

## 2nd CSI

# Development of hybrid finite element/neural network methods to help create digital surgical twins

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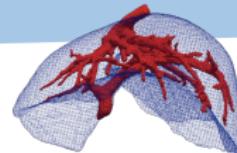
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★ - Update 2025

# Scientific context

**Context :** Create real-time digital twins of an organ (e.g. liver).



**Objective :** Develop an hybrid **finite element** / **neural network** method.  
accurate      quick + parameterized

★ **Parametric elliptic convection/diffusion PDE :** For one or several  $\mu \in \mathcal{M}$ , find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\mathcal{L}(u; \mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}, \boldsymbol{\mu}), \quad (\mathcal{P})$$

where  $\mathcal{L}$  is the parametric differential operator defined by

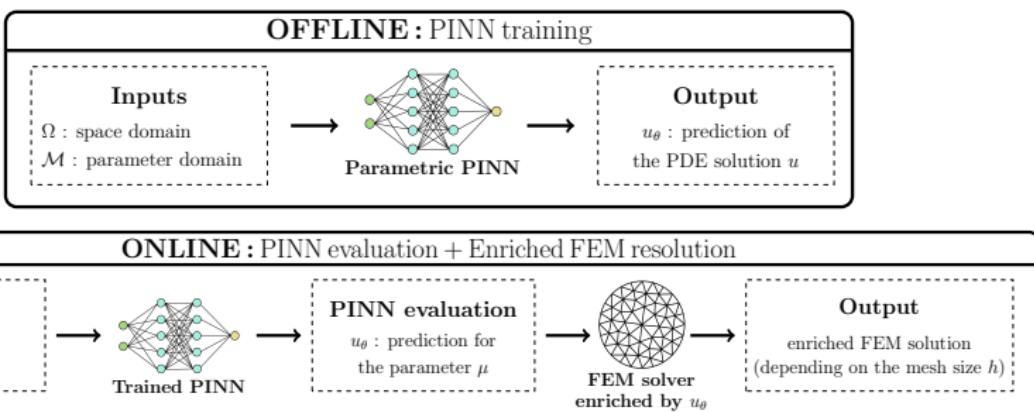
$$\mathcal{L}(\cdot; \mathbf{x}, \boldsymbol{\mu}) : u \mapsto R(\mathbf{x}, \boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{Pe} \nabla \cdot (D(\mathbf{x}, \boldsymbol{\mu}) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on  $\boldsymbol{\mu}$ ).

# Pipeline of the Enriched FEM

Enriched EEM = Combination of 2 standard methods

- **PINNs** : Physics Informed Neural Networks      Appendix 1.1
  - **FEMs** : Finite Element Methods      Appendix 1.2



**Remark:** The PINN prediction enriched Finite element approximation spaces.

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# Enriched finite element method using PINNs

Additive approach

★ Numerical results

This section is based on [F. Lecourtier et al., 2025].

# Enriched finite element method using PINNs

Additive approach

★ Numerical results

# Additive approach

**Variational Problem :** Let  $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ .

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(p_h^+, v_h) = I(v_h) - a(u_\theta, v_h), \quad (\mathcal{P}_h^+)$$

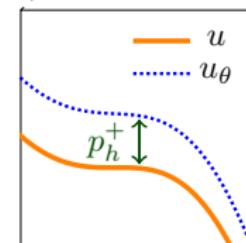
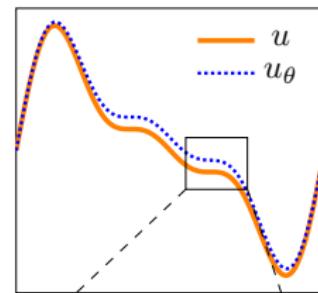
with the **enriched trial space**  $V_h^+$  defined by

$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0\}.$$

**General Dirichlet BC :** If  $u = g$  on  $\partial\Omega$ , then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with  $u_\theta$  the PINN prior.



# Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote  $u_h \in V_h$  the solution of  $(\mathcal{P}_h)$  with  $V_h$  the standard trial space. Then,

$$|u - u_h|_{H^1} \leq C_{H^1} h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [F. Lecourtier et al., 2025]

We denote  $u_h^+ \in V_h^+$  the solution of  $(\mathcal{P}_h^+)$  with  $V_h^+$  the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leq \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^+\|_{L^2} \leq \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

Gains of the additive approach.

LECOURTIER Frédérique

# Enriched finite element method using PINNs

Additive approach

★ Numerical results

# 1st problem considered

**Problem statement:** Considering an **Anisotropic Elliptic problem with Dirichlet BC**:

$$\begin{cases} -\operatorname{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with  $\Omega = [0, 1]^2$  and  $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$  ( $p = 4$ ).

**Right-hand side :**

$$f(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{0.025\sigma^2}\right).$$

**Diffusion matrix :** (symmetric and positive definite)

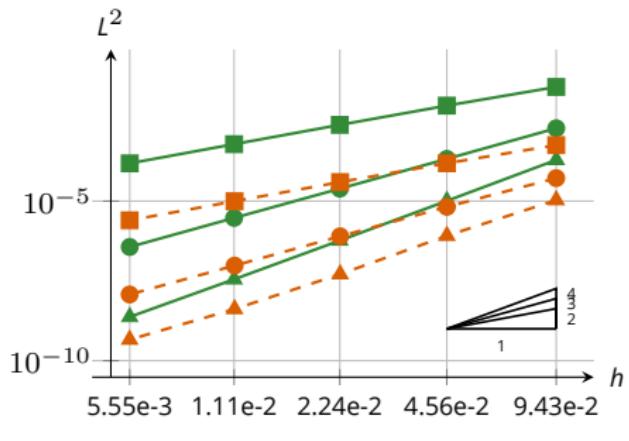
$$D(\mathbf{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

**PINN training:** Imposing BC exactly with a level-set function.

# Numerical results

Error estimates : 1 set of parameters.

