

DTE 2025

Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries

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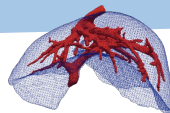
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Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.
accurate quick + parameterized

Parametric elliptic convection/diffusion PDE : For one or several $\mu \in \mathcal{M}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

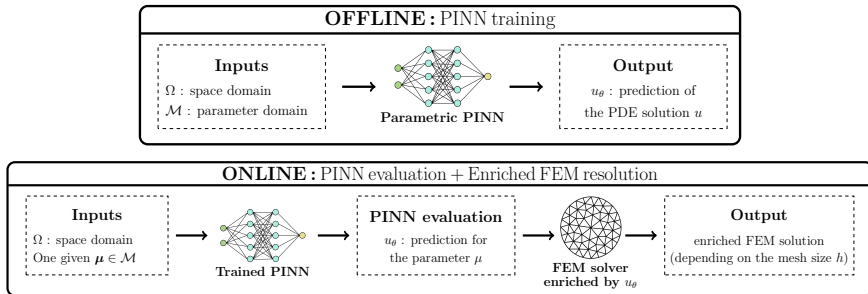
$$\mathcal{L}(u; \mathbf{x}, \mu) = f(\mathbf{x}, \mu), \quad (\mathcal{P})$$

where \mathcal{L} is the parametric differential operator defined by

$$\mathcal{L}(\cdot; \mathbf{x}, \mu) : u \mapsto R(\mathbf{x}, \mu)u + C(\mu) \cdot \nabla u - \frac{1}{\text{Pe}} \nabla \cdot (D(\mathbf{x}, \mu) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on μ).

Pipeline of the Enriched FEM



Remark : The PINN prediction enriched Finite element approximation spaces.

Physics-Informed Neural Networks

Standard PINNs¹ (Weak BC) : Find the optimal weights θ^* , such that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta) \right), \quad (\mathcal{P}_\theta)$$

with

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

boundary loss

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \boldsymbol{\mu}) - g(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In (\mathcal{P}_θ) , ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

¹[Raissi et al., 2019]

Physics-Informed Neural Networks

Improved PINNs¹ (Strong BC) : Find the optimal weights θ^* such that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)} \right),$$

with $\omega_r = 1$ and

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_{\theta}(\mathbf{x}, \mu); \mathbf{x}, \mu) - f(\mathbf{x}, \mu)|^2 d\mathbf{x} d\mu,$$

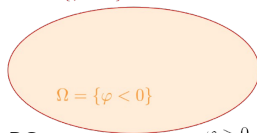
where u_{θ} is a neural network defined by

$$u_{\theta}(\mathbf{x}, \mu) = \varphi(\mathbf{x}) w_{\theta}(\mathbf{x}, \mu) + g(\mathbf{x}, \mu),$$

with φ a level-set function, w_{θ} a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_{\theta} = g$ on $\partial\Omega$.

$$\partial\Omega = \{\varphi = 0\}$$



$$\Omega = \{\varphi < 0\}$$

$$\varphi > 0$$

¹[Lagaris et al., 1998; Franck et al., 2024]

Finite Element Method¹

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_{\Omega} D \nabla u_h \cdot \nabla v_h + \int_{\Omega} R u_h v_h + \int_{\Omega} v_h C \cdot \nabla u_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \{v_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0\},$$

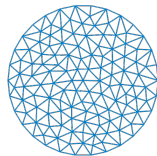
where \mathbb{P}_k is the space of polynomials of degree at most k .

Linear system : Let $(\phi_1, \dots, \phi_{N_h})$ a basis of V_h^0 .

$$\text{Find } U \in \mathbb{R}^{N_h} \text{ such that} \quad AU = b$$

with

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_h} \quad \text{and} \quad b = (l(\phi_j))_{1 \leq j \leq N_h}.$$



$$\mathcal{T}_h = \{K_1, \dots, K_{N_e}\}$$

(N_e : number of elements)

¹[Ern and Guermond, 2004]

How improve PINN prediction with FEM ?

Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

Find $p_h^+ \in V_h^0$ such that, $\forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h)$, (\mathcal{P}_h^+)

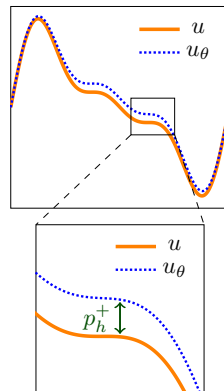
with the **enriched trial space** V_h^+ defined by

$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.



Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of (\mathcal{P}_h) with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leq C_{H^1} h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of (\mathcal{P}_h^+) with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leq \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^+\|_{L^2} \leq \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$ parameters).

