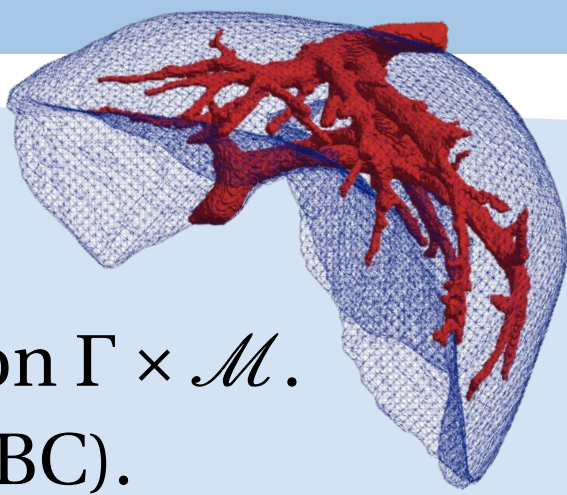
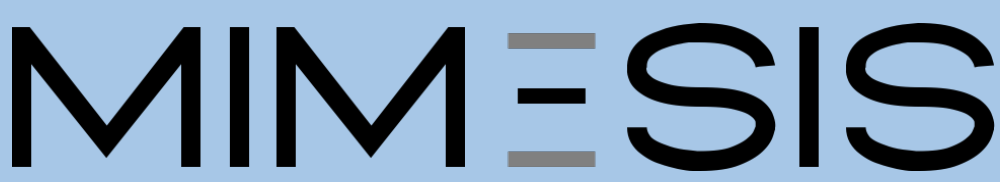


ENRICHING CONTINUOUS LAGRANGE FINITE ELEMENT APPROXIMATION SPACES USING NEURAL NETWORKS

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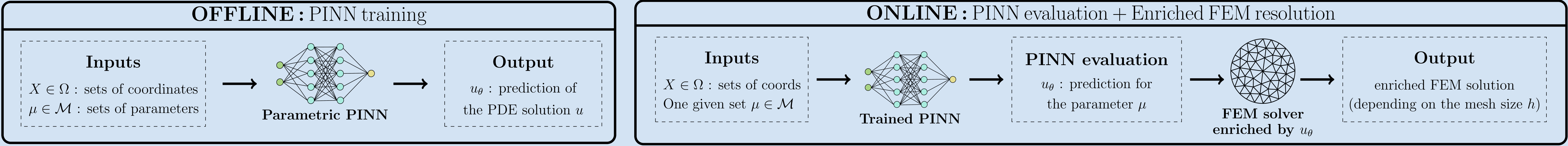
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Motivations

Current Objective : Develop hybrid **finite element** / **neural network** methods.
accurate quick + parameterized

Problem considered : $-\Delta u(X, \mu) = f(X, \mu)$ in $\Omega \times \mathcal{M}$, $u(x, \mu) = 0$ on $\Gamma \times \mathcal{M}$.
Poisson problem with homogeneous Dirichlet boundary conditions (BC).



Long term objective : Create real-time digital twins of an organ (e.g. liver).

