2nd CSI

Development of hybrid finite element/neural network methods to help create digital surgical twins

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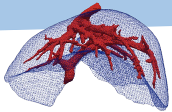
☆ - Update 2025

Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.

accurate quick + parameterized

☆ **Parametric elliptic convection/diffusion PDE :** For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{L}\big(u\,;\boldsymbol{x},\boldsymbol{\mu}\big) = f(\boldsymbol{x},\boldsymbol{\mu}), \qquad (\mathcal{P})$$

where $\mathcal{L}$ is the parametric differential operator defined by

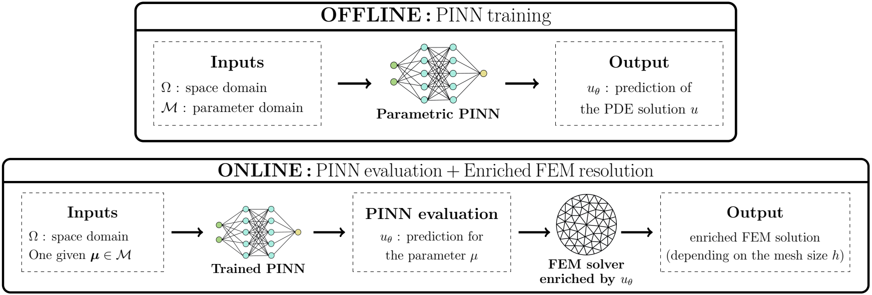
$$\mathcal{L}(\cdot;\boldsymbol{x},\boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x},\boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\mathrm{Pe}}\nabla \cdot (D(\boldsymbol{x},\boldsymbol{\mu})\nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Pipeline of the Enriched FEM

Enriched FEM = Combination of 2 standard methods

- **PINNs** : Physics Informed Neural Networks Appendix 1.1
- **FEMs** : Finite Element Methods Appendix 1.2



Remark : The PINN prediction enriched Finite element approximation spaces.

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Additive approach
☆ Numerical results

This section is based on [**F. Lecourtier** et al., 2025].

Enriched finite element method using PINNs

Additive approach

★ Numerical results

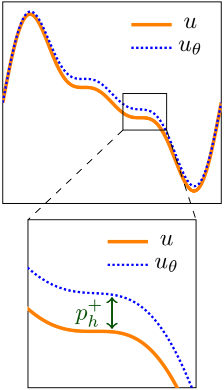
Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathcal{P}_h^+)$$

with the enriched trial space V_h^+ defined by

$$V_h^+ = \left\{ u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0 \right\}.$$



General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leqslant C_{H^1}\, h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [**F. Lecourtier** et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}} \right),$$

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}} \right).$$

Gains of the additive approach.

Enriched finite element method using PINNs

Additive approach

☆ Numerical results

1st problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0,1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$).

Right-hand side :

$$f(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{0.025\sigma^2}\right).$$

Diffusion matrix : (symmetric and positive definite)

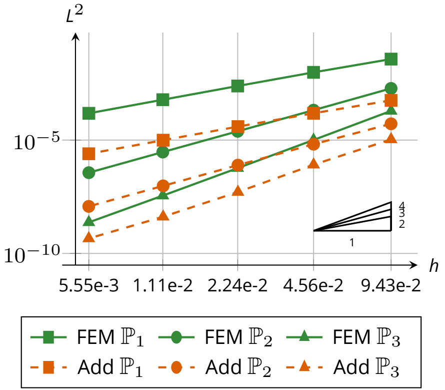
$$D(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

PINN training: Imposing BC exactly with a level-set function.

Numerical results

Error estimates : 1 set of parameters.

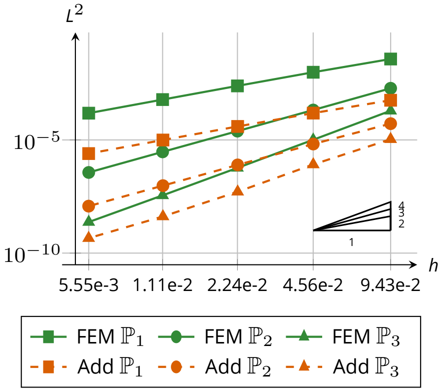
$$\boldsymbol{\mu} = (0.51, 0.54, 0.52, 0.55)$$



Numerical results

Error estimates : 1 set of parameters.

$$\boldsymbol{\mu} = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$N = 20$

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : N^2 nodes.

