# Combining Finite Element methods and Neural Networks to solve elliptic problem on complex 2D geometries

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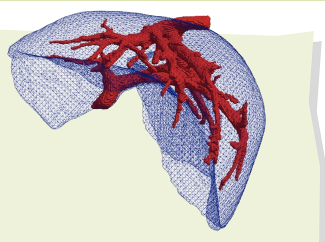
## Scientific context

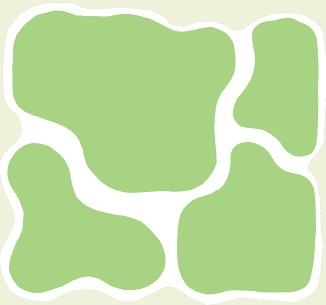


**Create real-time digital twins of an organ (e.g. liver)**

**Current Objective :** Develop hybrid **finite element** / **neural network** methods.
**accurate** **quick + parameterized**

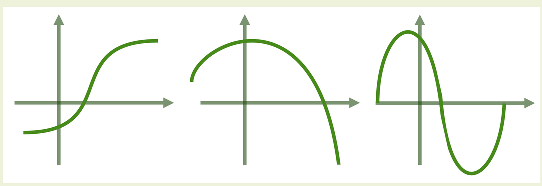
**OFFLINE**

Several Geometries

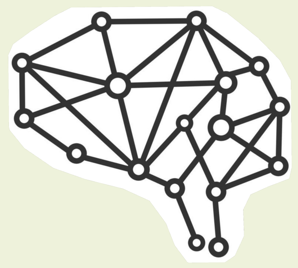


+

Several Forces

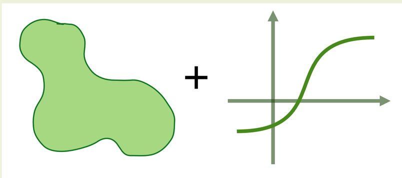


Train a PINNs

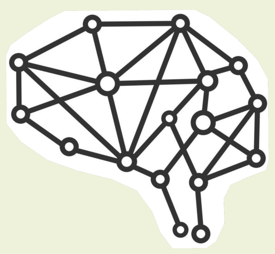


**ONLINE**

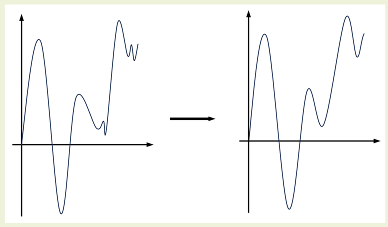
1 Geometry - 1 Force



Get PINNs prediction



Correct prediction with FEM



## Poisson problem with Dirichlet boundary conditions

Find $u : \Omega \to \mathbb{R}^d (d = 1,2,3)$ such that

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \Gamma \end{cases} \qquad (\mathcal{P})$$

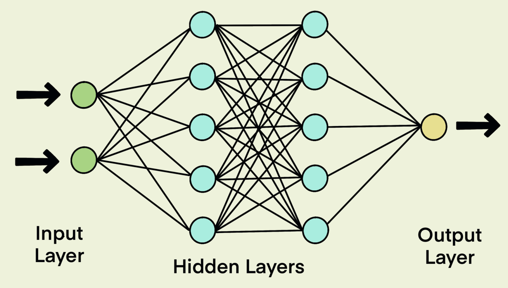
with $\Delta$ the Laplace operator, $\Omega$ a smooth bounded open set and $\Gamma$ its boundary.

## PINNs - Physics-Informed Neural Networks [RPK19]

**Implicit neural representation.**

$$u_\theta(x) = u_{NN}(x)$$

with $u_{NN}$ a neural network (e.g. a MLP).



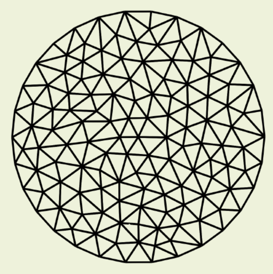

**DoFs Minimization Problem.**
Considering the least-square form of $(\mathcal{P})$, our discrete problem is

$$\bar{\theta} = \underset{\theta \in \mathbb{R}^m}{\operatorname{argmin}}\, \alpha J_{\text{in}}(\theta) + \beta J_{\text{bc}}(\theta)$$

with $m$ the number of parameters of the NN and

$$J_{\text{in}}(\theta) = \frac{1}{2}\int_\Omega (\Delta u_\theta + f)^2 \quad \text{and} \quad J_{\text{bc}}(\theta) = \frac{1}{2}\int_{\partial\Omega} (u_\theta - g)^2$$

## Finite Element methods



**Variational Problem.** Let's define $V$ as a Hilbert space.

$$\text{Find } u \in V \text{ such that, } \forall v \in V,\ a(u,v) = l(v)$$

with $a$ a bilinear form and $l$ a linear form defined as

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \quad \text{and} \quad l(v) = \int_\Omega f\,v.$$

**Approach Problem.** Let's define $V_h$ as a finite-dimensional subspace of $V$ ($V_h \subset V$).

$$\text{Find } u_h \in V \text{ such that, } \forall v_h \in V,\ a(u_h, v_h) = l(v_h)$$

with $u_h \in V_h$ an approximate solution of $u$ and $N_h = \dim(V_h)$.

