# Enriching continuous Lagrange finite element approximation spaces using neural networks

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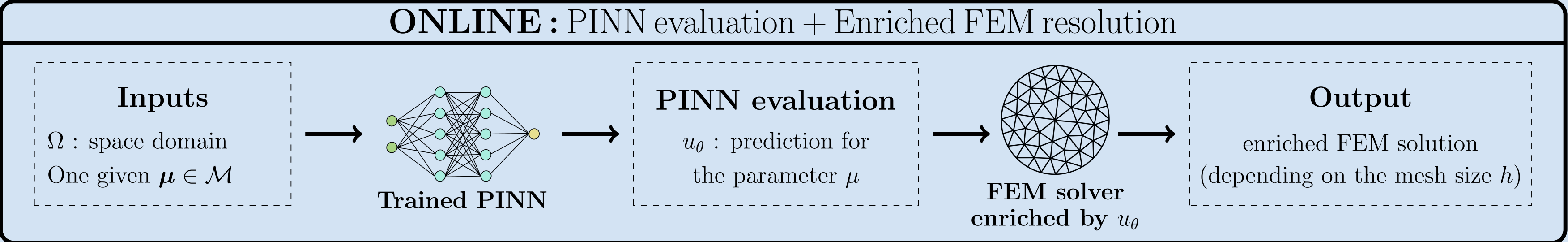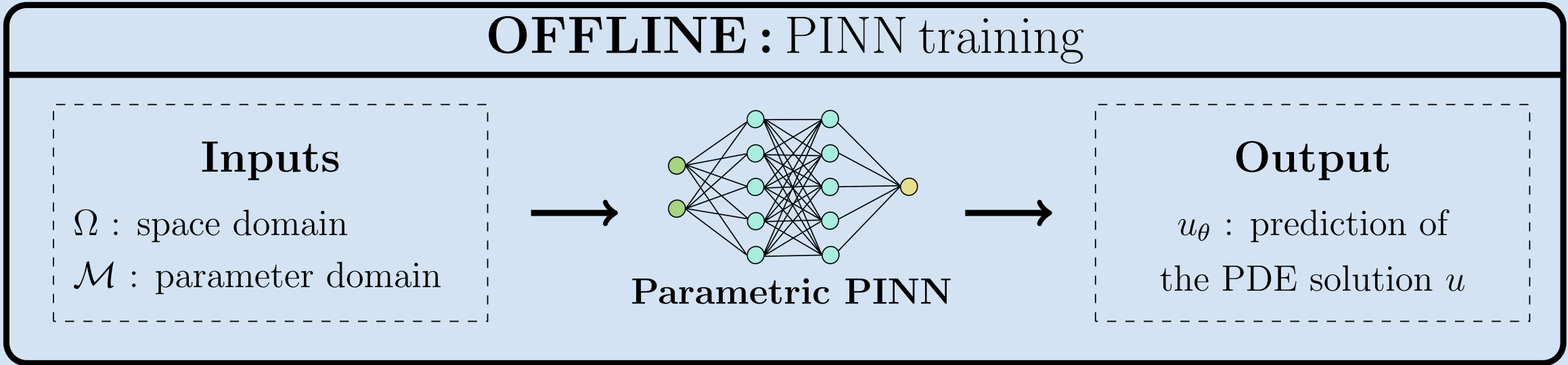
[1] Mimesis team, INRIA Nancy grand Est, Icube   [2] Macaron team, INRIA Nancy grand Est, IRMA   [3] Makutu team, INRIA Bordeaux, TotalEnergies   [4] University of Montpellier
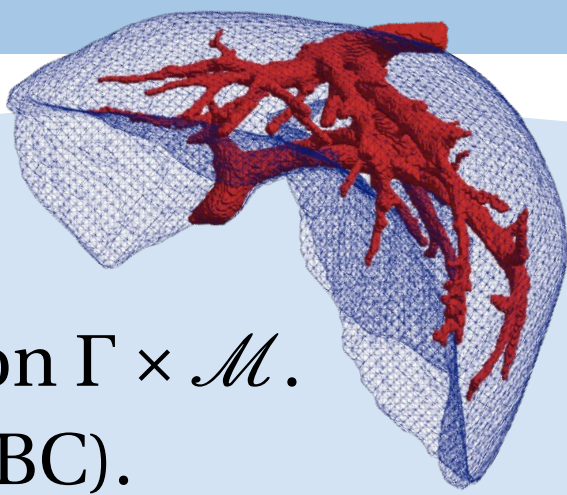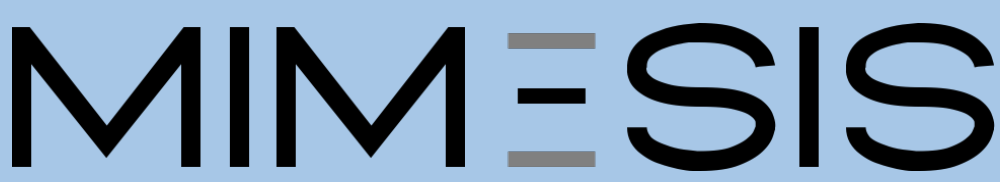
Motivations