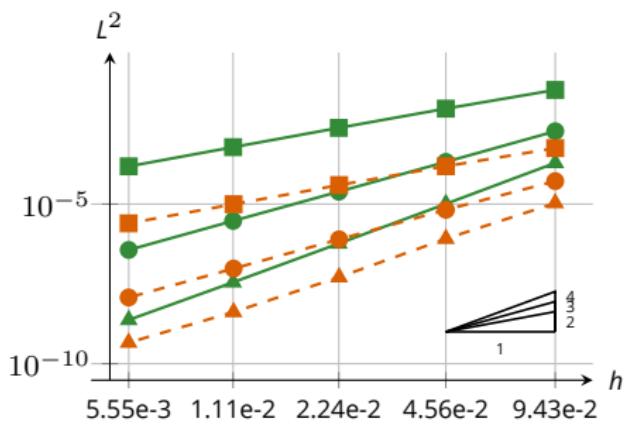
$$\mu = (0.51, 0.54, 0.52, 0.55)$$



# Numerical results

**Error estimates :** 1 set of parameters.

$$\mu = (0.51, 0.54, 0.52, 0.55)$$



**Gains achieved :**  $n_p = 50$  sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

Gains in $L^2$ rel error of our method w.r.t. FEM			
k	min	max	mean
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$N = 20$

$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Cartesian mesh :  $N^2$  nodes.

# 2nd problem considered

**Problem statement:** Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \frac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with  $\Omega = \{(x, y) \in \mathbb{R}^2, 0.25 \leq x^2 + y^2 \leq 1\}$  and  $\mathcal{M} = [2.4, 2.6]$  ( $\rho = 1$ ).  
 $\Gamma_E$  and  $\Gamma_I$  are the outer and inner boundaries of the annulus  $\Omega$ , respectively.

**Analytical solution :**

$$u(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1 \sqrt{x^2 + y^2})}{\ln(4)},$$

**Boundary conditions :**

$$g(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\mathbf{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

**PINN training:** Imposing mixed BC exactly in the PINN<sup>1</sup>.

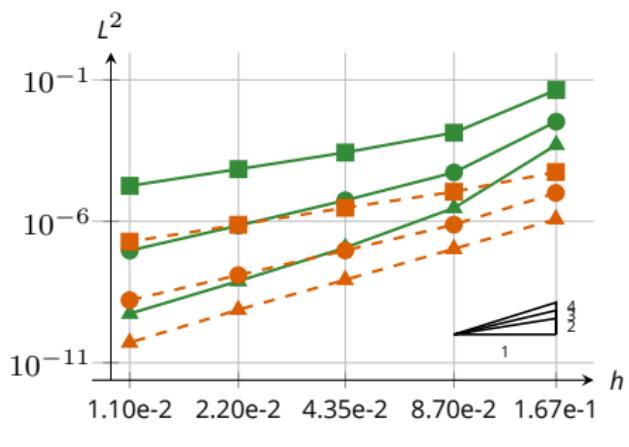
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<sup>1</sup>[Sukumar and Srivastava, 2022]

# Numerical results

Error estimates : 1 set of parameters.

$$\mu = 2.51$$



Gains achieved :  $n_p = 50$  sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

Gains in  $L^2$  rel error  
of our method w.r.t. FEM

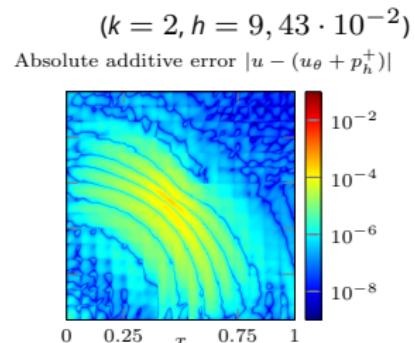
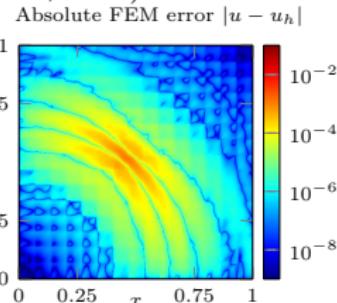
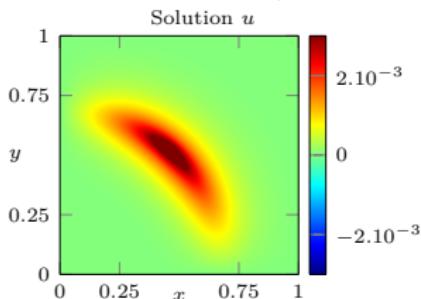
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

