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Combining Finite Element Methods and Neural Networks to Solve Elliptic Problems on 2D Geometries

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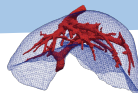
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Scientific context



Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.

accurate quick + parameterized

Parametric linear elliptic PDE : For one or several $\mu \in \mathcal{M}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\mathcal{L}(u; \mathbf{x}, \mu) = f(\mathbf{x}, \mu), \quad (\mathcal{P})$$

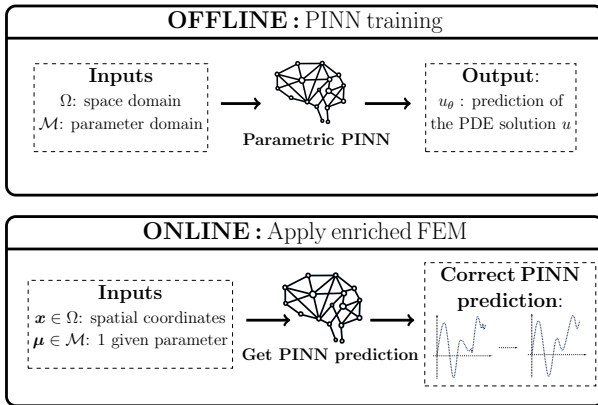
where \mathcal{L} is the parametric differential operator defined by

$$\mathcal{L}(\cdot; \mathbf{x}, \mu) : u \mapsto R(\mathbf{x}, \mu)u + C(\mu) \cdot \nabla u - \frac{1}{\text{Pe}} \nabla \cdot (D(\mathbf{x}, \mu) \nabla u),$$

and some Dirichlet, Neumann or Robin BC (which can also depend on μ).

Ω	Spatial domain	f	Right-hand side
d	Spatial dimension	R	Reaction coefficient
$\mathbf{x} = (x_1, \dots, x_d)$	Spatial coordinates	C	Convection coefficient
\mathcal{M}	Parameter space	D	Diffusion matrix
p	Number of parameters	Pe	Péclet number
$\mu = (\mu_1, \dots, \mu_p)$	Parameter vector		

Pipeline of the Enriched FEM



Correction : Enriched continuous Lagrange finite element approximation spaces using the PINN prediction.

Physics-Informed Neural Networks

Standard PINNs (Weak BC) : Find the optimal weights θ^* that satisfy

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta) \right), \quad (\mathcal{P}_\theta)$$

with the residual loss function and the boundary loss function defined by

$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_\theta(\mathbf{x}, \mu); \mathbf{x}, \mu) - f(\mathbf{x}, \mu)|^2 d\mathbf{x} d\mu,$$

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \mu) - g(\mathbf{x}, \mu)|^2 d\mathbf{x} d\mu,$$

where u_θ is a neural network, $g = 0$ is the Dirichlet BC. In (\mathcal{P}_θ) , the weights ω_r and ω_b (hyperparameters) are used to balance the different terms of the loss function.

Monte-Carlo method : Discretize the cost functions by random process.

Physics-Informed Neural Networks

Improved PINNs¹ (Strong BC) : Find the optimal weights θ^* that satisfy

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \cancel{\omega_b J_b(\theta)} \right),$$

with $\omega_r = 1$ and the residual loss function defined by

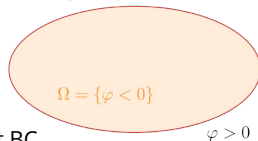
$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\mathcal{L}(u_{\theta}(\mathbf{x}, \mu); \mathbf{x}, \mu) - f(\mathbf{x}, \mu)|^2 d\mathbf{x} d\mu, \quad \partial\Omega = \{\varphi = 0\}$$

where u_{θ} is a neural network defined by

$$u_{\theta}(\mathbf{x}, \mu) = \varphi(\mathbf{x}) w_{\theta}(\mathbf{x}, \mu) + g(\mathbf{x}, \mu),$$

with φ a level-set function, w_{θ} a NN and $g = 0$ the Dirichlet BC.

