

2nd CSI

Development of hybrid finite element/neural network methods to help create digital surgical twins

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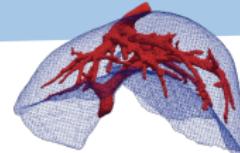
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★ - Update 2025

Scientific context

Context : Create real-time digital twins of an organ (e.g. liver).



Objective : Develop an hybrid finite element / neural network method.
accurate quick + parameterized

★ **Parametric elliptic convection/diffusion PDE :** For one or several $\mu \in \mathcal{M}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\mathcal{L}(u ; \boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}, \boldsymbol{\mu}), \quad (\mathcal{P})$$

where \mathcal{L} is the parametric differential operator defined by

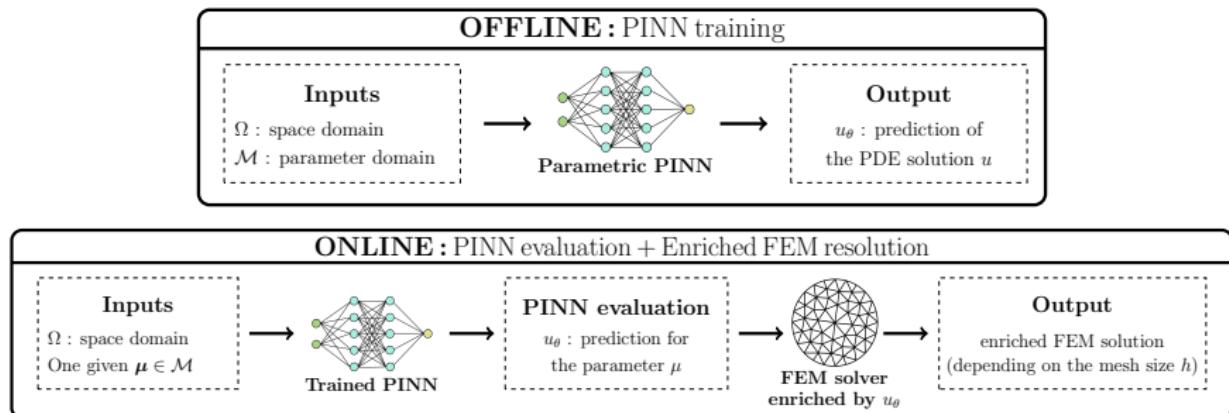
$$\mathcal{L}(\cdot ; \boldsymbol{x}, \boldsymbol{\mu}) : u \mapsto R(\boldsymbol{x}, \boldsymbol{\mu})u + C(\boldsymbol{\mu}) \cdot \nabla u - \frac{1}{\text{Pe}} \nabla \cdot (D(\boldsymbol{x}, \boldsymbol{\mu}) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on $\boldsymbol{\mu}$).

Pipeline of the Enriched FEM

Enriched FEM = Combination of 2 standard methods

- **PINNs** : Physics Informed Neural Networks Appendix 1.1
- **FEMs** : Finite Element Methods Appendix 1.2



Remark : The PINN prediction enriched Finite element approximation spaces.

Enriched finite element method

Additive approach

★ Numerical results

This section is based on [Lecourtier et al., 2025].

Enriched finite element method

Additive approach

★ Numerical results

Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

$$\text{Find } p_h^+ \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(p_h^+, v_h) = I(v_h) - a(u_\theta, v_h), \quad (\mathcal{P}_h^+)$$

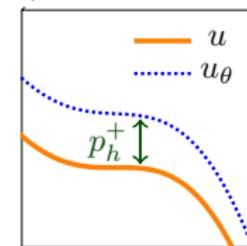
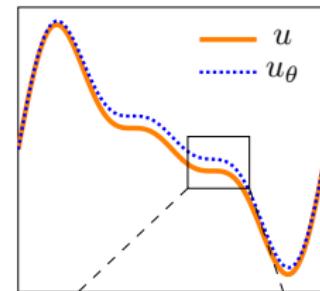
with the **enriched trial space** V_h^+ defined by

$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0\}.$$

General Dirichlet BC : If $u = g$ on $\partial\Omega$, then

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.



Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of (\mathcal{P}_h) with V_h the standard trial space. Then,

$$|u - u_h|_{H^1} \leq C_{H^1} h^k |u|_{H^{k+1}},$$

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of (\mathcal{P}_h^+) with V_h^+ the enriched trial space. Then,

$$|u - u_h^+|_{H^1} \leq \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} (C_{H^1} h^k |u|_{H^{k+1}}),$$

$$\|u - u_h^+\|_{L^2} \leq \boxed{\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

Gains of the additive approach.

LECOURTIER Frédérique

Enriched finite element method

Additive approach

★ Numerical results

1st problem considered

Problem statement: Considering an **Anisotropic Elliptic problem with Dirichlet BC**:

$$\begin{cases} -\operatorname{div}(D\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = [0, 1]^2$ and $\mathcal{M} = [0.4, 0.6] \times [0.4, 0.6] \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$).

Right-hand side :

$$f(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{0.025\sigma^2}\right).$$

Diffusion matrix : (symmetric and positive definite)

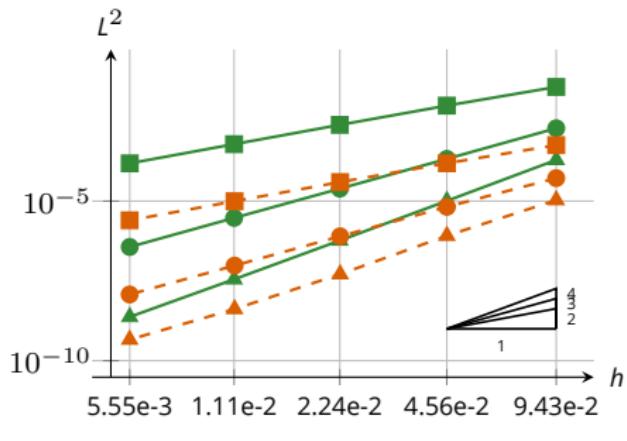
$$D(\mathbf{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

PINN training: Imposing BC exactly with a level-set function.

Numerical results

Error estimates : 1 set of parameters.

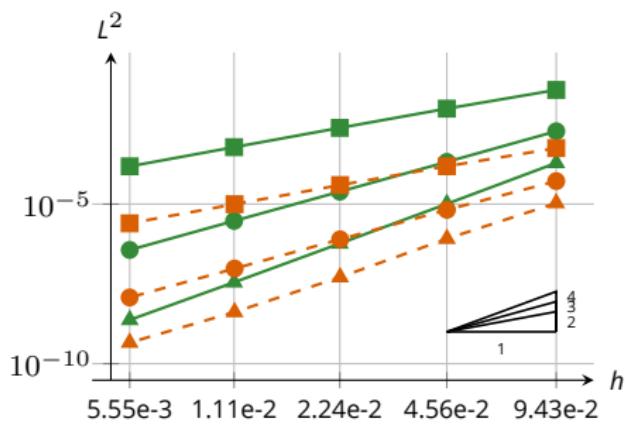
$$\mu = (0.51, 0.54, 0.52, 0.55)$$



Numerical results

Error estimates : 1 set of parameters.

$$\mu = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

k	min	max	mean
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

$N = 20$

$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Cartesian mesh : N^2 nodes.

2nd problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \frac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x, y) \in \mathbb{R}^2, 0.25 \leq x^2 + y^2 \leq 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($\rho = 1$).