2nd problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \dfrac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2,\ 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$).
Γ_E and Γ_I are the outer and inner boundaries of the annulus Ω, respectively.

Analytical solution :

$$u(\boldsymbol{x}; \boldsymbol{\mu}) = 1 - \frac{\ln\left(\mu_1 \sqrt{x^2 + y^2}\right)}{\ln(4)},$$

Boundary conditions :

$$g(\boldsymbol{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\boldsymbol{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

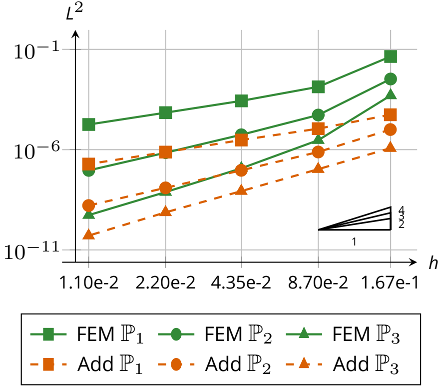
PINN training: Imposing mixed BC exactly in the PINN[1].

[1] [Sukumar and Srivastava, 2022]

Numerical results

Error estimates : 1 set of parameters.

$$\mu = 2.51$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

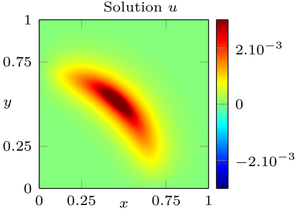
k	Gains in L^2 rel error of our method w.r.t. FEM		
	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

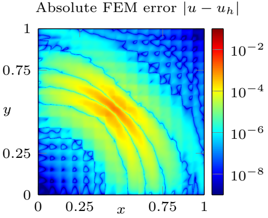
$$h = 1.33 \cdot 10^{-1}$$

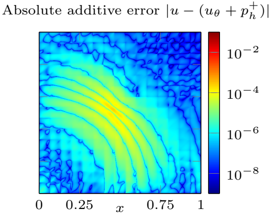
Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Numerical solutions

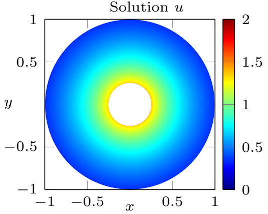
1st problem : $\boldsymbol{\mu} = (0.46, 0.52, 0.05, 0.12)$ $\quad$ ($k = 2, h = 9,43 \cdot 10^{-2}$)

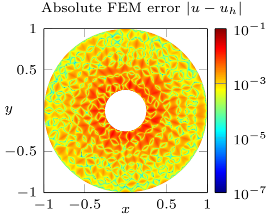


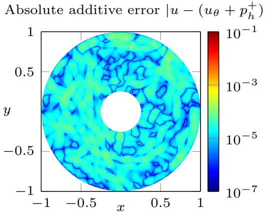




2nd problem : $\boldsymbol{\mu} = 2.51$ $\quad$ ($k = 1, h = 1,67 \cdot 10^{-1}$)







New lines of research

Complex geometries

★ A posteriori error estimates

★ Non linear PDEs

New lines of research

Complex geometries

★ A posteriori error estimates

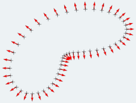
★ Non linear PDEs

Learn a regular levelset

Theorem 3: [Clémot and Digne, 2023]

If we have a boundary domain Γ, the SDF is solution to the Eikonal equation:

$$\begin{cases} ||\nabla \phi(X)|| = 1, \; X \in \mathcal{O} \\ \phi(X) = 0, \; X \in \Gamma \\ \nabla \phi(X) = n, \; X \in \Gamma \end{cases}$$



with $\mathcal{O}$ a box which contains Ω completely and n the exterior normal to Γ.

