Chapter 9
Graphs

MAD101

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1 G. & G.Models

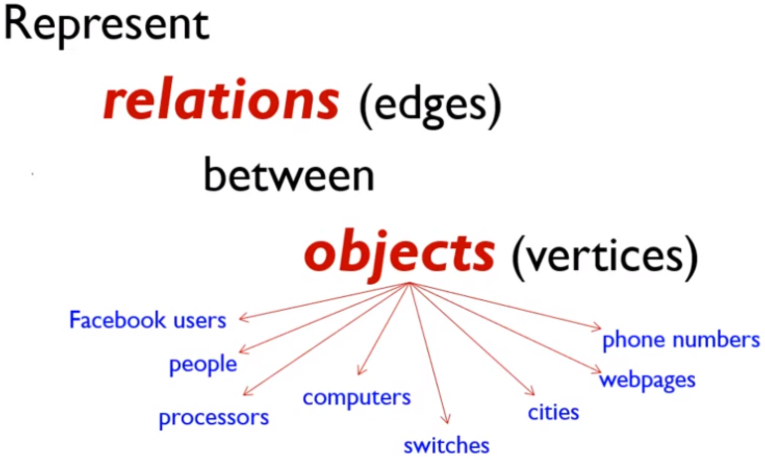
Introduction

1 G. & G.Models

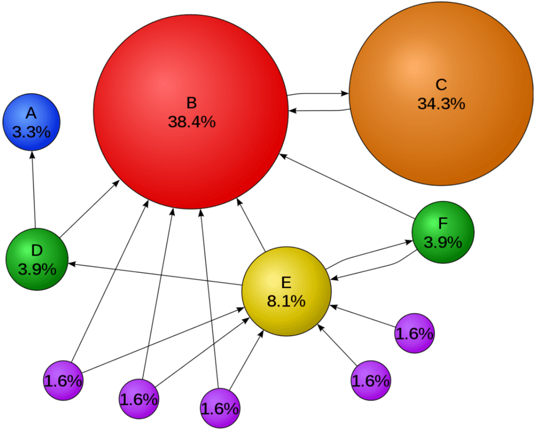
Relationship between

- Computers (in a computer network,...)
- Cities (paths, google map,..)
- Webpages (on internet)
- People (in society)
- ...

Introduction
1 G. & G.Models

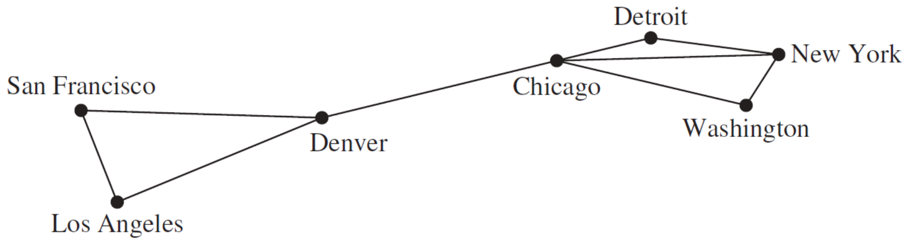

Represent
relations (edges)
between
objects (vertices)
Facebook users
people
processors
computers
switches
cities
webpages
phone numbers

Google Pagerank-Sergey Brin & Larry Page



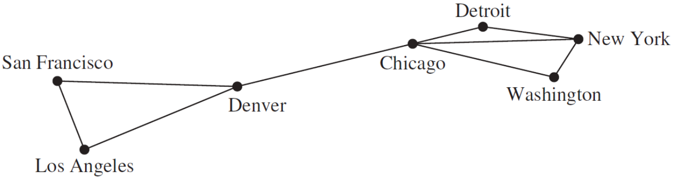
Definition

A graph G = (V, E) consists of V, a nonempty set of **vertices** (đỉnh) (or nodes) and E, a set of **edges** (cạnh). Each edge has either one or two vertices associated with it, called its **endpoints** (đầu mút). An edge is said to connect its endpoints.

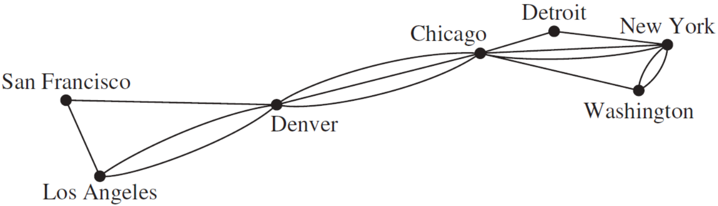


Some Kind of Graphs

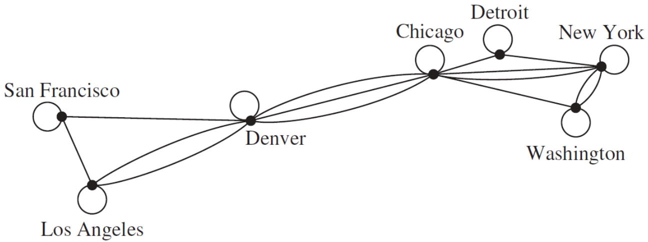
- **Simple graph:** No two edges connect the same pair of vertices



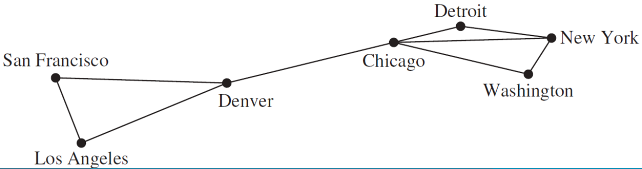
- **Multigraph:** Multiple edges connect the same pair of vertices



- **Pseudograph:** Multigraph may have **loops** (khuyên, vòng)



- **Undirected graph:** Each edge has no direction

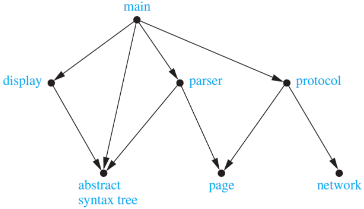


Directed Graph

1 G. & G.Models

A **directed graph** (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

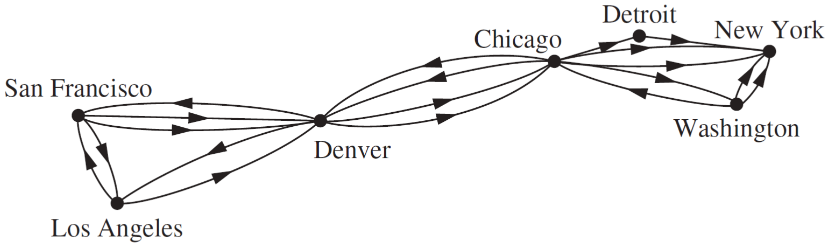
- **Simple directed graph** has no loops and has no multiple directed edges



Directed Graph

1 G. & G.Models

- **Directed multigraphs** (or multiple directed edges): have multiple directed edges from a vertex to a second (possibly the same) vertex.



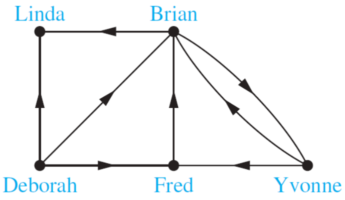
- **Mixed graph**: A graph with both directed and undirected edges.