How improve PINN prediction ? - Using enriched FEM

Additive approach

The enriched approximation space is defined by

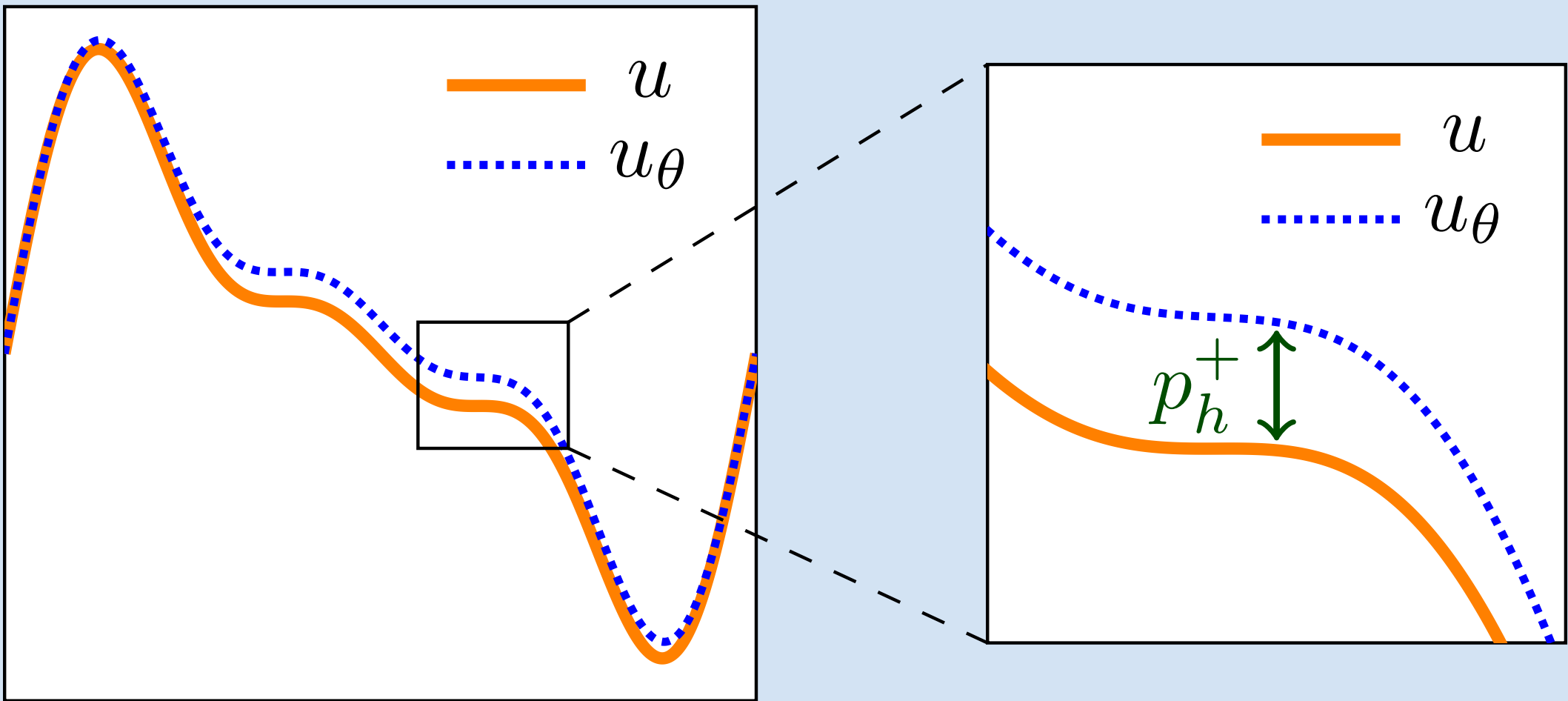
$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, p_h^+ \in V_h^0\}$$

with V_h^0 the standard continuous Lagrange FE space and the weak problem becomes

$$\text{Find } p_h^+ \in V_h^0, \forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \quad (\mathcal{D}_h^+)$$

with modified boundary conditions and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \quad l(v) = \int_\Omega f v.$$