Current Objective : Develop hybrid **finite element** / **neural network** methods.
 accurate   **quick + parameterized**

Problem considered : $-\Delta u(X,\mu) = f(X,\mu)$ in $\Omega \times \mathcal{M}$, $u(x,\mu) = 0$ on $\Gamma \times \mathcal{M}$.
Poisson problem with homogeneous Dirichlet boundary conditions (BC).

OFFLINE : PINN training

Inputs		Output
Ω : space domain $\mathcal{M}$: parameter domain	Parametric PINN	u_θ : prediction of the PDE solution u

ONLINE : PINN evaluation + Enriched FEM resolution

Inputs		PINN evaluation		Output
Ω : space domain One given $\boldsymbol{\mu} \in \mathcal{M}$	Trained PINN	u_θ : prediction for the parameter μ	FEM solver enriched by u_θ	enriched FEM solution (depending on the mesh size h)

Perspective : Create real-time digital twins of an organ (e.g. liver).

How improve PINN prediction ? - Using enriched FEM

Additive approach
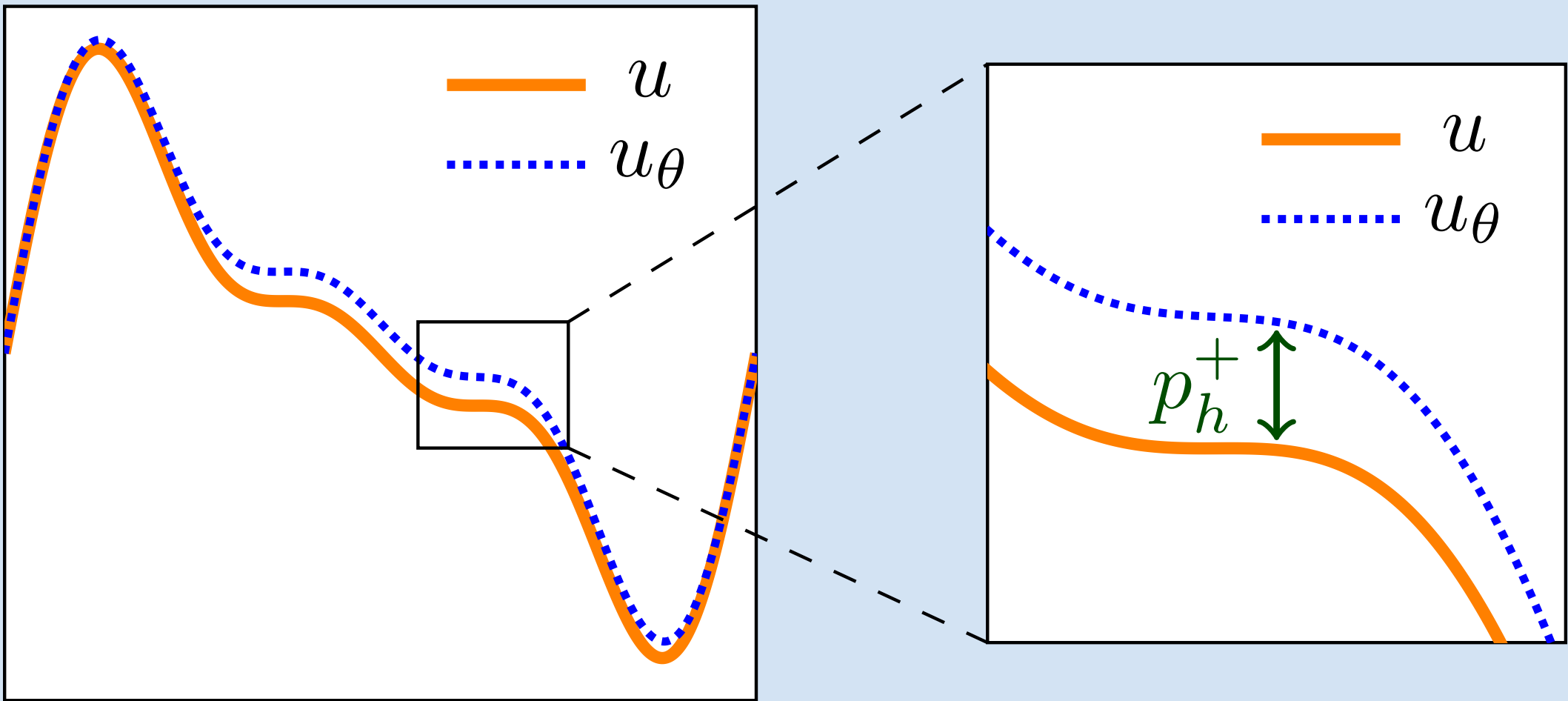
The enriched approximation space is defined by
$$V_h^+ = \{u_h^+ = u_\theta + p_h^+ \, , \; p_h^+ \in V_h^0\}$$
with V_h^0 the standard continuous Lagrange FE space and the weak problem becomes
$$\text{Find } p_h^+ \in V_h^0, \, \forall v_h \in V_h^0, \, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathscr{P}_h^+)$$
with modified boundary conditions and
$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad l(v) = \int_\Omega f \, v.$$

Convergence analysis

u : solution of the Poisson problem. u_θ : prediction of the PINN [RPK19].

Theorem 1: Convergence analysis of the standard FEM [EG]

We denote $u_h \in V_h^0$ the discrete solution of standard FEM with V_h^0 a $\mathbb{P}_k$ Lagrange space. Thus,
$$|u - u_h|_{H^1} \le C_{H^1} \, h^k |u|_{H^{k+1}},$$
$$\|u - u_h\|_{L^2} \le C_{L^2} \, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [F L+25]

We denote $u_h^+ \in V_h^+$ the discrete solution of $(\mathscr{P}_h^+)$ with V_h^+ a $\mathbb{P}_k$ Lagrange space. Thus
$$|u - u_h^+|_{H^1} \le \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} \left(C_{H^1} \, h^k |u|_{H^{k+1}}\right),$$
and
$$\|u - u_h^+\|_{L^2} \le \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} \left(C_{L^2} \, h^{k+1} |u|_{H^{k+1}}\right).$$