Graph Terminology

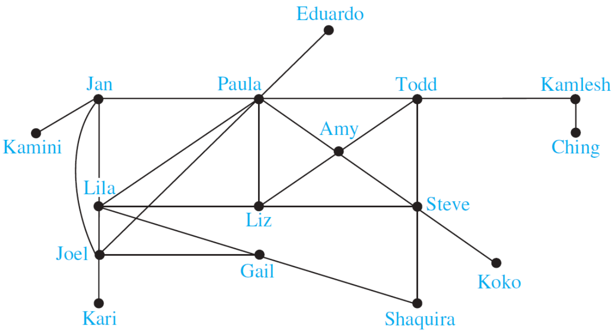
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Graph Models

- An influence graph



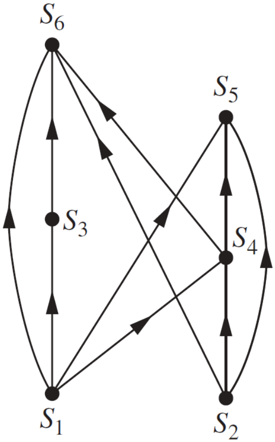
- An acquaintanceship graph



Graph Models

- A precedence graph

S_1	$a := 0$
S_2	$b := 1$
S_3	$c := a + 1$
S_4	$d := b + a$
S_5	$e := d + 1$
S_6	$e := c + d$



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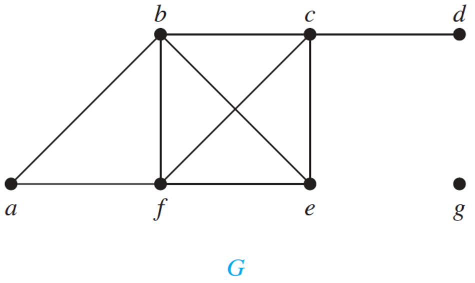
- Two vertices u and v in an undirected graph G are called **adjacent** (or neighbors)(kề nhau) in G if u and v are endpoints of an edge e of G. Such an edge e is called **incident** (cung/cạnh) with the vertices u and v and e is said to connect u and v.
- The **degree of a vertex** in an undirected graph is the number of edges incident with it, except that **a loop at a vertex contributes twice to the degree of that vertex**. The degree of the vertex v is denoted by **deg(v)**.

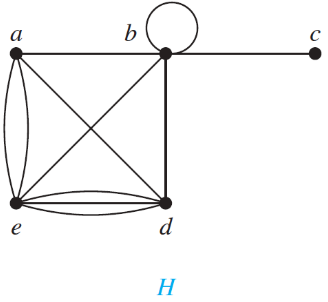
Notes.

- A vertex of degree zero is called **isolated** (đỉnh cô lập)
- A vertex is **pendant** (đỉnh treo) if and only if it has degree one.

Example.

What are the degrees of the vertices in the graphs G and H.





Solutions

Graph G: deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=1, deg(f)=4, deg(e)=3, deg(g)=0.

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Example. How many edges are there in a graph with 10 vertices each of degree six?

Solution. $2m = 10.6 \implies m = 30$

Theorem. An undirected graph has an even number of *vertices of odd degree*.

Example.

1. Is there an undirected graph with degree sequence $7, 5, 4, 3, 2, 2$. (No. 3 vertices of odd degree: 7,5,3)
2. How many edges does a graph have if its degree sequence is $5, 5, 4, 3, 2, 1, 0$. (10 edges)
3. Is there an simple undirected graph with degree sequence $4, 3, 2, 1, 0$? How many edges are there? Draw a such graph (if exist). (Yes, 5 edges)

Definition

$$a \longrightarrow b$$

- a is **initial vertex**
- b is **terminal vertex** (or end vertex)

Note. The initial vertex and terminal vertex of **a loop** are the same

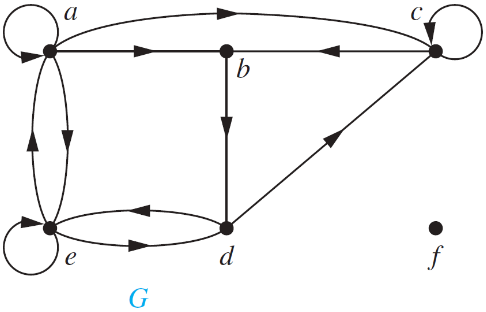
- **In-degree** ($deg^{-}(v)$): the number of edges with v as their terminal vertex,
- **Out-degree** ($deg^{+}(v)$): the number of edges with v as their initial vertex.

Note. A loop v have $deg^{-}(v) = deg^{+}(v) = 1$

Example.

2 G.Ter. & Spec. Types of G.

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure



- $deg^{-}(a) = 2, deg^{+}(a) = 4$
- $deg^{-}(b) = 2, deg^{+}(b) = 1$
- $deg^{-}(c) = 3, deg^{+}(c) = 2$
- $deg^{-}(d) = 2, deg^{+}(d) = 2$
- $deg^{-}(e) = 3, deg^{+}(e) = 3$
- $deg^{-}(f) = deg^{+}(f) = 0$

Theorem

Let G = (V, E) be a graph with directed edges. Then

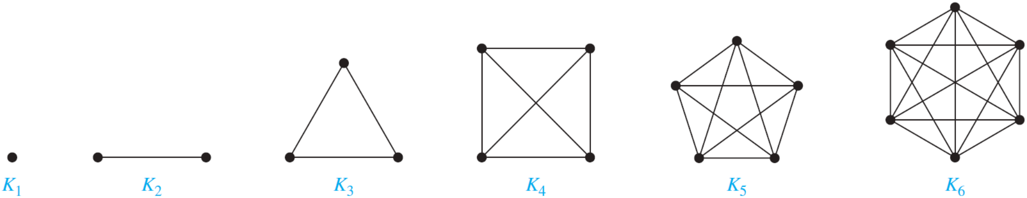
$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Example. In the previous example, we have

Vertex	In-degree	Out-degree
a	2	4
b	2	1
c	3	2
d	2	2
e	3	3
f	0	0
Sum	12	12

Complete Graphs (Đồ thị đầy đủ)

A **complete graph** on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

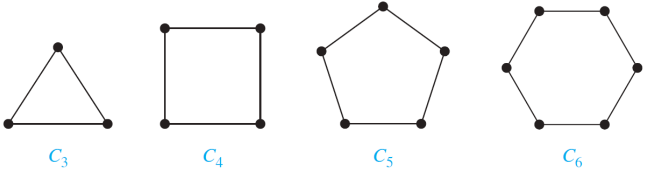


- How many vertices and edges in K_n? $n, C_n^2 = \dfrac{n(n-1)}{2}$ $\left(C_n^k = \dfrac{n!}{k!(n-k)!}\right)$
- What is the degree of each vertex? $n - 1$

Cycles (Đồ thị vòng)

2 G.Ter. & Spec. Types of G.

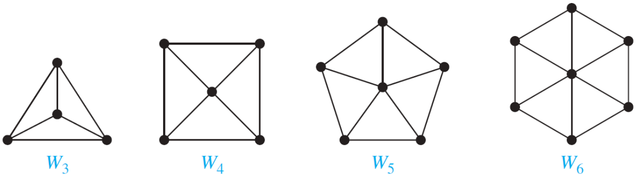
A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



- C_n has n vertices and n edges.
- The degree of each vertex is 2

Wheels (Đồ thị bánh xe)

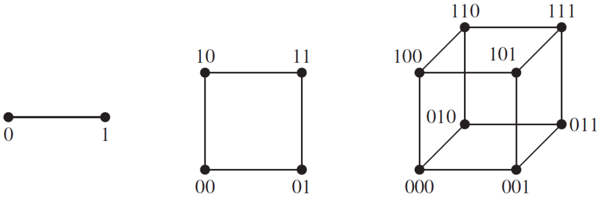
We obtain a **wheel** W_n when we add an additional vertex to a **cycle** C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



- W_n has $n+1$ vertices and $2n$ edges,
- There are 3-degree of n vertices and **one** vertex has n-degree.

n-Cubes (Đồ thị khối n-chiều)

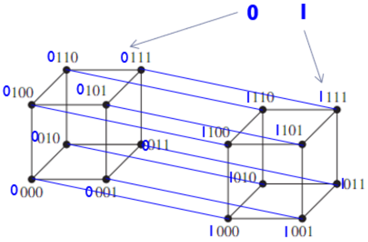
An n-dimensional hypercube, or **n-cube**, denoted by Q_n, is a graph that has vertices representing the 2_n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Q_1 Q_2 Q_3

n-Cubes (Đồ thị khối n-chiều)

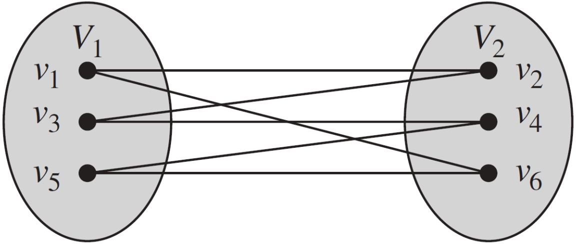
Contruct Q_4 from two copies of Q_3.