**Linear System.** Let's define $\{\varphi_1, \ldots, \varphi_{N_h}\}$ a basis of $V_h$.

$$AU = b$$

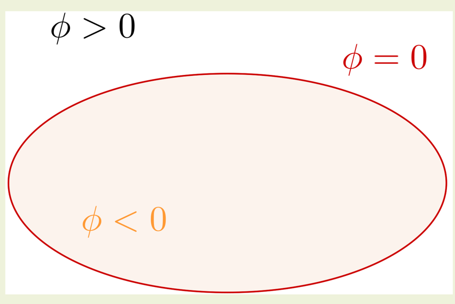
with

$$A = \big(a(\varphi_i, \varphi_j)\big)_{1 \le i,j \le N_h}, \quad U = (u_i)_{1 \le i \le N_h} \quad \text{and} \quad b = \big(l(\varphi_j)\big)_{1 \le j \le N_h}$$

## How to deal with complex geometry in PINNs ?

No mesh, so easy to go on complex geometry !

**Approach by levelset.** [SS22]




***Advantages :***
- ➙ Sample is easy in this case.
- ➙ Allow to impose in hard the BC (no more $J_{\text{bc}}$) :

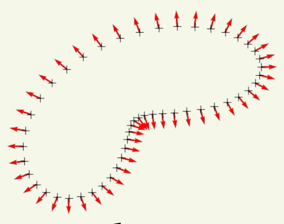
$$u_\theta(X) = \phi(X) w_\theta(X) + g(X)$$

with $\phi$ a levelset function and $w_\theta$ a NN.

**Levelset considered.** A regularized Signed Distance Function (SDF).

**Theorem 1: Eikonal equation.**
If we have a boundary domain $\Gamma$, the SDF is solution to:

$$\begin{cases} ||\nabla \phi(X)|| = 1, & X \in \mathcal{O} \\ \phi(X) = 0, & X \in \Gamma \\ \nabla \phi(X) = n, & X \in \Gamma \end{cases}$$

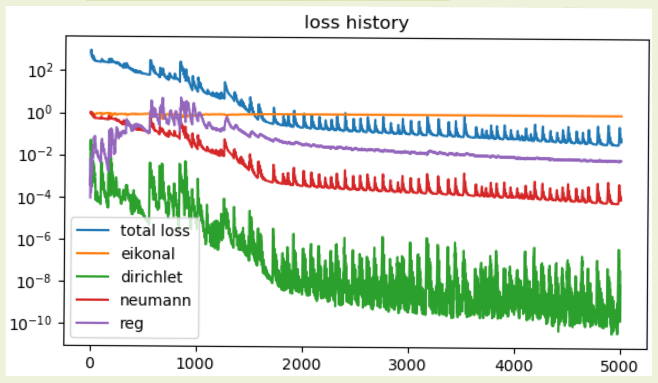


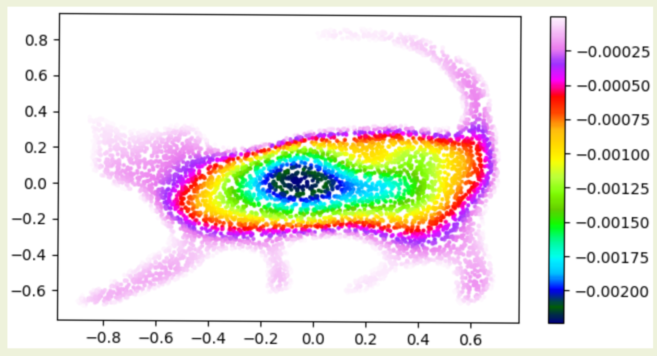
with $\mathcal{O}$ a box which contains $\Omega$ completely and $n$ the exterior normal to $\Gamma$.

**How to do that ?** with a PINNs [CD23], by adding the following regularization term

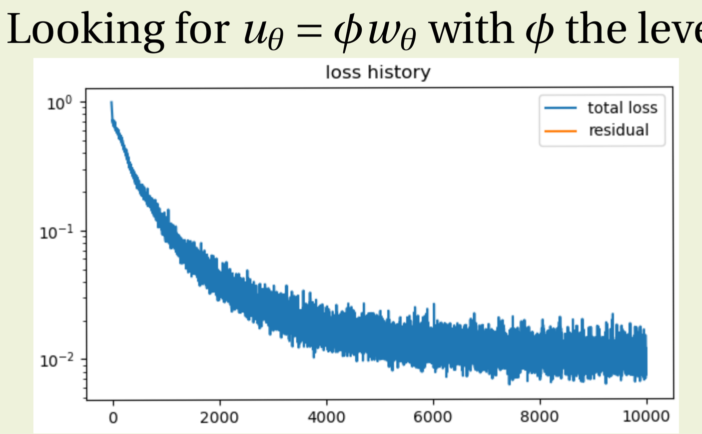
$$J_{\text{reg}} = \int_{\mathcal{O}} |\Delta \phi|^2.$$

