Enriching continuous Lagrange finite element approximation spaces using neural networks

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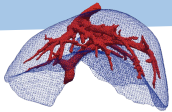
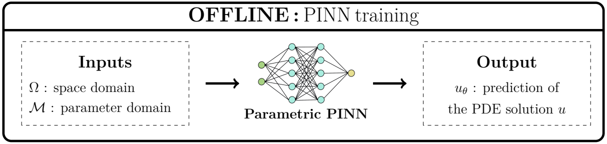
Scientific context

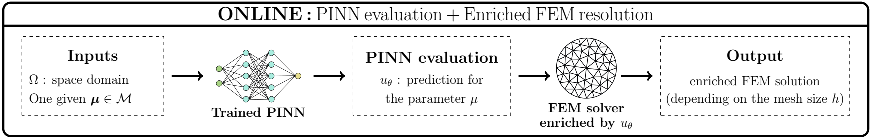


Context : Create real-time digital twins of an organ (e.g. liver).

Objective : Develop an hybrid finite element / neural network method.

accurate quick + parameterized





Problem considered

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :
We consider $\Omega = [-1, 1]^2$ a squared domain and $\mathbf{e}_y = (0, 1)$.
Find the velocity $\mathbf{u} = (u, v)$, the pressure p and the temperature T such that

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \;\text{ in } \Omega & \text{(incompressibility)} \\ (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mu \Delta \mathbf{u} - g(\beta T + 1)\mathbf{e}_y = 0 \;\text{ in } \Omega & \text{(momentum)} \\ \mathbf{u} \cdot \nabla T - k_f \Delta T = 0 \;\text{ in } \Omega & \text{(energy)} \end{cases} \qquad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, μ the viscosity and k_f the thermal conductivity. [Coulaud et al., 2024]

Boundary Conditions:

- $\mathbf{u} = 0$ on $\partial\Omega$
- $T = 1$ on the left wall ($x = -1$) and $T = -1$ on the right wall ($x = 1$)
  $\dfrac{\partial T}{\partial n} = 0$ on the top and bottom walls ($y = \pm 1$)

Problem considered

Objective: Simulate the flow for a range of $\boldsymbol{\mu} = (\mu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :
We consider $\boldsymbol{x} = (x, y) \in \Omega$ and $\mathbf{e}_y = (0, 1)$.
Find $U = (u, v, p, T)$ such that

$$\begin{cases} R_{inc}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(incompressibility)} \\ R_{mom}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(momentum)} \\ R_{ener}(U; \boldsymbol{x}, \boldsymbol{\mu}) = 0 \quad \text{in } \Omega & \text{(energy)} \end{cases} \qquad (\mathcal{P})$$

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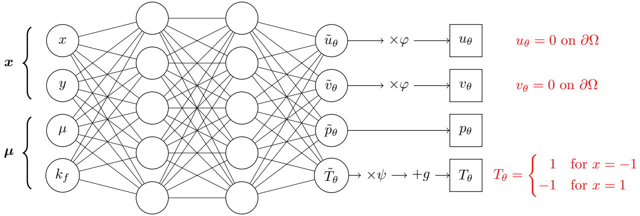
Physics-informed neural network (PINN)

Neural Network considered

We consider a parametric NN with 4 inputs and 4 outputs, defined by

$$U_\theta(\boldsymbol{x}, \boldsymbol{\mu}) = \big(u_\theta, v_\theta, p_\theta, T_\theta\big)(\boldsymbol{x}, \boldsymbol{\mu}).$$

The Dirichlet boundary conditions are imposed on the outputs of the MLP by a **post-processing** step. [Sukumar and Srivastava, 2022]



We consider two levelsets functions φ and ψ, and the linear function g defined by

$$\varphi(x,y) = (x-1)(x+1)(y-1)(y+1),$$

$$\psi(x,y) = (x-1)(x+1) \quad \text{and} \quad g(x,y) = 1-(x+1).$$

PINN losses

Approximation of the solution of $(\mathcal{P})$ by a PINN :

Find the optimal weights $\theta^\star$, such that

$$\theta^\star = \underset{\theta}{\operatorname{argmin}} \left(J_{inc}(\theta) + J_{mom}(\theta) + J_{ener}(\theta) + J_{ad}(\theta) \right), \qquad (\mathcal{P}_\theta)$$

where the different cost functions are defined by

adiabatic condition

$$\boxed{J_{ad}(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega|_{y=\pm 1}} \left| \frac{\partial T_\theta(\boldsymbol{x},\boldsymbol{\mu})}{\partial n} \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

3 residual losses

$$\boxed{J_\bullet(\theta) = \int_{\mathcal{M}} \int_{\Omega} \left| R_\bullet(U_\theta(\boldsymbol{x}, \boldsymbol{\mu}); \boldsymbol{x}, \boldsymbol{\mu}) \right|^2 d\boldsymbol{x} d\boldsymbol{\mu},}$$

with U_θ the parametric NN and $\bullet$ the PDE considered (i.e. *inc*, *mom* or *ener*).

Monte-Carlo method : Discretize the cost functions by random process.

PINN training

TODO (entrainement + solution pour 1 paramètre ?)

Finite element method (FEM)

Newton method

TODO

Finite Element Methods[1]

TODO

[1] [Ern and Guermond, 2004]

Enriched finite element method using PINN

Newton method - Additive approach

TODO

Numerical results

Numerical results

TODO

Conclusion

TODO

References

Guillaume Coulaud, Maxime Le, and Régis Duvigneau. Investigations on Physics-Informed Neural Networks for Aerodynamics, 2024.

A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. 2004.

N. Sukumar and A. Srivastava. Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks. 2022.