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Enriching continuous Lagrange finite element approximation spaces using neural networks

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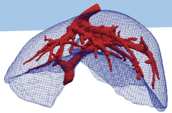
Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

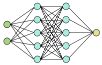
Objective : Develop an hybrid finite element / neural network method.

accurate (finite element) — quick + parameterized (neural network)

OFFLINE : PINN training

Inputs
Ω : space domain
$\mathcal{M}$: parameter domain

$\longrightarrow$


Parametric PINN

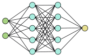
$\longrightarrow$

Output
u_θ : prediction of the PDE solution u

ONLINE : PINN evaluation + Enriched FEM resolution

Inputs
Ω : space domain
One given $\boldsymbol{\mu} \in \mathcal{M}$

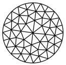
$\longrightarrow$


Trained PINN

$\longrightarrow$

PINN evaluation
u_θ : prediction for the parameter μ

$\longrightarrow$


FEM solver enriched by u_θ

$\longrightarrow$

Output
enriched FEM solution (depending on the mesh size h)

Complete ONLINE process : quick + accurate

Heated cavity test case

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity)[1] **:**
We consider $\Omega = [-1,1]^2$ a squared domain and $\boldsymbol{e}_y = (0,1)$.
Find the velocity $\boldsymbol{u} = (u_1, u_2)$, the pressure p and the temperature T such that

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \mu\Delta\boldsymbol{u} - g(\beta T + 1)\boldsymbol{e}_y = 0 \ \text{ in } \Omega & \text{(momentum)} \\ \nabla\cdot\boldsymbol{u} = 0 \ \text{ in } \Omega & \text{(incompressibility)} \\ \boldsymbol{u}\cdot\nabla T - k_f\Delta T = 0 \ \text{ in } \Omega & \text{(energy)} \\ + \text{ suitable BC} & \end{cases} \qquad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, μ the viscosity and k_f the thermal conductivity. [Coulaud et al., 2024]

[1] The approach will be shown on this example, but can be extended to other test cases.

Heated cavity test case

Objective: Simulation on a range of parameters $\boldsymbol{\mu} = (\mu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :
We consider $\boldsymbol{x} = (x, y) \in \Omega$ and $\boldsymbol{e}_y = (0, 1)$.
Find $\boldsymbol{U} = (\boldsymbol{u}, p, T) = (u_1, u_2, p, T)$ such that

$$\begin{cases} R_{mom}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(momentum)} \\ R_{inc}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(incompressibility)} \\ R_{ener}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(energy)} \\ + \text{ suitable BC} \end{cases} \qquad (\mathcal{P})$$

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Boundary Conditions:

No-slip BC : $\boldsymbol{u} = 0$ on $\partial\Omega$

Isothermal BC : $T = 1$ on the left wall ($x = -1$)
$T = -1$ on the right wall ($x = 1$)

Adiabatic BC : $\dfrac{\partial T}{\partial n} = 0$ on the top and bottom walls ($y = \pm 1$, denoted by Γ_{ad})

Evaluate quality of solutions

In the following, we are interested in three parameters (rising in complexity) :

$$\boldsymbol{\mu}^{(1)} = (0.1, 0.1),\ \boldsymbol{\mu}^{(2)} = (0.05, 0.05) \text{ and } \boldsymbol{\mu}^{(3)} = (0.01, 0.01)$$

We evaluate the quality of solutions by comparing them to a reference solution.[1]

[1] Computed on a over-refined mesh ($h = 7.10^{-3}$) on a $\mathbb{P}_3^2 \times \mathbb{P}_2 \times \mathbb{P}_3$ continuous Lagrange FE space.

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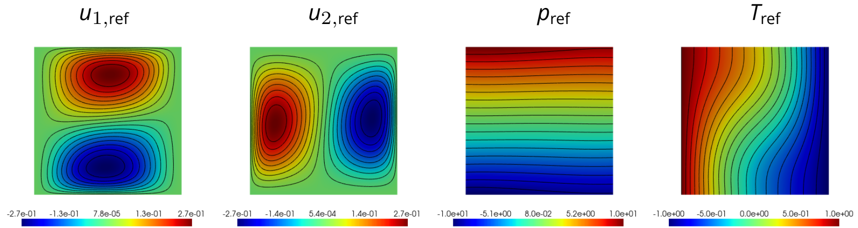
Reference solution - Rayleigh number : $Ra = 1\,569.6$



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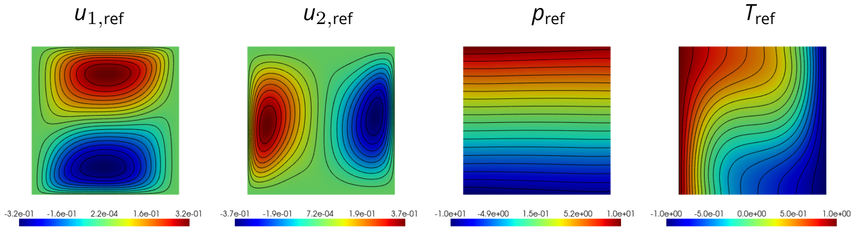
Reference solution - Rayleigh number : $Ra = 6\,278.4$



