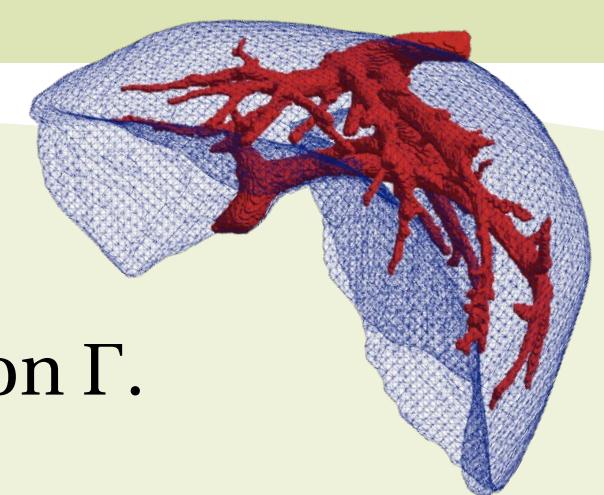


# COMBINING FINITE ELEMENT METHODS AND NEURAL NETWORKS TO SOLVE ELLIPTIC PROBLEM ON COMPLEX 2D GEOMETRIES

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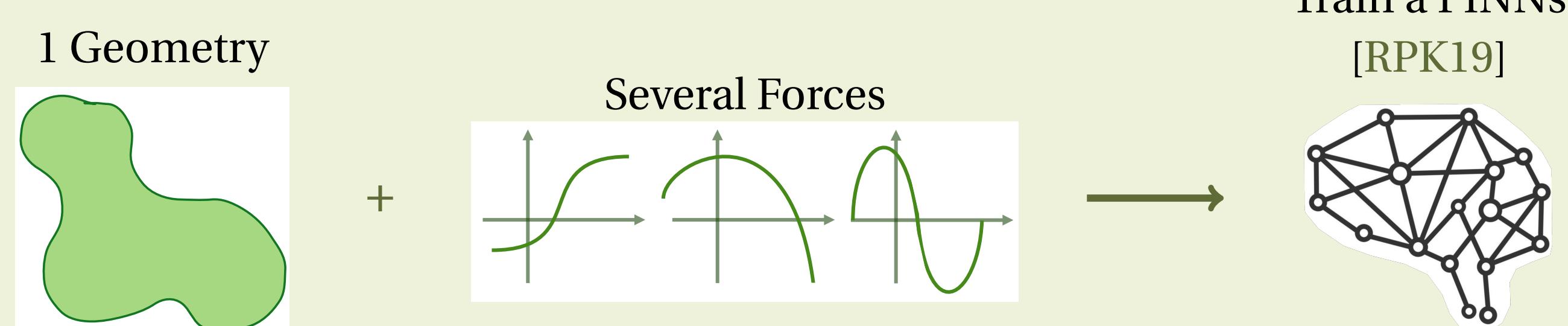


**Current Objective :** Develop hybrid **finite element / neural network** methods.  
accurate    quick + parameterized

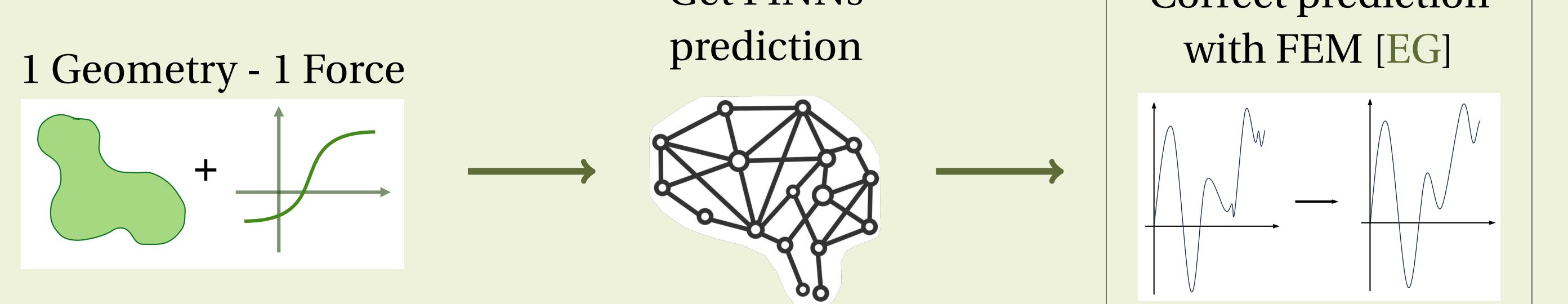
## Motivations

**Problem considered :**  $-\Delta u(x) = f(x)$  in  $\Omega$ ,  $u(x) = g(x)$  on  $\Gamma$ .  
Poisson problem with Dirichlet boundary conditions (BC).