Numerical results - Considered problem

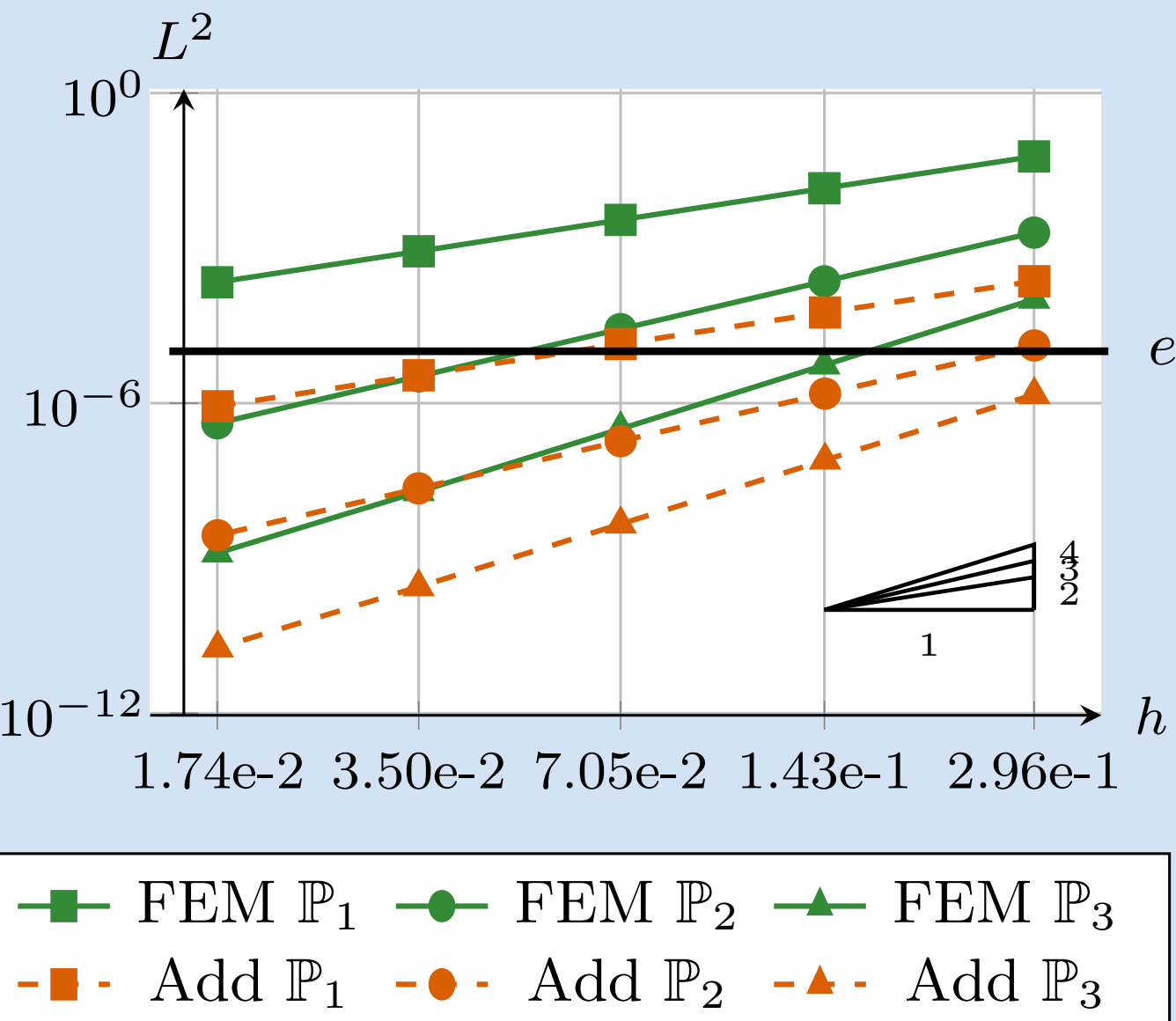
- Spatial domain : $\Omega = [-0.5\pi, 0.5\pi]^2$
- Parametric domain : $\mathcal{M} = [-0.5, 0.5]^2$
- Analytical solution :

$$u_{ex}((x, y), \mu) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y)$$

with $\mu = (\mu_1, \mu_2) \in \mathcal{M}$ (**parametric**) and the associated source term f .

Numerical results - Improve errors

Error estimates : $\mu = (0.05, 0.22)$.



Gains achieved : 50 sets of parameters.

$$S = \{\mu^{(1)}, \dots, \mu^{(50)}\}$$

Gains in L^2 rel error of our method w.r.t. FEM

k	min	max	mean
1	134.32	377.36	269.39
2	67.02	164.65	134.85
3	39.52	72.65	61.55

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : 20^2 nodes.

