2nd CSI

Development of hybrid finite element/neural network methods to help create digital surgical twins

Frédérique Lecourtier[1], Michel Duprez[1], Emmanuel Franck[2] and Vanessa Lleras[3]

[1]Project-Team MIMESIS, Inria, Strasbourg, France
[2]Project-Team MACARON, Inria, Strasbourg, France
[3]IMAG, University of Montpellier, Montpellier, France

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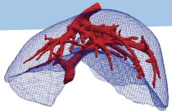
☆ - Update 2025

Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid $\boxed{\text{finite element}}$ / $\boxed{\text{neural network}}$ method.

accurate quick + parameterized

☆ **Parametric elliptic convection/diffusion PDE :** For one or several $\boldsymbol{\mu} \in \mathcal{M}$, find $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{L}\big(u\,;\boldsymbol{x},\boldsymbol{\mu}\big) = f(\boldsymbol{x},\boldsymbol{\mu}), \qquad (\mathcal{P})$$

where $\mathcal{L}$ is the parametric differential operator defined by

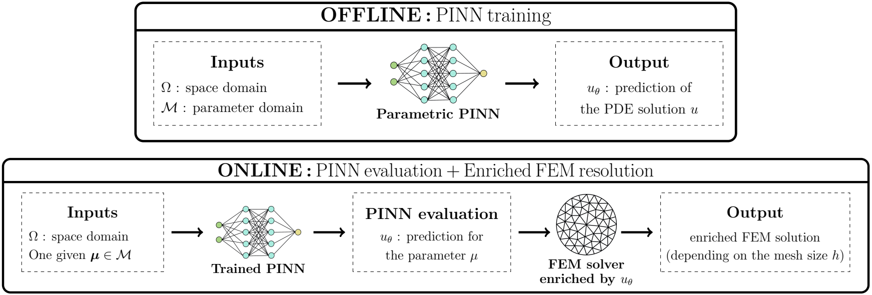
$$\mathcal{L}(\cdot;\boldsymbol{x},\boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x},\boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\text{Pe}} \nabla \cdot (D(\boldsymbol{x},\boldsymbol{\mu})\nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Pipeline of the Enriched FEM

Enriched FEM = Combination of 2 standard methods

- **PINNs** : Physics Informed Neural Networks Appendix 1.1
- **FEMs** : Finite Element Methods Appendix 1.2



Remark : The PINN prediction enriched Finite element approximation spaces.

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This section is based on [F. Lecourtier et al., 2025].

Enriched finite element method using PINNs

Additive approach

★ Numerical results

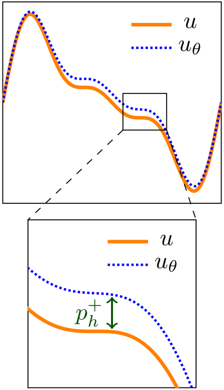
Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \qquad (\mathcal{P}_h^+)$$

with the enriched trial space V_h^+ defined by

$$V_h^+ = \left\{ u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0 \right\}.$$



General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of $(\mathcal{P}_h)$ with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leqslant C_{H^1}\, h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leqslant C_{L^2}\, h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [**F. Lecourtier** et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of $(\mathcal{P}_h^+)$ with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}}\right),$$

$$\|u - u_h^+\|_{L^2} \leqslant \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

Gains of the additive approach.

Enriched finite element method using PINNs

Additive approach

★ Numerical results

1st problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0,1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$).

Right-hand side :

$$f(\boldsymbol{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{0.025\sigma^2}\right).$$

Diffusion matrix : (symmetric and positive definite)

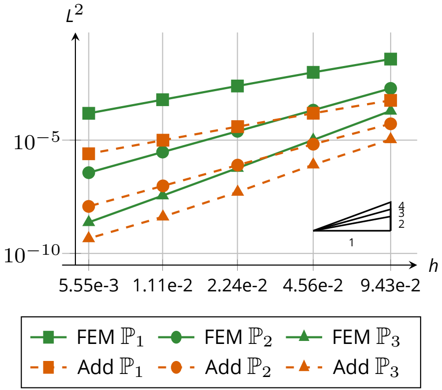
$$D(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

PINN training: Imposing BC exactly with a level-set function.