### OFFLINE



### ONLINE



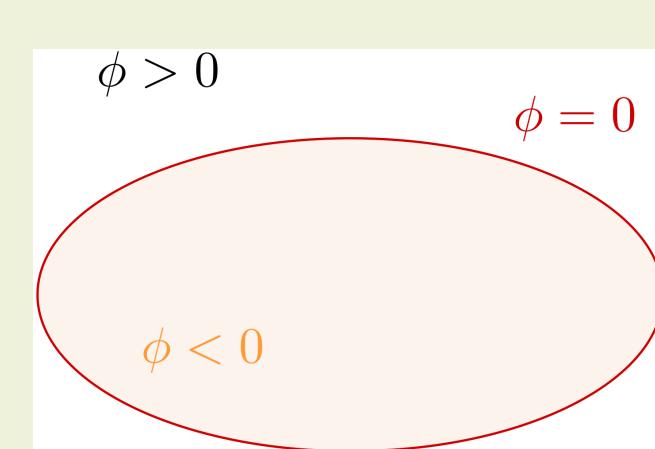
**Perspective :** Create real-time digital twins of an organ (e.g. liver).

## 1. How to deal with complex geometries in PINNs ?



No mesh, so easy to go on complex geometry!

### Approach by levelset. [SS22]



#### Advantages :

- Sample is easy in this case.
- Allow to impose in hard the BC (no BC loss) :  
 $u_\theta(X) = \phi(X) w_\theta(X) + g(X)$   
with  $\phi$  a levelset function and  $w_\theta$  a NN.

### Levelset considered. A regularized Signed Distance Function (SDF).

Theorem 1: Eikonal equation. [CD23]

If we have a boundary domain  $\Gamma$ , the SDF is solution to:

$$\begin{cases} \|\nabla\phi(X)\| = 1, X \in \mathcal{O} & (1) \\ \phi(X) = 0, X \in \Gamma & (2) \\ \nabla\phi(X) = n, X \in \Gamma & (3) \end{cases}$$