Objective: Move on to complex geometries by using a levelset function to

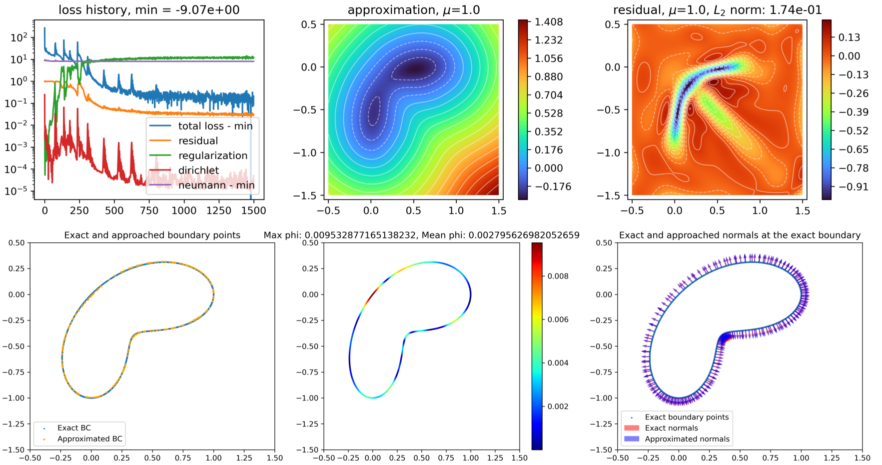
- Sample points in the domain Ω for the PINN training.
- Impose exactly the boundary condition in PINN [Sukumar and Srivastava, 2022].

How to learn a regular levelset ? with a PINN by adding a regularization term,

$$J_{reg} = \int_{\mathcal{O}} |\Delta \phi|^2,$$

and a sample of boundary points that considers the curvature of Γ. ☆

Numerical results



New lines of research

- Complex geometries
- ★ A posteriori error estimates
- ★ Non linear PDEs

Problem considered

Problem statement: Considering the Poisson problem with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \Gamma \times \mathcal{M}, \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$).

Analytical solution :

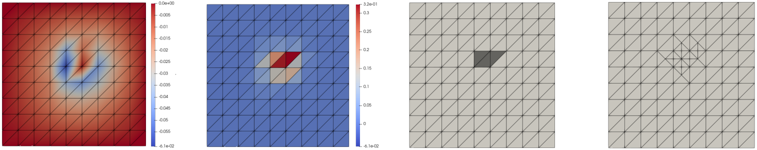
$$u(\boldsymbol{x}; \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2(0.15)^2}\right) \sin(2x)\sin(2y).$$

PINN training: Imposing Dirichlet BC exactly in the PINN.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy. Appendix 4.1

Standard FEM



$\cdots \longrightarrow$ SOLVE on u_h $\longrightarrow$ ESTIMATE $\eta_{res,T}$ $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

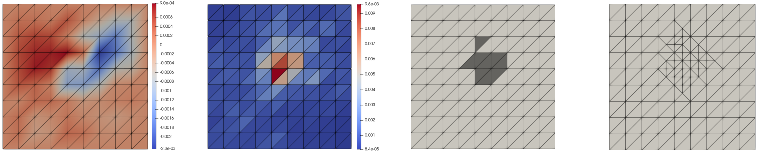
$$\eta_{res,T}^2 = h_T^2 \|\Delta u_h + f_h\|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E \|[\nabla u_h \cdot n]\|_{L^2(E)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach



$\cdots \longrightarrow$ SOLVE on p_h^+ $\longrightarrow$ ESTIMATE $\eta_{res,T}$ $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

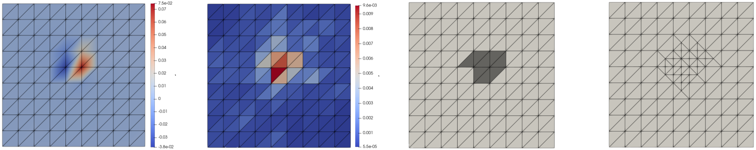
$$\eta_{res,T}^2 = h_T^2 \| \left((\Delta u_\theta)_h + \Delta p_h^+ \right) + f_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E \| [\nabla p_h^+ \cdot n] \|_{L^2(E)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach - No resolution



$\cdots \longrightarrow$ INTERPOLATE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

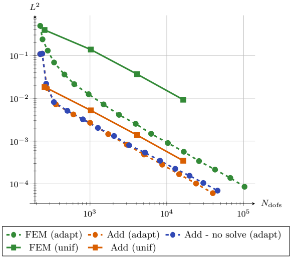
u_θ $\eta_{res,T}$

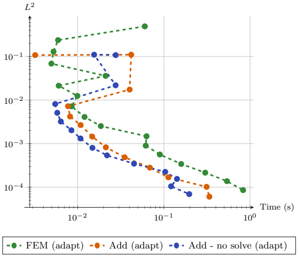
Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

$$\eta_{res,T}^2 = h_T^2 \|(\Delta u_\theta)_h + f_h\|_{L^2(T)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Numerical results





⚠ Results obtained on a laptop GPU (Time measurements polluted by external factors).