$$h = 1.33 \cdot 10^{-1}$$

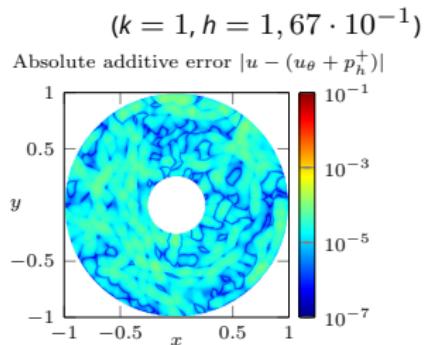
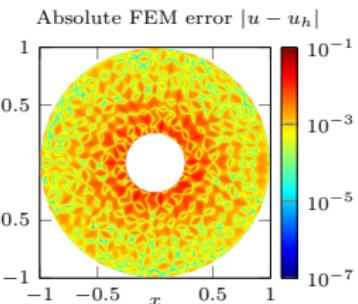
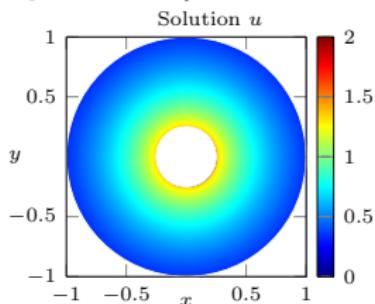
$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

# Numerical solutions

**1st problem :  $\mu = (0.46, 0.52, 0.05, 0.12)$**



**2nd problem :  $\mu = 2.51$**



# New lines of research

Complex geometries

- ★ A posteriori error estimates
- ★ Non linear PDEs

# New lines of research

## Complex geometries

- ★ A posteriori error estimates
- ★ Non linear PDEs

# Learn a regular levelset

Theorem 3: [Clémot and Digne, 2023]

If we have a boundary domain  $\Gamma$ , the SDF is solution to the Eikonal equation:

$$\begin{cases} \|\nabla\phi(x)\| = 1, & x \in \mathcal{O} \\ \phi(x) = 0, & x \in \Gamma \\ \nabla\phi(x) = n, & x \in \Gamma \end{cases}$$



with  $\mathcal{O}$  a box which contains  $\Omega$  completely and  $n$  the exterior normal to  $\Gamma$ .

**Objective:** Move on to complex geometries by using a levelset function to

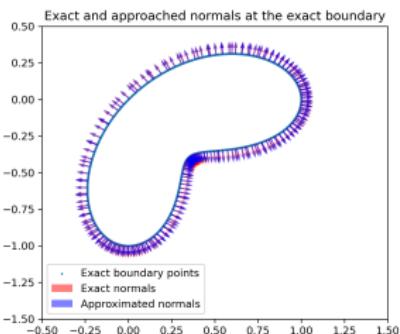
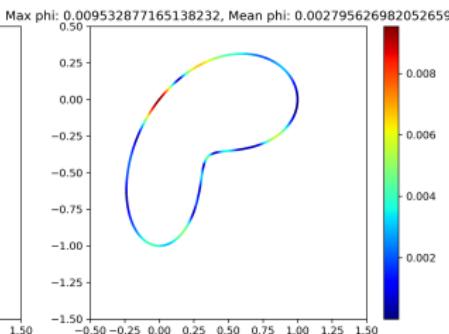
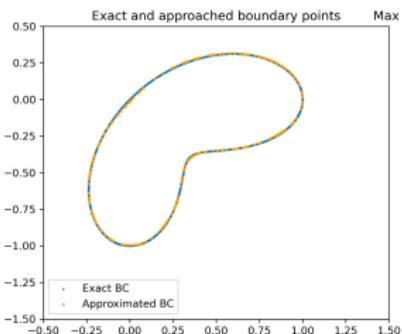
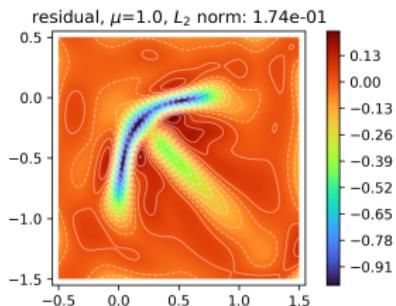
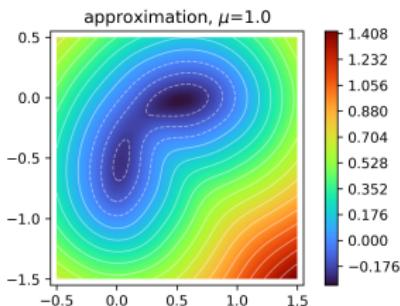
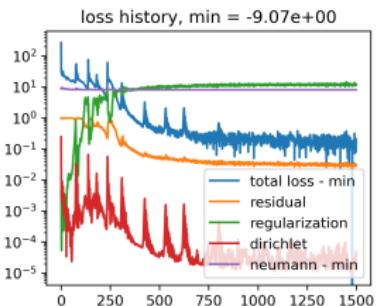
- Sample points in the domain  $\Omega$  for the PINN training.
- Impose exactly the boundary condition in PINN [Sukumar and Srivastava, 2022].

**How to learn a regular levelset ?** with a PINN by adding a regularization term,

$$J_{reg} = \int_{\mathcal{O}} |\Delta\phi|^2,$$

and a sample of boundary points that considers the curvature of  $\Gamma$ . ★

# Numerical results



# New lines of research

Complex geometries

★ A posteriori error estimates

★ Non linear PDEs

# Problem considered

**Problem statement:** Considering the Poisson problem with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

with  $\Omega = [-0.5\pi, 0.5\pi]^2$  and  $\mathcal{M} = [-0.5, 0.5]^2$  ( $p = 2$ ).

