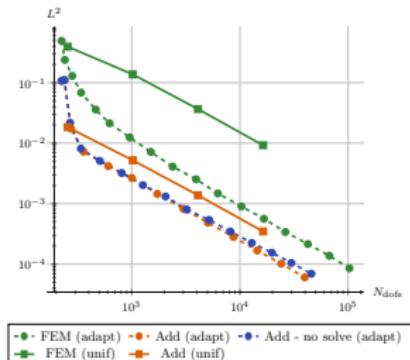


Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

$$\eta_{res,T}^2 = h_T^4 \|(\Delta u_\theta)_h + f_h\|_{L^2(T)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Numerical results



⚠ Results obtained on a laptop GPU (probably due to external factors).

Ideas for improving results : Additive approach (no resolution).

time Interpolate only mesh points added in the refinement process.

error Use another metric such as curvature, rather than residual error.

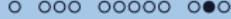
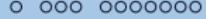
$$\Delta u_\theta + f \quad \neq \quad u - u_\theta$$

New lines of research

Complex geometries

★ Posteriori error estimates

★ Non linear PDEs



Problem considered

TODO



Numerical results

TODO

Supplementary work

Supplementary work I

Teaching

- ▶ 2024/2025 :
 - ▶ 64h of Computer Science Practical Work - L1S2 and L2S3 (Python) / L3S6 (C++)
 - ▶ 3 days supervising a group of high school girls in RJMI ("Rendez-vous des Jeunes Mathématiciennes et Informaticiennes")
- ▶ 2023/2024 : 50h of Computer Science Practical Work - L2S3 (Python) / L3S6 (C++)

Training courses (Total : 176h35)

- ▶ A dozen seminars organized by IRMA ($\approx 10h$)
- ▶ 1 Deep Learning introductory course - FIDLE ($\approx 40h$)
- ▶ 2 workshops on Scientific Machine Learning ($\approx 2 \times 21h$)
- ▶ 1 summer school on "New Trend in computing" ($\approx 27h$)
- ▶ several cross-disciplinary courses - Methodology, scientific English, etc. ($\approx 58h$)



Supplementary work II

Talks

- ▶ **ICOSAHOM 2025, Montréal** - July 2025 (*Coming soon...*)
"Enriching continuous Lagrange finite element approximation spaces using neural networks"
- ▶ **DTE & AICOMAS 2025, Paris** - February 20, 2025
"Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- ▶ **Exama project, WP2 reunion** - March 26, 2024
"How to work with complex geometries in PINNs ?"
- ▶ **Retreat (Macaron/Tonus)** - February 6, 2024
"Mesh-based methods and physically informed learning"
- ▶ **Team meeting (Mimesis)** - December 12, 2023
"Development of hybrid finite element/neural network methods to help create digital surgical twins"

Supplementary work III

Posters

- ▶ **EMS-TAG-SciML 2025, Milan** - March 24, 2025 - "Enriching continuous Lagrange finite element approximation spaces using neural networks"
- ▶ **CJC-MA 2024, Lyon** - October 29, 2024 - "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- ▶ **MSII poster day, Strasbourg** - October 24, 2024
- ▶ **SciML 2024, Strasbourg** - July 08, 2024

Publications

- ▶ Enriching continuous lagrange finite element approximation spaces using neural networks. (*submitted in February 2025, M2AN journal*)
H. Barucq, M. Duprez, F. Faucher, E. Franck, **F. Lecourtier**, V. Lleras, V. Michel-Dansac, and N. Victorion.

Conclusion

Enriched finite element method using PINNs :

- PINNs are good candidates for the enriched approach. [Appendix 2](#)
- Numerical validation of the theoretical results.
- The enriched approach provides the same results as the standard FEM method, but with coarser meshes. ⇒ Reduction of the computational cost.

We have also tested a multiplicative approach. [Appendix 3](#)

New lines of research :

- The treatment of complex geometries is progressing.
- New PDEs begin to be considered, in particular non-linear problems.
- Other methods for improving the additive approach are being studied, including posterior error estimators.