Numerical results

Error estimates : 1 set of parameters.

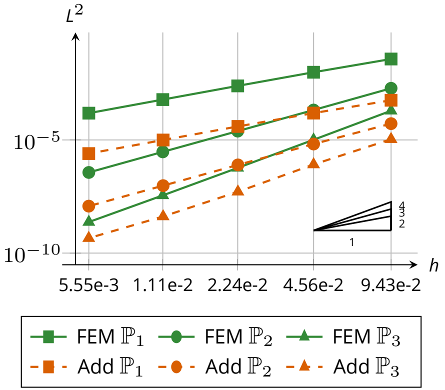
$$\boldsymbol{\mu} = (0.51, 0.54, 0.52, 0.55)$$



Numerical results

Error estimates : 1 set of parameters.

$$\boldsymbol{\mu} = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

	Gains in L^2 rel error of our method w.r.t. FEM		
k	**min**	**max**	**mean**
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$N = 20$

Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Cartesian mesh : N^2 nodes.

2nd problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \dfrac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2,\ 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$).
Γ_E and Γ_I are the outer and inner boundaries of the annulus Ω, respectively.

Analytical solution :

$$u(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln\left(\mu_1 \sqrt{x^2+y^2}\right)}{\ln(4)},$$

Boundary conditions :

$$g(\boldsymbol{x};\boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\boldsymbol{x};\boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

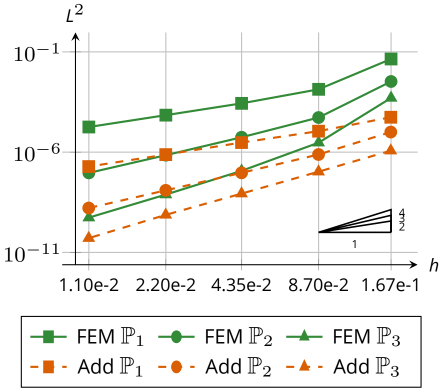
PINN training: Imposing mixed BC exactly in the PINN[1].

[1] [Sukumar and Srivastava, 2022]

Numerical results

Error estimates : 1 set of parameters.

$$\boldsymbol{\mu} = 2.51$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

Gains in L^2 rel error of our method w.r.t. FEM

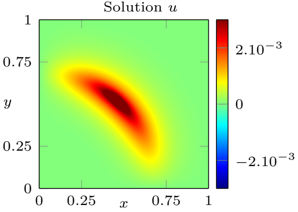
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

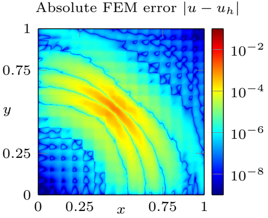
$$h = 1.33 \cdot 10^{-1}$$

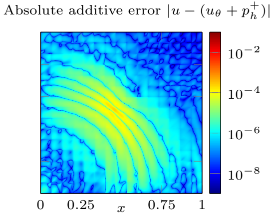
Gain : $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Numerical solutions

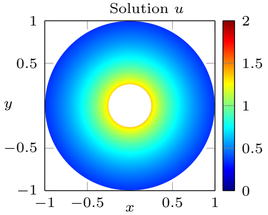
1st problem : $\mu = (0.46, 0.52, 0.05, 0.12)$ $(k = 2, h = 9,43 \cdot 10^{-2})$

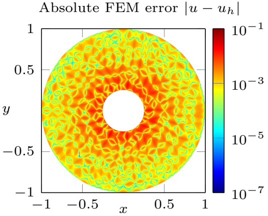


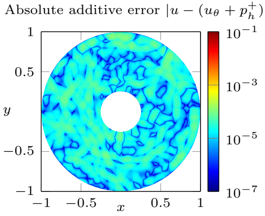




2nd problem : $\mu = 2.51$ $(k = 1, h = 1,67 \cdot 10^{-1})$







New lines of research

Complex geometries
★ A posteriori error estimates
★ Non linear PDEs

New lines of research

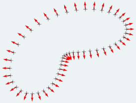
Complex geometries
- ⭐ A posteriori error estimates
- ⭐ Non linear PDEs

Learn a regular levelset

Theorem 3: [Clémot and Digne, 2023]

If we have a boundary domain Γ, the SDF is solution to the Eikonal equation:

$$\begin{cases} ||\nabla\phi(X)|| = 1, \; X \in \mathcal{O} \\ \phi(X) = 0, \; X \in \Gamma \\ \nabla\phi(X) = n, \; X \in \Gamma \end{cases}$$



with $\mathcal{O}$ a box which contains Ω completely and n the exterior normal to Γ.