Thus, the Dirichlet BC is imposed exactly in the PINN : $u_{\theta} = g$ on $\partial\Omega$.



Monte-Carlo method : Discretize the residual cost function by random process.

¹Lagaris et al. [1998]; Franck et al. [2024]

Finite Element Method

Variational Problem :

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

with h the characteristic mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \frac{1}{\text{Pe}} \int_{\Omega} D \nabla u_h \cdot \nabla v_h + \int_{\Omega} R u_h v_h + \int_{\Omega} v_h C \cdot \nabla u_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h the finite element space of dimension N_h defined by

$$V_h = \{v_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k, v_h|_{\partial\Omega} = 0\},$$

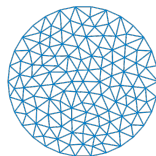
where \mathbb{P}_k is the space of polynomials of degree at most k .

Linear system : Let $(\phi_1, \dots, \phi_{N_h})$ a basis of V_h .

$$\text{Find } U \in \mathbb{R}^{N_h} \text{ such that} \quad AU = b$$

with

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_h} \quad \text{and} \quad b = (l(\phi_j))_{1 \leq j \leq N_h}.$$



$$\mathcal{T}_h = \{K_1, \dots, K_{N_e}\}$$

(N_e : number of elements)

How improve PINN prediction with FEM ?

Additive approach

Variational Problem : Let $u_\theta \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$.

Find $p_h^+ \in V_h^0$ such that, $\forall v_h \in V_h^0, a(p_h^+, v_h) = l(v_h) - a(u_\theta, v_h)$, (\mathcal{P}_h^+)

with the **enriched trial space** V_h^+ defined by

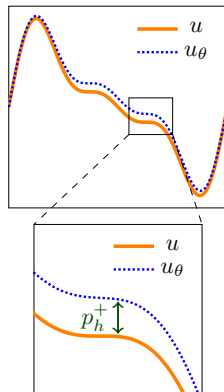
$$V_h^+ = \{u_h^+ = u_\theta + p_h^+, \quad p_h^+ \in V_h^0\}.$$

Impose BC : If our problem satisfies $u = g$ on $\partial\Omega$, then p_h^+ has to satisfy

$$p_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior (weak BC).

Considering the strong BC, $p_h^+ = 0$ on $\partial\Omega$.



Theoretical results

Let α and γ respectively the coercivity and continuity constants of a . Let u the solution of (\mathcal{P}) .

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

We denote $u_h \in V_h$ the solution of (\mathcal{P}_h) with V_h the standard trial space.

For all $1 \leq q \leq k$,

$$\|u - u_h\|_{L^2} \leq C \frac{\gamma^2}{\alpha} h^{q+1} |u|_{H^{q+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Barucq et al., 2025]

We denote $u_h^+ \in V_h^+$ the solution of (\mathcal{P}_h^+) with V_h^+ the enriched trial space.

For all $1 \leq q \leq k$,

$$\|u - u_h^+\|_{L^2} \leq \boxed{\frac{|u - u_\theta|_{H^{q+1}}}{|u|_{H^{q+1}}}} \left(C \frac{\gamma^2}{\alpha} h^{q+1} |u|_{H^{q+1}} \right).$$

The same type of estimates holds for the H^1 norm.

Numerical results - 2D Poisson problem

2D Poisson problem

TODO

Numerical results - 2D anysotropic Elliptic problem

2D anisotropic Elliptic problem

TODO

Conclusion

Conclusion

TODO

References

- Hélène Barucq, Michel Duprez, Florian Faucher, Emmanuel Franck, Frédérique Lecourtier, Vanessa Lleras, Victor Michel-Dansac, and Nicolas Victorion. Enriching continuous lagrange finite element approximation spaces using neural networks. 2025.
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Appendix

Appendix 1 : Standard FEM

Appendix 1 : General Idea

TODO