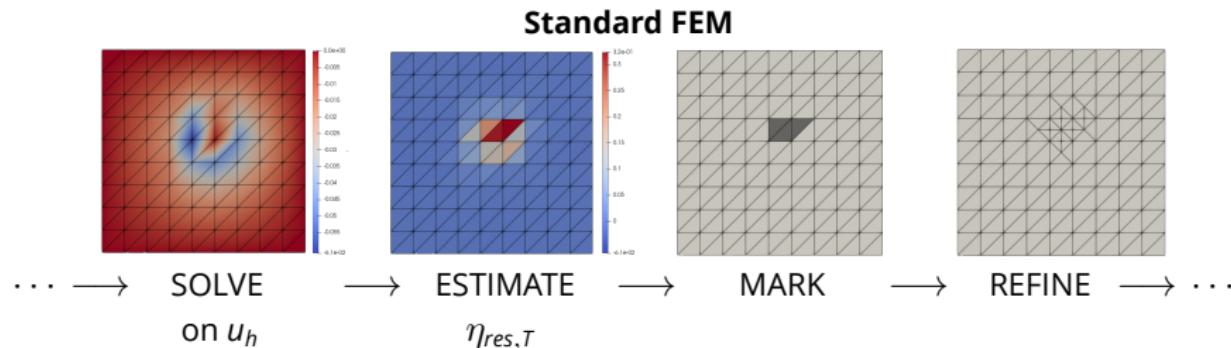
**Analytical solution :**

$$u(\mathbf{x}; \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2(0.15)^2}\right) \sin(2x) \sin(2y).$$

**PINN training:** Imposing Dirichlet BC exactly in the PINN.

## Adaptive mesh refinement

**Adaptive refinement loop** using Dorfler marking strategy. Appendix 4.1



**Local residual estimator (in  $L^2$  norm):** Let  $T$  be a cell of  $\mathcal{T}_h$ .

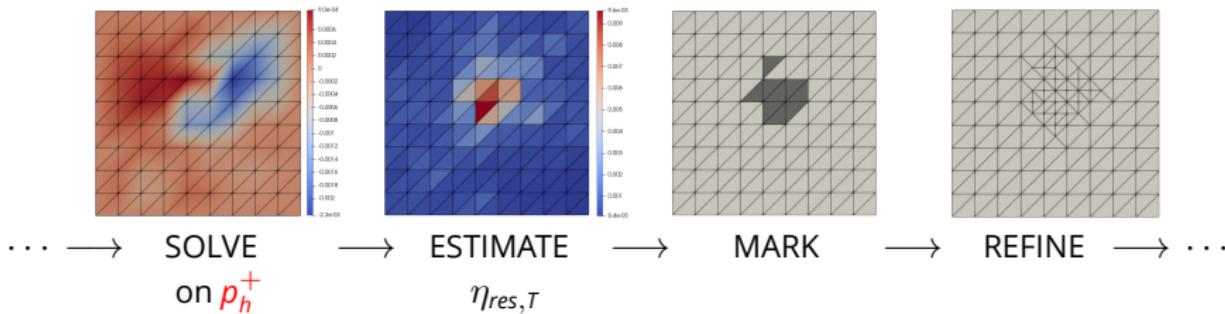
$$\eta_{res,T}^2 = h_T^2 \|\Delta u_h + f_h\|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E \|[\nabla u_h \cdot n]\|_{L^2(E)}^2$$

with  $h_\bullet$  the size of  $\bullet$  and considering the Poisson problem.

# Adaptive mesh refinement

**Adaptive refinement loop** using Dorfler marking strategy.

Additive Approach



**Local residual estimator (in  $L^2$  norm):** Let  $T$  be a cell of  $\mathcal{T}_h$ .

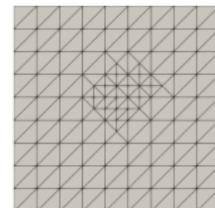
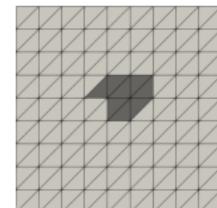
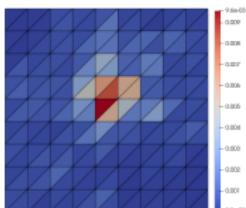
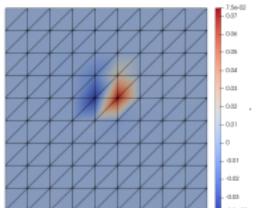
$$\eta_{res,T}^2 = h_T^2 \| ((\Delta u_\theta)_h + \Delta p_h^+) + f_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E \| [\nabla p_h^+ \cdot n] \|_{L^2(E)}^2$$

with  $h_\bullet$  the size of  $\bullet$  and considering the Poisson problem.

# Adaptive mesh refinement

**Adaptive refinement loop** using Dorfler marking strategy.

## Additive Approach - No resolution

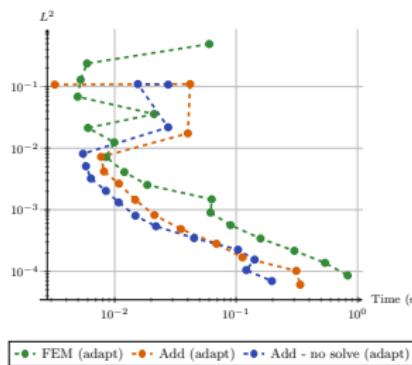
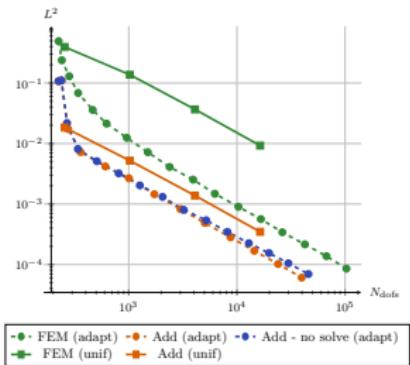


**Local residual estimator (in  $L^2$  norm):** Let  $T$  be a cell of  $\mathcal{T}_h$ .

$$\eta_{res,T}^2 = h_T^2 \|(\Delta u_\theta)_h + f_h\|_{L^2(T)}^2$$

with  $h_\bullet$  the size of • and considering the Poisson problem.