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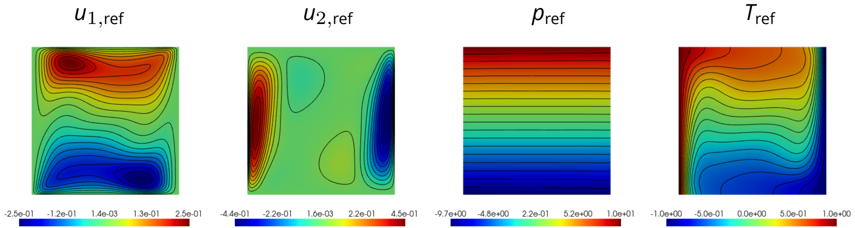
Reference solution - Rayleigh number : $Ra = 156\,960$



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Parametric Physics-Informed Neural Network (PINN)

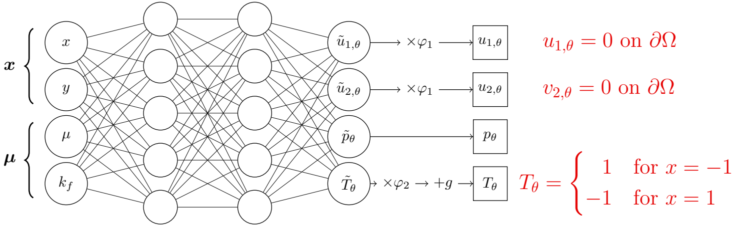
The PINN is parametrized by the $\boldsymbol{\mu}$ parameter.

Neural Network considered

We consider a parametric NN with 4 inputs and 4 outputs, defined by

$$U_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \big(u_{1,\theta}, u_{2,\theta}, p_\theta, T_\theta\big)(\boldsymbol{x}, \boldsymbol{\mu}).$$

The Dirichlet boundary conditions are imposed on the outputs of the MLP by a **post-processing** step. [Sukumar and Srivastava, 2022]



We consider two levelsets functions φ_1 and φ_2, and the linear function g defined by

$$\varphi_1(x,y) = (x-1)(x+1)(y-1)(y+1),$$

$$\varphi_2(x,y) = (x-1)(x+1) \quad \text{and} \quad g(x,y) = 1-(x+1).$$

PINN training

Approximate the solution of ($\mathcal{P}$) by a PINN : Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(J_{inc}(\theta) + J_{mom}(\theta) + J_{ener}(\theta) + J_{ad}(\theta) \right), \qquad (\mathcal{P}_\theta)$$

where the different cost functions[1] are defined by

adiabatic condition

$$\boxed{J_{ad}(\theta) = \int_{\mathcal{M}} \int_{\Gamma_{\text{ad}}} \left| \frac{\partial T_\theta(\boldsymbol{x},\boldsymbol{\mu})}{\partial n} \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

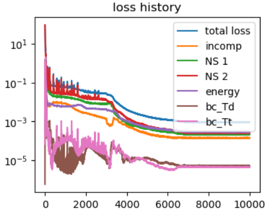
3 residual losses

$$\boxed{J_\bullet(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| R_\bullet(U_\theta(\boldsymbol{x},\boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

with U_θ the parametric NN and $\bullet$ the PDE considered (i.e. *inc*, *mom* or *ener*).

Network - MLP	
layers	40, 60, 60, 60, 40
σ	sine

Training (ADAM / LBFGs)			
lr	7e-3	N_{col}	40000
n_{epochs}	10000	N_{bc}	30000

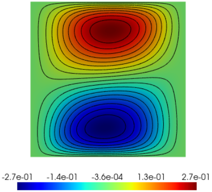


[1] Discretized by a random process using Monte-Carlo method.

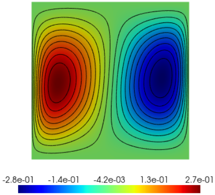
Prediction on $\mu^{(1)} = (0.1, 0.1)$

Prediction :

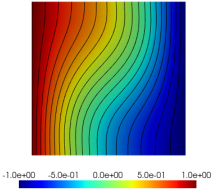
$u_{1,\theta}$



$u_{2,\theta}$

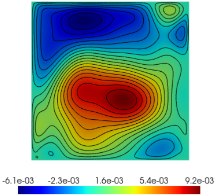


T_{θ}

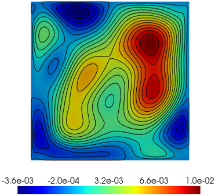


Error map :

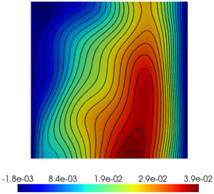
$u_{1,\text{ref}} - u_{1,\theta}$



$u_{2,\text{ref}} - u_{2,\theta}$



$T_{\text{ref}} - T_{\theta}$



L^2 error : (relative)	2.98×10^{-2}	3.17×10^{-2}	3.90×10^{-2}

Finite element method (FEM)

The $\boldsymbol{\mu}$ parameter is fixed in the FE resolution.