Q_n has 2^n vertices and $n2^{n-1}$ edges.

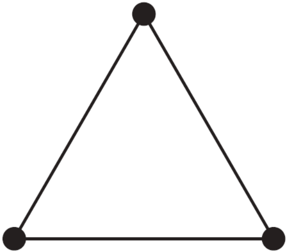
Lưu ý: Mỗi đỉnh nằm trong đồ thị Q_n có bậc là n, tổng số bậc nằm trong đồ thị Q_n là $n2^n$, áp dụng định lý HANDSHAKING ta có kết quả trên.

Bipartite Graphs (Đồ thị lưỡng phân)



A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Example. Non-Bipartite



Theorem

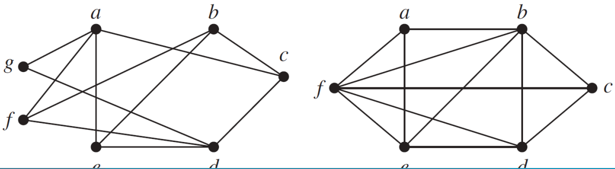
A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that **no two adjacent vertices are assigned the same color**.

How to check?

- Let color the vertices using 2 different colors,
- **Two adjacent vertices** must have **different colors** (e.g., **red** and **black**)

Are the graphs G and H displayed in the Figure bipartite?

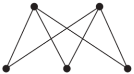
G: Biparite ({a,b,d},{c,e,f,g}); H: Non-Bipatite.



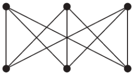
Complete Bipartite Graphs

2 G.Ter. & Spec. Types of G.

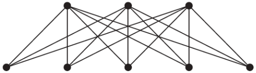
A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset



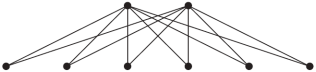
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

$K_{m,n}$ has $m + n$ vertices and mn edges.

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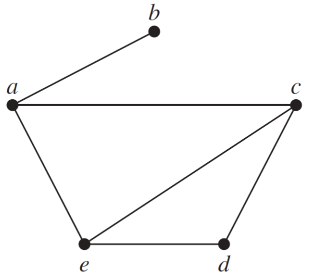
Adjacency List

Using **adjacency list**: For each vertex u in the graph, there is a list of vertex v which there is an edge between u and v.

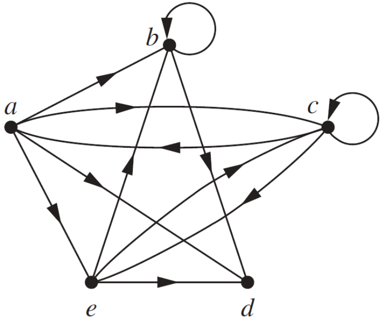
Example 1. Use adjacency lists to describe the simple graph given in Figure



Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Adjacency List

Example 2. Represent the directed graph shown in Figure by listing all the vertices that are the **terminal vertices** of edges starting at each vertex of the graph.



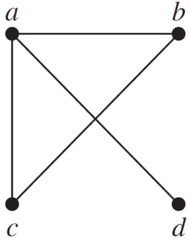
Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

Adjacency matrix $A = [a_{ij}]$ with

$$a_{ij} = \begin{cases} \text{the number of edges that are associated to } \{v_i, v_j\} \\ 0, \ \text{Otherwise} \end{cases}$$

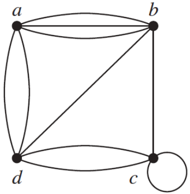
Example. Use an adjacency matrix to represent the graph shown in Figure.



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrices

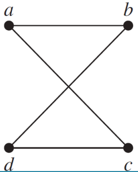
Example. Use an adjacency matrix to represent the pseudograph shown in Figure



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

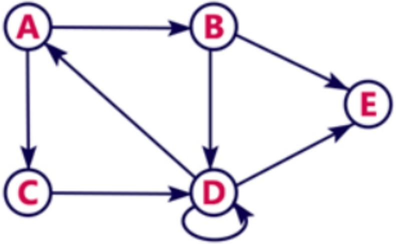
Example. Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency Matrices

Example. Use an adjacency matrix to represent the directed graph shown in Figure



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Exercise.

3 Repre. G. & G. Iso.

Given the ***adjacency matrix*** representing an undirected graph G.

Find the *deg(c)*.

	a	b	c	d
a	1	2	1	0
b	2	0	2	2
c	1	2	1	0
d	0	2	0	0

$deg(c) = 1 + 2 + 1.2 + 0 = 5$ (c is loop).

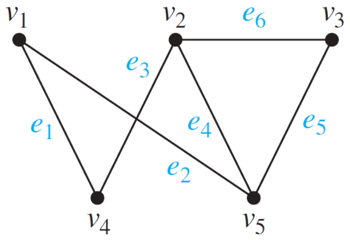
How many edges are there? (9).

Incidence Matrices (ma trận liên thuộc)

$M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i \\ 0, & \text{Otherwise} \end{cases}$$

Example. Represent the graph shown in Figure with an incidence matrix.



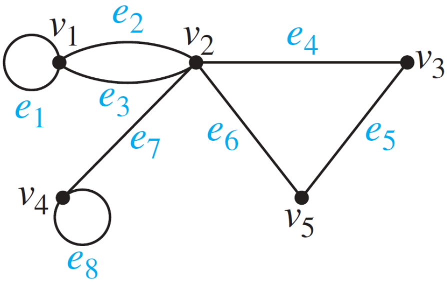
Solution.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example.

Represent the graph shown in Figure with an incidence matrix

Solution.

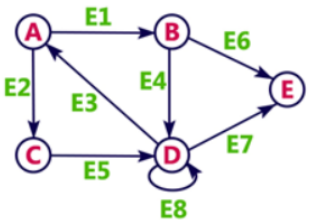


$$
\begin{array}{c|cccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
\hline
v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
v_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
$$

Example.

In directed graph,

- 0 is used to represent row edge which is **not connected** to column vertex.
- 1 is used to represent row edge which is connected as **outgoing** edge to column vertex.
- -1 is used to represent row edge which is connected as **incoming** edge to column vertex.



$$
\begin{array}{c}
\\ A \\ B \\ C \\ D \\ E
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} E1 & E2 & E3 & E4 & E5 & E6 & E7 & E8 \end{array} \\
\begin{pmatrix}
1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0
\end{pmatrix}
\end{array}
$$

Exercises.

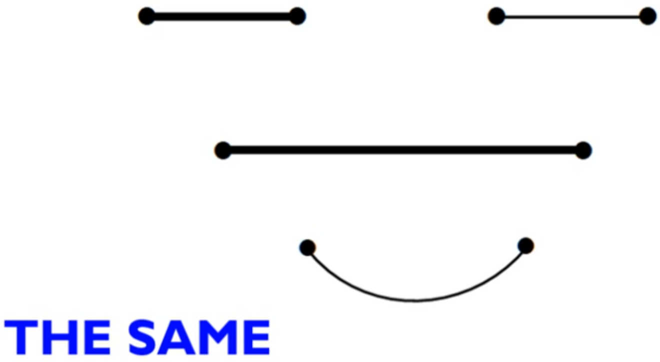
1. How many column are there in the incidence matrix of W_9. (18)
2. How many 1s are there in the incidence matrix representing the complete graph K_{10}. (90)
3. .