Ideas for improving results : Additive approach (no resolution).

time ↘ Interpolate only mesh points added in the refinement process.

error ↘ Use another metric such as curvature, rather than residual error.

New lines of research

- Complex geometries
- ☆ A posteriori error estimates
- ★ Non linear PDEs

Problem considered

Objective: Extend the additive approach to non linear PDEs.

Problem statement: Considering the non linear Poisson problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}\big((1+4u^4)\nabla u\big) = f, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega. \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$).

Analytical solution :

$$u(\boldsymbol{x};\boldsymbol{\mu}) = 1 + \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right) \sin(2x)\sin(2y)$$

PINN training: Imposing BC exactly with a level-set function.

Newton method

We want to solve the non linear system:

N_h : number of degrees of freedom.

$$F(u) = 0 \tag{1}$$

with $F : \mathbb{R}^{N_h} \rightarrow \mathbb{R}^{N_h}$ a non linear operator and $u \in \mathbb{R}^{N_h}$ the unknown vector.

Algorithm 1: Newton's method to solve (1) [Aghili et al., 2025]

Initialization step: set $u^{(0)} = u_0$;
for $k \geq 0$ **do**
  Solve the linear system $F(u^{(k)}) + F'(u^{(k)})\delta^{(k+1)} = 0$ for $\delta^{(k+1)}$;
  Update $u^{(k+1)} = u^{(k)} + \delta^{(k+1)}$;
end

Standard version:

Initialization with a constant value u_0. For instance, $u_0 = 1$.

DeepPhysics version: [Odot et al., 2021]

Initialization with a PINN solution $u_0 = u_\theta$.

Newton method

We want to solve the non linear system:

N_h : number of degrees of freedom.

$$F(p_+ + u_\theta) = 0 \tag{1}$$

with $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ a non linear operator and $p_+ \in \mathbb{R}^{N_h}$ the unknown vector.

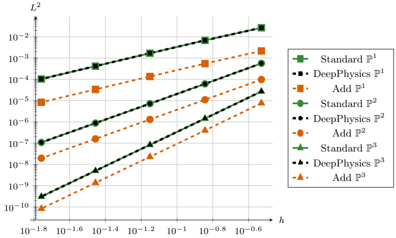
Algorithm 2: Additive approach to solve (1)

Initialization step: set $p_+^{(0)} = 0$;
for $k \geq 0$ **do**
  Solve the linear system $F(p_+^{(k)} + u_\theta) + F'(p_+^{(k)} + u_\theta)\delta^{(k+1)} = 0$ for $\delta^{(k+1)}$;
  Update $p_+^{(k+1)} = p_+^{(k)} + \delta^{(k+1)}$;
end

Advantage compared to DeepPhysics:

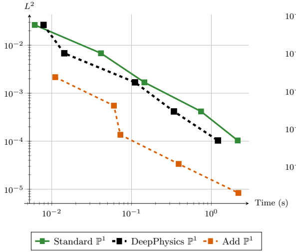
u_θ is not required to live in the same space as p_+.

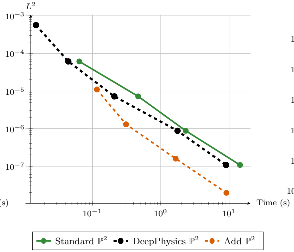
Numerical results

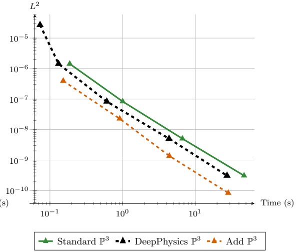


Number of iterations :

- Standard Newton: 8 iterations.
- DeepPhysics: 4 iterations.
- Additive approach: 4 iterations.