Discrete weak formulation

We consider a mixed finite element space $\boxed{M_h = [V_h^0]^2 \times Q_h \times W_h}$ and

$$\left.\begin{array}{ccccccc} \boldsymbol{u}_h & \in & [V_h^0]^2 & \subset & [H_0^1(\Omega)]^2 & : & \mathbb{P}_2 \\ p_h & \in & Q_h & \subset & L_0^2(\Omega) & : & \mathbb{P}_1 \end{array}\right\} \text{(Taylor–Hood spaces)}$$

$$\begin{array}{ccccccc} T_h & \in & W_h & \subset & W & : & \mathbb{P}_2 \end{array}$$

with $W = \{w \in H^1(\Omega),\ w|_{x=-1} = 1,\ w|_{x=1} = -1\}$.

Discrete weak formulation

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with $W = \{w \in H^1(\Omega),\ w|_{x=-1} = 1,\ w|_{x=1} = -1\}$.

Weak problem : Find $U_h = (\boldsymbol{u}_h, p_h, T_h) \in M_h$ s.t., $\forall(\boldsymbol{v}_h, q_h, w_h) \in M_h^0$,

$$\begin{aligned} &\int_\Omega (\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \mu \int_\Omega \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} \\ &\qquad - \int_\Omega p_h \nabla \cdot \boldsymbol{v}_h \, d\boldsymbol{x} - g \int_\Omega (1 + \beta T_h)\boldsymbol{e}_y \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0, \qquad \text{(momentum)} \\ &\int_\Omega q_h \nabla \cdot \boldsymbol{u}_h \, d\boldsymbol{x} + 10^{-4} \int_\Omega q_h \, p_h \, d\boldsymbol{x} = 0, \qquad \text{(incompressibility + pressure penalization)} \\ &\int_\Omega (\boldsymbol{u}_h \cdot \nabla T_h) \, w_h \, d\boldsymbol{x} + \int_\Omega k_f \nabla T_h \cdot \nabla w_h \, d\boldsymbol{x} = 0, \qquad \text{(energy)} \end{aligned} \qquad (\mathcal{P}_h)$$

where $M_h^0 = [V_h^0]^2 \times Q_h \times W_h^0$ with $W_h^0 \subset \{w \in H^1[\Omega],\ w|_{x=\pm 1} = 0\}$.

Newton method

We consider the following three parameters:

$$\boldsymbol{\mu}^{(1)} = (0.1, 0.1),\ \boldsymbol{\mu}^{(2)} = (0.05, 0.05) \text{ and } \boldsymbol{\mu}^{(3)} = (0.01, 0.01).$$

Denoting N_h the dimension of M_h, we want to solve the non linear system:

$$F(\vec{U}_k) = 0$$

with $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ a non linear operator and $\vec{U}_k \in \mathbb{R}^{N_h}$ the unknown vector associated to the k-th parameter $\boldsymbol{\mu}^{(k)}$ ($k = 1, 2, 3$). Appendix 1

Algorithm 1: Newton algorithm

Initialization step: set $\vec{U}_k^{(0)} = \vec{U}_{k,0}$;

for $n \geq 0$ **do**

  Solve the linear system $F(\vec{U}_k^{(n)}) + F'(\vec{U}_k^{(n)})\delta_k^{(n+1)} = 0$ for $\delta_k^{(n+1)}$;

  Update $\vec{U}_k^{(n+1)} = \vec{U}_k^{(n)} + \delta_k^{(n+1)}$;

end

Newton method

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end

How to initialize the Newton solver?

3 types of initialization

- **Natural :**
- **PINN :**
- **Continuation method :**

3 types of initialization

- **Natural :** Using constant or linear function.
  Considering a fixed parameter with $k \in \{1,2,3\}$, we can use the following initialization:
  $$\vec{U}_{k,0} = \left(\vec{0}, \vec{0}, \vec{0}, \vec{T}_0\right)$$
  where for $i = 1, \dots, \dim(W_h)$,
  $$(\vec{T}_0)_i = g(\boldsymbol{x}^{(i)}) = 1 - (x^{(i)} + 1)$$
  with $\boldsymbol{x}^{(i)} = \left(x^{(i)}, y^{(i)}\right)$ the i-th dofs coordinates of W_h.
- **PINN :**
- **Continuation method :**

3 types of initialization

- **Natural :** Using constant or linear function.
- **PINN :** Using PINN prediction.
  (UNet : [Odot et al., 2021] ; FNO : [Aghili et al., 2025])
  Considering a fixed parameter with $k \in \{1, 2, 3\}$, we can use the following initialization for $i = 1, \dots, N_h$,

  $$\left(\vec{U}_{k,0}\right)_i = U_\theta\left(\boldsymbol{x}^{(i)}, \boldsymbol{\mu}^{(k)}\right)$$

  with $\boldsymbol{x}^{(i)} = \left(x^{(i)}, y^{(i)}\right)$ the i-th dofs coordinates of M_h and U_θ the PINN.
- **Continuation method :**

3 types of initialization

- **Natural :** Using constant or linear function.
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  (UNet : [Odot et al., 2021] ; FNO : [Aghili et al., 2025])
- **Continuation method :** Using a coarse FE solution of a simpler parameter.
  - We consider a fixed parameter with $k \in \{2, 3\}$.
  - We consider a coarse grid (16×16 grid) and compute the FE solution of ($\mathcal{P}_h$) for the parameter $\boldsymbol{\mu}^{(k-1)}$.
  - We interpolate the coarse solution to the current mesh.
  - We use it as an initialization for the Newton method, i.e.

$$\vec{U}_{k,0} = \left(\vec{u}_{k-1}, \vec{v}_{k-1}, \vec{p}_{k-1}, \vec{T}_{k-1}\right)$$

where $\vec{u}_{k-1}$, $\vec{v}_{k-1}$, $\vec{p}_{k-1}$ and $\vec{T}_{k-1}$ are the FE solutions for the parameter $\boldsymbol{\mu}^{(k-1)}$.