How many ***loops*** are there?

Find deg(v_4).

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$loops: 2; deg(v_4) = 1 + 1.2 = 3$ (loop)



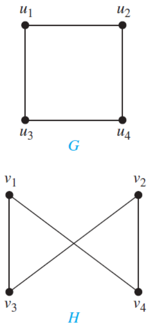
THE SAME

Two Simple Graphs are isomorphic. In this way, they have:

- The same number of vertices,
- The same number of edges,
- The same degrees of the vertices (deg(v)=deg[f(v)]),
- a and b are adjacent if $f(a)$ and $f(b)$ are adjacent.

Definition. The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a **bijection** f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

Show that the graphs G = (V, E) and H = (W, F), displayed in Figure, are isomorphic.

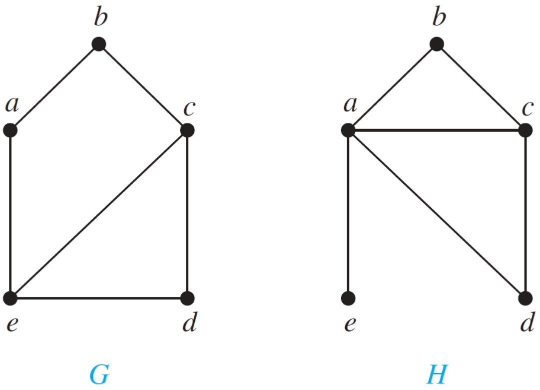


$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$

Example.

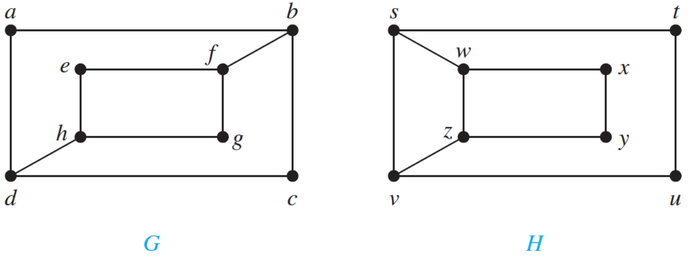
3 Repre. G. & G. Iso.

Show that the graphs displayed in Figure are not isomorphic



Example

Determine whether the graphs shown in Figure are isomorphic.



Not isomorphic because vertex a (2-degree) in G adjacent with b, d (all of them have 3-degree) and in H, there is not exist a vertex which have the same characteristic.

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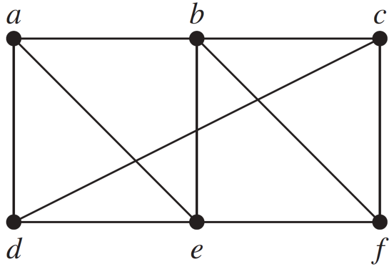
Example.

4 Connectivity



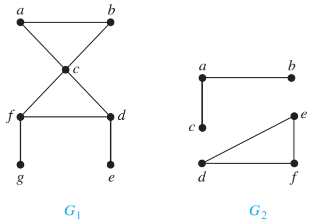
- a, d, c, f, e is a **simple path** of length 4.
- d, e, c, a is not a path.
- b, c, f, e, b is a **circuit** of length 4.
- The path a, b, e, d, a, b, which is of length 5, is **not simple** because it contains the edge {a, b} twice.

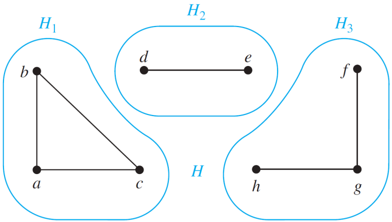
Paths (đường đi)

Defintion 1. Let n be a nonnegative integer and G an **undirected graph**.

- A **path** (đường đi) of length n from u to v in G is a sequence of n edges $e_1, ..., e_n$ of G for which there exists a sequence $x_0 = u, x_1, ..., x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, ..., n$, the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, ..., x_n$.
- The path is a **circuit** (chu trình) if it begins and ends at the same vertex.
- A path or circuit is **simple** (đơn) if it does not contain the same edge more than once.

Connectedness (liên thông) In Undirected Graphs





G_1 is called **connected** and G_2 is **disconnected**.

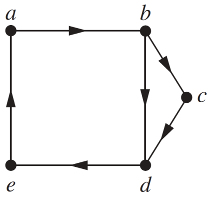
A disconnected with 3 **connected components**

Definition. An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

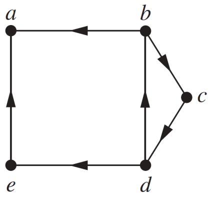
Theorem

There is a simple path between every pair of distinct vertices of a connected undirected graph.

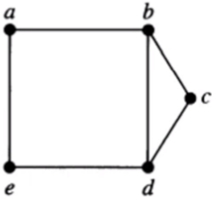
Example.
4 Connectivity



G



H



underlying undirected graph of *H*

- G is strongly connected → weakly connected.
- H is not strongly connected (There is no directed path from a to b in this graph) but the underlying undirected graph of H is connected → weakly connected.

Connectedness In Directed Graphs

Definition.

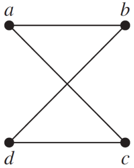
- A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- A directed graph is **weakly connected** if and only if there is always a path between two vertices when the directions of the edges are disregarded (không quan tâm)*(underlying undirected)*.

Notes.

- Any strongly connected directed graph is also weakly connected.

Counting Paths Between Vertices

Example. How many paths of length four are there from a to d in the simple graph G in figure



Theorem. *(with directed or undirected edges, with multiple edges and loops allowed).*

- Let G be a graph with **adjacency matrix A** with respect to the ordering $v_1, v_2, ..., v_n$ of the vertices of the graph,
- **The number of different paths of length** r from v_i to v_j **equals the** (i, j) **th entry of** A^r, where r is a positive integer.

Solution.

- The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

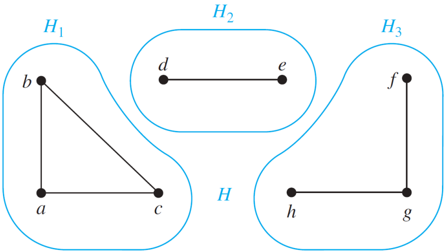
- Thus, we have

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

- There are 8 paths of length four from a to d.

CONNECTED COMPONENTS

- A **connected component** (thành phần liên thông) of a graph G is a connected subgraph of G that *is not a proper subgraph*(không là đồ thị con) of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G.
- A graph G that is not connected has two or more connected components that are disjoint and have G as their union(hợp).

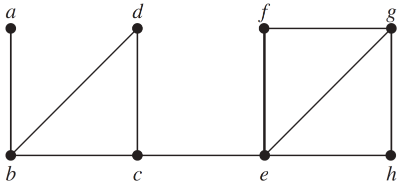


Definition

4 Connectivity

- **Cut vertex** (articulation point) (điểm cắt, đỉnh khớp): It's removal will produce disconnected subgraph from original connected graph.
- Cut edge **(bridge)** (cạnh cắt, cầu): It's removal will produce subgraphs which are more connected components than in the original graph.

Example. Find the cut vertices and cut edges in the graph shown in Figure.