Objective: Move on to complex geometries by using a levelset function to

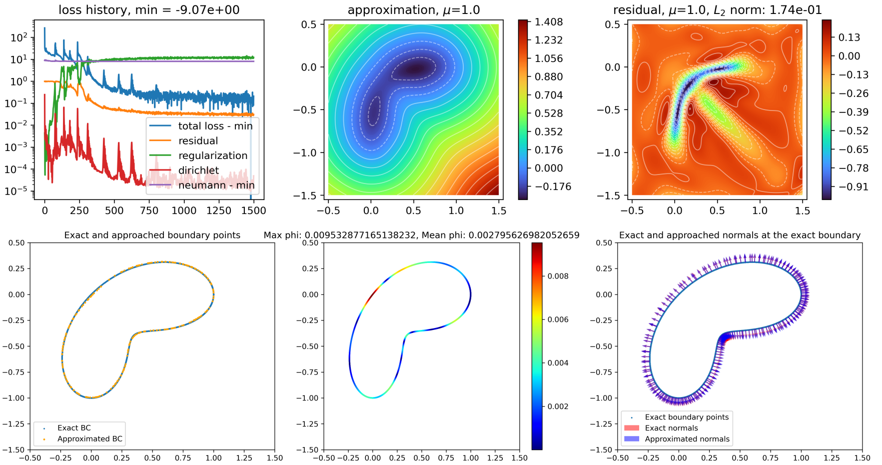
- Sample points in the domain Ω for the PINN training.
- Impose exactly the boundary condition in PINN [Sukumar and Srivastava, 2022].

How to learn a regular levelset ? with a PINN by adding a regularization term,

$$J_{reg} = \int_{\mathcal{O}} |\Delta\phi|^2,$$

and a sample of boundary points that considers the curvature of Γ. ☆

Numerical results



New lines of research

Complex geometries

★ A posteriori error estimates

★ Non linear PDEs

Problem considered

Problem statement: Considering the Poisson problem with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \Gamma \times \mathcal{M}, \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$).

Analytical solution :

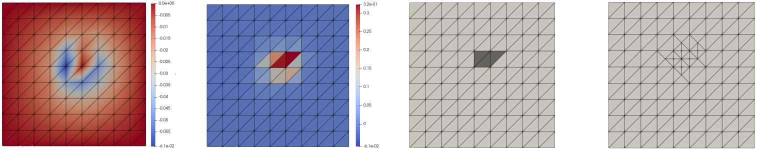
$$u(\boldsymbol{x}; \boldsymbol{\mu}) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2(0.15)^2}\right) \sin(2x)\sin(2y).$$

PINN training: Imposing Dirichlet BC exactly in the PINN.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy. Appendix 4.1

Standard FEM



$\cdots \longrightarrow$ SOLVE on u_h $\longrightarrow$ ESTIMATE $\eta_{res,T}$ $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

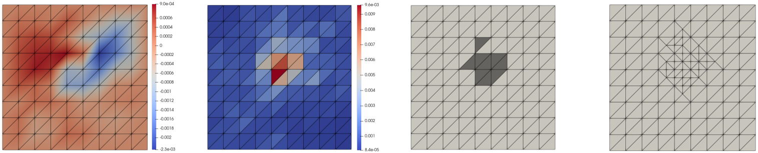
$$\eta_{res,T}^2 = h_T^2 ||\Delta u_h + f_h||_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E ||[\nabla u_h \cdot n]||_{L^2(E)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach



$\cdots \longrightarrow$ SOLVE on p_h^+ $\longrightarrow$ ESTIMATE $\eta_{res,T}$ $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

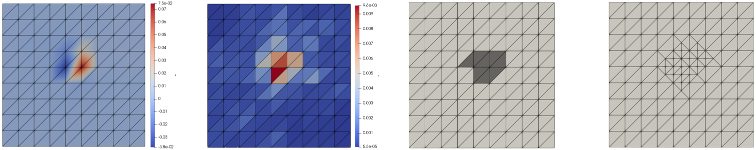
$$\eta_{res,T}^2 = h_T^2 \| \left((\Delta u_\theta)_h + \Delta p_h^+ \right) + f_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E \| [\nabla p_h^+ \cdot n] \|_{L^2(E)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach - No resolution



$\cdots \longrightarrow$ INTERPOLATE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE $\longrightarrow \cdots$

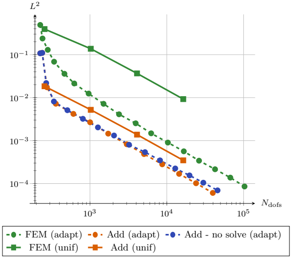
u_θ $\quad$ $\eta_{res,T}$

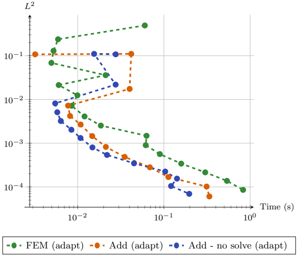
Local residual estimator (in L^2 norm): Let T be a cell of $\mathcal{T}_h$.

$$\eta_{res,T}^2 = h_T^2 \|(\Delta u_\theta)_h + f_h\|_{L^2(T)}^2$$

with $h_\bullet$ the size of $\bullet$ and considering the Poisson problem.

Numerical results





⚠ Results obtained on a laptop GPU (Time measurements polluted by external factors).

Ideas for improving results : Additive approach (no resolution).

↘ **time** Interpolate only mesh points added in the refinement process.

↘ **error** Use another metric such as curvature, rather than residual error.

New lines of research

Complex geometries

☆ A posteriori error estimates

★ Non linear PDEs

Problem considered

Objective: Extend the additive approach to non linear PDEs.

Problem statement: Considering the non linear Poisson problem with Dirichlet BC:

$$\begin{cases} -\mathrm{div}\big((1+4u^4)\nabla u\big) = f, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega. \end{cases}$$

with $\Omega = [-0.5\pi, 0.5\pi]^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$).

Analytical solution :

$$u(\boldsymbol{x};\boldsymbol{\mu}) = 1 + \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right)\sin(2x)\sin(2y)$$

PINN training: Imposing BC exactly with a level-set function.

Newton method

We want to solve the non linear system:

N_h : number of degrees of freedom.

$$F(u) = 0 \tag{1}$$

with $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ a non linear operator and $u \in \mathbb{R}^{N_h}$ the unknown vector.

Algorithm 1: Newton's method to solve (1) [Aghili et al., 2025]

Initialization step: set $u^{(0)} = u_0$;
for $k \geq 0$ **do**
  Solve the linear system $F(u^{(k)}) + F'(u^{(k)})\delta^{(k+1)} = 0$ for $\delta^{(k+1)}$;
  Update $u^{(k+1)} = u^{(k)} + \delta^{(k+1)}$;
end

Standard version:

Initialization with a constant value u_0. For instance, $u_0 = 1$.

DeepPhysics version: [Odot et al., 2021]

Initialization with a PINN solution $u_0 = u_\theta$.

Newton method

We want to solve the non linear system:

N_h : number of degrees of freedom.

$$F(p_+ + u_\theta) = 0 \tag{1}$$

with $F : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ a non linear operator and $p_+ \in \mathbb{R}^{N_h}$ the unknown vector.