Convergence analysis

u : solution of the Poisson problem. u_θ : prediction of the PINN [RPK19].

Theorem 1: Convergence analysis of the standard FEM [EG]

We denote $u_h \in V_h^0$ the discrete solution of standard FEM with V_h^0 a \mathbb{P}_k Lagrange space. Thus,

$$|u - u_h|_{H^1} \leq C_{H^1} h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [F L+25]

We denote $u_h^+ \in V_h^+$ the discrete solution of (\mathcal{D}_h^+) with V_h^+ a \mathbb{P}_k Lagrange space. Thus

$$|u - u_h^+|_{H^1} \leq \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

and

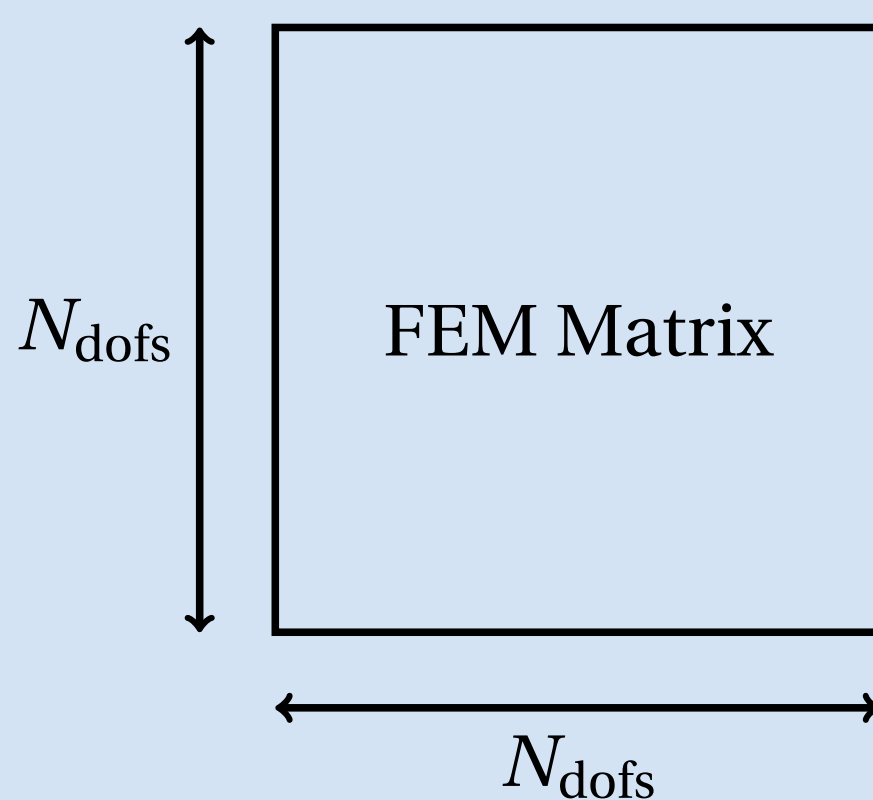
$$\|u - u_h^+\|_{L^2} \leq \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

Theoretical gain of the additive approach.

Numerical results - Improve numerical costs

N_{dofs} required to reach the same error e : $\mu = (0.05, 0.22)$.

k	e	N_{dofs}	
		FEM	Add
1	$1 \cdot 10^{-3}$	14,161	64
	$1 \cdot 10^{-4}$	143,641	576
2	$1 \cdot 10^{-4}$	6,889	225
	$1 \cdot 10^{-5}$	31,329	1,089
3	$1 \cdot 10^{-5}$	6,724	784
	$1 \cdot 10^{-6}$	20,164	2,704