## Numerical results



 Results obtained on a laptop GPU (Time measurements polluted by external factors).

**Ideas for improving results :** Additive approach (no resolution).

time

Interpolate only mesh points added in the refinement process.

## error

Use another metric such as curvature, rather than residual error.

# New lines of research

Complex geometries

★ A posteriori error estimates

★ Non linear PDEs

# Problem considered

**Objective:** Extend the additive approach to non linear PDEs.

**Problem statement:** Considering the **non linear Poisson problem with Dirichlet BC**:

$$\begin{cases} -\operatorname{div}\left((1 + 4u^4)\nabla u\right) = f, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega. \end{cases}$$

with  $\Omega = [-0.5\pi, 0.5\pi]^2$  and  $\mathcal{M} = [-0.5, 0.5]^2$  ( $p = 2$ ).

**Analytical solution :**

$$u(\mathbf{x}; \boldsymbol{\mu}) = 1 + \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y)$$

**PINN training:** Imposing BC exactly with a level-set function.

# Newton method

We want to solve the non linear system:

$N_h$  : number of degrees of freedom.

$$F(u) = 0 \quad (1)$$

with  $F : \mathbb{R}^{N_h} \rightarrow \mathbb{R}^{N_h}$  a non linear operator and  $u \in \mathbb{R}^{N_h}$  the unknown vector.

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**Algorithm 1:** Newton's method to solve (1) [Aghili et al., 2025]

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**Initialization step:** set  $u^{(0)} = u_0$ ;

**for**  $k \geq 0$  **do**

Solve the linear system  $F(u^{(k)}) + F'(u^{(k)})\delta^{(k+1)} = 0$  for  $\delta^{(k+1)}$ ;

Update  $u^{(k+1)} = u^{(k)} + \delta^{(k+1)}$ ;

**end**

---

**Standard version:**

Initialization with a constant value  $u_0$ . For instance,  $u_0 = 1$ .

**DeepPhysics version:** [Odote et al., 2021]

Initialization with a PINN solution  $u_0 = u_\theta$ .

# Newton method

We want to solve the non linear system:

$N_h$  : number of degrees of freedom.

$$F(u) = 0 \quad (1)$$

with  $F : \mathbb{R}^{N_h} \rightarrow \mathbb{R}^{N_h}$  a non linear operator and  $u \in \mathbb{R}^{N_h}$  the unknown vector.

## Algorithm 2: Additive approach to solve (1)

**Initialization step:** set  $p_+^{(0)} = 0$ ;

**for**  $k \geq 0$  **do**

Solve the linear system  $F(p_+^{(k)} + u_\theta) + F'(p_+^{(k)} + u_\theta)\delta^{(k+1)} = 0$  for  $\delta^{(k+1)}$ ;

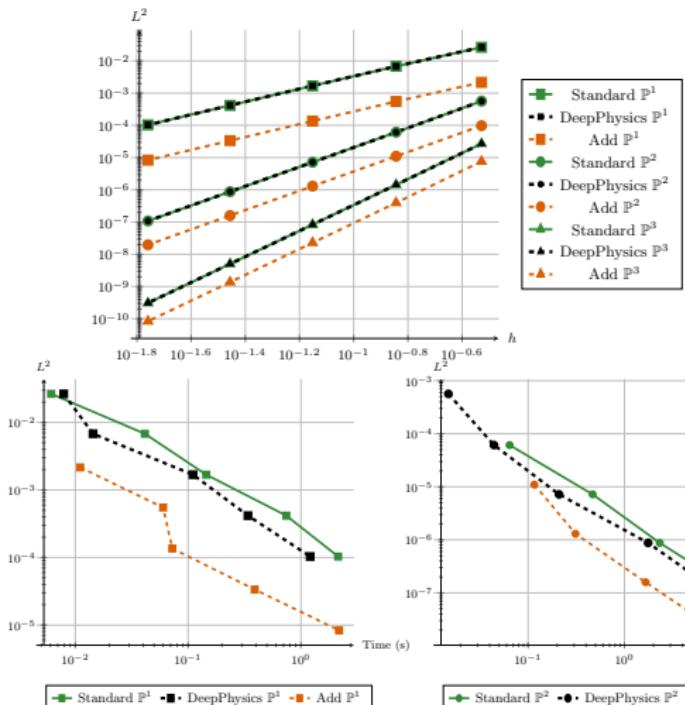
Update  $p_+^{(k+1)} = p_+^{(k)} + \delta^{(k+1)}$ ;

**end**

## Advantage compared to DeepPhysics:

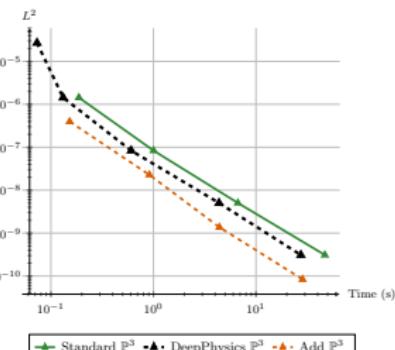
$u_\theta$  is not required to live in the same space as  $p_h^+$ .