Algorithm 2: Additive approach to solve (1)

Initialization step: set $p_+^{(0)} = 0$;

for $k \geq 0$ **do**

  Solve the linear system $F(p_+^{(k)} + u_\theta) + F'(p_+^{(k)} + u_\theta)\delta^{(k+1)} = 0$ for $\delta^{(k+1)}$;

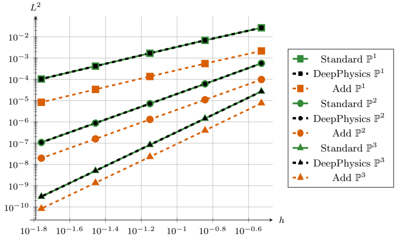
  Update $p_+^{(k+1)} = p_+^{(k)} + \delta^{(k+1)}$;

end

Advantage compared to DeepPhysics:

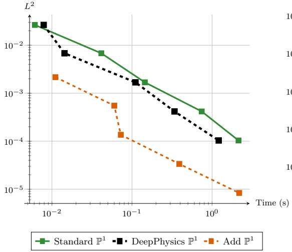
u_θ is not required to live in the same space as p_+.

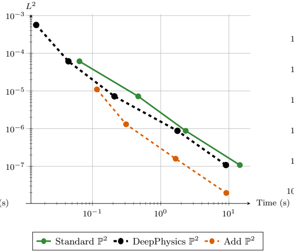
Numerical results

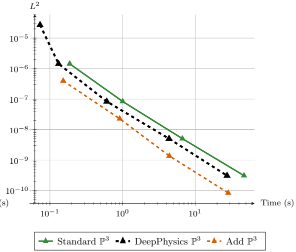


Number of iterations :

- Standard Newton: 8 iterations.
- DeepPhysics: 4 iterations.
- Additive approach: 4 iterations.







Supplementary work

Supplementary work I

Teaching

- 2024/2025 :
  - 64h of Computer Science Practical Work - L1S2 and L2S3 (Python) / L3S6 (C++)
  - 3 days supervising a group of high school girls in RJMI ("Rendez-vous des Jeunes Mathématiciennes et Informaticiennes")
- 2023/2024 : 50h of Computer Science Practical Work - L2S3 (Python) / L3S6 (C++)

Training courses (Total : 176h35)

- A dozen seminars organized by IRMA ($\approx 10h$)
- 1 Deep Learning introductory course - FIDLE ($\approx 40h$)
- 2 workshops on Scientific Machine Learning ($\approx 2 \times 21h$)
- 1 summer school on "New Trend in computing" ($\approx 27h$)
- several cross-disciplinary courses - Methodology, scientific English, etc. ($\approx 58h$)

Supplementary work II

Talks

- **ICOSAHOM 2025, Montréal** - July 2025 *(Coming soon...)*
  "Enriching continuous Lagrange finite element approximation spaces using neural networks"
- **DTE & AICOMAS 2025, Paris** - February 20, 2025
  "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- **Exama project, WP2 reunion** - March 26, 2024
  "How to work with complex geometries in PINNs ?"
- **Retreat (Macaron/Tonus)** - February 6, 2024
  "Mesh-based methods and physically informed learning"
- **Team meeting (Mimesis)** - December 12, 2023
  "Development of hybrid finite element/neural network methods to help create digital surgical twins"

Supplementary work III

Posters

- **EMS-TAG-SciML 2025, Milan** - March 24, 2025 - "Enriching continuous Lagrange finite element approximation spaces using neural networks"
- **CJC-MA 2024, Lyon** - October 29, 2024 - "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
- **MSII poster day, Strasbourg** - October 24, 2024
- **SciML 2024, Strasbourg** - July 08, 2024

Publications

- Enriching continuous lagrange finite element approximation spaces using neural networks. *(submitted in February 2025, M2AN journal)*
  H. Barucq, M. Duprez, F. Faucher, E. Franck, **F. Lecourtier**, V. Lleras, V. Michel-Dansac, and N. Victorion.

Conclusion

Enriched finite element method using PINNs :

- PINNs are good candidates for the enriched approach. Appendix 2
- Numerical validation of the theoretical results.
- The enriched approach provides the same results as the standard FEM method, but with coarser meshes. $\Rightarrow$ Reduction of the computational cost.

We have also tested a multiplicative approach. Appendix 3

New lines of research :

- The treatment of complex geometries is progressing.
- New PDEs begin to be considered, in particular non-linear problems.
- Other methods for improving the additive approach are being studied, including a posteriori error estimators.