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Appendix 1 : Standard methods



Appendix 2 : Data-driven vs Physics-Informed training



Appendix 3 : Multiplicative approach



Appendix 4 : More



Appendix 1 : Standard methods

LECOURTIER Frédérique

Development of an hybrid finite element and neural network method

A1.1 – Physics-Informed Neural Networks

Standard PINNs¹ (Weak BC) : Find the optimal weights θ^* , such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \omega_b J_b(\theta)), \quad (\mathcal{P}_\theta)$$

with

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

boundary loss

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \boldsymbol{\mu}) - g(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In (\mathcal{P}_θ) , ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

¹[Raissi et al., 2019]

A1.1 – Physics-Informed Neural Networks

Improved PINNs¹ (Strong BC) : Find the optimal weights θ^* such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)}),$$

with $\omega_r = 1$ and

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_{\theta}(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

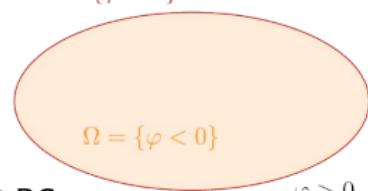
$$\partial\Omega = \{\varphi = 0\}$$

where u_{θ} is a neural network defined by

$$u_{\theta}(\mathbf{x}, \boldsymbol{\mu}) = \varphi(\mathbf{x}) w_{\theta}(\mathbf{x}, \boldsymbol{\mu}) + g(\mathbf{x}, \boldsymbol{\mu}),$$

with φ a level-set function, w_{θ} a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_{\theta} = g$ on $\partial\Omega$.



¹[Lagaris et al., 1998; Franck et al., 2024]

A1.2 – Finite Element Methods¹

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

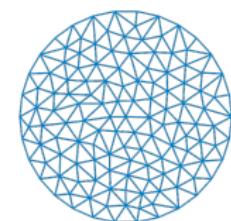
with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_{\Omega} D \nabla u_h \cdot \nabla v_h + \int_{\Omega} R u_h v_h + \int_{\Omega} v_h C \cdot \nabla u_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where \mathbb{P}_k is the space of polynomials of degree at most k .



Linear system : Let $(\phi_1, \dots, \phi_{N_h})$ a basis of V_h^0 .

Find $U \in \mathbb{R}^{N_h}$ such that $AU = b$

with

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_h} \quad \text{and} \quad b = (l(\phi_j))_{1 \leq j \leq N_h}.$$

$$\mathcal{T}_h = \{K_1, \dots, K_{N_e}\}$$

(N_e : number of elements)

¹[Ern and Guermond, 2004]

Appendix 2 : Data-driven vs Physics-Informed training

A2 – Problem considered

Problem statement: Consider the Poisson problem in 1D with Dirichlet BC:

$$\begin{cases} -\partial_{xx}u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [0, 1]^2$ and $\mathcal{M} = [0, 1]^3$ ($p = 3$ parameters).

Analytical solution : $u(x; \boldsymbol{\mu}) = \mu_1 \sin(2\pi x) + \mu_2 \sin(4\pi x) + \mu_3 \sin(6\pi x)$.

Construction of two priors: MLP of 6 layers; Adam optimizer (10000 epochs).

Imposing the Dirichlet BC exactly in the PINN with $\varphi(x) = x(x - 1)$.

- **Physics-informed training:** $N_{\text{col}} = 5000$ collocation points.