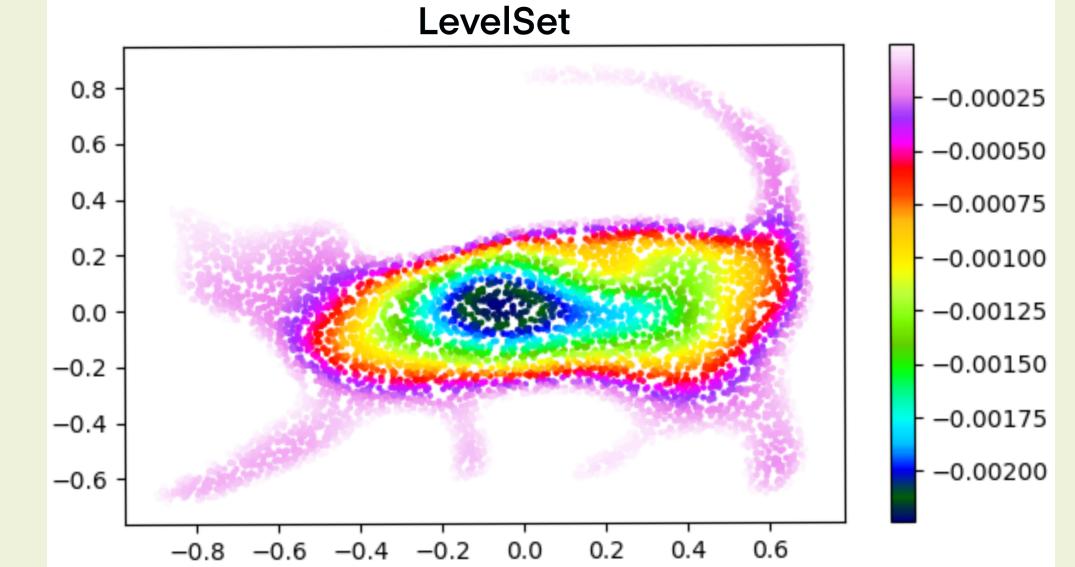
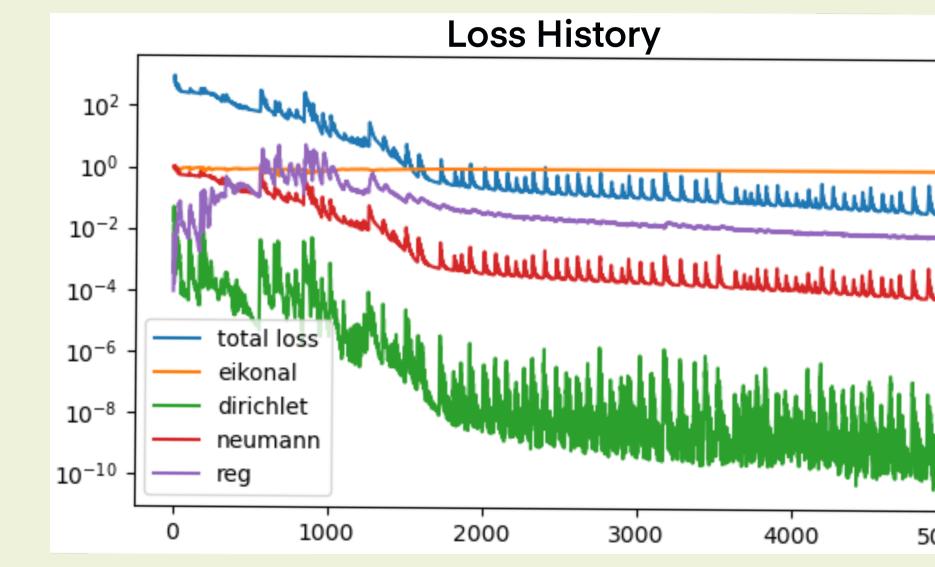
with  $\mathcal{O}$  a box which contains  $\Omega$  completely and  $n$  the exterior normal to  $\Gamma$ .

Approximate  $\phi$ ? with a PINNs [CD23], by adding the following regularization term

$$\mathcal{L} = \underbrace{\int_{\mathcal{O}} (1 - \|\nabla\phi(x)\|)^2 dx}_{(1)} + \underbrace{\int_{\Gamma} |\phi(x)|^2 dx}_{(2)} + \underbrace{\int_{\Gamma} 1 - \frac{n(x) \cdot \nabla\phi(x)}{\|n(x)\| \|\nabla\phi(x)\|} dx}_{(3)} + \underbrace{\int_{\mathcal{O}} |\Delta\phi(x)|^2 dx}_{\text{reg}}$$