Supplementary work

Supplementary work I

Teaching

- 2024/2025 :
  - 64h of Computer Science Practical Work - L1S2 and L2S3 (Python) / L3S6 (C++)
  - 3 days supervising a group of high school girls in RJMI ("Rendez-vous des Jeunes Mathématiciennes et Informaticiennes")
- 2023/2024 : 50h of Computer Science Practical Work - L2S3 (Python) / L3S6 (C++)

Training courses (Total : 176h35)

- A dozen seminars organized by IRMA ($\approx 10h$)
- 1 Deep Learning introductory course - FIDLE ($\approx 40h$)
- 2 workshops on Scientific Machine Learning ($\approx 2 \times 21h$)
- 1 summer school on "New Trend in computing" ($\approx 27h$)
- several cross-disciplinary courses - Methodology, scientific English, etc. ($\approx 58h$)

Supplementary work II

Talks

- **ICOSAHOM 2025, Montréal** - July 2025 *(Coming soon...)*
  ”Enriching continuous Lagrange finite element approximation spaces using neural networks”
- **DTE & AICOMAS 2025, Paris** - February 20, 2025
  ”Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries”
- **Exama project, WP2 reunion** - March 26, 2024
  ”How to work with complex geometries in PINNs ?”
- **Retreat (Macaron/Tonus)** - February 6, 2024
  ”Mesh-based methods and physically informed learning”
- **Team meeting (Mimesis)** - December 12, 2023
  ”Development of hybrid finite element/neural network methods to help create digital surgical twins”

Supplementary work III

Posters

- **EMS-TAG-SciML 2025, Milan** - March 24, 2025 - "Enriching continuous Lagrange finite element approximation spaces using neural networks"
- **CJC-MA 2024, Lyon** - October 29, 2024 - "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- **MSII poster day, Strasbourg** - October 24, 2024
- **SciML 2024, Strasbourg** - July 08, 2024

Publications

- Enriching continuous lagrange finite element approximation spaces using neural networks. *(submitted in February 2025, M2AN journal)*
  H. Barucq, M. Duprez, F. Faucher, E. Franck, **F. Lecourtier**, V. Lleras, V. Michel-Dansac, and N. Victorion.

Conclusion

Enriched finite element method using PINNs :

- PINNs are good candidates for the enriched approach. Appendix 2
- Numerical validation of the theoretical results.
- The enriched approach provides the same results as the standard FEM method, but with coarser meshes. $\Rightarrow$ Reduction of the computational cost.

We have also tested a multiplicative approach. Appendix 3

New lines of research :

- The treatment of complex geometries is progressing.
- New PDEs begin to be considered, in particular non-linear problems.
- Other methods for improving the additive approach are being studied, including a posteriori error estimators.

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Appendix 1 : Standard methods

A1.1 – Physics-Informed Neural Networks

Standard PINNs[1] **(Weak BC) :** Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r\, J_r(\theta) + \omega_b\, J_b(\theta)\right), \qquad (\mathcal{P}_\theta)$$

with

residual loss $\boxed{J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left|\mathcal{L}\left(u_\theta(\boldsymbol{x},\boldsymbol{\mu}); \boldsymbol{x},\boldsymbol{\mu}\right) - f(\boldsymbol{x},\boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$

boundary loss $\boxed{J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \left|u_\theta(\boldsymbol{x},\boldsymbol{\mu}) - g(\boldsymbol{x},\boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In $(\mathcal{P}_\theta)$, ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

[1] [Raissi et al., 2019]

A1.1 – Physics-Informed Neural Networks

Improved PINNs[1] (Strong BC) : Find the optimal weights $\theta^\star$ such that

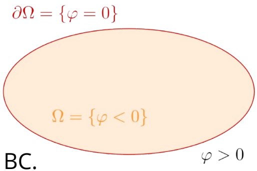
$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)} \right),$$

with $\omega_r = 1$ and

residual loss $$\boxed{J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| \mathcal{L}\left(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\right) - f(\boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

where u_θ is a neural network defined by

$$u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \varphi(\boldsymbol{x}) w_\theta(\boldsymbol{x}, \boldsymbol{\mu}) + g(\boldsymbol{x}, \boldsymbol{\mu}),$$



with φ a level-set function, w_θ a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_\theta = g$ on $\partial\Omega$.

