DTE 2025

Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries

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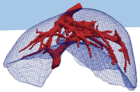
Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.

finite element: accurate

neural network: quick + parameterized

Parametric linear elliptic PDE : For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{L}\big(u; \boldsymbol{x}, \boldsymbol{\mu}\big) = f(\boldsymbol{x}, \boldsymbol{\mu}),$$

where $\mathcal{L}$ is the parametric differential operator defined by

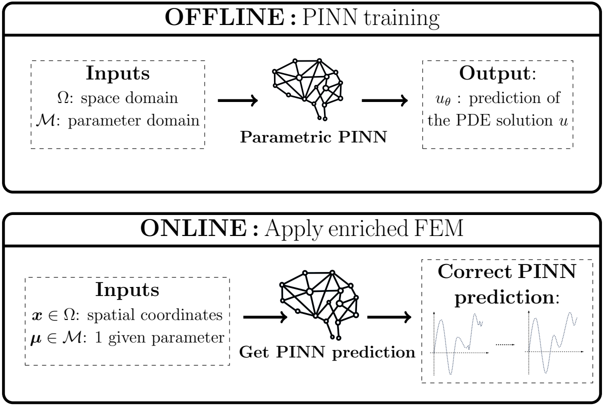
$$\mathcal{L}(\cdot; \boldsymbol{x}, \boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x}, \boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\mathrm{Pe}} \nabla \cdot (D(\boldsymbol{x}, \boldsymbol{\mu}) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Ω	Spatial domain
d	Spatial dimension
$\boldsymbol{x} = (x_1, \ldots, x_d)$	Spatial coordinates
$\mathcal{M}$	Parameter space
p	Number of parameters
$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$	Parameter vector

f	Right-hand side
R	Reaction coefficient
C	Convection coefficient
D	Diffusion matrix
Pe	Péclet number

Pipeline of the Enriched FEM



Correction : Enriched continuous Lagrange finite element approximation spaces using the PINN prediction.

Physics-Informed Neural Networks

Standard PINNs (Weak BC) : Find the optimal weights $\theta^\star$ that satisfy

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta)\right), \tag{1}$$

with the residual loss function and the boundary loss function defined by

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left|\mathcal{L}\left(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\right) - f(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \left|u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) - g(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC. In (1), the weights ω_r and ω_b (hyperparameters) are used to balance the different terms of the loss function.

Monte-Carlo method : Discretize the cost functions by random process.

Physics-Informed Neural Networks

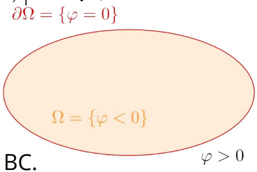
Improved PINNs[1] (Strong BC) : Find the optimal weights $\theta^\star$ that satisfy

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)} \right), \qquad (2)$$

with $\omega_r = 1$ and the residual loss function defined by

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| \mathcal{L}\big(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\big) - f(\boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},$$

where u_θ is a neural network defined by



$$u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \varphi(\boldsymbol{x}) w_\theta(\boldsymbol{x}, \boldsymbol{\mu}) + g(\boldsymbol{x}, \boldsymbol{\mu}),$$

with φ a level-set function, w_θ a NN and $g = 0$ the Dirichlet BC.
Thus, the Dirichlet BC is imposed exactly in the PINN : $u_\theta = g$ on $\partial\Omega$.

Monte-Carlo method : Discretize the residual cost function by random process.

[1] Lagaris et al. [1998]; Franck et al. [2024]

Finite Element Method

Variational Problem : Find $u_h \in V_h^0$ such that, $\forall v_h \in V_h^0, a(u_h, v_h) = l(v_h)$,

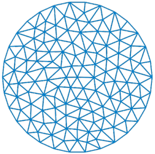
with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\mathrm{Pe}} \int_\Omega D\nabla u_h \cdot \nabla v_h + \int_\Omega R\, u_h\, v_h + \int_\Omega v_h\, C \cdot \nabla u_h, \quad l(v_h) = \int_\Omega f v_h,$$

and V_h the finite element space of dimension N_h defined by

$$V_h = \left\{ v_h \in C^0(\Omega),\ \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where $\mathbb{P}_k$ is the space of polynomials of degree at most k.



$\mathcal{T}_h = \{K_1, \ldots, K_{N_e}\}$
(N_e : number of elements)

Linear system : Let $(\phi_1, \ldots, \phi_{N_h})$ a basis of V_h.

Find $U \in \mathbb{R}^{N_h}$ such that

$$AU = b$$

with

$$A = \left(a(\phi_i, \phi_j)\right)_{1 \le i,j \le N_h} \quad \text{and} \quad b = \left(l(\phi_j)\right)_{1 \le j \le N_h}.$$

How improve PINN prediction with FEM ?

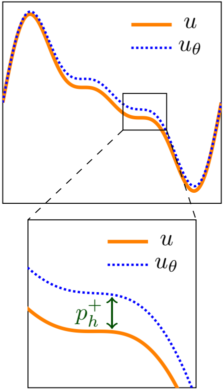
Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0,\ a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h),$$

with the enriched trial space V_h^+ defined by

$$V_h^+ = \left\{ u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0 \right\}.$$



Impose BC : If our problem satisfies $u = g$ on $\partial\Omega$, then p_h^+ has to satisfy

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior (weak BC).

Considering the strong BC, $p_h^+ = 0$ on $\partial\Omega$.

Theorerical results

TODO

Numerical results - 2D Poisson problem

2D Poisson problem

TODO

Numerical results - 2D anysotropic Elliptic problem

2D anysotropic Elliptic problem

TODO

Conclusion

Conclusion

TODO

References

E. Franck, V. Michel-Dansac, and L. Navoret. Approximately well-balanced Discontinuous Galerkin methods using bases enriched with Physics-Informed Neural Networks. *J. Comput. Phys.*, 512:113144, 2024. ISSN 0021-9991. doi: 10.1016/j.jcp.2024.113144.

I. E. Lagaris, A. Likas, and D. I. Fotiadis. Artificial neural networks for solving ordinary and partial differential equations. *IEEE Trans. Neural Netw.*, 9(5):987–1000, 1998. ISSN 1045-9227. doi: 10.1109/72.712178.

Appendix

Appendix 1 : Standard FEM

Appendix 1 : General Idea

TODO