Analytical solution :

$$u(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1 \sqrt{x^2 + y^2})}{\ln(4)},$$

Boundary conditions :

$$g(\mathbf{x}; \boldsymbol{\mu}) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\mathbf{x}; \boldsymbol{\mu}) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

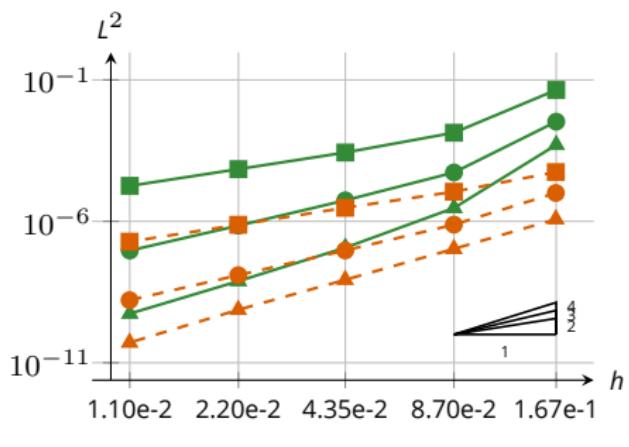
PINN training: Imposing mixed BC exactly in the PINN¹.

¹[Sukumar and Srivastava, 2022]

Numerical results

Error estimates : 1 set of parameters.

$$\mu = 2.51$$



Gains achieved : $n_p = 50$ sets of parameters.

$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}$$

Gains in L^2 rel error
of our method w.r.t. FEM

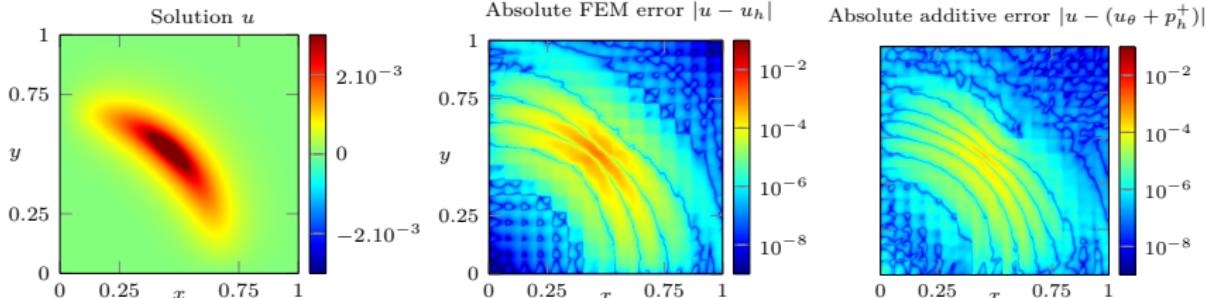
k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

$$h = 1.33 \cdot 10^{-1}$$

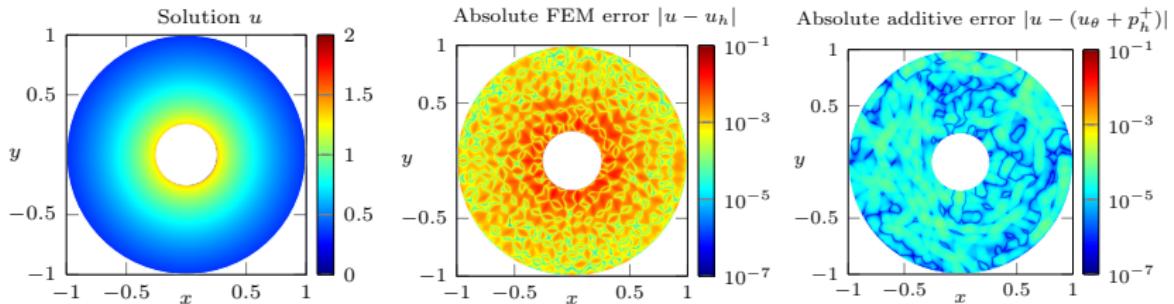
$$\text{Gain : } \|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$$