[1] [Lagaris et al., 1998; Franck et al., 2024]

A1.2 – Finite Element Methods[1]

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

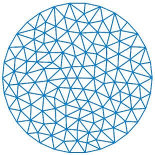
with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\mathrm{Pe}} \int_\Omega D\nabla u_h \cdot \nabla v_h + \int_\Omega R\, u_h\, v_h + \int_\Omega v_h\, C \cdot \nabla u_h, \quad l(v_h) = \int_\Omega f\, v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega),\ \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where $\mathbb{P}_k$ is the space of polynomials of degree at most k.



$\mathcal{T}_h = \{K_1, \ldots, K_{N_e}\}$
(N_e : number of elements)

Linear system : Let $(\phi_1, \ldots, \phi_{N_h})$ a basis of V_h^0.

Find $U \in \mathbb{R}^{N_h}$ such that $\qquad AU = b$

with

$$A = \big(a(\phi_i, \phi_j)\big)_{1 \le i,j \le N_h} \quad \text{and} \quad b = \big(l(\phi_j)\big)_{1 \le j \le N_h}.$$

[1] [Ern and Guermond, 2004]

Appendix 2 : Data-driven vs Physics-Informed training

A2 – Problem considered

Problem statement: Consider the Poisson problem in 1D with Dirichlet BC:

$$\begin{cases} -\partial_{xx} u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [0,1]^2$ and $\mathcal{M} = [0,1]^3$ ($p = 3$ parameters).

Analytical solution : $u(x;\boldsymbol{\mu}) = \mu_1 \sin(2\pi x) + \mu_2 \sin(4\pi x) + \mu_3 \sin(6\pi x)$.

Construction of two priors: MLP of 6 layers; Adam optimizer (10000 epochs).
Imposing the Dirichlet BC exactly in the PINN with $\varphi(x) = x(x-1)$.

- **Physics-informed training:** $N_{\text{col}} = 5000$ collocation points.

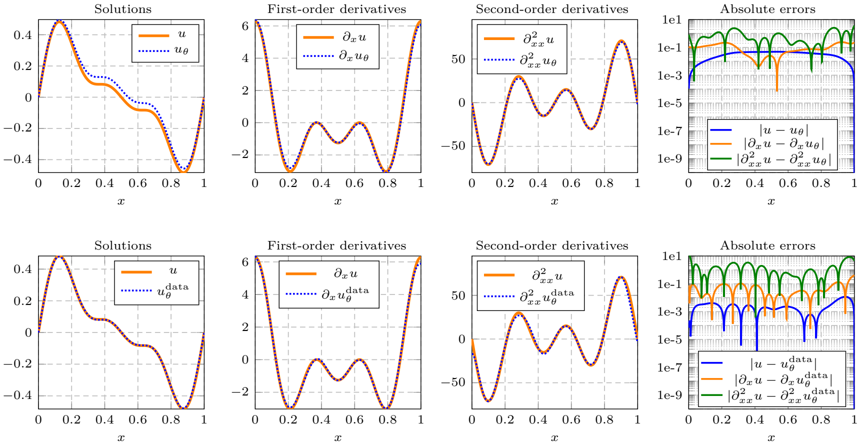
$$J_r(\theta) \simeq \frac{1}{N_{\text{col}}} \sum_{i=1}^{N_{\text{col}}} \left| \partial_{xx} u_\theta\left(\boldsymbol{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}\right) + f\left(\boldsymbol{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}\right) \right|^2.$$

- **Data-driven training:** $N_{\text{data}} = 5000$ data.

$$J_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \left| u_\theta^{\text{data}}\left(\boldsymbol{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}\right) - u\left(\boldsymbol{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}\right) \right|^2.$$

A2 – Priors derivatives

$$\boldsymbol{\mu}^{(1)} = (0.3, 0.2, 0.1)$$



A2 – Additive approach in $\mathbb{P}_1$

1 set of parameters: $\boldsymbol{\mu}^{(1)} = (0.3, 0.2, 0.1)$

	FEM
N	**error**
16	$5.18 \cdot 10^{-2}$
32	$1.24 \cdot 10^{-2}$

	PINN prior u_θ		Data prior u_θ^{data}	
N	**error**	**gain**	**error**	**gain**
16	$1.29 \cdot 10^{-3}$	40.34	$3.51 \cdot 10^{-3}$	14.78
32	$3.49 \cdot 10^{-4}$	35.41	$8.8 \cdot 10^{-4}$	14.06

50 set of parameters:

	Gains in L^2 rel error of our method w.r.t. FEM					
	PINN prior u_θ			Data prior u_θ^{data}		
N	**min**	**max**	**mean**	**min**	**max**	**mean**
20	26.49	271.92	140.74	6.91	60.85	26.12
40	23.4	258.37	134.11	7.13	39.34	20.55

N : Nodes.