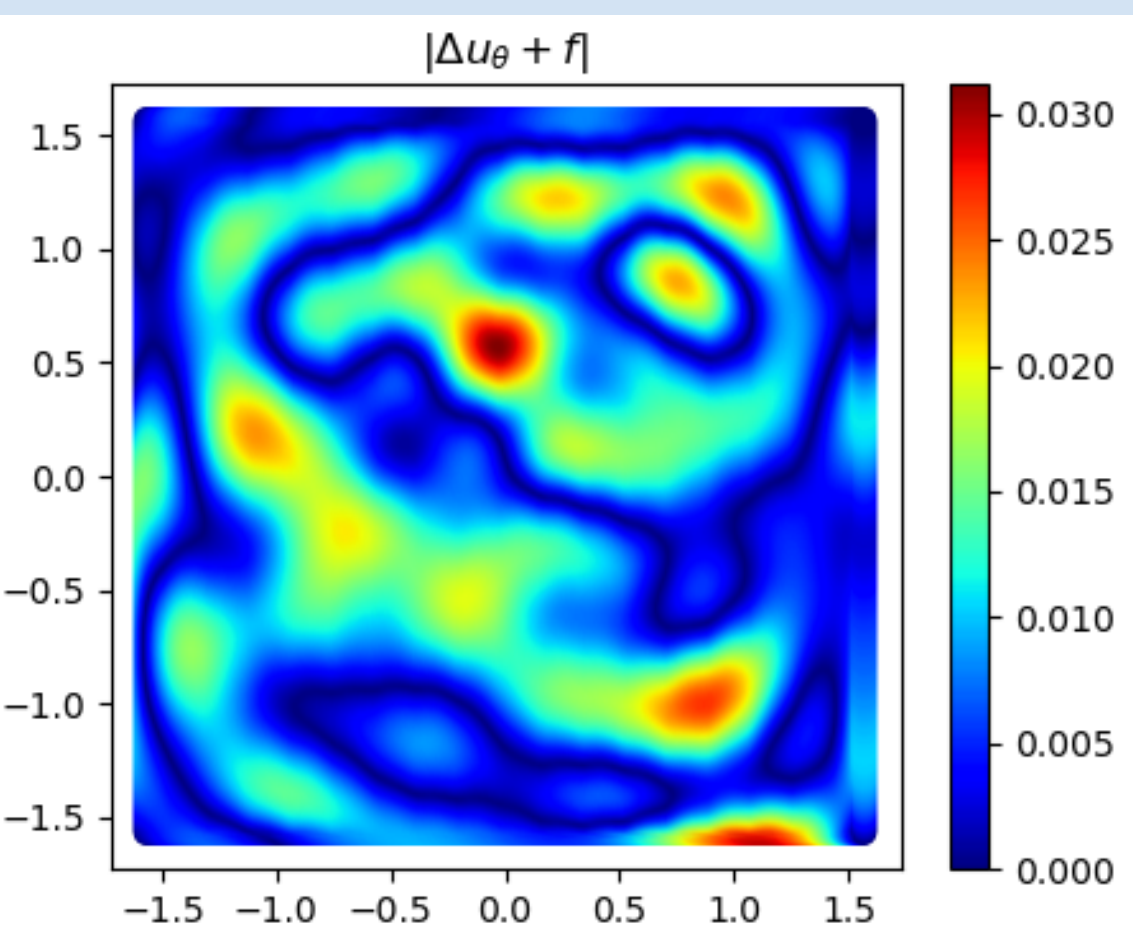


Less degrees of freedom \Rightarrow $\begin{cases} \text{Lower numerical cost} \\ \text{Faster simulation} \end{cases}$