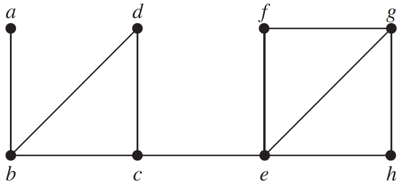
G_1

Definition

4 Connectivity

- **Cut vertex** (articulation point) (điểm cắt, đỉnh khớp): It's removal will produce disconnected subgraph from original connected graph.
- Cut edge **(bridge)** (cạnh cắt, cầu): It's removal will produce subgraphs which are more connected components than in the original graph.

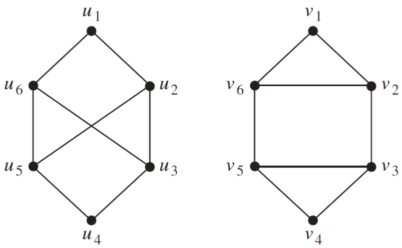
Example. Find the cut vertices and cut edges in the graph shown in Figure.



G_1

Solution: Cut vertices: b,c,e; Cut edges: $\{a, b\}$ and $\{c, e\}$

Using path to determine whether two graphs are isomorphic.



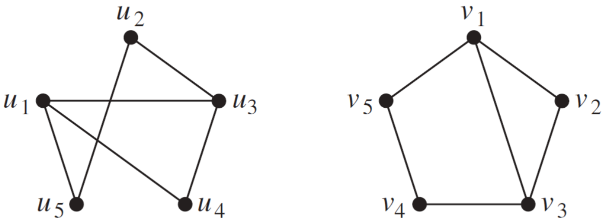
G *H*

G and H have the same

- Number of vertices
- Number of edges
- 2 vertices degree 2
- 4 vertices degree 3

But, H has a simple circuit with length 3 and G has a simple circuit with length 4. Thus, they are not isomorphic.

Using path to determine whether two graphs are isomorphic



G H

They are isomorphic.

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Introduction

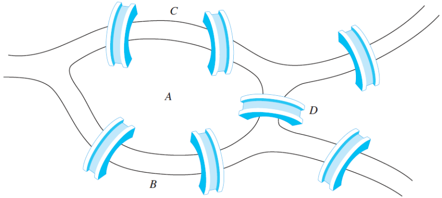
5 Euler & Hamilton Paths

The town of Konigsberg, Prussia (now called Kaliningrad and part of the Russian republic),was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure depicts these regions and bridges.

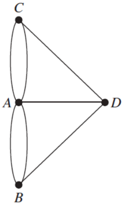
The townspeople took long walks through town on Sundays. They wondered whether it was possible to **start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.**

Introduction

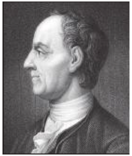
5 Euler & Hamilton Paths



Hình: The seven bridges of Konigsberg



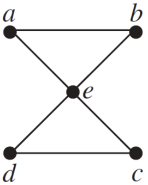
Hình: Multigraph model



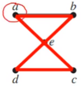
Can once travel across all the bridges once and return to starting point?

His answer: All of vertices must have even-degree.

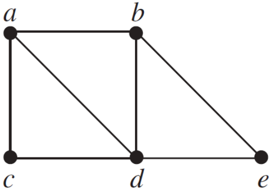
Euler Paths and Circuits

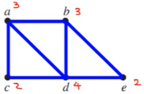


G_1



G_1: has an Euler circuit: a,e,c,d,e,b,a



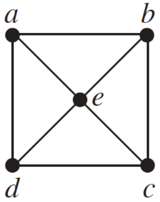


G_3: has no Euler circuit but it has Euler path a,c,d,e,b,d,a,b

Euler Paths and Circuits

5 Euler & Hamilton Paths

G_2:



G_2

- An **Euler circuit** in a graph G is a simple circuit containing every edge of G,
- An **Euler path** in G is a simple path containing every edge of G.

has no Euler circuit and also has no Euler path.

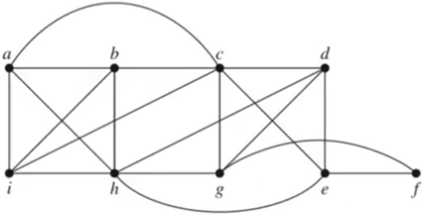
Theorem

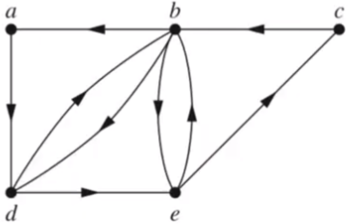
1. A connected multigraph with at least two vertices has an **Euler circuit** if and only if each of its vertices has **even degree.**
2. A connected multigraph has an **Euler path** but not an Euler circuit if and only if it has exactly two vertices of **odd degree.**

Exercises

5 Euler & Hamilton Paths

- Does the given graph have an ***Euler circuit*** ?

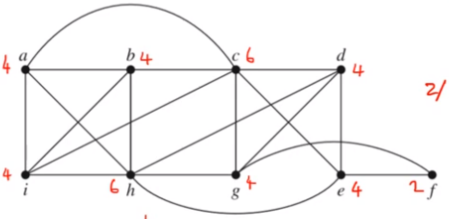




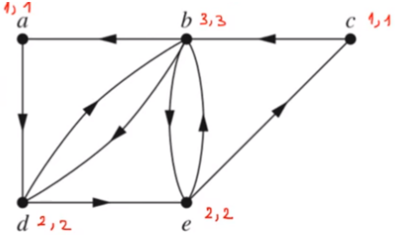
$$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$K_{4,12}$

1/



2/



3/

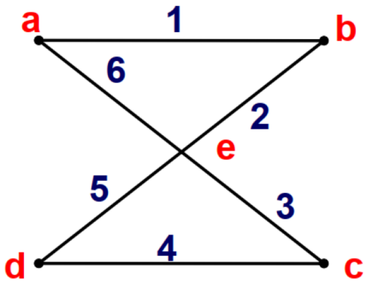
$$\begin{array}{c c c}
 & \begin{matrix} & b & & & \end{matrix} & \\
\begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & \boxed{2} & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix} & \begin{matrix} 8 \\ 9 \;\; \times \\ \\ \\ \\ \end{matrix}
\end{array}$$

adjacency matrix

4/ $K_{4,12}$ ✓

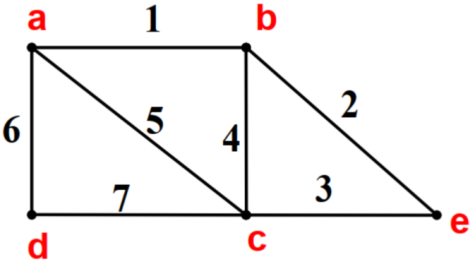
Examples.

The graph have $deg(a) = deg(b) = deg(c) = deg(d) = deg(e) = 2$. So, there is an Euler circuit a, b, e, d, c, e, a.



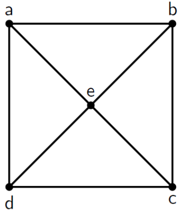
Examples.

The graph have $deg(a) = deg(b) = 3; deg(c) = 4; deg(d) = deg(e) = 2$. So, there is an Euler path a, d, c, e, b, c, a, b.



Examples.

The graph have neither Euler circuit nor Euler path.

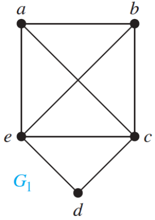


Hamilton Paths and Circuits

Definition.

- A simple path in a graph G that passes through every vertex exactly once is called a **Hamilton path**.
- A simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**.

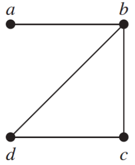
Examples.



G_1 has Hamilton circuit: a, b, c, d, e, a.

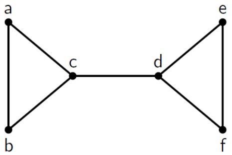
Examples.

- G_2 has no Hamilton circuit.