Theoretical gain of the additive approach.
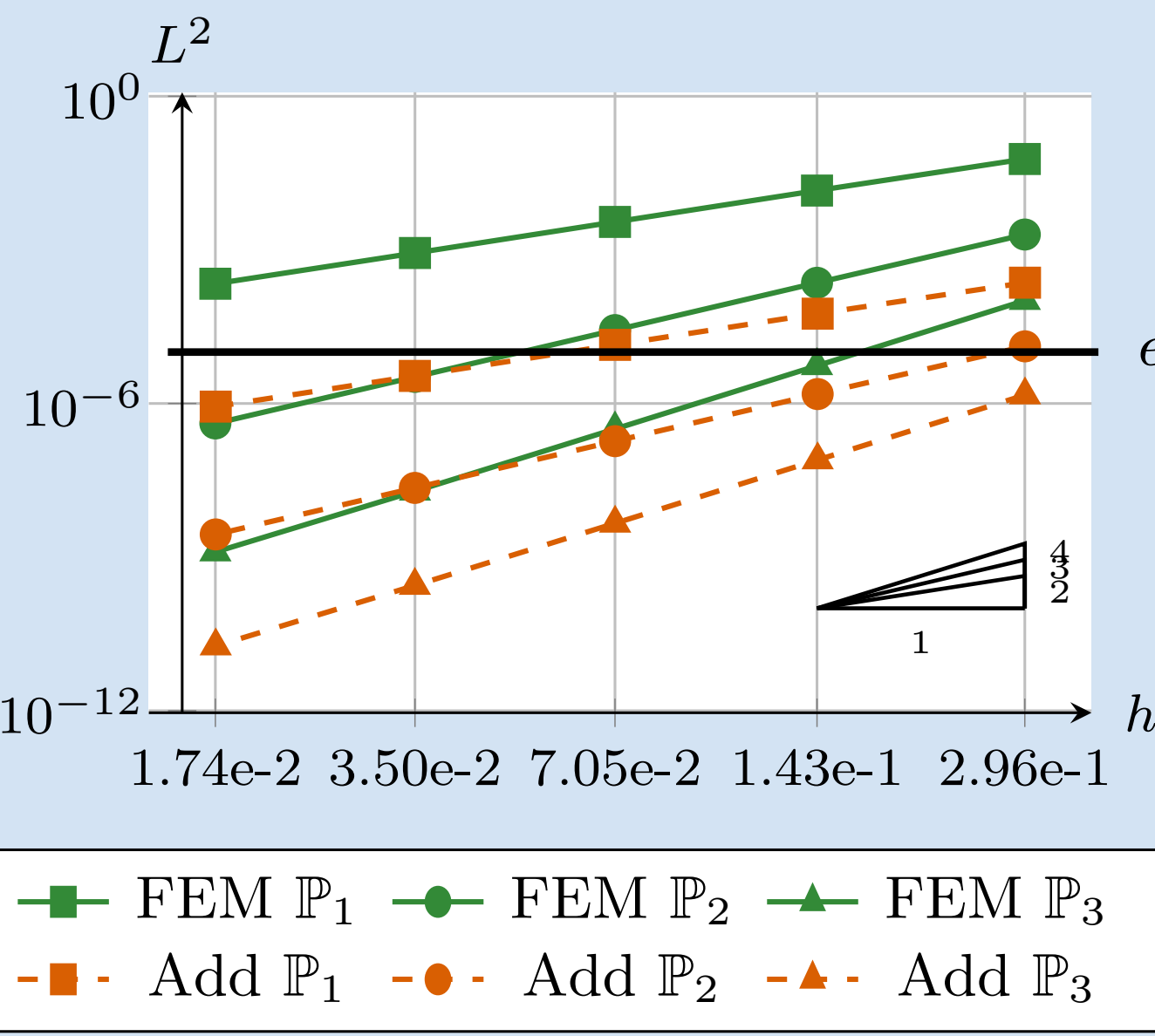
Problem considered - Numerical results

➜ Spatial domain : $\Omega = [-0.5\pi, 0.5\pi]^2$
➜ Parametric domain : $\mathcal{M} = [-0.5, 0.5]^2$
➜ Analytical solution :
$$u_{ex}\big((x,y),\mu\big) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right)\sin(2x)\sin(2y)$$
with $\mu = (\mu_1, \mu_2) \in \mathcal{M}$ (**parametric**) and the associated source term f.