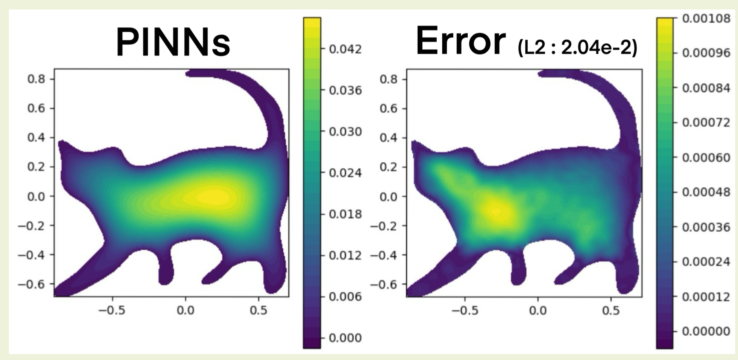
**Result : Levelset learning.**



**Result : Poisson on Cat.**
- ➙ Solving $(\mathcal{P})$ with $f = 1$ (non parametric) and homogeneous Dirichlet BC ($g = 0$).
- ➙ Looking for $u_\theta = \phi w_\theta$ with $\phi$ the levelset learned.




## How can we improve PINNs prediction ? - Using FEM-type methods

**Additive approach.** Considering $u_{NN}$ as the prediction of our PINNs for $(\mathcal{P})$, the correction problem consists in writing the solution as

$$\tilde{u} = u_\theta + \underbrace{\boxed{\tilde{C}}}_{\ll 1}$$

and searching $\tilde{C} : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} -\Delta \tilde{C} = \tilde{f}, & \text{in } \Omega, \\ \tilde{C} = 0, & \text{on } \Gamma, \end{cases} \qquad (\mathcal{P}^+)$$

with $\tilde{f} = f + \Delta u_\theta$.

**Problem considered.** Poisson on Square
- ➙ Solving $(\mathcal{P})$ with homogeneous Dirichlet BC ($g = 0$) and $\Omega = [-0.5\pi, 0.5\pi]^2$.
- ➙ Analytical levelset function : $\phi(x,y) = (x - 0.5\pi)(x + 0.5\pi)(y - 0.5\pi)(y + 0.5\pi)$
- ➙ Analytical solution :

$$u_{ex}(x,y) = \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2}\right)\sin(2x)\sin(2y)$$

with $\mu_1, \mu_2 \in [-0.5, 0.5]$ (**parametric**).

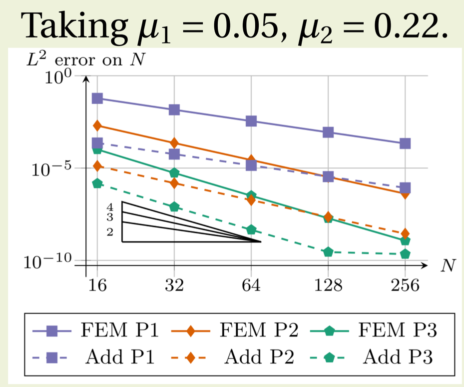
**Theoretical results.**
We denote $u$ the solution of $(\mathcal{P})$ and $u_h$ the discrete solution of $(\mathcal{P}^+)$.

**Theorem 2: [Lec+ss]**
Considering $V_h$ as a $\mathbb{P}_k$ Lagrange space, we have

$$||u - u_h||_0 \lesssim \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} h^{k+1} |u|_{H^{k+1}}$$

*Remark :* We note N the number of nodes in each direction of the square.

Taking $\mu_1 = 0.05$, $\mu_2 = 0.22$.




**Result : Gains using additive approach.**
Considering a set of $n_p = 50$ parameters : $\left\{(\mu_1^{(1)}, \mu_2^{(1)}), \ldots, (\mu_1^{(n_p)}, \mu_2^{(n_p)})\right\}$.

Solution $\mathbb{P}_1$

| | Gains on PINNs | | | | Gains on FEM | | | |
|---|---|---|---|---|---|---|---|---|
| N | min | max | mean | std | min | max | mean | std |
| 20 | 15.7 | 48.35 | 33.64 | 5.57 | 134.31 | 377.36 | 269.4 | 43.67 |
| 40 | 61.47 | 195.75 | 135.41 | 23.21 | 131.18 | 362.09 | 262.12 | 41.67 |

[CD23] Clémot and Digne. "Neural skeleton: Implicit neural representation away from the surface". In: *Computers and Graphics* (2023).
[Lec+ss] **Lecourtier** et al. "Enhanced finite element methods using neural networks". In: (in progress).
[RPK19] Raissi, Perdikaris, and Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *Journal of Computational Physics* (2019).