Analytical solution :

$$u(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y).$$

PINN training: MLP of 5 layers; LBFGs optimizer (5000 epochs).

Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

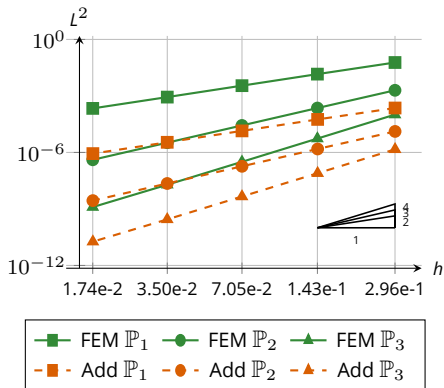
$$\varphi(\mathbf{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi).$$

Training time : less than 1 hour on a laptop GPU.

Numerical results

Error estimates : 1 set of parameters.

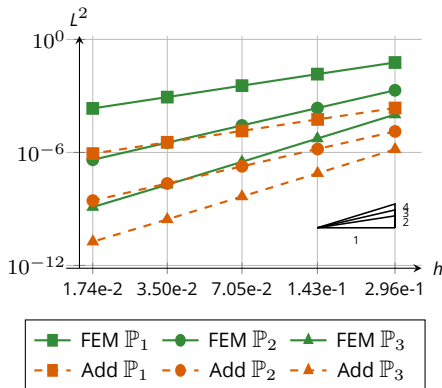
$$\mu^{(1)} = (0.05, 0.22)$$



Numerical results

Error estimates : 1 set of parameters.

$$\mu^{(1)} = (0.05, 0.22)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

k	min	max	mean
1	134.32	377.36	269.39
2	67.02	164.65	134.85
3	39.52	72.65	61.55

$N = 20$

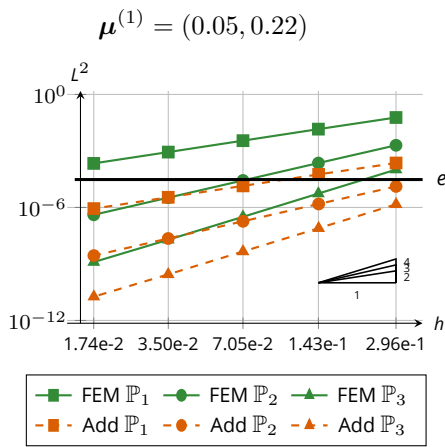
$$\text{Gain} : \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Cartesian mesh : N^2 nodes.

Numerical results

Error estimates : 1 set of parameters.

N_{dofs} required to reach the same error e :



k	e	N_{dofs}	
		FEM	Add
1	$1 \cdot 10^{-3}$	14,161	64
	$1 \cdot 10^{-4}$	143,641	576
2	$1 \cdot 10^{-4}$	6,889	225
	$1 \cdot 10^{-5}$	31,329	1,089
3	$1 \cdot 10^{-5}$	6,724	784
	$1 \cdot 10^{-6}$	20,164	2,704

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\operatorname{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0, 1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$).

Right-hand side :

$$f(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{0.025\sigma^2}\right).$$

Diffusion matrix : (symmetric and positive definite)

$$D(\mathbf{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

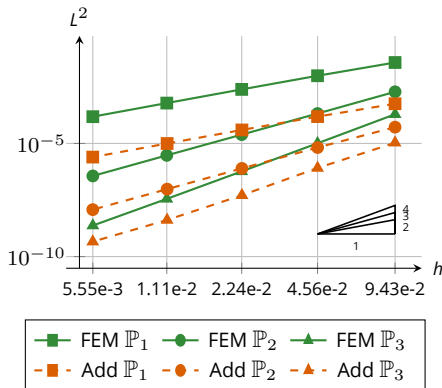
PINN training: MLP with Fourier Features¹ of 5 layers; Adam optimizer (15000 epochs). Imposing the Dirichlet BC exactly in the PINN with a level-set function.

¹[Tancik et al., 2020]

Numerical results

Error estimates : 1 set of parameters.