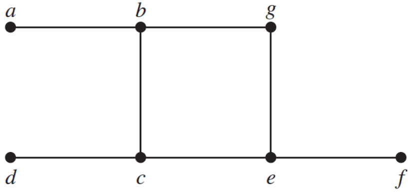
G_2

- G has Hamilton path a,b,c,d,e,f.



Examples.

- G_3 has neither a Hamilton circuit nor a Hamilton path.



G_3

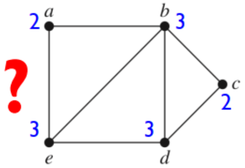
Hamilton circuit - Sufficient conditions

Sufficient conditions for the existence of Hamilton circuits

Dirac's theorem.

G is a graph:
- *simple*
- n ($\geq$ 3) vertices
- $\forall v_i$, **$deg(v_i) \geq \frac{n}{2}$**

$\Rightarrow$ G has a ***Hamilton circuit***



Ore's theorem.

G is a graph:
- *simple*
- n ($\geq$ 3) vertices
- $\forall u, \forall v$, *non-adjacent* **$deg(u) + deg(v) \geq n$**

$\Rightarrow$ G has a ***Hamilton circuit***

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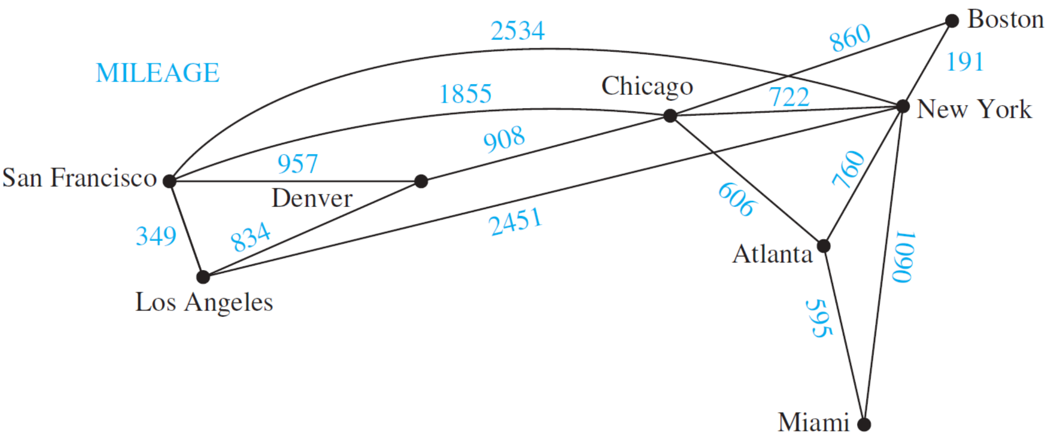
Introduction

6 S. Path Prob.



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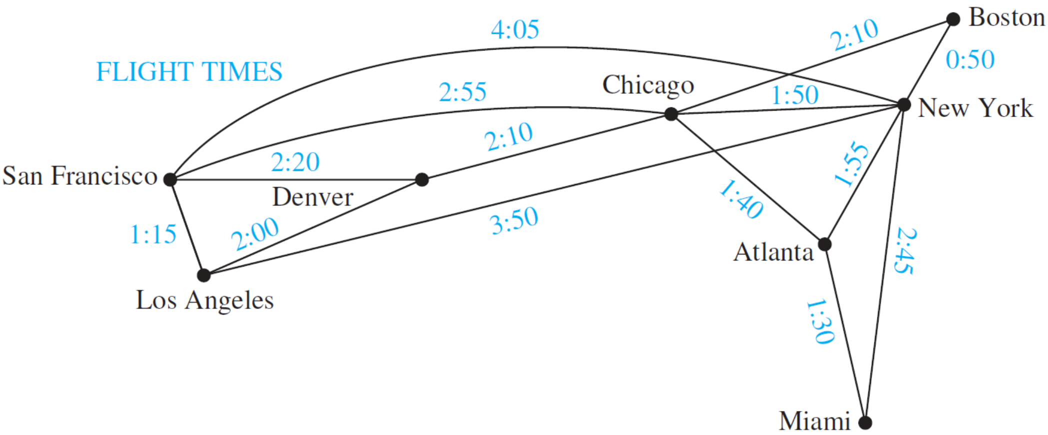
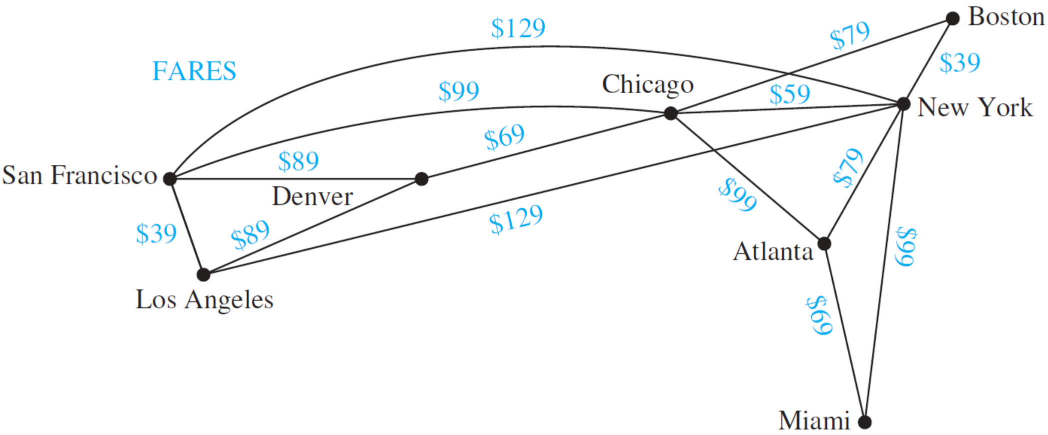
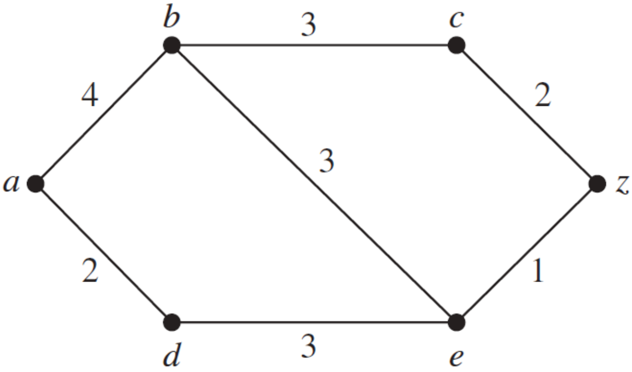


Introduction

6 S. Path Prob.

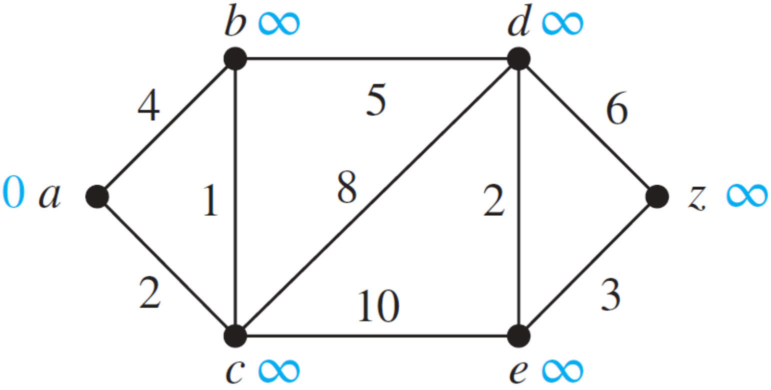


A weighted simple graph



Example.
6 S. Path Prob.