Enriched finite element method using PINN

Very simple linear test case

The heated cavity test case considered

Enriched finite element method using PINN

Very simple linear test case

The heated cavity test case considered

What is the purpose of enrichment?

Poisson problem (with Dirichlet BC) **:** Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Variational Problem : We consider V_h^0 a $\mathbb{P}_k$ continuous Lagrange FE space ($k \geq 1$).

$$\text{Find } u_h \in V_h^0 \text{ such that}, \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

with h the characteristic mesh size, a and l the associated bilinear and linear forms.

What is the purpose of enrichment?

Modified Poisson problem : Find $C_{h,u}^{+} : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta C_{h,u}^{+} = f + \Delta u_\theta, & \text{in } \Omega, \\ C_{h,u}^{+} = 0, & \text{on } \partial\Omega, \end{cases}$$

with u_θ a PINN prediction.

Modified variational Problem :

$$\text{Find } C_{h,u}^{+} \in V_h^0 \text{ such that}, \forall v_h \in V_h^0, a(C_{h,u}^{+}, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathcal{P}_h^{+})$$

with the enriched trial space V_h^{+} defined by

$$V_h^{+} = \left\{ u_h^{+} = u_\theta + C_{h,u}^{+}, \quad C_{h,u}^{+} \in V_h^0 \right\}.$$

What is the purpose of enrichment?

Modified variational Problem :

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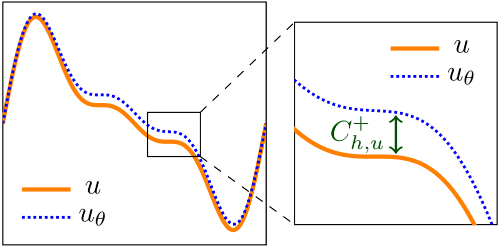


We hope that the modified problem will give the same results as the standard one on coarser meshes.

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leqslant C_{H^1}\, h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [**F. Lecourtier** et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}}\right),$$

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

Gains of the additive approach.

Enriched finite element method using PINN

Very simple linear test case

The heated cavity test case considered

Enriched space using PINN

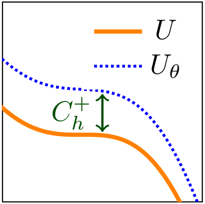
Considering the PINN prior $U_\theta = (\boldsymbol{u}_\theta, p_\theta, T_\theta)$, we define the mixed finite element space additively enriched by the PINN as follows:

$$\boxed{M_h^+ = \left\{ U_h^+ = U_\theta + C_h^+, \quad C_h^+ \in M_h^0 \right\}}$$

with $M_h^0 = [V_h^0]^2 \times Q_h \times W_h^0$, $U_h^+ = (\boldsymbol{u}_h^+, p_h^+, T_h^+) \in M_h^+$ and $C_h^+ = (\boldsymbol{C}_{h,\boldsymbol{u}}^+, C_{h,p}^+, C_{h,T}^+)$.

We can then define the three finite element subspaces of M_h^+ as follows:

$$\begin{aligned} \boldsymbol{V}_h^+ &= \left\{ \boldsymbol{u}_h^+ = \boldsymbol{u}_\theta + \boldsymbol{C}_{h,\boldsymbol{u}}^+, \ \boldsymbol{C}_{h,\boldsymbol{u}}^+ \in [V_h^0]^2 \right\}, \\ Q_h^+ &= \left\{ p_h^+ = p_\theta + C_{h,p}^+, \ C_{h,p}^+ \in Q_h \right\}, \\ W_h^+ &= \left\{ T_h^+ = T_\theta + C_{h,T}^+, \ C_{h,T}^+ \in W_h^0 \right\}, \end{aligned}$$



where $\boldsymbol{C}_{h,\boldsymbol{u}}^+$, $C_{h,p}^+$ and $C_{h,T}^+$ becomes the unknowns.