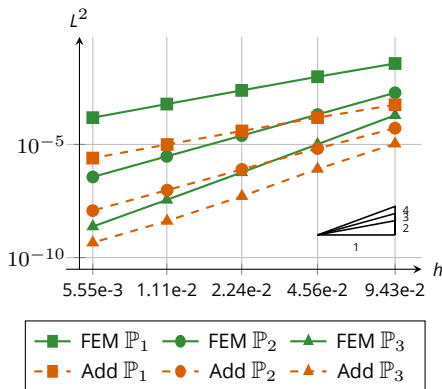
$$\mu^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Numerical results

Error estimates : 1 set of parameters.

$$\mu^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

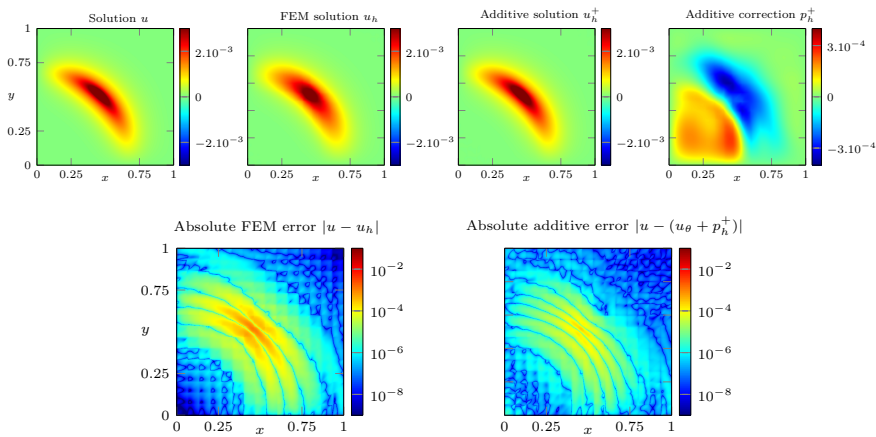
k	min	max	mean
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$N = 20$

$$\text{Gain} : \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Cartesian mesh : N^2 nodes.

Numerical results



$$\mu^{(2)} = (0.46, 0.52, 0.05, 0.12)$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

2D Anisotropic Elliptic problem on a Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \frac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x, y) \in \mathbb{R}^2, 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$).

Analytical solution :

$$u(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1 \sqrt{x^2 + y^2})}{\ln(4)},$$

Boundary conditions :

$$g(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\mathbf{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

PINN training: MLP of 5 layers; LBFGs optimizer (4000 epochs).

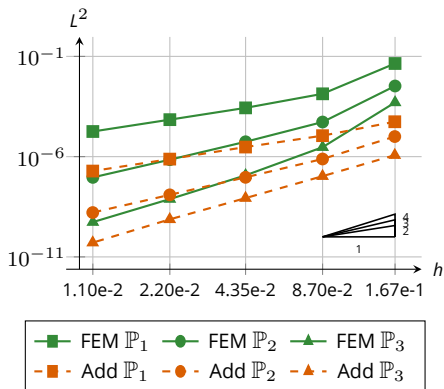
Imposing the mixed BC exactly in the PINN¹.

¹[Sukumar and Srivastava, 2022]

Numerical results

Error estimates : 1 set of parameters.

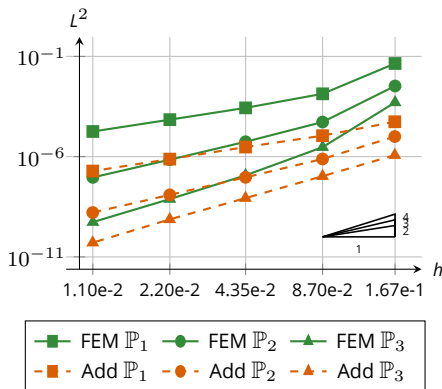
$$\mu^{(1)} = 2.51$$



Numerical results

Error estimates : 1 set of parameters.

$$\mu^{(1)} = 2.51$$



Gains achieved : $n_p = 50$ sets of parameters.

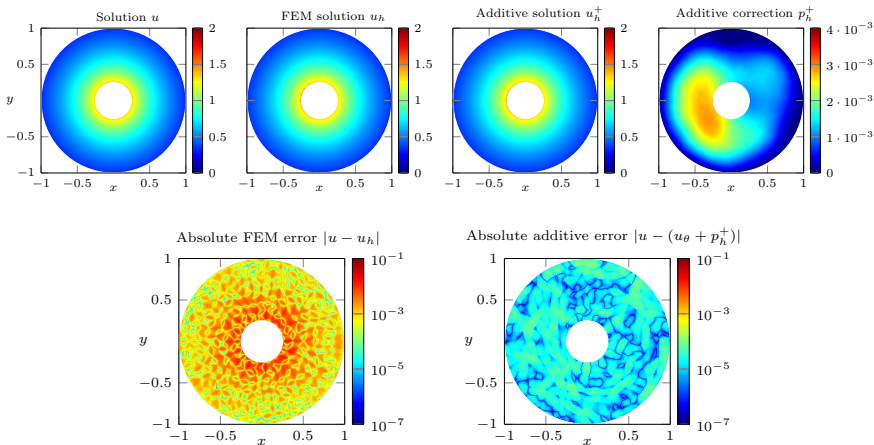
$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