Numerical results - Improve errors

Error estimates : $\mu = (0.05, 0.22)$.

Gains achieved : 50 sets of parameters.

$$S = \{\mu^{(1)}, \ldots, \mu^{(50)}\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	min	max	mean
1	134.32	377.36	269.39
2	67.02	164.65	134.85
3	39.52	72.65	61.55

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : 20^2 nodes.

Numerical results - Improve numerical costs

N_{dofs} required to reach the same error e : $\mu = (0.05, 0.22)$.

k	e	N_{dofs} FEM	N_{dofs} Add
1	$1 \cdot 10^{-3}$	14,161	64
	$1 \cdot 10^{-4}$	143,641	576
2	$1 \cdot 10^{-4}$	6,889	225
	$1 \cdot 10^{-5}$	31,329	1,089
3	$1 \cdot 10^{-5}$	6,724	784
	$1 \cdot 10^{-6}$	20,164	2,704
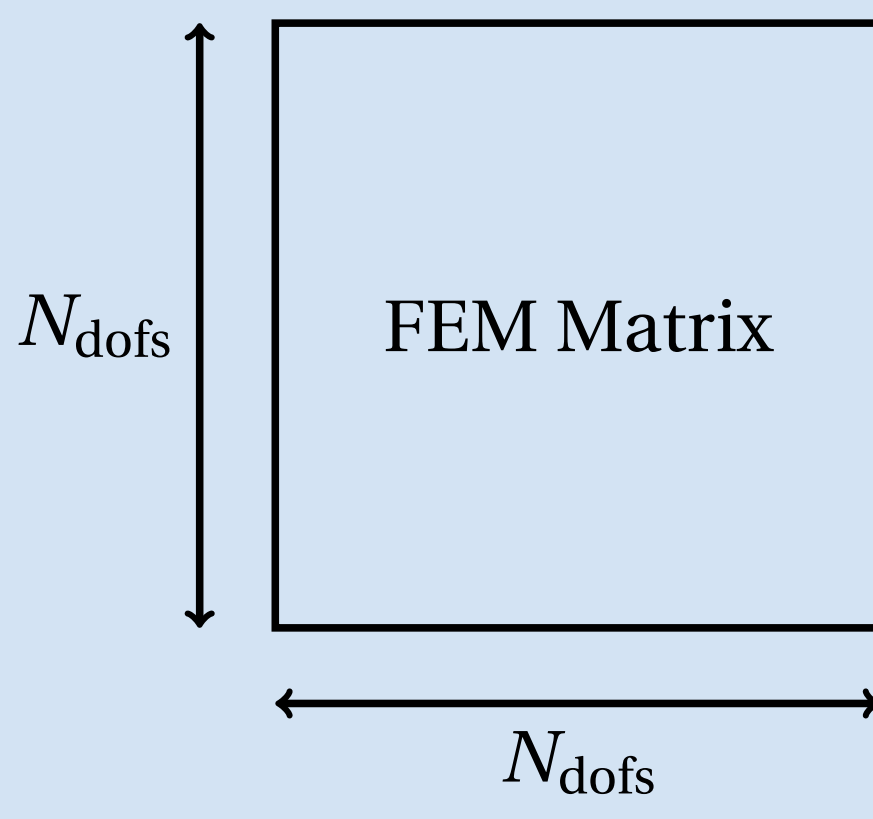
Less degrees of freedom $\Rightarrow$ Lower numerical cost / Faster simulation
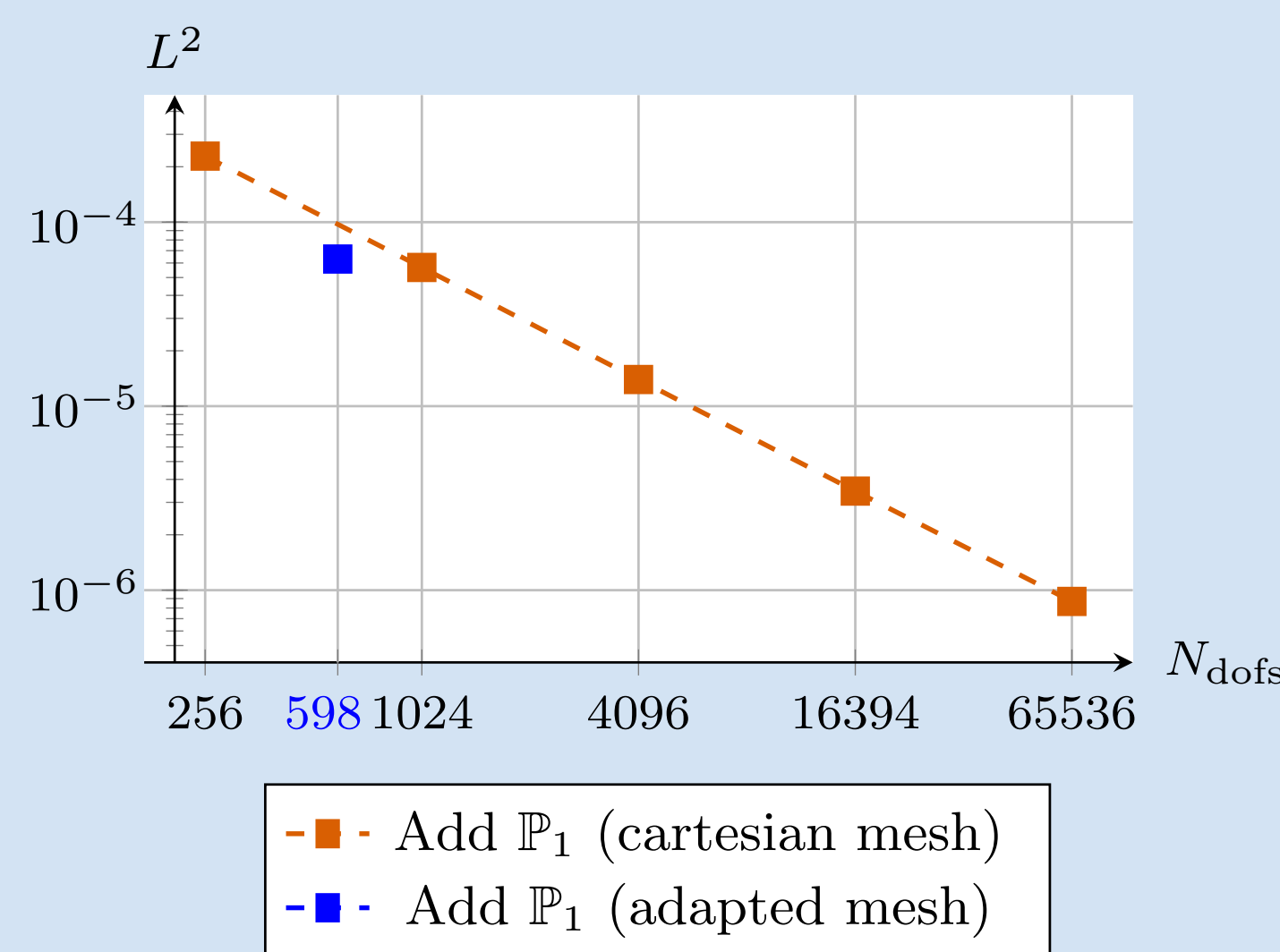
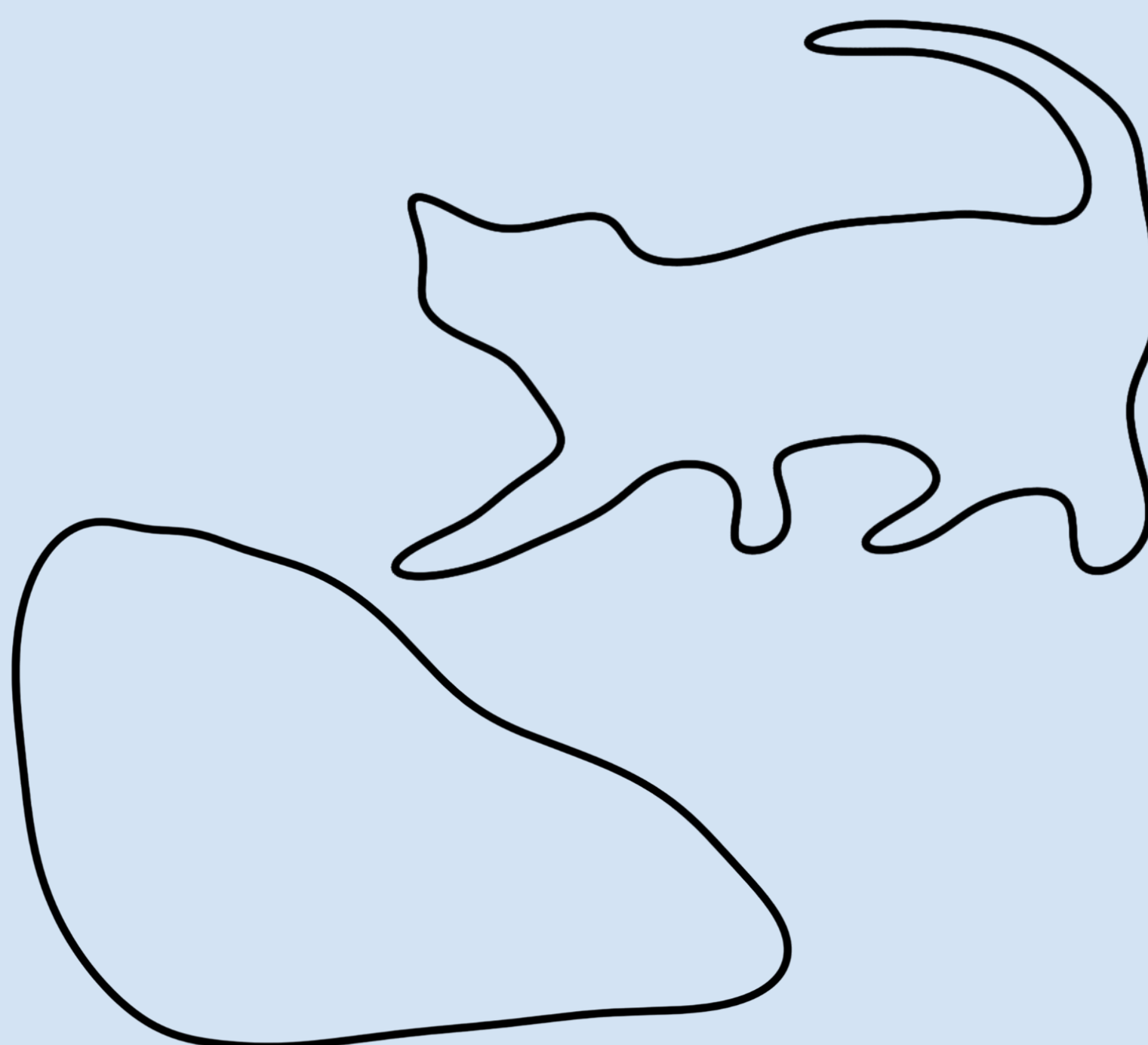## Perspectives