Weak formulation - Additive approach

Weak problem : Find $C_h^+ = (C_{h,\boldsymbol{u}}^+, C_{h,p}^+, C_{h,T}^+) \in M_h^0$ s.t., $\forall (\boldsymbol{v}_h, q_h, w_h) \in M_h^0$,

$$\begin{aligned}
&\int_\Omega \left[(\boldsymbol{u}_\theta \cdot \nabla)\boldsymbol{u}_\theta + (\boldsymbol{u}_\theta \cdot \nabla) C_{h,\boldsymbol{u}}^+ + (C_{h,\boldsymbol{u}}^+ \cdot \nabla)\boldsymbol{u}_\theta + (C_{h,\boldsymbol{u}}^+ \cdot \nabla) C_{h,\boldsymbol{u}}^+ \right] \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \\
&\quad + \mu \left(\int_\Omega \nabla \boldsymbol{u}_\theta : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} + \int_\Omega \nabla C_{h,\boldsymbol{u}}^+ : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} \right) + \left(\int_\Omega \nabla p_\theta \cdot \boldsymbol{v}_h \, d\boldsymbol{x} - \int_\Omega C_{h,p}^+ \nabla \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \right) \\
&\qquad - g \int_\Omega (1 + \beta(T_\theta + C_{h,T}^+)) \boldsymbol{e}_y \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0, \ \text{(momentum)} \\
&\int_\Omega q_h \left[\nabla \cdot \boldsymbol{u}_\theta + \nabla \cdot C_{h,\boldsymbol{u}}^+ \right] d\boldsymbol{x} + 10^{-4} \int_\Omega q_h \, (p_\theta + C_{h,p}^+) \, d\boldsymbol{x} = 0, \ \text{(incompressibility + penal)} \\
&\int_\Omega \left[\boldsymbol{u}_\theta \cdot \nabla T_\theta + \boldsymbol{u}_\theta \cdot \nabla C_{h,T}^+ + C_{h,\boldsymbol{u}}^+ \cdot \nabla T_\theta + C_{h,\boldsymbol{u}}^+ \cdot \nabla C_{h,T}^+ \right] w_h \, d\boldsymbol{x} \\
&\quad + k_f \left(\int_\Omega \nabla T_\theta \cdot \nabla w_h \, d\boldsymbol{x} + \int_\Omega \nabla C_{h,T}^+ \cdot \nabla w_h \, d\boldsymbol{x} \, w_h \, d\boldsymbol{s} \right) = 0, \ \text{(energy)}
\end{aligned} \qquad (\mathcal{P}_h^+)$$

with $U_\theta = (\boldsymbol{u}_\theta, p_\theta, T_\theta)$ the PINN prior and some modified boundary conditions.

Newton method - Additive approach

We want to solve the non linear system:

$$F_\theta(\vec{C}) = 0$$

with $F_\theta : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ the non linear operator associated to the weak problem $(\mathcal{P}_h^+)$ and $\vec{C} \in \mathbb{R}^{N_h}$ the correction vector (unknown).

Algorithm 2: Newton algorithm [Aghili et al., 2025]

Initialization step: set $\vec{C}^{(0)} = 0$;
for $n \geq 0$ **do**
  Solve the linear system $F_\theta(\vec{C}^{(n)}) + F'_\theta(\vec{C}^{(n)})\delta^{(n+1)} = 0$ for $\delta^{(n+1)}$;
  Update $\vec{C}^{(n+1)} = \vec{C}^{(n)} + \delta^{(n+1)}$;
end

Advantage compared to PINN initialization[1]:

u_θ is not required to live in the same discrete space as C_h^+.

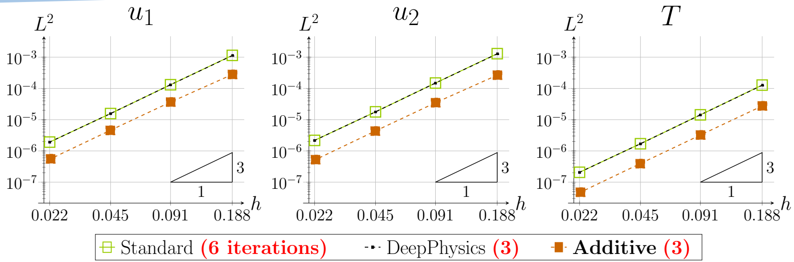
[1] Taking U_θ and C_h^+ in the same space, additive approach is exactly the same as the PINN initialization.

Numerical results

- Results obtained with a laptop GPU.
- The newton solver is the same for all methods (rtol$=10^{-10}$, atol$=10^{-10}$, max_it$=30$).
- Additive approach : we consider u_θ in a $\mathbb{P}_3^2 \times \mathbb{P}_2 \times \mathbb{P}_3$ continuous Lagrange FE space (defined on the current mesh).

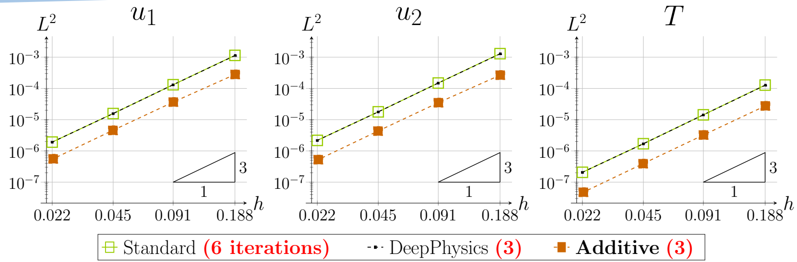
Error estimates I

$\boldsymbol{\mu}^{(1)}$:

