Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

DTE 2025

Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries

Hélène Barucq[1], Michel Duprez[2], Florian Faucher[1], Emmanuel Franck[3], **Frédérique Lecourtier**[2], Vanessa Lleras[4], Victor Michel-Dansac[3] and Nicolas Victorion[1]

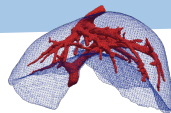[1]Project-Team Makutu, Inria, TotalEnergies, Pau, France
[2]Project-Team MIMESIS, Inria, Strasbourg, France
[3]Project-Team MACARON, Inria, Strasbourg, France
[4]IMAG, University of Montpellier, Montpellier, France

February 20, 2025

anr®   MIMESIS   Inria

Scientific context

Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid $\boxed{\text{finite element}}$ / $\boxed{\text{neural network}}$ method.

accurate quick + parameterized

Parametric elliptic convection/diffusion PDE : For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that
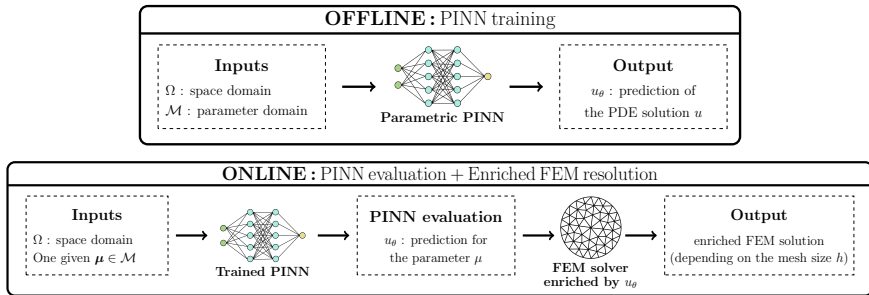
$$\mathcal{L}\big(u\,;\boldsymbol{x},\boldsymbol{\mu}\big) = f(\boldsymbol{x},\boldsymbol{\mu}), \qquad (\mathcal{P})$$

where $\mathcal{L}$ is the parametric differential operator defined by

$$\mathcal{L}(\cdot\,;\boldsymbol{x},\boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x},\boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\text{Pe}}\nabla \cdot (D(\boldsymbol{x},\boldsymbol{\mu})\nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Introduction
○●○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

Pipeline of the Enriched FEM

Remark : The PINN prediction enriched Finite element approximation spaces.

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Physics-Informed Neural Networks

Standard PINNs[1] **(Weak BC) :** Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta) \right), \qquad (\mathcal{P}_\theta)$$

with

residual loss
$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| \mathcal{L}\left(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\right) - f(\boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

boundary loss
$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \left| u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) - g(\boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In $(\mathcal{P}_\theta)$, ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

---

[1][Raissi et al., 2019]

Introduction
○○○●○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

Physics-Informed Neural Networks

Improved PINNs[1] **(Strong BC)** : Find the optimal weights $\theta^\star$ such that

$$\theta^\star = \operatorname*{argmin}_{\theta} \left(\omega_r \, J_r(\theta) + \underline{\omega_b \, J_b(\theta)} \right),$$
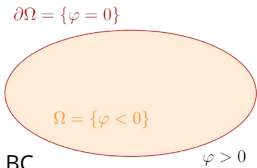
with $\omega_r = 1$ and

residual loss $\boxed{J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| \mathcal{L}\left(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\right) - f(\boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$

where u_θ is a neural network defined by

$$u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \varphi(\boldsymbol{x}) w_\theta(\boldsymbol{x}, \boldsymbol{\mu}) + g(\boldsymbol{x}, \boldsymbol{\mu}),$$

$\partial\Omega = \{\varphi = 0\}$

$\Omega = \{\varphi < 0\}$

$\varphi > 0$

with φ a level-set function, w_θ a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_\theta = g$ on $\partial\Omega$.

---

[1][Lagaris et al., 1998; Franck et al., 2024]

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
○○○○●

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

Finite Element Method[1]

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_\Omega D\nabla u_h \cdot \nabla v_h + \int_\Omega R\, u_h\, v_h + \int_\Omega v_h\, C \cdot \nabla u_h, \quad l(v_h) = \int_\Omega f v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega), \ \forall K \in \mathcal{T}_h, \ v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$
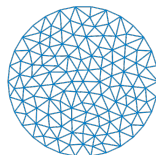
where $\mathbb{P}_k$ is the space of polynomials of degree at most k.