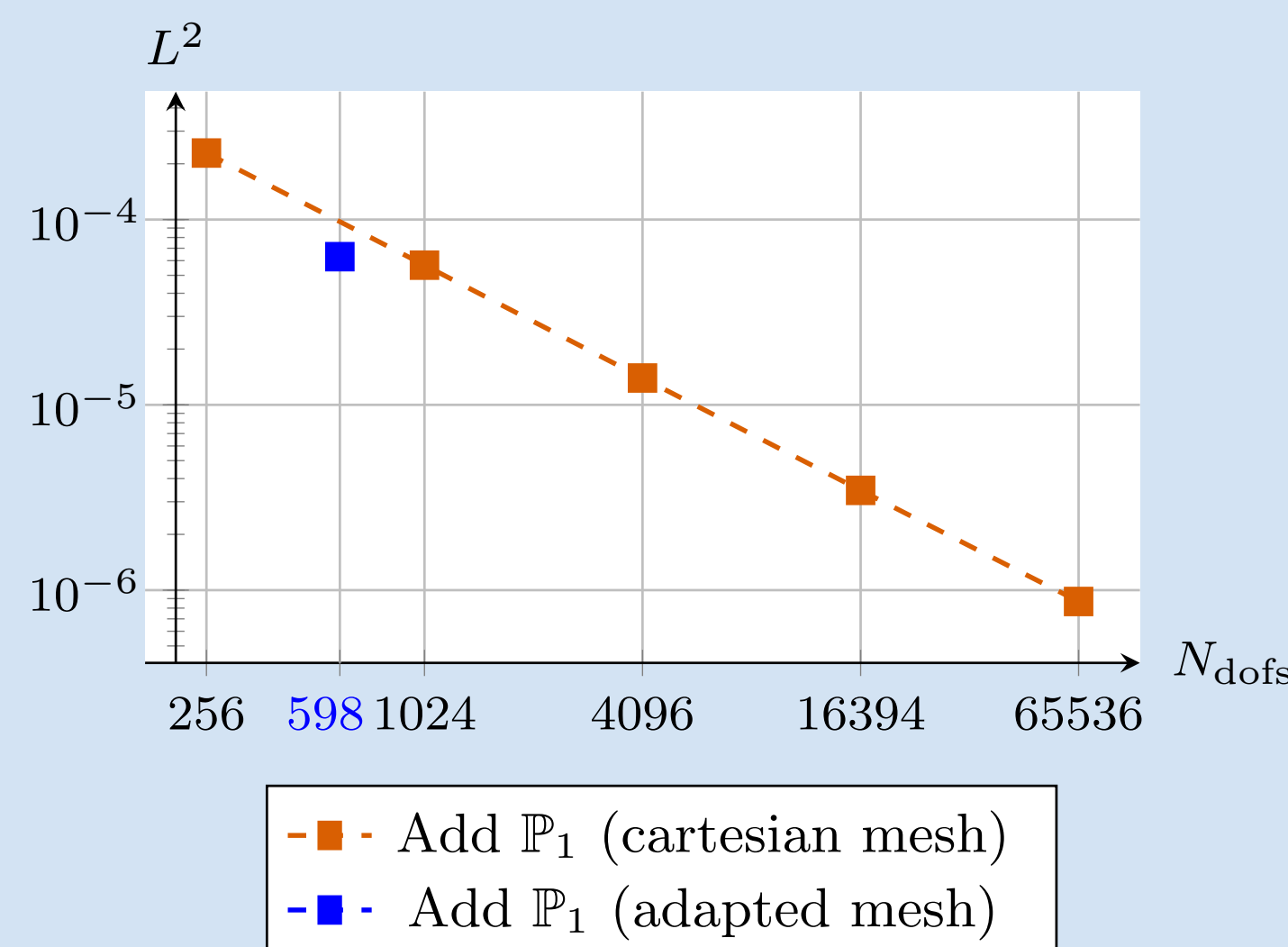
Perspectives

Mesh adaptation

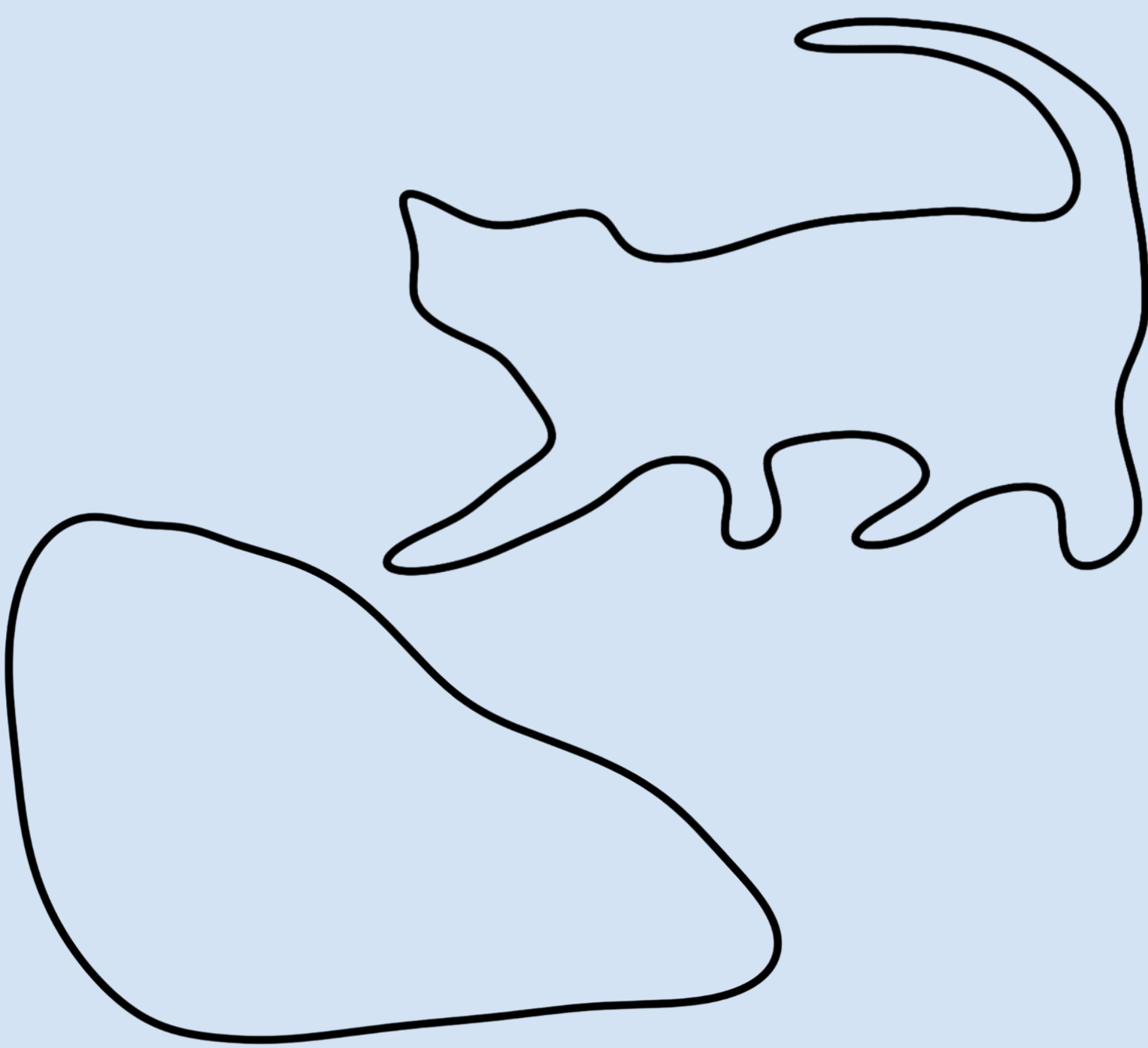
Construction of the mesh: $N_{\text{dofs}} = 598$.



Apply enriched FEM:



More complex geometries



[EG] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. Springer New York (2004).

[F L+25] **F. Lecourtier** et al. *Enriching continuous Lagrange finite element approximation spaces using neural networks*. 2025.

[RPK19] M. Raissi, P. Perdikaris, and G. E. Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *J. Comput. Phys.* 378 (2019), pp. 686–707.