Gains in L^2 rel error of our method w.r.t. FEM			
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

$$h = 1.33 \cdot 10^{-1}$$

$$\text{Gain} : \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Numerical results



$$\mu^{(1)} = 2.51$$

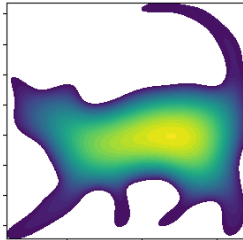
Conclusion

Conclusion and Perspectives

- PINNs are good candidates for the enriched approach. Appendix 2
- Numerical validation of the theoretical results.
- The enriched approach provides the same results as the standard FEM method, but with coarser meshes. \Rightarrow Reduction of the computational cost.

Perspectives :

- Consider non-linear problems. Appendix 1
- Use PINN prediction to build an optimal mesh, via a posteriori error estimates.
- Validate the additive approach on more complex geometry.



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Appendix

Appendix 1 : Non-linear problems

TODO

Appendix 2 : Data-driven vs Physics-Informed training

Appendix 2 : Problem considered

Problem statement: Consider the Poisson problem in 1D with Dirichlet BC:

$$\begin{cases} -\partial_{xx}u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [0, 1]^2$ and $\mathcal{M} = [0, 1]^3$ ($p = 3$ parameters).

Analytical solution : $u(x; \mu) = \mu_1 \sin(2\pi x) + \mu_2 \sin(4\pi x) + \mu_3 \sin(6\pi x)$.

Construction of two priors: MLP of 6 layers; Adam optimizer (10000 epochs).

Imposing the Dirichlet BC exactly in the PINN with $\varphi(x) = x(x - 1)$.

- **Physics-informed training:** $N_{\text{col}} = 5000$ collocation points.

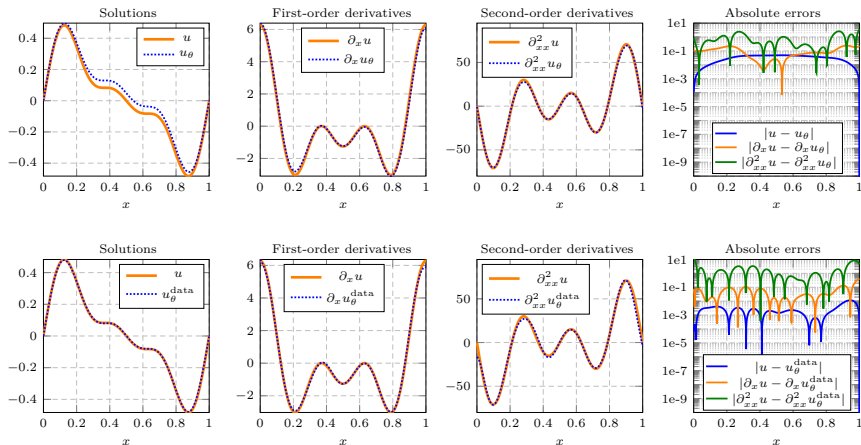
$$J_r(\theta) \simeq \frac{1}{N_{\text{col}}} \sum_{i=1}^{N_{\text{col}}} \left| \partial_{xx} u_{\theta}(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) + f(\mathbf{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) \right|^2.$$

- **Data-driven training:** $N_{\text{data}} = 5000$ data.

$$J_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \left| u_{\theta}^{\text{data}}(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) - u(\mathbf{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) \right|^2.$$

Priors derivatives

$$\mu^{(1)} = (0.3, 0.2, 0.1)$$



Additive approach in \mathbb{P}_1

1 set of parameters: $\mu^{(1)} = (0.3, 0.2, 0.1)$

FEM		PINN prior u_θ			Data prior u_θ^{data}	
N	error	N	error	gain	error	gain
16	$5.18 \cdot 10^{-2}$	16	$1.29 \cdot 10^{-3}$	40.34	$3.51 \cdot 10^{-3}$	14.78
32	$1.24 \cdot 10^{-2}$	32	$3.49 \cdot 10^{-4}$	35.41	$8.8 \cdot 10^{-4}$	14.06

50 set of parameters:

Gains in L^2 rel error of our method w.r.t. FEM						
N	PINN prior u_θ			Data prior u_θ^{data}		
	min	max	mean	min	max	mean
20	26.49	271.92	140.74	6.91	60.85	26.12
40	23.4	258.37	134.11	7.13	39.34	20.55

N : Nodes.

Appendix 3 : Multiplicative approach

Appendix 3 : Multiplicative approach

TODO