Appendix 3 : Multiplicative approach

A3 – Multiplicative approach

Liffted problem : Considering M such that $u_M = u + M > 0$ on Ω,

$$\begin{cases} \mathcal{L}(u_M) = f, & \text{in } \Omega, \\ u_M = M, & \text{on } \partial\Omega. \end{cases}$$

Variational Problem : Let $u_{\theta,M} = u_\theta + M \in M + H^{k+1}(\Omega) \cap H^1_0(\Omega)$.

$$\text{Find } p_h^\times \in 1 + V_h^0 \text{ such that}, \forall v_h \in V_h^0, a\big(u_{\theta,M}\, p_h^\times, u_{\theta,M} v_h\big) = l(u_{\theta,M} v_h), \qquad (\mathcal{P}_h^\times)$$

with the enriched trial space $V_h^\times$ defined by

$$\left\{ u_{h,M}^\times = u_{\theta,M}\, p_h^\times, \quad p_h^\times \in 1 + V_h^0 \right\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^\times = \frac{g+M}{u_{\theta,M}} \quad \text{on } \partial\Omega,$$

with $u_{\theta,M}$ the PINN prior.

A3 – Convergence analysis

Theorem 4: Convergence analysis of the enriched FEM [F. Lecourtier et al., 2025]

We denote $u_{h,M}^\times \in V_h^\times$ the solution of $(\mathcal{P}_h^\times)$ with $V_h^\times$ the enriched trial space.
Then, denoting $u_h^\times = u_{h,M}^\times - M$,

$$|u - u_h^\times|_{H^1} \leqslant \boxed{\left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}}\right),$$

$$\|u - u_h^\times\|_{L^2} \leqslant \boxed{C_{\theta,M} \left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{|u|_{H^{q+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

with

$$C_{\theta,M} = \|u_{\theta,M}^{-1}\|_{L^\infty} + 2|u_{\theta,M}^{-1}|_{W^{1,\infty}} + |u_{\theta,M}^{-1}|_{W^{2,\infty}}.$$

A3 – Additive vs Multiplicative

Theorem 5: [**F. Lecourtier** et al., 2025]

We have

$$\boxed{\left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \underset{M\to\infty}{\longrightarrow} \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}},$$

in H^1 semi-norm and

$$\boxed{C_{\theta,M} \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|^2_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \underset{M\to\infty}{\longrightarrow} \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}},$$

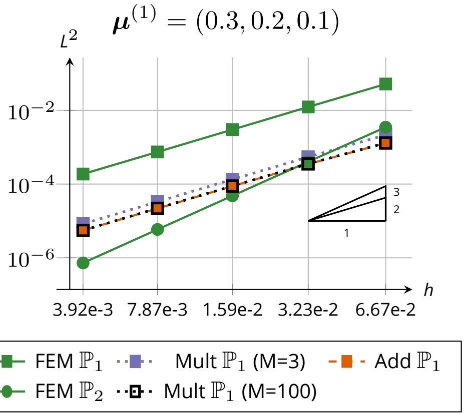
in L^2 norm.

Multiplicative and Additive approaches.

A3 – Numerical results

Considering the 1D Poisson problem of Appendix 2.

Error estimates : 1 set of parameters.



Appendix 4 : More

A4.1 – Adaptive mesh refinement

Dorfler marking strategy : [Dörfler, 1996]
Find $\mathcal{M}_h \subset \mathcal{T}_h$ of minimal cardinality such that

$$\sum_{T \in \mathcal{M}_h} \eta_{\bullet,T}^2 \geq \theta \sum_{T \in \mathcal{T}_h} \eta_{\bullet,T}^2,$$

with $\eta_{\bullet,T}$ a local estimator [1] and $\theta \in (0,1)$.

[1] For instance, the residual estimator. [Ainsworth and Oden, 1997]