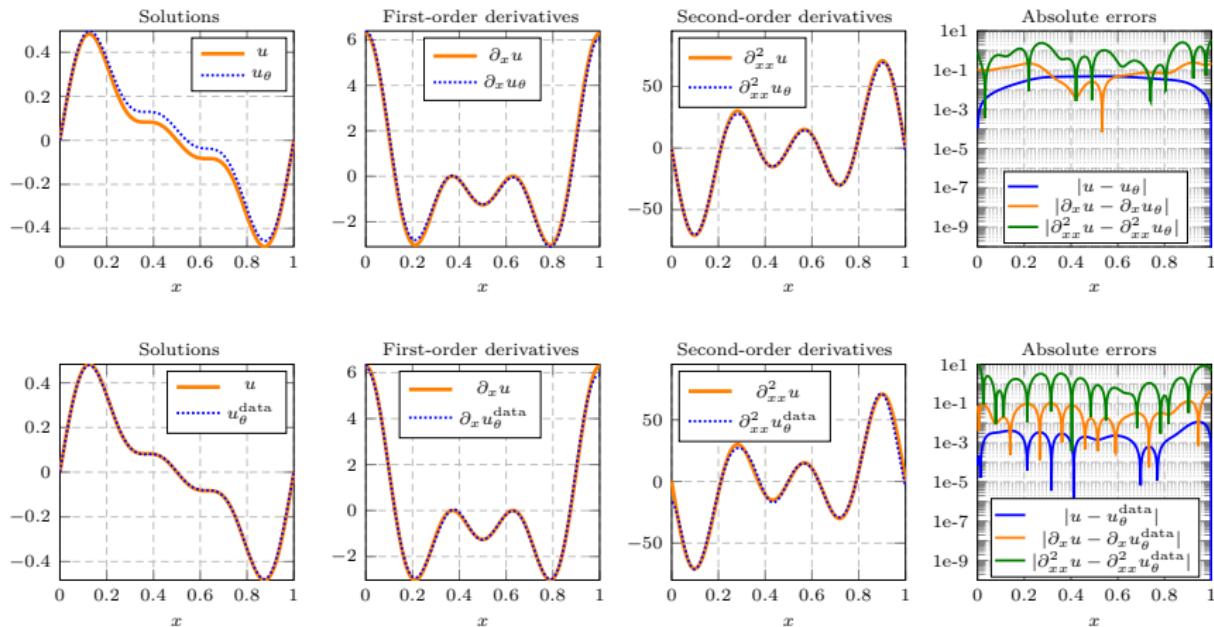
$$J_r(\theta) \simeq \frac{1}{N_{\text{col}}} \sum_{i=1}^{N_{\text{col}}} \left| \partial_{xx}u_\theta(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) + f(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) \right|^2.$$

- **Data-driven training:** $N_{\text{data}} = 5000$ data.

$$J_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \left| u_\theta^{\text{data}}(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) - u(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) \right|^2.$$

A2 – Priors derivatives

$$\mu^{(1)} = (0.3, 0.2, 0.1)$$



A2 – Additive approach in \mathbb{P}_1

1 set of parameters: $\mu^{(1)} = (0.3, 0.2, 0.1)$

FEM		PINN prior u_θ			Data prior u_θ^{data}	
N	error	N	error	gain	error	gain
16	$5.18 \cdot 10^{-2}$	16	$1.29 \cdot 10^{-3}$	40.34	$3.51 \cdot 10^{-3}$	14.78
32	$1.24 \cdot 10^{-2}$	32	$3.49 \cdot 10^{-4}$	35.41	$8.8 \cdot 10^{-4}$	14.06

50 set of parameters:

Gains in L^2 rel error of our method w.r.t. FEM						
PINN prior u_θ				Data prior u_θ^{data}		
N	min	max	mean	min	max	mean
20	26.49	271.92	140.74	6.91	60.85	26.12
40	23.4	258.37	134.11	7.13	39.34	20.55

N : Nodes.

Appendix 1 : Standard methods

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Appendix 2 : Data-driven vs Physics-Informed training

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Appendix 3 : Multiplicative approach

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Appendix 4 : More

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Appendix 3 : Multiplicative approach

LECOURTIER Frédérique

Development of an hybrid finite element and neural network method

A3 – Multiplicative approach

Lifted problem : Considering M such that $u_M = u + M > 0$ on Ω ,

$$\begin{cases} \mathcal{L}(u_M) = f, & \text{in } \Omega, \\ u_M = M, & \text{on } \partial\Omega. \end{cases}$$

Variational Problem : Let $u_{\theta,M} = u_\theta + M \in M + H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