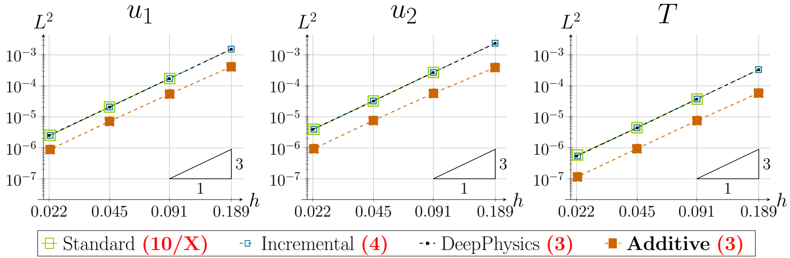


Error estimates I

$\boldsymbol{\mu}^{(1)}$:

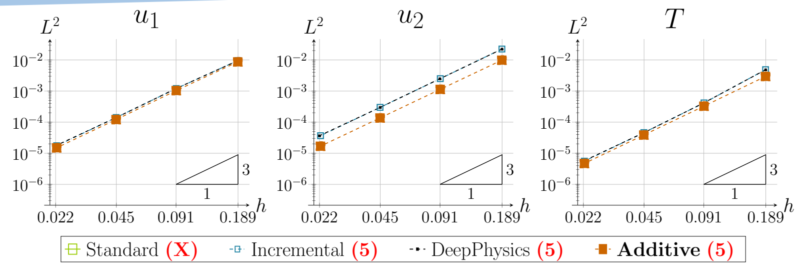


$\boldsymbol{\mu}^{(2)}$:



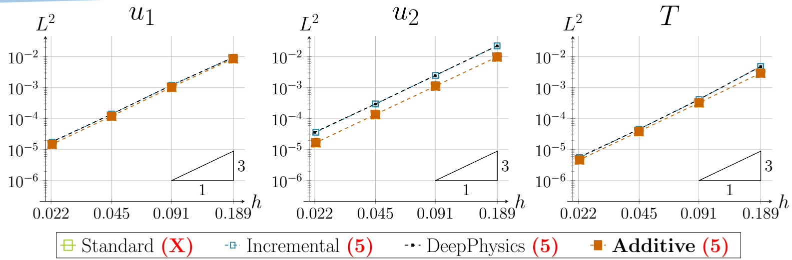
Error estimates II

$\boldsymbol{\mu}^{(3)}$:

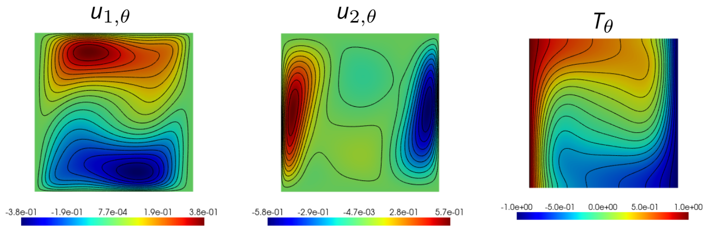


Error estimates II

$\boldsymbol{\mu}^{(3)}$:

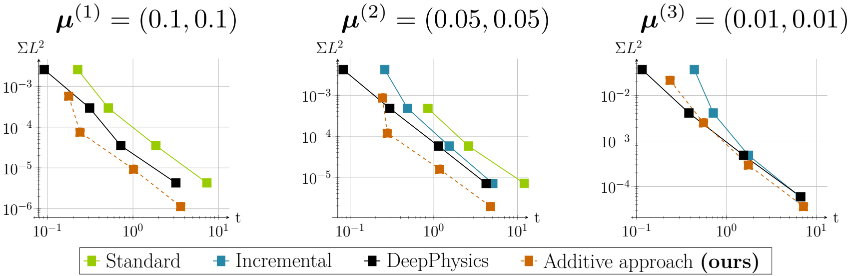


Prediction :



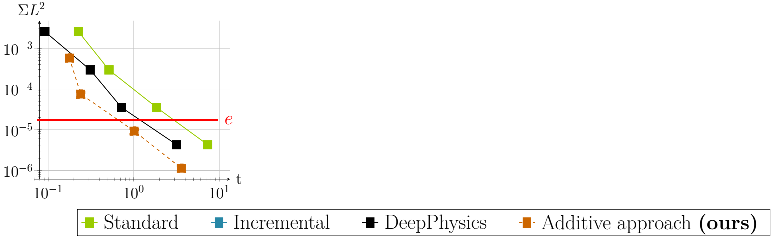
L^2 **error : (relative)**	5.75×10^{-1}	4.89×10^{-1}	2.57×10^{-1}

Numerical costs



Numerical costs

$\boldsymbol{\mu}^{(1)} = (0.1, 0.1)$



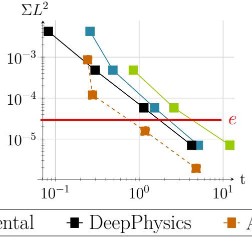
N_{dofs} and execution time required to reach the same global L^2 relative error[1] e :

	Number of DoFs		Execution times		
e	**Std/DPhy**	**Add**	**Std**	**DPhy**	**Add**
$1 \cdot 10^{-3}$	6,031	2,044	0.32	0.16	0.16
$1 \cdot 10^{-4}$	26,959	10,588	0.99	0.48	0.23
$1 \cdot 10^{-5}$	121,156	49,231	4.21	1.75	0.96

[1] Defined as the sum of the L^2 relatives errors on $\boldsymbol{u}$ and T.