Numerical solutions

1st problem : $\mu = (0.46, 0.52, 0.05, 0.12)$



2nd problem : $\mu = 2.51$



New lines of research

Complex geometries

★ Posteriori error estimates

★ Non linear PDEs

New lines of research

Complex geometries

★ Posteriori error estimates

★ Non linear PDEs

Learn a regular levelset

Theorem 3: [Clémot and Digne, 2023]

If we have a boundary domain Γ , the SDF is solution to the Eikonal equation:

$$\begin{cases} \|\nabla\phi(x)\| = 1, & x \in \mathcal{O} \\ \phi(x) = 0, & x \in \Gamma \\ \nabla\phi(x) = n, & x \in \Gamma \end{cases}$$



with \mathcal{O} a box which contains Ω completely and n the exterior normal to Γ .

Objective: Move on to complex geometries by using a levelset function to

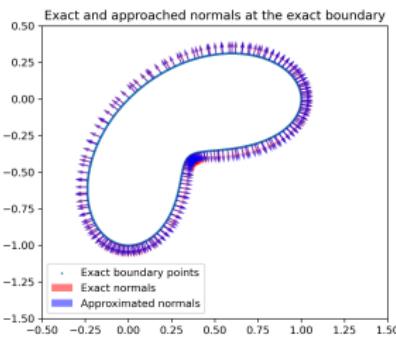
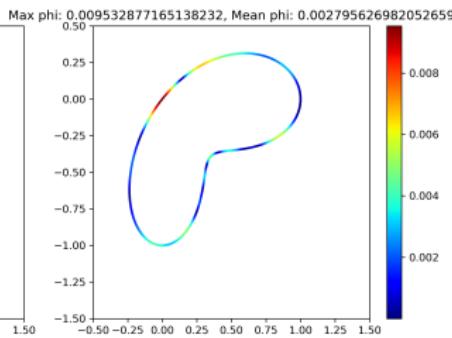
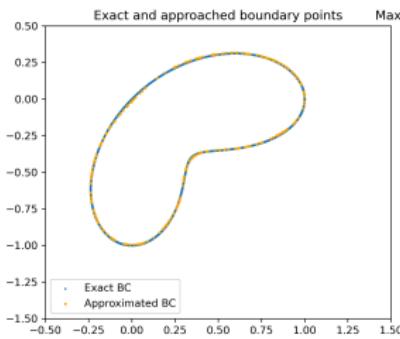
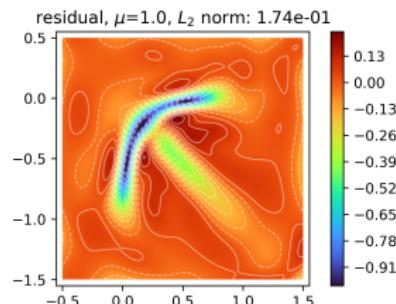
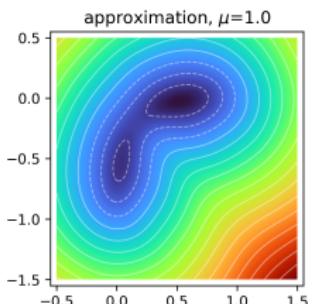
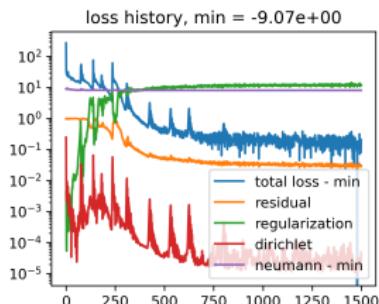
- Sample points in the domain Ω for the PINN training.
- Impose exactly the boundary condition in PINN [Sukumar and Srivastava, 2022].

How to learn a regular levelset ? with a PINN by adding a regularization term,

$$J_{reg} = \int_{\mathcal{O}} |\Delta\phi|^2,$$

and a sample of boundary points that considers the curvature of Γ . ★

Numerical results



TODO : Ajouter résultats "Poisson on Bean" + Mettre au propre les images.