References

J. Aghili, E. Franck, R. Hild, V. Michel-Dansac, and V. Vigon. Accelerating the convergence of newton's method for nonlinear elliptic pdes using fourier neural operators. 2025.

M. Ainsworth and J. Tinsley Oden. A posteriori error estimation in finite element analysis. 1997.

M. Clémot and J. Digne. Neural skeleton: Implicit neural representation away from the surface. 2023.

W. Dörfler. A convergent adaptive algorithm for poisson's equation. 1996.

A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. 2004.

E. Franck, V. Michel-Dansac, and L. Navoret. Approximately well-balanced Discontinuous Galerkin methods using bases enriched with Physics-Informed Neural Networks. 2024.

I. E. Lagaris, A. Likas, and D. I. Fotiadis. Artificial neural networks for solving ordinary and partial differential equations. 1998.

A. Odot, R. Haferssas, and S. Cotin. Deepphysics: a physics aware deep learning framework for real-time simulation, 2021.

M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. 2019.

N. Sukumar and A. Srivastava. Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks. 2022.

M. Tancik, P. Srinivasan, and al. Fourier Features Let Networks Learn High Frequency Functions in Low Dimensional Domains. 2020.

F. Lecourtier, H. Barucq, M. Duprez, F. Faucher, E. Franck, V. Lleras, V. Michel-Dansac, and N. Victorion. Enriching continuous lagrange finite element approximation spaces using neural networks, 2025.

Appendix 1 : Standard methods

A1.1 – Physics-Informed Neural Networks

Standard PINNs[1] **(Weak BC) :** Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r \, J_r(\theta) + \omega_b \, J_b(\theta)\right), \qquad (\mathcal{P}_\theta)$$

with

residual loss
$$\boxed{J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left|\mathcal{L}\big(u_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}\big) - f(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

boundary loss
$$\boxed{J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} \left|u_\theta(\boldsymbol{x}, \boldsymbol{\mu}) - g(\boldsymbol{x}, \boldsymbol{\mu})\right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In ($\mathcal{P}_\theta$), ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

[1] [Raissi et al., 2019]

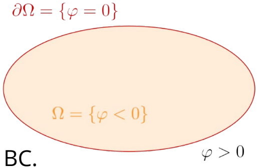
A1.1 – Physics-Informed Neural Networks

Improved PINNs[1] (Strong BC) : Find the optimal weights $\theta^\star$ such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)} \right),$$

with $\omega_r = 1$ and

residual loss $$\boxed{J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| \mathcal{L}\big(u_\theta(\boldsymbol{x},\boldsymbol{\mu}); \boldsymbol{x},\boldsymbol{\mu}\big) - f(\boldsymbol{x},\boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$



where u_θ is a neural network defined by

$$u_\theta(\boldsymbol{x},\boldsymbol{\mu}) = \varphi(\boldsymbol{x}) w_\theta(\boldsymbol{x},\boldsymbol{\mu}) + g(\boldsymbol{x},\boldsymbol{\mu}),$$

with φ a level-set function, w_θ a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_\theta = g$ on $\partial\Omega$.

[1] [Lagaris et al., 1998; Franck et al., 2024]

A1.2 – Finite Element Methods[1]

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \qquad (\mathcal{P}_h)$$

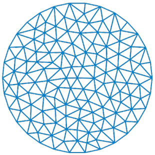
with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_\Omega D\nabla u_h \cdot \nabla v_h + \int_\Omega R\, u_h\, v_h + \int_\Omega v_h\, C \cdot \nabla u_h, \quad l(v_h) = \int_\Omega f v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega),\ \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where $\mathbb{P}_k$ is the space of polynomials of degree at most k.



$\mathcal{T}_h = \{K_1, \ldots, K_{N_e}\}$
(N_e : number of elements)

Linear system : Let $(\phi_1, \ldots, \phi_{N_h})$ a basis of V_h^0.