# Numerical results



## Number of iterations :

- Standard Newton: 8 iterations.
- DeepPhysics: 4 iterations.
- Additive approach: 4 iterations.



# Supplementary work

# Supplementary work I

## Teaching

- ▶ 2024/2025 :
  - ▶ 64h of Computer Science Practical Work - L1S2 and L2S3 (Python) / L3S6 (C++)
  - ▶ 3 days supervising a group of high school girls in RJMI ("Rendez-vous des Jeunes Mathématiciennes et Informaticiennes")
- ▶ 2023/2024 : 50h of Computer Science Practical Work - L2S3 (Python) / L3S6 (C++)

## Training courses (Total : 176h35)

- ▶ A dozen seminars organized by IRMA ( $\approx 10h$ )
- ▶ 1 Deep Learning introductory course - FIDLE ( $\approx 40h$ )
- ▶ 2 workshops on Scientific Machine Learning ( $\approx 2 \times 21h$ )
- ▶ 1 summer school on "New Trend in computing" ( $\approx 27h$ )
- ▶ several cross-disciplinary courses - Methodology, scientific English, etc. ( $\approx 58h$ )

# Supplementary work II

## Talks

- ▶ **ICOSAHOM 2025, Montréal** - July 2025 (*Coming soon...*)  
"Enriching continuous Lagrange finite element approximation spaces using neural networks"
- ▶ **DTE & AICOMAS 2025, Paris** - February 20, 2025  
"Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- ▶ **Exama project, WP2 reunion** - March 26, 2024  
"How to work with complex geometries in PINNs ?"
- ▶ **Retreat (Macaron/Tonus)** - February 6, 2024  
"Mesh-based methods and physically informed learning"
- ▶ **Team meeting (Mimesis)** - December 12, 2023  
"Development of hybrid finite element/neural network methods to help create digital surgical twins"

# Supplementary work III

## Posters

- ▶ **EMS-TAG-SciML 2025, Milan** - March 24, 2025 - "Enriching continuous Lagrange finite element approximation spaces using neural networks"
- ▶ **CJC-MA 2024, Lyon** - October 29, 2024 - "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- ▶ **MSII poster day, Strasbourg** - October 24, 2024
- ▶ **SciML 2024, Strasbourg** - July 08, 2024

## Publications

- ▶ Enriching continuous lagrange finite element approximation spaces using neural networks. (*submitted in February 2025, M2AN journal*)  
H. Barucq, M. Duprez, F. Faucher, E. Franck, **F. Lecourtier**, V. Lleras, V. Michel-Dansac, and N. Victorion.

## Conclusion

## Enriched finite element method using PINNs :

- PINNs are good candidates for the enriched approach. [Appendix 2](#)
  - Numerical validation of the theoretical results.
  - The enriched approach provides the same results as the standard FEM method, but with coarser meshes.  $\Rightarrow$  Reduction of the computational cost.

We have also tested a multiplicative approach. Appendix 3

#### New lines of research :

- The treatment of complex geometries is progressing.
  - New PDEs begin to be considered, in particular non-linear problems.
  - Other methods for improving the additive approach are being studied, including a posteriori error estimators.

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Appendix 1 : Standard methods



Appendix 2 : Data-driven vs Physics-Informed training



Appendix 3 : Multiplicative approach



Appendix 4 : More



# Appendix 1 : Standard methods

LECOURTIER Frédérique

Development of an hybrid finite element and neural network method

# A1.1 – Physics-Informed Neural Networks

**Standard PINNs<sup>1</sup> (Weak BC)** : Find the optimal weights  $\theta^*$ , such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \omega_b J_b(\theta)), \quad (\mathcal{P}_\theta)$$

with

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

boundary loss

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \boldsymbol{\mu}) - g(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

where  $u_\theta$  is a neural network,  $g = 0$  is the Dirichlet BC.

In  $(\mathcal{P}_\theta)$ ,  $\omega_r$  and  $\omega_b$  are some weights.

**Monte-Carlo method** : Discretize the cost functions by random process.

<sup>1</sup>[Raissi et al., 2019]

# A1.1 – Physics-Informed Neural Networks

**Improved PINNs<sup>1</sup> (Strong BC)** : Find the optimal weights  $\theta^*$  such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)}),$$

with  $\omega_r = 1$  and

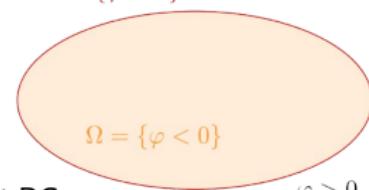
residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_{\theta}(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

$$\partial\Omega = \{\varphi = 0\}$$

where  $u_{\theta}$  is a neural network defined by

$$u_{\theta}(\mathbf{x}, \boldsymbol{\mu}) = \varphi(\mathbf{x}) w_{\theta}(\mathbf{x}, \boldsymbol{\mu}) + g(\mathbf{x}, \boldsymbol{\mu}),$$



with  $\varphi$  a level-set function,  $w_{\theta}$  a NN and  $g = 0$  the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN :  $u_{\theta} = g$  on  $\partial\Omega$ .