Numerical costs

$$\mu^{(2)} = (0.05, 0.05)$$

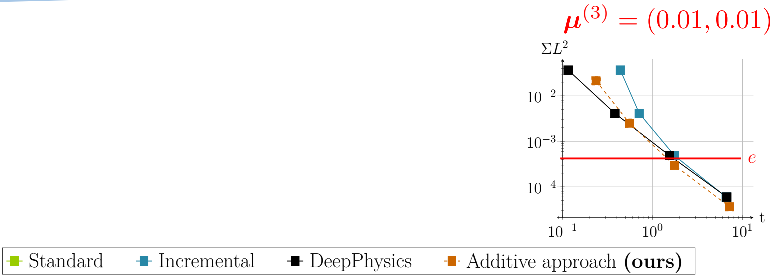


N_{dofs} and execution time required to reach the same global L^2 relative error[1] e :

e	Number of DoFs		Execution times			
	Std/Inc/DPhy	Add	Std	Inc	DPhy	Add
$1 \cdot 10^{-3}$	7,828	2,748	0.58	0.39	0.19	0.24
$1 \cdot 10^{-4}$	35,884	14,623	1.95	1.14	0.8	0.32
$1 \cdot 10^{-5}$	167,583	70,303	9.39	4.16	3.4	1.59

[1] Defined as the sum of the L^2 relatives errors on $\boldsymbol{u}$ and T.

Numerical costs



N_{dofs} and execution time required to reach the same global L^2 relative error[1] e :

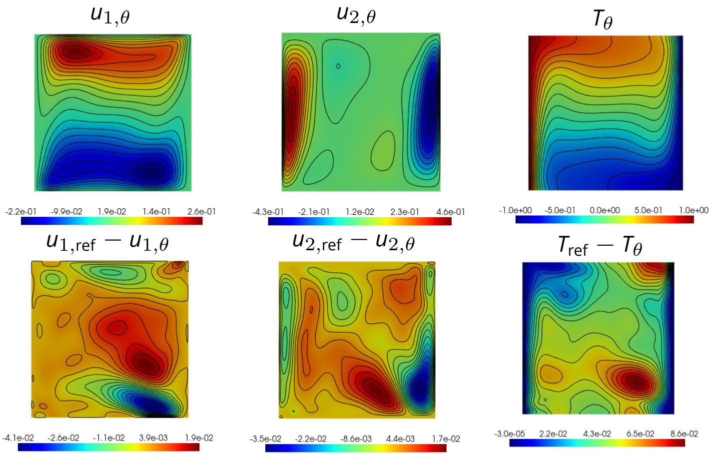
	Number of DoFs			Execution times			
e	**Std**	**Inc/DPhy**	**Add**	**Std**	**Inc**	**DPhy**	**Add**
$1 \cdot 10^{-3}$	X	33,204	23,524	X	1.29	0.96	0.91
$1 \cdot 10^{-4}$	X	150,339	108,931	X	4.76	4.67	3.65
$1 \cdot 10^{-5}$	X	690,924	502,156	X	20.34	23.3	17.23

[1] Defined as the sum of the L^2 relatives errors on $\boldsymbol{u}$ and T.

Non parametric PINN[1] for $\mu^{(3)}$

Prediction :

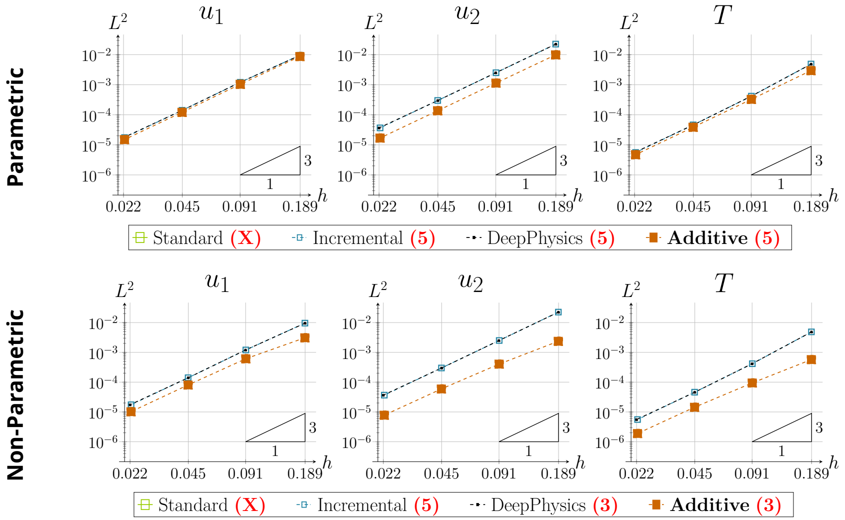
Error map :



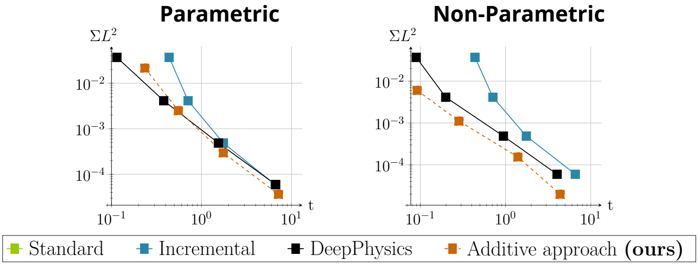
	$u_{1,\theta}$	$u_{2,\theta}$	T_θ
L^2 error : (relative)	7.60×10^{-2}	5.38×10^{-2}	9.63×10^{-2}

[1] We consider exactly the same architecture, but this time we train the PINN non-parametrically.

