How improve PINNs ?

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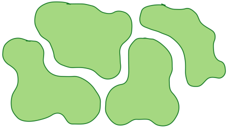
Problem considered



Poisson problem with homogeneous Dirichlet conditions :
Find $u : \Omega \to \mathbb{R}^d (d = 1, 2, 3)$ such that

$$\begin{cases} -\Delta u(X) = f \quad \text{in } \Omega, \\ u(X) = g \quad \text{on } \partial\Omega \end{cases}$$

with Δ the Laplace operator, Ω a smooth bounded open set and Γ its boundary.
Standard PINNs : We are looking for θ such that

$$\theta_u = \underset{\theta}{\operatorname{argmin}}\, w_r\, J_r(\theta) + w_{bc}\, J_{bc}(\theta)$$

where w_r and w_{bc} are the respective weights associated with

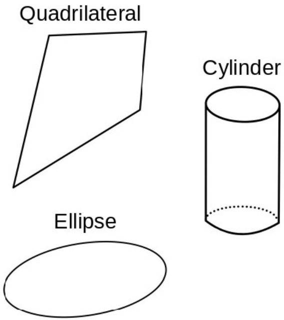
$$J_r = \int_\Omega (\Delta u + 1)^2 \;\text{ and }\; J_{bc} = \int_{\partial\Omega} u^2.$$

Remark : In practice, we use a Monte-Carlo method to discretize the cost function by random process.

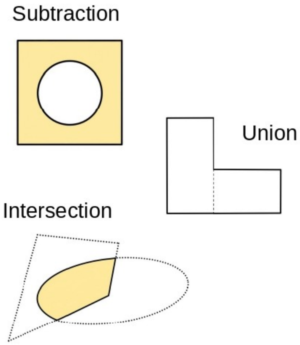
Simple geometry

Claim on PINNs : No mesh, so easy to go on complex geometry !

Easy-to-sample shape



Shape composition



In practice : Not so easy ! We need to find how to sample in the geometry.

Complex geometry

1st approach : Mapping	**2nd approach :** LevelSet function

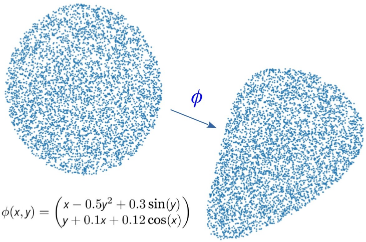
Idea :

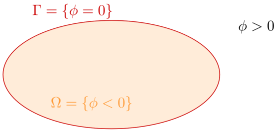
➙ Ω_0 a simple domain (as circle)

➙ Ω a target domain

➙ A mapping from Ω_0 to Ω :

$$\Omega = \phi(\Omega_0)$$





Advantages :

➙ Sample is easy in this case.

➙ Allow to impose in hard the BC :

$$u_\theta(X) = \phi(X) w_\theta(X) + g(X)$$

Natural LevelSet :

Signed Distance Function (SDF)

Problem : SDF is a $\mathcal{C}^0$ function

$\Rightarrow$ its derivatives explodes

$\Rightarrow$ We need a regular levelset

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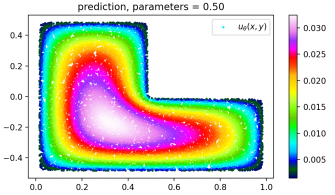
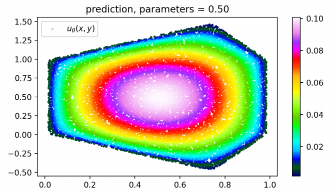
Construct smooth SDF I

1st solution : Approximation theory REFERENCE !!

Δu can be singular at the boundary. Sampling at ϵ to it solve the problem.

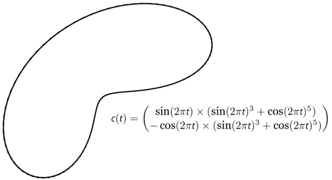
Polygonal domain Appendix 1





Curved domain Appendix 2

Minus : Use of a parametric curve $c(t)$.