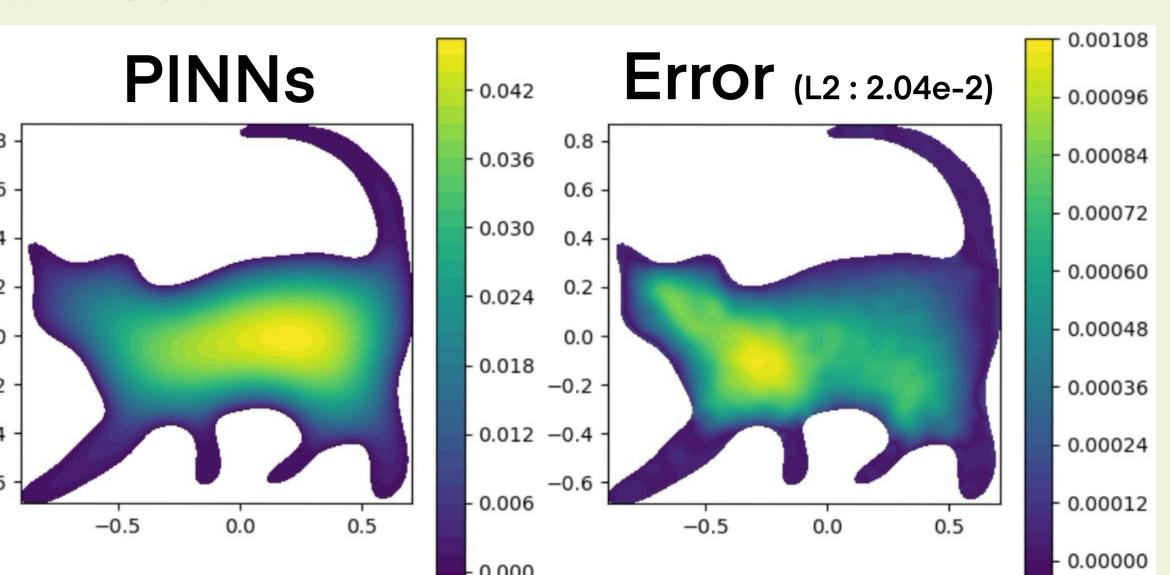
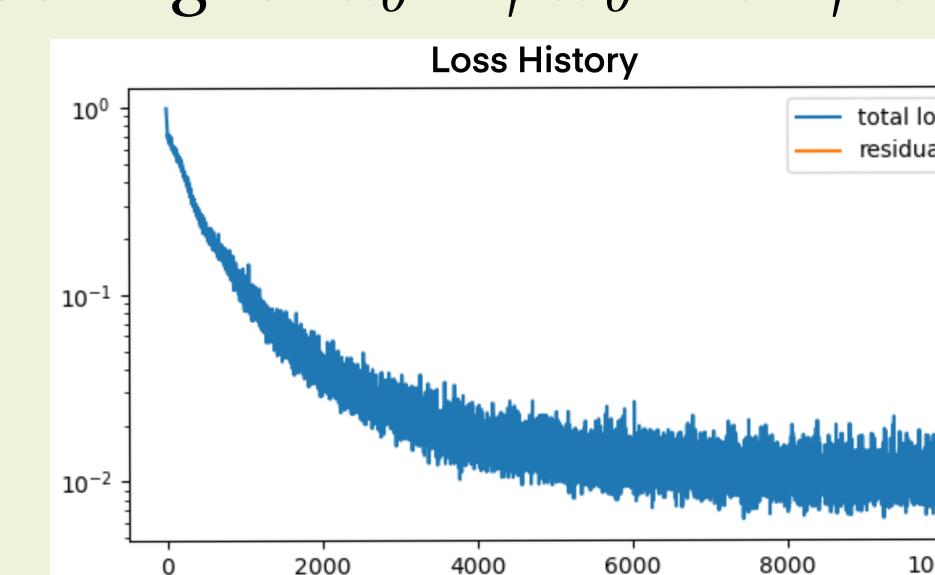
## Numerical Results - Learn a complex levelset

### Levelset learning.



### Poisson problem on Cat.

- Taking  $f = 1$  (**non parametric**) and homogeneous Dirichlet BC ( $g = 0$ ).
- Looking for  $u_\theta = \phi w_\theta$  with  $\phi$  the levelset learned.



## 2. How can we improve PINNs prediction ? - Using FEM

**Additive approach.** Considering  $u_\theta$  as the prediction of our PINNs for the Poisson problem, the correction problem consists in writing the solution as

$$\tilde{u} = u_\theta + \tilde{C}$$

and searching  $\tilde{C} : \Omega \rightarrow \mathbb{R}^d$  such that

$$\begin{cases} -\Delta \tilde{C} = \tilde{f}, \text{ in } \Omega, \\ \tilde{C} = 0, \text{ on } \Gamma, \end{cases} \quad (\mathcal{P}^+)$$

with  $\tilde{f} = f + \Delta u_\theta$ .

**Error estimation.** Considering  $u_\theta$  as the prediction of our PINNs.

Theorem 2: [Lec+]

We denote  $u$  the solution of the Poisson problem and  $u_h$  the discrete solution of the correction problem  $(\mathcal{P}^+)$  with  $V_h$  a  $\mathbb{P}_k$  Lagrange space. Thus