New lines of research

Complex geometries

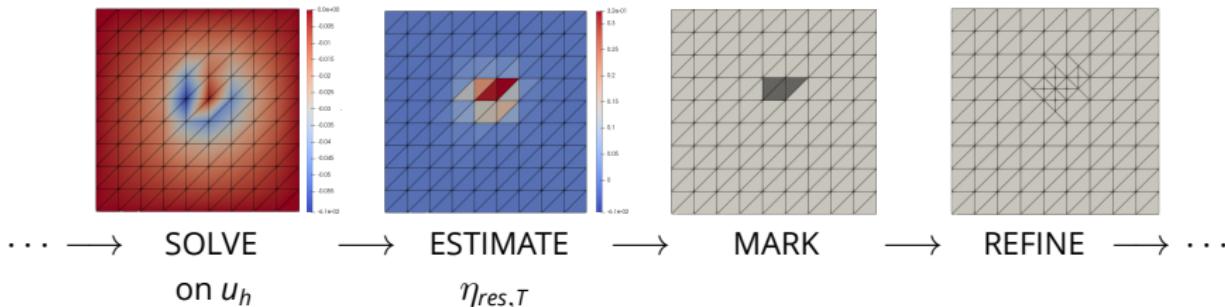
★ Posteriori error estimates

★ Non linear PDEs

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy. Appendix 2.1

Standard FEM



Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

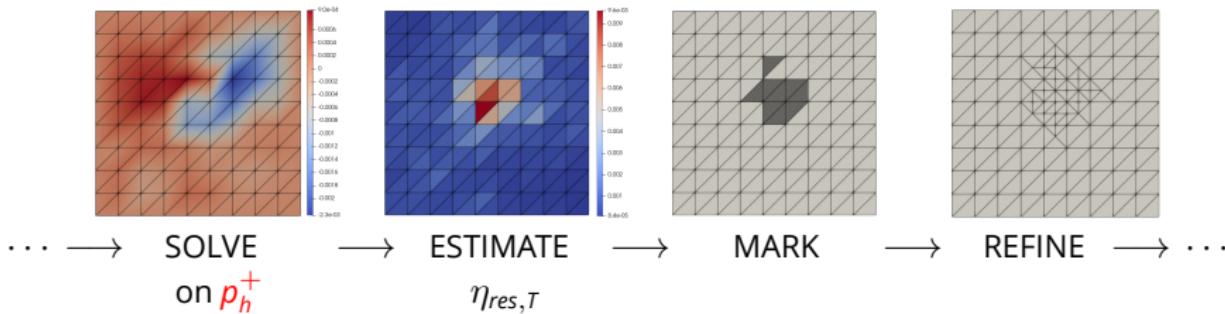
$$\eta_{res,T}^2 = h_T^4 \|\Delta u_h + f_h\|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E^2 \|[\nabla u_h \cdot n]\|_{L^2(E)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach



Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

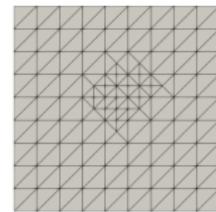
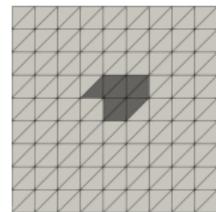
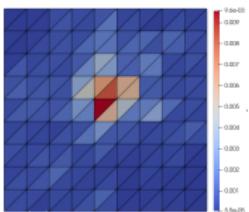
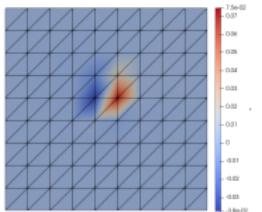
$$\eta_{res,T}^2 = h_T^4 \| (\Delta u_\theta)_h + \Delta p_h^+ + f_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{E \in \partial T} h_E^2 \| [\nabla p_h^+ \cdot n] \|_{L^2(E)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Adaptive mesh refinement

Adaptive refinement loop using Dorfler marking strategy.