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Construct smooth SDF II

2nd solution : Learn the levelset.

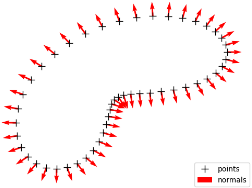
How make that ? with a PINNs. REFERENCE !!

If we have a boundary domain Γ, the SDF is solution to the Eikonal equation:

$$\begin{cases} ||\nabla\phi(X)|| = 1, \; X \in \mathcal{O} \\ \phi(X) = 0, \; X \in \Gamma \\ \nabla\phi(X) = n, \; X \in \Gamma \end{cases}$$

with $\mathcal{O}$ a box which contains Ω completely and n the exterior normal to Γ.

Advantage : No need for parametric curves.



➞ set of boundary points

➞ exterior normals at Γ

(evaluated at this points)

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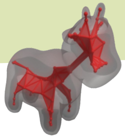
Learn LevelSet I



Objective of the paper :

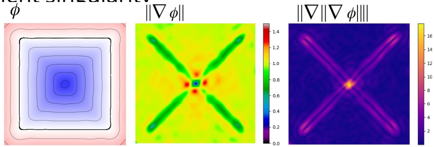
Learn topological Skeleton (by learning SDF) Appendix 3

→ Skeleton correspond exactly to the gradient singularity

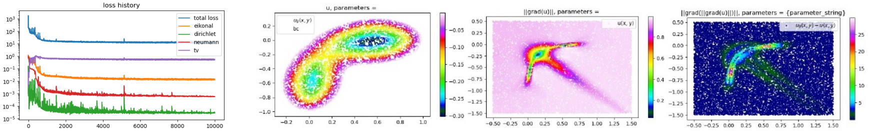
→ Adding the following term in the loss

$$\int_{\mathcal{O}} ||\nabla||\nabla\phi(X)||(p)||dp$$

(Total Variation Regularization)



1st test : Eikonal equation with TV Regularization REFERENCE !!



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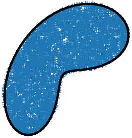
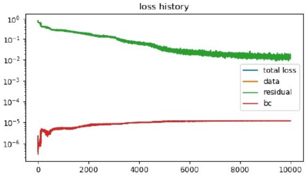
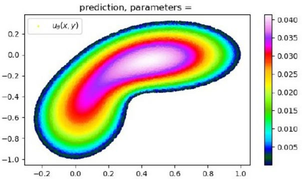
Learn LevelSet I

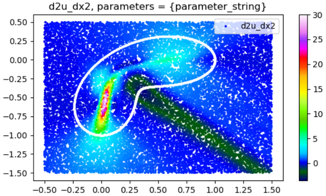
Classical PINNs :

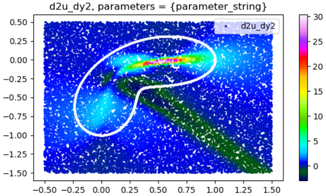






Derivatives $\Rightarrow$ we can't impose in hard boundary conditions





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Learn LevelSet II

2nd test : We replace the tv term by a penalization on the laplacian of the levelset

$$\int_{\mathcal{O}} |\nabla \phi|^2$$

TO COMPLETE !

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Learn LevelSet II

TO COMPLETE !

Conclusion

2 main questions :

➞ How to sample in complex domains?

- Using mapping
- Using Levelset (Approximation theory/Learning)

➞ How can we obtain a levelset that usable for imposing boundary conditions in hard ?
By learning the Eikonal equation with penalisation of the levelset Laplacian

To go further : We can combine the options (Mapping for the big domain. Level set for the hole.)

Thank you !

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Appendix 1 : Polygonal domain

TO COMPLETE !

Appendix 2 : Curved domain

TO COMPLETE !

Appendix 3 : Neural Skeleton

TO COMPLETE !