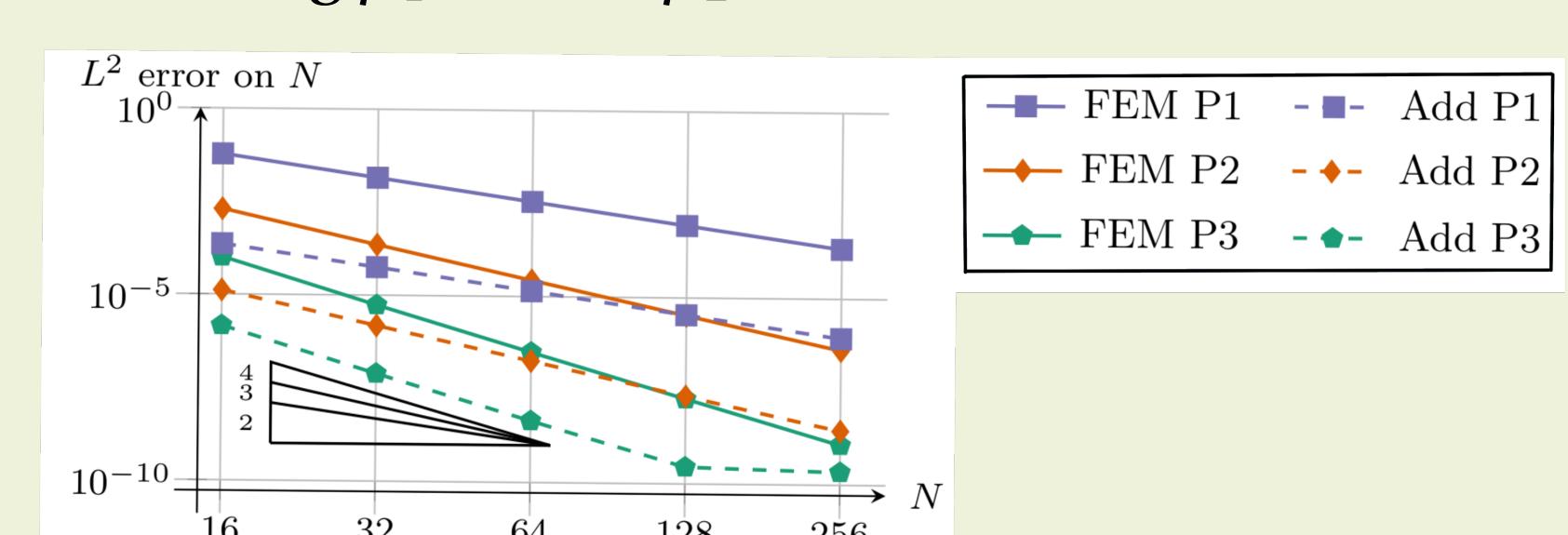
$$\|u - u_h\|_0 \lesssim \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} h^{k+1} |u|_{H^{k+1}}$$

$C_{\text{gain}}$

Remark : The constant  $C_{\text{gain}}$  shows that the closer the prior is to the solution, the lower the error constant associated with the method.

## Numerical results - Improve errors

**Theoretical results.** Taking  $\mu_1 = 0.05, \mu_2 = 0.22$ .



Remark : We note  $N$  the number of nodes in each direction of the square (Total :  $N^2$ ).

### Gains on error using additive approach.

Considering a set of  $n_p = 50$  parameters :  $\{(\mu_1^{(1)}, \mu_2^{(1)}), \dots, (\mu_1^{(n_p)}, \mu_2^{(n_p)})\}$ .

Solution $\mathbb{P}_1$	Gains on PINNs				Gains on FEM					
	N	min	max	mean	std	N	min	max	mean	std
	20	15.7	48.35	33.64	5.57	134.31	377.36	269.4	43.67	
	40	61.47	195.75	135.41	23.21	131.18	362.09	262.12	41.67	

## Numerical results - Improve times

**Time/error ratio.** Training time for PINNs :  $t_{\text{PINNs}} \approx 240$ s.

→ After training, how long does each method take to solve 1 problem ?

Precision	N		time (s)	
	FEM	Add	FEM	Add
$1e-3$	120	8	43	0.24
$1e-4$	373	25	423.89	1.93

→ Including training, how long does each method take to solve  $n_p$  problems ?

Total time of Additive approach :

$$Tot_{\text{Add}} = t_{\text{PINNs}} + n_p t_{\text{Add}}$$

Total time of FEM :

$$Tot_{\text{FEM}} = n_p t_{\text{FEM}}$$

**How many parameters  $n_p$  to make our method faster than FEM ?**

Let's suppose we want to achieve an **error of  $1e-3$** .

$$Tot_{\text{Add}} < Tot_{\text{FEM}} \Rightarrow n_p > \frac{t_{\text{PINNs}}}{t_{\text{FEM}} - t_{\text{Add}}} \approx 5.61 \Rightarrow n_p = 6$$

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[SS22] Sukumar and Srivastava. "Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks". In: *Computer Methods in Applied Mechanics and Engineering* (2022).