Additive Approach - No resolution

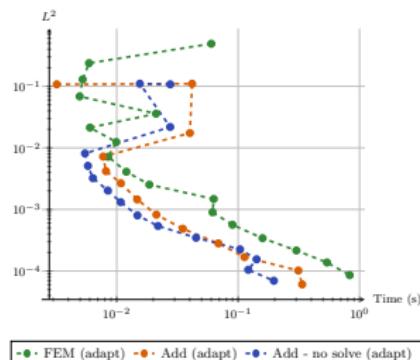
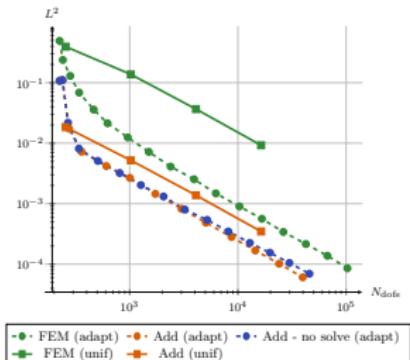


Local residual estimator (in L^2 norm): Let T be a cell of \mathcal{T}_h .

$$\eta_{res,T}^2 = h_T^4 \|(\Delta u_\theta)_h + f_h\|_{L^2(T)}^2$$

with h_\bullet the size of \bullet and considering the Poisson problem.

Numerical results



 Results obtained on a laptop GPU (probably due to external factors).

Ideas for improving results : Additive approach (no resolution).

Interpolate only mesh points added in the refinement process.

error Use another metric such as curvature, rather than residual error.

$$\Delta u_\theta + f \quad \neq \quad u - u_\theta$$

New lines of research

Complex geometries

★ Posteriori error estimates

★ Non linear PDEs

Problem considered

TODO

Numerical results

TODO

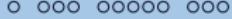
Introduction



Enriched finite element method



New lines of research



Supplementary work



Conclusion



References

Supplementary work

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Development of an hybrid finite element and neural network method

Supplementary work I

Teaching

- ▶ 2024/2025 :
 - ▶ 64h of Computer Science Practical Work - L1S2 and L2S3 (Python) / L3S6 (C++)
 - ▶ 3 days supervising a group of high school girls in RJMI ("Rendez-vous des Jeunes Mathématiciennes et Informaticiennes")
- ▶ 2023/2024 : 50h of Computer Science Practical Work - L2S3 (Python) / L3S6 (C++)

Training courses (Total : 176h35)

- ▶ A dozen seminars organized by IRMA ($\approx 10h$)
- ▶ 1 Deep Learning introductory course - FIDLE ($\approx 40h$)
- ▶ 2 workshops on Scientific Machine Learning ($\approx 2 \times 21h$)
- ▶ 1 summer school on "New Trend in computing" ($\approx 27h$)
- ▶ several cross-disciplinary courses - Methodology, scientific English, etc. ($\approx 58h$)

Supplementary work II

Talks

- ▶ **ICOSAHOM 2025, Montréal** - July 2025 (*Coming soon...*)
"Enriching continuous Lagrange finite element approximation spaces using neural networks"
 - ▶ **DTE & AICOMAS 2025, Paris** - February 20, 2025
"Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
 - ▶ **Exama project, WP2 reunion** - March 26, 2024
"How to work with complex geometries in PINNs ?"
 - ▶ **Retreat (Macaron/Tonus)** - February 6, 2024
"Mesh-based methods and physically informed learning"
 - ▶ **Team meeting (Mimesis)** - December 12, 2023
"Development of hybrid finite element/neural network methods to help create digital surgical twins"

Supplementary work III

Posters

- ▶ **EMS-TAG-SciML 2025, Milan** - March 24, 2025 - "Enriching continuous Lagrange finite element approximation spaces using neural networks"
 - ▶ **CJC-MA 2024, Lyon** - October 29, 2024 - "Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries"
 - ▶ **MSII poster day, Strasbourg** - October 24, 2024
 - ▶ **SciML 2024, Strasbourg** - July 08, 2024

Publications

- ▶ Enriching continuous lagrange finite element approximation spaces using neural networks. (*submitted in February 2025, M2AN journal*)
H. Barucq, M. Duprez, F. Faucher, E. Franck, **F. Lecourtier**, V. Lleras, V. Michel-Dansac, and N. Victorion.