Find $U \in \mathbb{R}^{N_h}$ such that $\qquad AU = b$

with

$$A = \left(a(\phi_i, \phi_j)\right)_{1 \le i,j \le N_h} \quad \text{and} \quad b = \left(l(\phi_j)\right)_{1 \le j \le N_h}.$$

[1] [Ern and Guermond, 2004]

Appendix 2 : Data-driven vs Physics-Informed training

A2 – Problem considered

Problem statement: Consider the Poisson problem in 1D with Dirichlet BC:

$$\begin{cases} -\partial_{xx} u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = [0,1]^2$ and $\mathcal{M} = [0,1]^3$ ($p = 3$ parameters).

Analytical solution : $u(x; \boldsymbol{\mu}) = \mu_1 \sin(2\pi x) + \mu_2 \sin(4\pi x) + \mu_3 \sin(6\pi x)$.

Construction of two priors: MLP of 6 layers; Adam optimizer (10000 epochs).
Imposing the Dirichlet BC exactly in the PINN with $\varphi(x) = x(x-1)$.

- **Physics-informed training:** $N_{\text{col}} = 5000$ collocation points.

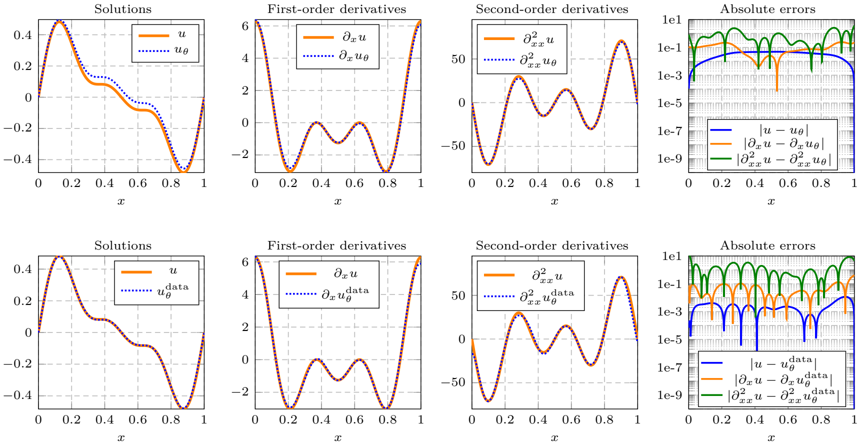
$$J_r(\theta) \simeq \frac{1}{N_{\text{col}}} \sum_{i=1}^{N_{\text{col}}} \left| \partial_{xx} u_\theta(\boldsymbol{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}) + f\left(\boldsymbol{x}_{\text{col}}^{(i)}; \boldsymbol{\mu}_{\text{col}}^{(i)}\right) \right|^2 .$$

- **Data-driven training:** $N_{\text{data}} = 5000$ data.

$$J_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \left| u_\theta^{\text{data}}(\boldsymbol{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) - u(\boldsymbol{x}_{\text{data}}^{(i)}; \boldsymbol{\mu}_{\text{data}}^{(i)}) \right|^2 .$$

A2 – Priors derivatives

$$\boldsymbol{\mu}^{(1)} = (0.3, 0.2, 0.1)$$



A2 – Additive approach in $\mathbb{P}_1$

1 set of parameters: $\mu^{(1)} = (0.3, 0.2, 0.1)$

	FEM
N	**error**
16	$5.18 \cdot 10^{-2}$
32	$1.24 \cdot 10^{-2}$

	PINN prior u_θ		Data prior u_θ^{data}	
N	**error**	**gain**	**error**	**gain**
16	$1.29 \cdot 10^{-3}$	40.34	$3.51 \cdot 10^{-3}$	14.78
32	$3.49 \cdot 10^{-4}$	35.41	$8.8 \cdot 10^{-4}$	14.06

50 set of parameters:

	Gains in L^2 rel error of our method w.r.t. FEM					
	PINN prior u_θ			Data prior u_θ^{data}		
N	**min**	**max**	**mean**	**min**	**max**	**mean**
20	26.49	271.92	140.74	6.91	60.85	26.12
40	23.4	258.37	134.11	7.13	39.34	20.55

N : Nodes.