Error estimates on $\mu^{(3)}$



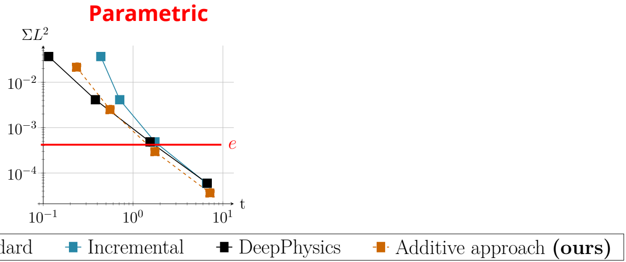
Numerical costs on $\mu^{(3)}$



N_{dofs} and execution time required to reach the same global L^2 relative error e :

A modifier

Numerical costs on $\mu^{(3)}$

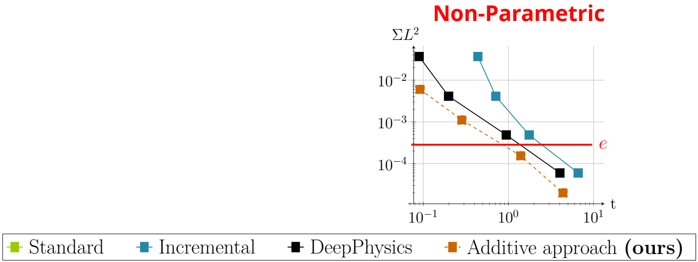


N_{dofs} and execution time required to reach the same global L^2 relative error e :

	Number of DoFs			Execution times			
e	**Std**	**Inc/DPhy**	**Add**	**Std**	**Inc**	**DPhy**	**Add**
$1 \cdot 10^{-3}$	X	33,204	23,524	X	1.29	0.96	0.91
$1 \cdot 10^{-4}$	X	150,339	108,931	X	4.76	4.67	3.65
$1 \cdot 10^{-5}$	X	690,924	502,156	X	20.34	23.3	17.23

A modifier

Numerical costs on $\mu^{(3)}$



N_{dofs} and execution time required to reach the same global L^2 relative error e :

	Number of DoFs			Execution times			
e	Std	Inc/DPhy	Add	Std	Inc	DPhy	Add
$1 \cdot 10^{-3}$	X	33,204	13,764	X	1.29	0.56	0.31
$1 \cdot 10^{-4}$	X	150,339	70,303	X	4.76	2.82	1.78
$1 \cdot 10^{-5}$	X	690,924	339,231	X	20.34	13.84	6.42

A modifier

Conclusion

TODO

Parler du papier en linéaire et dire que dans ce cadre on a des résultats théoriques de convergence.

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Appendix 1 : Finite element method (FEM)

A1 – Construction of the unknown vector

Considering $(\phi_i)_{i=1}^{N_u}$, $(\psi_j)_{j=1}^{N_p}$ and $(\eta_k)_{k=1}^{N_T}$ the basis functions of the finite element spaces V_h^0, Q_h and W_h respectively, we can write the discrete solutions as:

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{i=1}^{N_u} \begin{pmatrix} u_i \\ v_i \end{pmatrix} \phi_i(\boldsymbol{x}), \quad p_h(\boldsymbol{x}) = \sum_{j=1}^{N_p} p_j \psi_j(\boldsymbol{x}) \quad \text{and} \quad T_h(\boldsymbol{x}) = \sum_{k=1}^{N_T} T_k \eta_k(\boldsymbol{x}),$$

with the unknown vectors for velocity, pressure and temperature defined by

$$\vec{u} = \left(u_i\right)_{i=1}^{N_u} \in \mathbb{R}^{N_u}, \quad \vec{v} = \left(v_i\right)_{i=1}^{N_u} \in \mathbb{R}^{N_u},$$

$$\vec{p} = \left(p_j\right)_{j=1}^{N_p} \in \mathbb{R}^{N_p} \text{ and } \vec{T} = \left(T_k\right)_{k=1}^{N_T} \in \mathbb{R}^{N_T}.$$

Considering $N_h = 2N_u + N_p + N_T$, we can define the global vector of unknowns as:

$$\vec{U} = \left(\vec{u}, \vec{v}, \vec{p}, \vec{T}\right) \in \mathbb{R}^{N_h}.$$

and $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ the nonlinear operator associated to the weak formulation $(\mathcal{P}_h)$.

Appendix 2 : DeepPhysics / Additive approach

A2 – ??