Conclusion

TODO (parler de data-driven / multiplicative approach)

References

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Appendix 1 : Standard methods

A1.1 – Physics-Informed Neural Networks

Standard PINNs¹ (Weak BC) : Find the optimal weights θ^* , such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \omega_b J_b(\theta)), \quad (\mathcal{P}_\theta)$$

with

residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

boundary loss

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \boldsymbol{\mu}) - g(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC.

In (\mathcal{P}_θ) , ω_r and ω_b are some weights.

Monte-Carlo method : Discretize the cost functions by random process.

¹[Raissi et al., 2019]

A1.1 – Physics-Informed Neural Networks

Improved PINNs¹ (Strong BC) : Find the optimal weights θ^* such that

$$\theta^* = \operatorname{argmin}_{\theta} (\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)}),$$

with $\omega_r = 1$ and

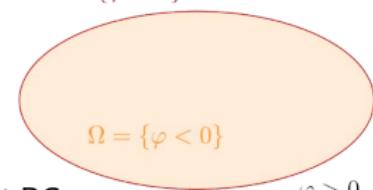
residual loss

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_{\theta}(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu}) - f(\mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

$$\partial\Omega = \{\varphi = 0\}$$

where u_{θ} is a neural network defined by

$$u_{\theta}(\mathbf{x}, \boldsymbol{\mu}) = \varphi(\mathbf{x}) w_{\theta}(\mathbf{x}, \boldsymbol{\mu}) + g(\mathbf{x}, \boldsymbol{\mu}),$$



with φ a level-set function, w_{θ} a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_{\theta} = g$ on $\partial\Omega$.

¹[Lagaris et al., 1998; Franck et al., 2024]

A1.2 – Finite Element Methods¹

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

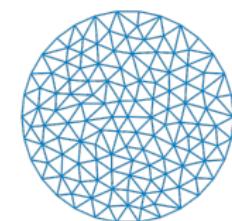
with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_{\Omega} D \nabla u_h \cdot \nabla v_h + \int_{\Omega} R u_h v_h + \int_{\Omega} v_h C \cdot \nabla u_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 the finite element space defined by

$$V_h^0 = \left\{ v_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0 \right\},$$

where \mathbb{P}_k is the space of polynomials of degree at most k .



Linear system : Let $(\phi_1, \dots, \phi_{N_h})$ a basis of V_h^0 .

Find $U \in \mathbb{R}^{N_h}$ such that $AU = b$

with

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_h} \quad \text{and} \quad b = (l(\phi_j))_{1 \leq j \leq N_h}.$$

$$\mathcal{T}_h = \{K_1, \dots, K_{N_e}\}$$

(N_e : number of elements)

¹[Ern and Guermond, 2004]

Appendix 2 : More

A2.1 – Adaptive mesh refinement

Dorfler marking strategy : [Dörfler, 1996]

Find $\mathcal{M}_h \subset \mathcal{T}_h$ of minimal cardinality such that

$$\sum_{T \in \mathcal{M}_h} \eta_{\bullet,T}^2 \geq \theta \sum_{T \in \mathcal{T}_h} \eta_{\bullet,T}^2,$$

with $\eta_{\bullet,T}$ a local estimator¹ and $\theta \in (0, 1)$.

¹For instance, the residual estimator. [Ainsworth and Oden, 1997]