Linear system : Let $(\phi_1, \ldots, \phi_{N_h})$ a basis of V_h^0.
Find $U \in \mathbb{R}^{N_h}$ such that
$$AU = b$$
with

$$A = \left(a(\phi_i, \phi_j)\right)_{1 \leq i,j \leq N_h} \quad \text{and} \quad b = \left(l(\phi_j)\right)_{1 \leq j \leq N_h}.$$

$\mathcal{T}_h = \{K_1, \ldots, K_{N_e}\}$
(N_e : number of elements)

[1][Ern and Guermond, 2004]

Introduction
00000

How improve PINN prediction with FEM ?
●OO

Numerical results
O 00000 0000 0000

Conclusion
OO

References

How improve PINN prediction with FEM ?

Introduction
○○○○○

How improve PINN prediction with FEM ?
○●○

Numerical results
○ ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

Find $p_h^+ \in V_h^0$ such that, $\forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h),$ $(\mathcal{P}_h^+)$
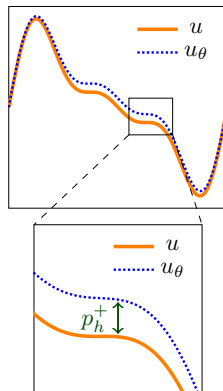
with the enriched trial space V_h^+ defined by

$$V_h^+ = \left\{ u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0 \right\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
00000

How improve PINN prediction with FEM ?
00●

Numerical results
0 00000 0000 0000

Conclusion
00

References

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leqslant C_{H^1} \, h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leqslant C_{L^2} \, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leqslant \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{H^1} \, h^k |u|_{H^{k+1}} \right),$$

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2} \, h^{k+1} |u|_{H^{k+1}} \right).$$

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
● ○○○○○ ○○○○ ○○○○

Conclusion
○○

References

Numerical results

2D Poisson problem on Square - Dirichlet BC
2D Anisotropic Elliptic problem on a Square - Dirichlet BC
2D Poisson problem on Annulus - Mixed BC

Introduction
00000

How improve PINN prediction with FEM ?
000

Numerical results
○ ●0000 0000 0000

Conclusion
00

References

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: We consider the Poisson problem in 2D with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$ parameters).

We define the right-hand side f such that the solution is given by

$$u(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y),$$

with $\boldsymbol{x} = (x, y) \in \Omega$ and some parameters $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{M}$.

PINN training: MLP of 5 layers, trained with an LBFGs optimizer (5000 epochs). Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by
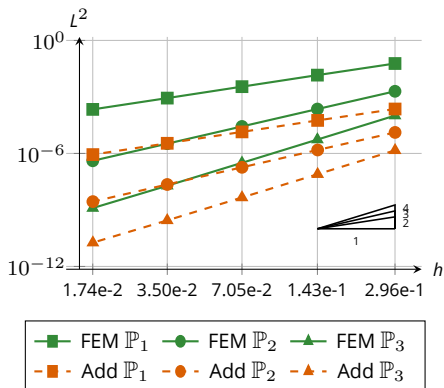
$$\varphi(\boldsymbol{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi).$$

Training time : less than 1 hour on a laptop GPU.

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○●○○ ○○○○ ○○○○

Conclusion
○○

References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22)$$

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
○○○○○
How improve PINN prediction with FEM ?
○○○
Numerical results
○ ○○○●○ ○○○○ ○○○○
Conclusion
○○
References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22)$$

Gains achieved : $n_p = 50$ parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(n_p)} \right\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	134.32	377.36	269.39
2	67.02	164.65	134.85
3	39.52	72.65	61.55

$$N = 20$$

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : N^2 nodes.

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○   ○○○○○● ○○○○ ○○○○

Conclusion
○○

References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22)$$
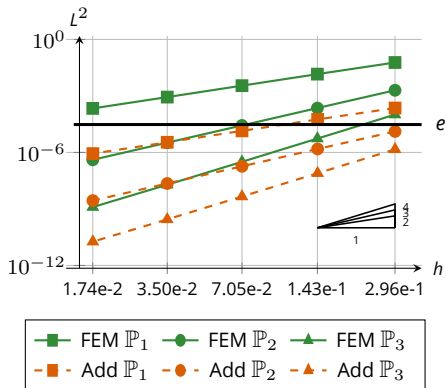
Numerical costs of the two approaches :

N required to reach the same error e.