Appendix 3 : Multiplicative approach

A3 – Multiplicative approach

Liffted problem : Considering M such that $u_M = u + M > 0$ on Ω,

$$\begin{cases} \mathcal{L}(u_M) = f, & \text{in } \Omega, \\ u_M = M, & \text{on } \partial\Omega. \end{cases}$$

Variational Problem : Let $u_{\theta,M} = u_\theta + M \in M + H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^\times \in 1 + V_h^0 \text{ such that, } \forall v_h \in V_h^0, a\big(u_{\theta,M}\, p_h^\times, u_{\theta,M} v_h\big) = l(u_{\theta,M} v_h), \qquad (\mathcal{P}_h^\times)$$

with the enriched trial space $V_h^\times$ defined by

$$\left\{ u_{h,M}^\times = u_{\theta,M}\, p_h^\times, \quad p_h^\times \in 1 + V_h^0 \right\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^\times = \frac{g + M}{u_{\theta,M}} \quad \text{on } \partial\Omega,$$

with $u_{\theta,M}$ the PINN prior.

A3 – Convergence analysis

Theorem 4: Convergence analysis of the enriched FEM [**F. Lecourtier** et al., 2025]

We denote $u_{h,M}^{\times} \in V_h^{\times}$ the solution of $(\mathcal{P}_h^{\times})$ with $V_h^{\times}$ the enriched trial space. Then, denoting $u_h^{\times} = u_{h,M}^{\times} - M$,

$$|u - u_h^{\times}|_{H^1} \leqslant \boxed{\left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \left(C_{H^1}\, h^k |u|_{H^{k+1}}\right),$$

$$\|u - u_h^{\times}\|_{L^2} \leqslant \boxed{C_{\theta,M} \left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}^2}{|u|_{H^{q+1}}}} \left(C_{L^2}\, h^{k+1} |u|_{H^{k+1}}\right).$$

with

$$C_{\theta,M} = \|u_{\theta,M}^{-1}\|_{L^\infty} + 2|u_{\theta,M}^{-1}|_{W^{1,\infty}} + |u_{\theta,M}^{-1}|_{W^{2,\infty}}.$$

A3 – Additive vs Multiplicative

Theorem 5: [**F. Lecourtier** et al., 2025]

We have

$$\boxed{\left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \xrightarrow[M\to\infty]{} \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}},$$

in H^1 semi-norm and

$$\boxed{C_{\theta,M}\left|\frac{u_M}{u_{\theta,M}}\right|_{H^{q+1}} \frac{\|u_{\theta,M}\|^2_{W^{1,\infty}}}{|u|_{H^{q+1}}}} \xrightarrow[M\to\infty]{} \boxed{\frac{|u-u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}},$$

in L^2 norm.

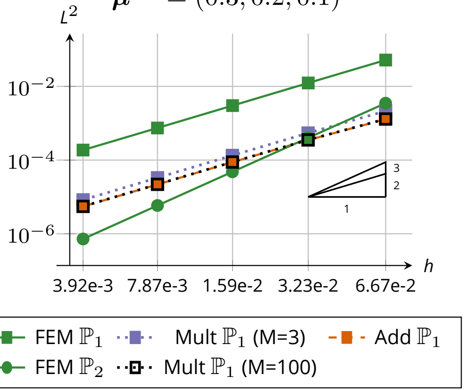
Multiplicative and Additive approaches.

A3 – Numerical results

Considering the 1D Poisson problem of Appendix 2.

Error estimates : 1 set of parameters.

$$\boldsymbol{\mu}^{(1)} = (0.3, 0.2, 0.1)$$



Appendix 4 : More

A4.1 – Adaptive mesh refinement

Dorfler marking strategy : [Dörfler, 1996]
Find $\mathcal{M}_h \subset \mathcal{T}_h$ of minimal cardinality such that

$$\sum_{T\in\mathcal{M}_h} \eta_{\bullet,T}^2 \geq \theta \sum_{T\in\mathcal{T}_h} \eta_{\bullet,T}^2,$$

with $\eta_{\bullet,T}$ a local estimator [1] and $\theta \in (0,1)$.

[1] For instance, the residual estimator. [Ainsworth and Oden, 1997]