Use Dijkstra's algorithm to find the length of a shortest path between the vertices a and z in the weighted graph displayed in Figure

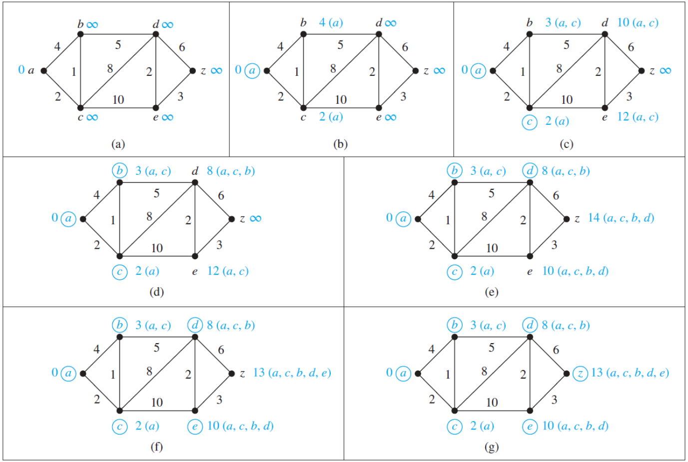


L(a)	L(b)	L(c)	L(d)	L(e)	L(z)	S
0	∞	∞	∞	∞	∞	$\emptyset$
	∞	∞	∞	∞	∞	$\{a\}$
	4	2	∞	∞	∞	$\{a, c\}$
	3		10	12	∞	$\{a, c, b\}$
			8	12	∞	$\{a, c, b, d\}$
				10	14	$\{a, c, b, d, e\}$
					13	$\{a, c, b, d, e, z\}$

$$z(13) \longleftarrow e(10) \longleftarrow d(8) \longleftarrow b(3) \longleftarrow c(2) \longleftarrow a(0)$$

Solution

6 S. Path Prob.



Dijkstra's Algorithm

procedure *Dijkstra*(G: weighted connected simple graph, with
  all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
  where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
  $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
  other labels are ∞, and S is the empty set}
while $z \notin S$
  $u :=$ a vertex not in S with $L(u)$ minimal
  $S := S \cup \{u\}$
  for all vertices v not in S
    if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
    {this adds a vertex to S with minimal label and updates the
    labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

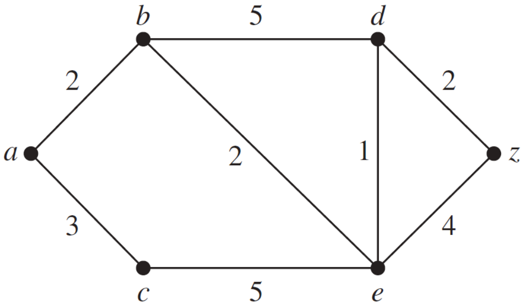
Theorem

1. Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
2. Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

Example.

6 S. Path Prob.

Find the length of a shortest path between a and z in the given weighted graph.

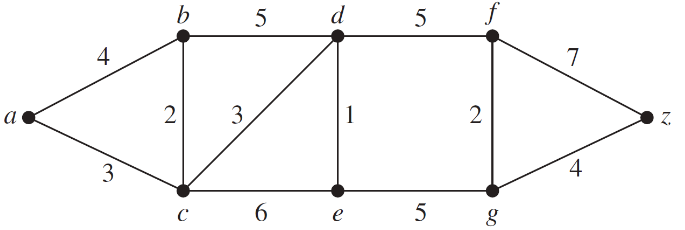


A shortest path between a and z is a, b, e, d, z and the length of the shortest path from a to z is 7.

Example.

6 S. Path Prob.

Find the length of a shortest path between a and z in the given weighted graph.

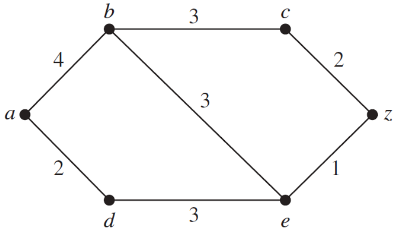


A shortest path between a and z is a, c, d, e, g, z and the length of the shortest path from a to z is 16.

Example.

6 S. Path Prob.

Find the length of a shortest path between a and z in the given weighted graph.



A shortest path between a and z is a, d, e, z and the length of the shortest path from a to z is 6

Example.

6 S. Path Prob.

Find the length of a shortest path between a and z in the given weighted graph.

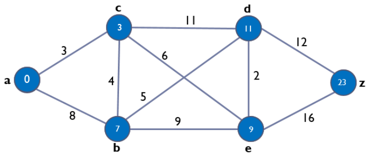


Table of Contents

7 Problems

Study a simple graph having the degree sequence { 5,5,4,4,4,3,3,2,2,2,2, 1,1}.
If the graph has n edges, then n = ...

Select one:

- a. 38
- b. 19
- c. 20
- d. 17

Ans: 19

For which values of n do K_n have an Euler path but no Euler circuit?

Select one:

- a. n = 2
- b. n = 1
- c. n = 3
- d. 2m for all m > 1

Ans: $n = 2$

Quizz

7 Problems

Every Euler circuit in K_{11} is a path of length ___

Select one:

- a. 45
- b. 22
- c. 11
- d. 55
- e.

Ans: 55

Quizz

If K_n has x edges and y vertices, then x = ... and y = ...

Select one:

- a. n + 1 , n
- b. 2n, n+1
- c. n(n-1)/2, n
- d. n(n -1)/2, 2^n

Ans: n(n-1)/2, n

The incidence matrix of the graph $K_{2,3}$ has the size of ...

Select one:

- a. 30
- b. 5 x 6
- c. 2 x 3
- d. 6 x 5

Ans: 5×6

Study the statements:

(i) $K_{2,3}$ has no an Euler circuit, but has an Euler path.

(ii) $K_{2,3}$ has an Euler circuit.

Which statement is true?

Select one:

- a. None
- b. (ii)
- c. (i)
- d. Both

Ans: (i)

Given the adjacency matrix of an undirected graph with vertices {m, n, p}

	m	n	p
m	2	1	3
n	1	1	1
p	3	1	0

How many **paths of length 2** are there from the vertex n to the vertex m in this graph?

Select one:

- a. 6
- b. 5
- c. 4
- d. 3

Ans: 6

Choose correct answer.

A simple graph having n vertices is bipartile if ...

(i) n is even

(ii) it is K_n

(iii) it is C_n

Select one:

- a. (iii)
- b. (ii)
- c. (i)
- d. None of the others

Ans: None of the others

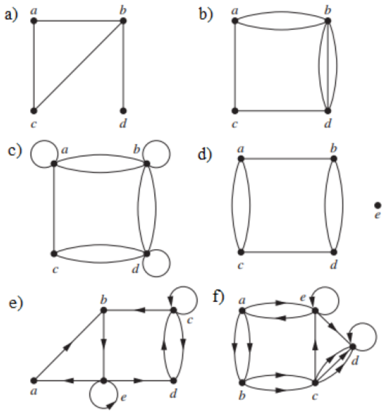
Suppose G and K_5 are isomorphic. Select correct statement(s).

Select one or more:

- ☐ a. G has Euler circuit
- ☐ b. G has no Euler circuit
- ☐ c. G has Hamilton circuit
- ☐ d. G has 10 edges

Ans: G has Euler circuit, G has Hamilton circuit, G has 10 edges

1. Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops

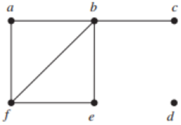


Graph Terminology and Special Types of Graphs

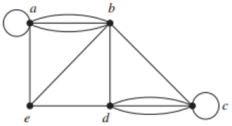
7 Problems

1. find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

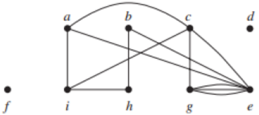
a)



b)



c)



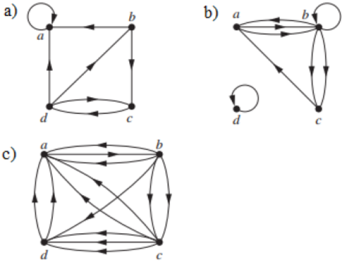
Graph Terminology and Special Types of Graphs

7 Problems

2. Find the sum of the degrees of the vertices of each graph in Exercises 1 and verify that it equals twice the number of edges in the graph.