		N	
k	**e**	**FEM**	**Add**
1	$1 \cdot 10^{-3}$	119	8
	$1 \cdot 10^{-4}$	379	24
2	$1 \cdot 10^{-4}$	42	8
	$1 \cdot 10^{-5}$	89	17
3	$1 \cdot 10^{-5}$	28	10
	$1 \cdot 10^{-6}$	48	18

LECOURTIER Frédérique

Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
00000

How improve PINN prediction with FEM ?
000

Numerical results
○ 00000 ●000 0000

Conclusion
00

References

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: We consider the Poisson problem in 2D with mixed BC:
$$\begin{cases} -\text{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0,1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ (*p* = 4).

We define the right-hand side *f* by
$$f(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{0.025\sigma^2}\right).$$
with $\boldsymbol{x} = (x,y) \in \Omega$ and some parameters $\boldsymbol{\mu} = (\mu_1, \mu_2, \epsilon, \sigma) \in \mathcal{M}$.
The diffusion matrix *D* (symmetric and positive definite) is given by

$$D(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$
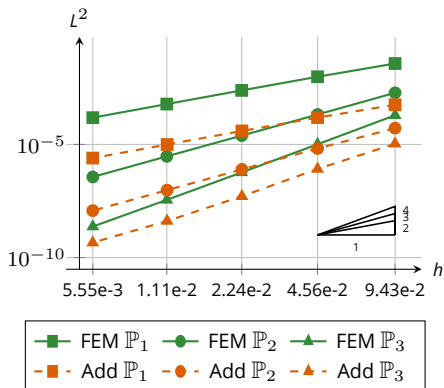
PINN training: MLP with Fourier Features[1] of 5 layers, trained with an Adam optimizer
(15000 epochs). Imposing the Dirichlet BC exactly in the PINN with a levelset function.

[1][Tancik et al., 2020]

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○●○ ○○○○

Conclusion
○○

References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = (0.51, 0.54, 0.52, 0.55)$$

LECOURTIER Frédérique

Combine FEM and NN to solve Elliptic Problems on 2D geometries

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = (0.51, 0.54, 0.52, 0.55)$$

FEM $\mathbb{P}_1$ • FEM $\mathbb{P}_2$ ▲ FEM $\mathbb{P}_3$
▪ Add $\mathbb{P}_1$ • Add $\mathbb{P}_2$ ▲ Add $\mathbb{P}_3$
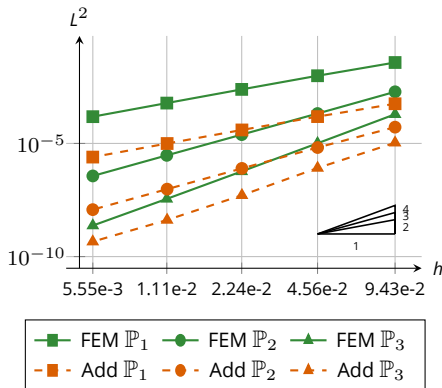
Gains achieved : $n_p = 50$ parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(n_p)} \right\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$$N = 20$$

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : N^2 nodes.

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ●○○○

Conclusion
○○

References

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Introduction
00000

How improve PINN prediction with FEM ?
000

Numerical results
0 00000 0000 0●00

Conclusion
00

References

Problem considered

Problem statement: We consider the Poisson problem in 2D with mixed BC:
$$
\begin{cases}
-\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\
u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\
\dfrac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M},
\end{cases}
$$
with $\Omega = \{(x,y) \in \mathbb{R}^2, \ 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$).

We define the right-hand side f such that the solution is given by
$$
u(\boldsymbol{x}; \boldsymbol{\mu}) = 1 - \frac{\ln\left(\mu_1 \sqrt{x^2 + y^2}\right)}{\ln(4)},
$$

with $\boldsymbol{x} = (x,y) \in \Omega$ and some parameters $\boldsymbol{\mu} = \mu_1 \in \mathcal{M}$. The BC are given by
$$
g(\boldsymbol{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\boldsymbol{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.
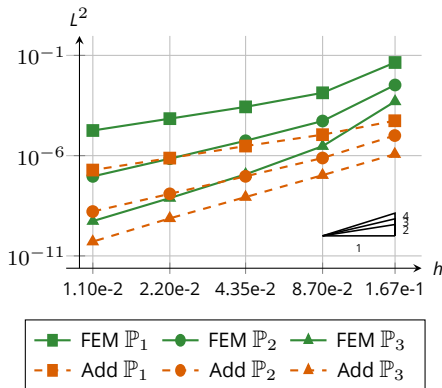$$

PINN training: MLP of 5 layers, trained with an LBFGs optimizer (4000 epochs).
Imposing the mixed BC exactly in the PINN[1].