Mesh adaptation

Construction of the mesh: $N_{\text{dofs}} = 598$.
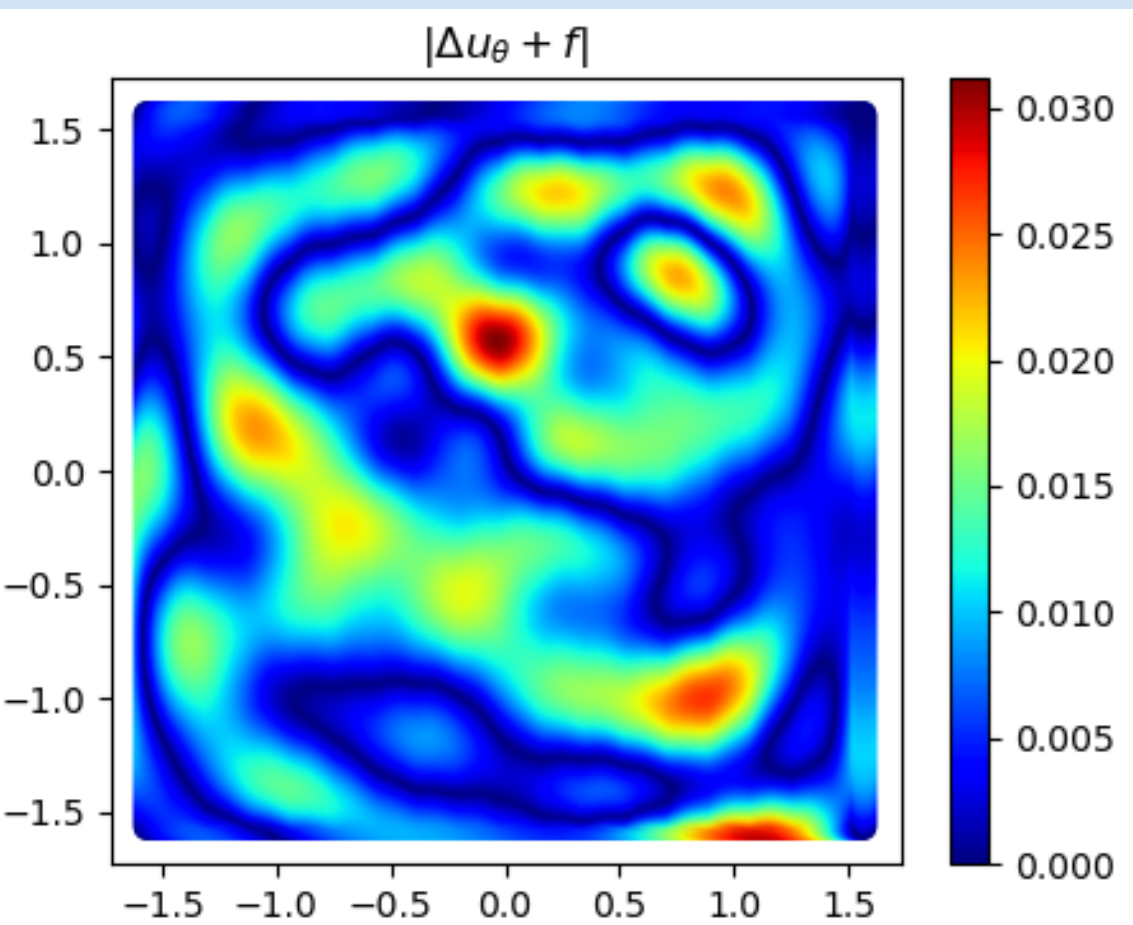
Apply enriched FEM:

Complex geometries

[EG] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements.* Springer New York (2004).
[F L+25] **F. Lecourtier** et al. *Enriching continuous Lagrange finite element approximation spaces using neural networks.* 2025.
[RPK19] M. Raissi, P. Perdikaris, and G. E. Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *J. Comput. Phys.* 378 (2019), pp. 686–707.