---

<sup>1</sup>[Lagaris et al., 1998; Franck et al., 2024]

# A1.2 – Finite Element Methods<sup>1</sup>

## Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

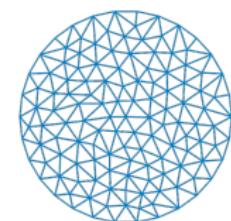
with  $h$  the characteristic mesh size,  $a$  and  $l$  the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_{\Omega} D \nabla u_h \cdot \nabla v_h + \int_{\Omega} R u_h v_h + \int_{\Omega} v_h C \cdot \nabla u_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and  $V_h^0$  the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where  $\mathbb{P}_k$  is the space of polynomials of degree at most  $k$ .



**Linear system :** Let  $(\phi_1, \dots, \phi_{N_h})$  a basis of  $V_h^0$ .

Find  $U \in \mathbb{R}^{N_h}$  such that  $AU = b$

with

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_h} \quad \text{and} \quad b = (l(\phi_j))_{1 \leq j \leq N_h}.$$

$$\mathcal{T}_h = \{K_1, \dots, K_{N_e}\}$$

( $N_e$  : number of elements)

---

<sup>1</sup>[Ern and Guermond, 2004]

# Appendix 2 : Data-driven vs Physics-Informed training

# A2 – Problem considered

**Problem statement:** Consider the Poisson problem in 1D with Dirichlet BC:

$$\begin{cases} -\partial_{xx}u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with  $\Omega = [0, 1]^2$  and  $\mathcal{M} = [0, 1]^3$  ( $p = 3$  parameters).

**Analytical solution :**  $u(x; \boldsymbol{\mu}) = \mu_1 \sin(2\pi x) + \mu_2 \sin(4\pi x) + \mu_3 \sin(6\pi x)$ .

**Construction of two priors:** MLP of 6 layers; Adam optimizer (10000 epochs).

Imposing the Dirichlet BC exactly in the PINN with  $\varphi(x) = x(x - 1)$ .

- **Physics-informed training:**  $N_{\text{col}} = 5000$  collocation points.

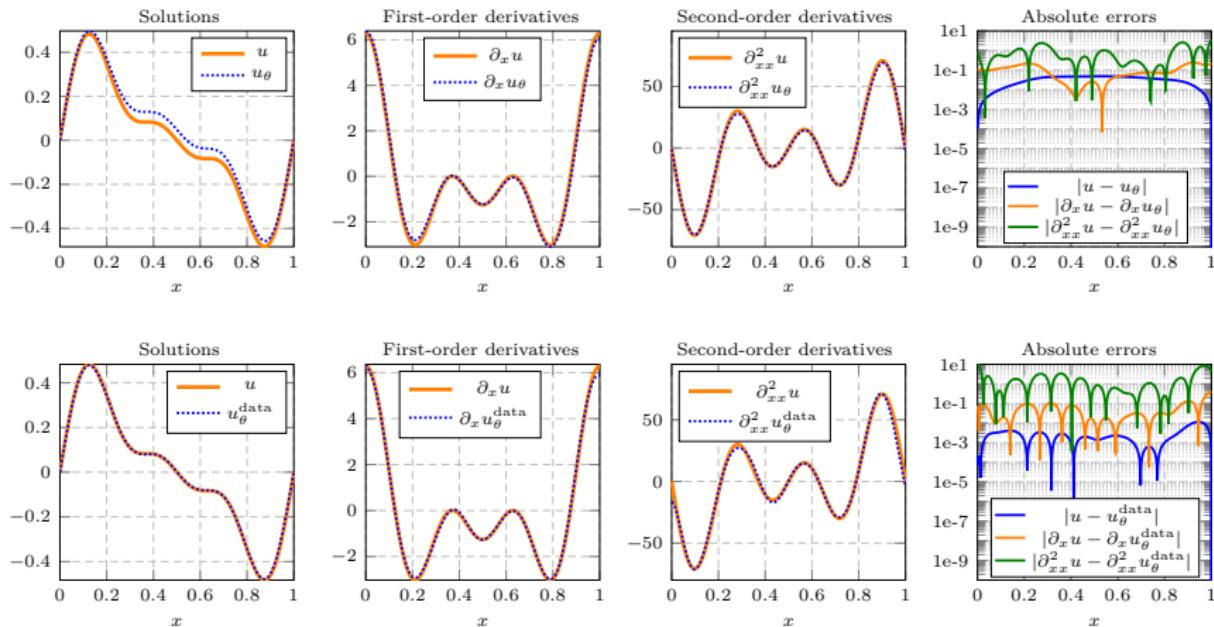
$$J_r(\theta) \simeq \frac{1}{N_{\text{col}}} \sum_{i=1}^{N_{\text{col}}} \left| \partial_{xx}u_\theta(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) + f(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) \right|^2.$$

- **Data-driven training:**  $N_{\text{data}} = 5000$  data.

$$J_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \left| u_\theta^{\text{data}}(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) - u(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) \right|^2.$$

# A2 – Priors derivatives

$$\mu^{(1)} = (0.3, 0.2, 0.1)$$



# A2 – Additive approach in $\mathbb{P}_1$

**1 set of parameters:**  $\mu^{(1)} = (0.3, 0.2, 0.1)$

<b>FEM</b>		<b>PINN prior <math>u_\theta</math></b>			<b>Data prior <math>u_\theta^{\text{data}}</math></b>	
<b>N</b>	<b>error</b>	<b>N</b>	<b>error</b>	<b>gain</b>	<b>error</b>	<b>gain</b>
16	$5.18 \cdot 10^{-2}$	16	$1.29 \cdot 10^{-3}$	40.34	$3.51 \cdot 10^{-3}$	14.78
32	$1.24 \cdot 10^{-2}$	32	$3.49 \cdot 10^{-4}$	35.41	$8.8 \cdot 10^{-4}$	14.06

**50 set of parameters:**

<b>Gains in <math>L^2</math> rel error of our method w.r.t. FEM</b>						
<b>PINN prior <math>u_\theta</math></b>				<b>Data prior <math>u_\theta^{\text{data}}</math></b>		
<b>N</b>	<b>min</b>	<b>max</b>	<b>mean</b>	<b>min</b>	<b>max</b>	<b>mean</b>
20	26.49	271.92	140.74	6.91	60.85	26.12
40	23.4	258.37	134.11	7.13	39.34	20.55

*N* : Nodes.