Find $p_h^\times \in 1 + V_h^0$ such that, $\forall v_h \in V_h^0$, $a(u_{\theta,M} p_h^\times, u_{\theta,M} v_h) = I(u_{\theta,M} v_h)$, (\mathcal{P}_h^\times)

with the **enriched trial space** V_h^\times defined by

$$\{u_{\theta,M}^\times = u_{\theta,M} p_h^\times, \quad p_h^\times \in 1 + V_h^0\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^\times = \frac{g + M}{u_{\theta,M}} \quad \text{on } \partial\Omega,$$

with $u_{\theta,M}$ the PINN prior.

A3 – Convergence analysis

Theorem 4: Convergence analysis of the enriched FEM [F. Lecourtier et al., 2025]

We denote $u_{h,M}^X \in V_h^X$ the solution of (\mathcal{P}_h^X) with V_h^X the enriched trial space.
Then, denoting $u_h^X = u_{h,M}^X - M$,

$$|u - u_h^X|_{H^1} \leqslant \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^X\|_{L^2} \leqslant C_{\theta,M} \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{|u|_{H^{q+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

with

$$C_{\theta,M} = \|u_{\theta,M}^{-1}\|_{L^\infty} + 2|u_{\theta,M}^{-1}|_{W^{1,\infty}} + |u_{\theta,M}^{-1}|_{W^{2,\infty}}.$$

A3 – Additive vs Multiplicative

Theorem 5: [F. Lecourtier et al., 2025]

We have

$$\left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{\|u\|_{H^{q+1}}} \xrightarrow[M \rightarrow \infty]{} \frac{|u - u_\theta|_{H^{k+1}}}{\|u\|_{H^{k+1}}},$$

in H^1 semi-norm and

$$C_{\theta,M} \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{\|u\|_{H^{q+1}}} \xrightarrow[M \rightarrow \infty]{} \frac{|u - u_\theta|_{H^{k+1}}}{\|u\|_{H^{k+1}}},$$

in L^2 norm.

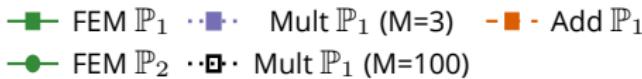
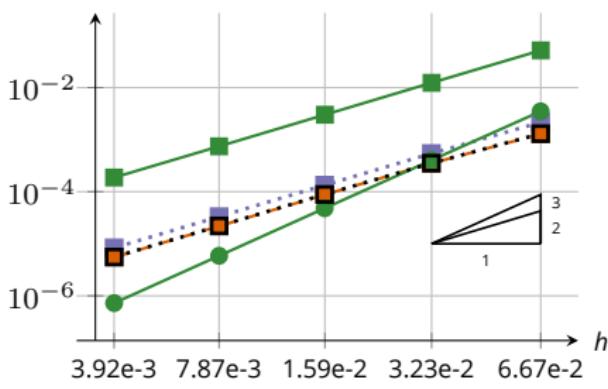
Multiplicative and Additive approaches.

A3 – Numerical results

Considering the 1D Poisson problem of [Appendix 2](#).

Error estimates : 1 set of parameters.

$$L^2 \quad \mu^{(1)} = (0.3, 0.2, 0.1)$$



Appendix 1 : Standard methods



Appendix 2 : Data-driven vs Physics-Informed training



Appendix 3 : Multiplicative approach



Appendix 4 : More



Appendix 4 : More

A4.1 – Adaptive mesh refinement

Dorfler marking strategy : [Dörfler, 1996]

Find $\mathcal{M}_h \subset \mathcal{T}_h$ of minimal cardinality such that

$$\sum_{T \in \mathcal{M}_h} \eta_{\bullet,T}^2 \geq \theta \sum_{T \in \mathcal{T}_h} \eta_{\bullet,T}^2,$$

with $\eta_{\bullet,T}$ a local estimator¹ and $\theta \in (0, 1)$.

¹For instance, the residual estimator. [Ainsworth and Oden, 1997]