---

[1] [Sukumar and Srivastava, 2022]

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○●○

Conclusion
○○

References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = \mu_1 = 2.51$$

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
○○○○○

How improve PINN prediction with FEM ?
○○○

Numerical results
○ ○○○○○ ○○○○ ○○○●

Conclusion
○○

References

Numerical results

Error estimates : 1 given parameter.

$$\boldsymbol{\mu}^{(1)} = \mu_1 = 2.51$$
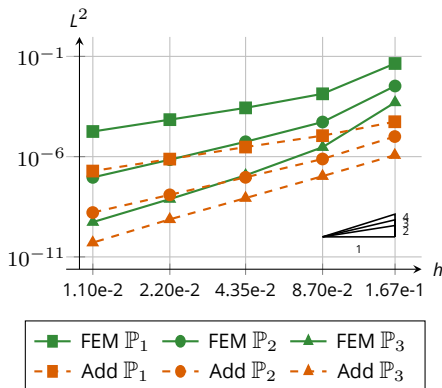
Gains achieved : $n_p = 50$ parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(n_p)} \right\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

$$h = 1.33 \cdot 10^{-1}$$

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

LECOURTIER Frédérique

Combine FEM and NN to solve Elliptic Problems on 2D geometries

Introduction
00000

How improve PINN prediction with FEM ?
000

Numerical results
O 00000 0000 0000

Conclusion
●○

References

Conclusion

Introduction
00000

How improve PINN prediction with FEM ?
000

Numerical results
O 00000 0000 0000

Conclusion
O●

References

Conclusion and Perspectives

- PINNs are good candidates for the enriched approach.
- Numerical validation of the theoretical results.
- The enriched approach provides the same results as the standard FEM method, but with coarser meshes. $\Rightarrow$ Reduction of the computational cost.

Perspectives :

- Validate the additive approach on more complex geometry.
- Consider non-linear problems.
- Use the PINN prediction to build an optiaml mesh, wia a posteriori error estimates.

Add QR code with the paper + Image of the bean testcase

Introduction
ooooo

How improve PINN prediction with FEM ?
ooo

Numerical results
o ooooo oooo oooo

Conclusion
oo

References

References

A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. Springer New York, 2004. doi: 10.1007/978-1-4757-4355-5.

E. Franck, V. Michel-Dansac, and L. Navoret. Approximately well-balanced Discontinuous Galerkin methods using bases enriched with Physics-Informed Neural Networks. *J. Comput. Phys.*, 512:113144, 2024. ISSN 0021-9991. doi: 10.1016/j.jcp.2024.113144.

I. E. Lagaris, A. Likas, and D. I. Fotiadis. Artificial neural networks for solving ordinary and partial differential equations. *IEEE Trans. Neural Netw.*, 9(5):987–1000, 1998. ISSN 1045-9227. doi: 10.1109/72.712178.

F. Lecourtier, H. Barucq, M. Duprez, F. Faucher, E. Franck, V. Lleras, V. Michel-Dansac, and N. Victorion. Enriching continuous lagrange finite element approximation spaces using neural networks, 2025.

M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *J. Comput. Phys.*, 378:686–707, 2019. doi: 10.1016/j.jcp.2018.10.045.

N. Sukumar and A. Srivastava. Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks. *Comput. Method. Appl. M.*, 389:114333, 2022. ISSN 0045-7825. doi: 10.1016/j.cma.2021.114333.

M. Tancik, P. Srinivasan, and al. Fourier Features Let Networks Learn High Frequency Functions in Low Dimensional Domains. In *Advances in Neural Information Processing Systems*, volume 33, pages 7537–7547. Curran Associates, Inc., 2020.

LECOURTIER Frédérique
Combine FEM and NN to solve Elliptic Problems on 2D geometries