3. Can a simple graph exist with 15 vertices each of degree five?

4. determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph



Graph Terminology and Special Types of Graphs

7 Problems

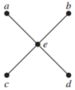
5. Draw these graphs.

a) K_7 b) $K_{1,8}$ c) $K_{4,4}$

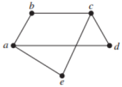
d) C_7 e) W_7 f) Q_4

6. Determine whether the graph is bipartite

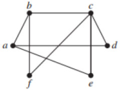
a)



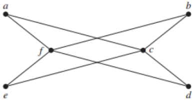
b)



c)



d)



7. For which values of n are these graphs bipartite?

a) K_n b) C_n c) W_n d) Q_n

8. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

9. Find the degree sequences for each of the graphs in Exercises 6.

10. Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4 d) $K_{2,3}$ e) Q_3

11. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

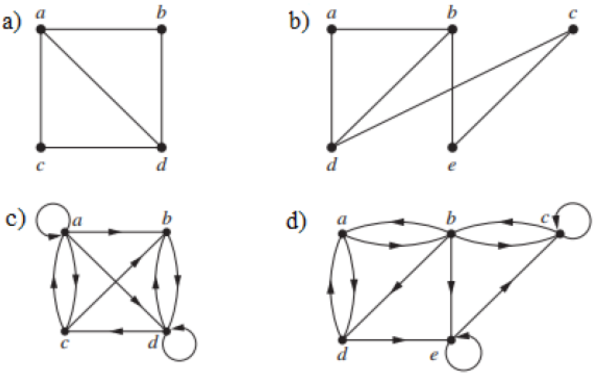
12. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

13. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 4, 4, 3, 2, 1

d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 3, 2, 2, 1, 0

1. use an adjacency list to represent the given graph.



2. Represent the graph in Exercise 1 with an adjacency matrix.

3. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$ d) C_4 e) W_4 f) Q_3

4. Draw a graph with the given adjacency matrix

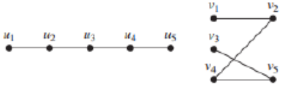
a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Representing Graphs and Graph Isomorphism

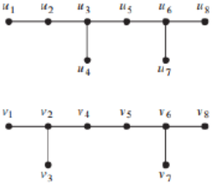
7 Problems

5. Determine whether the given pair of graphs is isomorphic

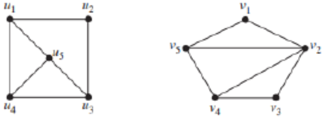
a)



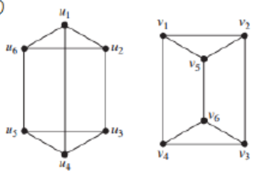
b)



c)



d)



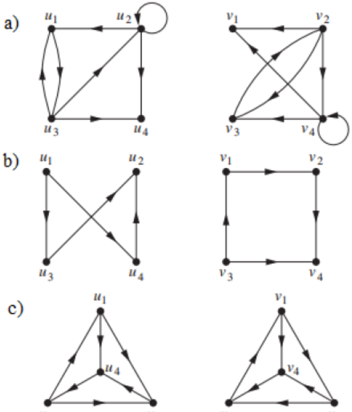
6. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

7. Determine whether the given pair of directed graphs are isomorphic.

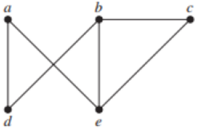


Connectivity

7 Problems

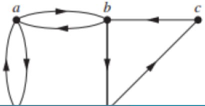
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b   b) a, e, a, d, b, c, a   c) e, b, a, d, b, e   d) c, b, d, a, e, c



2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b   b) a, d, a, d, a   c) a, d, b, e, a   d) a, b, e, c, b, d, a



Connectivity

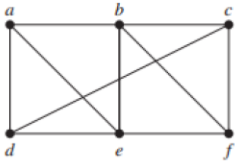
7 Problems

3. Find the number of paths of length n between two different vertices in K_4 if n is

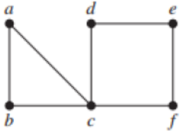
a) 2 b) 3 c) 4 d) 5

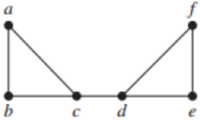
4. Find the number of paths between c and d in the graph of length

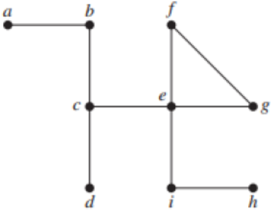
a) 2 b) 3 c) 4 d) 5 e) 6 f) 7



5. Find all the cut vertices of the given graph

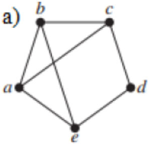
a) 

b) 

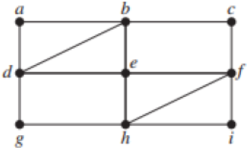
c) 

1. Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists.

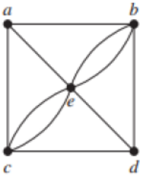
a)



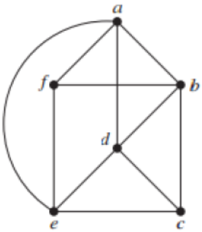
b)



c)

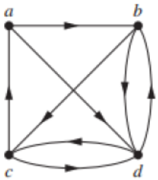


d)

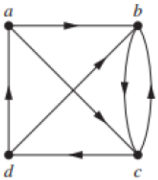


2. Determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists.

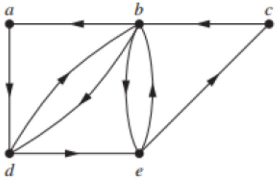
a)



b)



c)



3. For which values of n do these graphs have an Euler circuit?

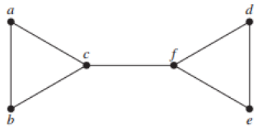
a) K_n b) C_n c) W_n d) Q_n

4. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

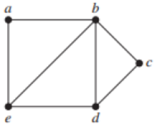
a) Euler circuit? b) Euler path?

5. Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.

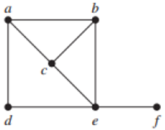
a)



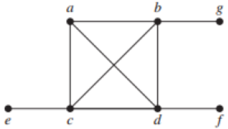
b)



c)

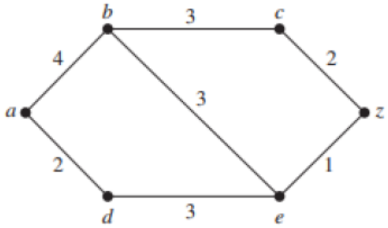


d)



Shortest-Path Problems

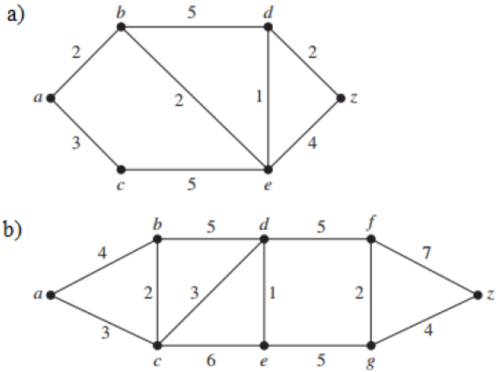
1. What is the length of a shortest path between a and z in the weighted graph shown in the Figure?



Shortest-Path Problems

7 Problems

2. Find a shortest path between a and z in the given weighted graph.



Q&A

Thank you for listening!