Appendix 1 : Standard methods

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Appendix 2 : Data-driven vs Physics-Informed training

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Appendix 3 : Multiplicative approach

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Appendix 4 : More

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# Appendix 3 : Multiplicative approach

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Development of an hybrid finite element and neural network method

# A3 – Multiplicative approach

**Lifted problem :** Considering  $M$  such that  $u_M = u + M > 0$  on  $\Omega$ ,

$$\begin{cases} \mathcal{L}(u_M) = f, & \text{in } \Omega, \\ u_M = M, & \text{on } \partial\Omega. \end{cases}$$

**Variational Problem :** Let  $u_{\theta,M} = u_\theta + M \in M + H^{k+1}(\Omega) \cap H_0^1(\Omega)$ .

Find  $p_h^\times \in 1 + V_h^0$  such that,  $\forall v_h \in V_h^0$ ,  $a(u_{\theta,M} p_h^\times, u_{\theta,M} v_h) = I(u_{\theta,M} v_h)$ ,  $(\mathcal{P}_h^\times)$

with the **enriched trial space**  $V_h^\times$  defined by

$$\{u_{\theta,M}^\times = u_{\theta,M} p_h^\times, \quad p_h^\times \in 1 + V_h^0\}.$$

**General Dirichlet BC :** If  $u = g$  on  $\partial\Omega$ , then

$$p_h^\times = \frac{g + M}{u_{\theta,M}} \quad \text{on } \partial\Omega,$$

with  $u_{\theta,M}$  the PINN prior.

# A3 – Convergence analysis

Theorem 4: Convergence analysis of the enriched FEM [F. Lecourtier et al., 2025]

We denote  $u_{h,M}^X \in V_h^X$  the solution of  $(\mathcal{P}_h^X)$  with  $V_h^X$  the enriched trial space.  
Then, denoting  $u_h^X = u_{h,M}^X - M$ ,

$$|u - u_h^X|_{H^1} \leqslant \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^X\|_{L^2} \leqslant C_{\theta,M} \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{|u|_{H^{q+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

with

$$C_{\theta,M} = \|u_{\theta,M}^{-1}\|_{L^\infty} + 2|u_{\theta,M}^{-1}|_{W^{1,\infty}} + |u_{\theta,M}^{-1}|_{W^{2,\infty}}.$$

# A3 – Additive vs Multiplicative

Theorem 5: [F. Lecourtier et al., 2025]

We have

$$\left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}} \xrightarrow[M \rightarrow \infty]{} \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}},$$

in  $H^1$  semi-norm and

$$C_{\theta,M} \left| \frac{u_M}{u_{\theta,M}} \right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{|u|_{H^{q+1}}} \xrightarrow[M \rightarrow \infty]{} \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}},$$

in  $L^2$  norm.

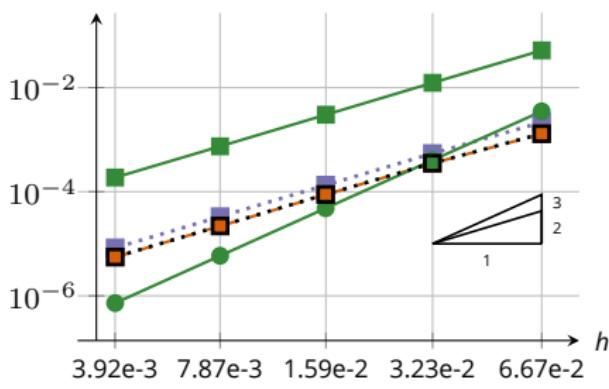
Multiplicative and Additive approaches.

# A3 – Numerical results

Considering the 1D Poisson problem of [Appendix 2](#).

**Error estimates :** 1 set of parameters.

$$L^2 \quad \mu^{(1)} = (0.3, 0.2, 0.1)$$



- FEM  $\mathbb{P}_1$    ···□··· Mult  $\mathbb{P}_1$  ( $M=3$ )   -■- Add  $\mathbb{P}_1$
- FEM  $\mathbb{P}_2$    ···□··· Mult  $\mathbb{P}_1$  ( $M=100$ )

Appendix 1 : Standard methods



Appendix 2 : Data-driven vs Physics-Informed training



Appendix 3 : Multiplicative approach



Appendix 4 : More



# Appendix 4 : More

## A4.1 – Adaptive mesh refinement

**Dorfler marking strategy :** [Dörfler, 1996]

Find  $\mathcal{M}_h \subset \mathcal{T}_h$  of minimal cardinality such that

$$\sum_{T \in \mathcal{M}_h} \eta_{\bullet,T}^2 \geq \theta \sum_{T \in \mathcal{T}_h} \eta_{\bullet,T}^2,$$

with  $\eta_{\bullet,T}$  a local estimator<sup>1</sup> and  $\theta \in (0, 1)$ .

---

<sup>1</sup>For instance, the residual estimator. [Ainsworth and Oden, 1997]