Diagonal Simplicial Tensor Modules and Algebraic n-Hypergroupoids

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Abstract

Let A be a commutative ring, $k \in \mathbb{Z}^+$, and $\vec{s} = (n_1, \ldots, n_k) \in (\mathbb{Z}^+)^k$ with $n = \min(\vec{s}) - 1$. We attach to $\vec{s}$ a diagonal simplicial tensor module $X_\bullet(\vec{s}; A)$ whose p-simplices are functions on a cosimplicial index set $I_p(\vec{s}) \subseteq \mathbb{N}^k$. This extends Quillen's diagonal on double simplicial groups to k commuting directions in $\mathbf{Mod}_A$, compatibly with Dold–Kan normalization.

We analyze the horn kernels $R_{p,j}(X)$ via "missing indices" and show that $R_{p,j}(X) \neq 0$ if and only if $k \geq p$. Thus $X_\bullet(\vec{s}; A)$ is a strict algebraic n-hypergroupoid (in the sense of Duskin–Glenn) if and only if $k = n$, and a Horn Non-Degeneracy Lemma yields a decomposition $X_n = R_{n,j}(X) \oplus D_n$. A shift-and-truncate chain homotopy, $\mathrm{Stab}(\vec{s})$-equivariant and filtration-preserving, shows that $X_\bullet(\vec{s}; A)$ is contractible and forces the associated spectral sequence to collapse at E_1.

Over an infinite field K we classify simplicial submodules generated by a single tensor via kernel sequences and a moduli map to a product of Grassmannians, obtaining an irreducible and unirational incidence variety.

Keywords: Diagonal simplicial tensor module, horn kernels, strict algebraic n-hypergroupoids, incidence varieties, moduli spaces
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1. Introduction

Let A be a commutative ring and $k \in \mathbb{Z}^+$. Fix a **shape** $\vec{s} = (n_1, \ldots, n_k) \in (\mathbb{Z}^+)^k$ and set $n = \min(\vec{s}) - 1$. We construct the diagonal simplicial tensor module $X_\bullet(\vec{s}; A)$ (DSTM), a simplicial A-module whose p-simplices are functions on cosimplicial index sets $I_p(\vec{s}) \subseteq \mathbb{N}^k$.

This construction generalizes Quillen's diagonal on a double simplicial group [1] to k commuting simplicial directions in the category $\mathbf{Mod}_A$ and is compatible with Dold–Kan normalization in the sense of Dold, Kan, May, Weibel, and Goerss–Jardine [2, 3, 4, 5, 6].

In particular, the diagonal simplicial module $X_\bullet(\vec{s}; A)$ is obtained from $X_\bullet(\vec{s}; \mathbb{Z})$ by base change, and the normalized Moore complexes satisfy

$$N_\bullet\big(X_\bullet(\vec{s}; A)\big) \cong N_\bullet\big(X_\bullet(\vec{s}; \mathbb{Z})\big) \otimes_{\mathbb{Z}} A$$

for all commutative rings A.

We analyze the horn kernels

$$R_{p,j}(X) = \bigcap_{i \neq j} \ker\big(d_i : X_p \to X_{p-1}\big),$$

and show that their combinatorics controls the homotopy-theoretic behavior of the DSTM. For the simplicial dimension $n = \min(\vec{s}) - 1$ we prove that $X_\bullet(\vec{s}; A)$ is a strict algebraic n-hypergroupoid (in the sense of Duskin–Glenn [7, 8]) if and only if the tensor order satisfies $k = n$. The transition from non-unique to unique horn fillers occurs exactly at $p = n$, and this threshold is determined by the tensor order.

We briefly outline the main aspects of the paper: combinatorics and classification of the DSTM, and the algebraic geometry of generated submodules,.

Combinatorics and Classification

In the abelian setting of simplicial modules, horn fillers always exist (the Kan condition is automatic). The obstruction to the uniqueness of fillers is captured by the horn kernel $R_{p,j}$. Following terminology from Duskin–Glenn [7, 8] and the nLab, a simplicial module is a **strict algebraic n-hypergroupoid** if fillers are unique in dimensions strictly greater than n ($R_{p,j} = 0$ for $p > n$), and non-unique in dimension n ($R_{n,j} \neq 0$).

We characterize $R_{n,j}$ combinatorially via a basis of "missing indices." The Horn Non-Degeneracy Lemma proves $R_{n,j} \cap D_n = \{0\}$, where D_n denotes the degenerate submodule in degree n, yielding the decomposition $X_n = R_{n,j} \oplus D_n$. Counting missing indices gives exact rank formulas for $R_{p,j}$ via inclusion–exclusion on the product sets $I_p = \prod_a [M_a(p)]$. In the constant shape case $n_a = n + 1$ for all a, so $M_a(p) = p$ and $I_p = [p]^k$, and these rank formulas specialize to finite differences of x^k and can be expressed using Stirling numbers of the second kind $\left\{ {k \atop m} \right\}$ [9]. By analyzing when the horn kernel vanishes, we obtain our main classification theorem: $X_\bullet(\vec{s}; A)$ is a strict algebraic n-hypergroupoid if and only if $k = n$.

Algebraic Geometry of Generated Submodules

We work throughout with submodules generated by a single tensor $T \in X_n(\vec{s}; K)$ over an infinite field K; we do not attempt to parametrize arbitrary subcomplexes. Such a generated submodule $\langle T \rangle$ is encoded by its kernel sequence

$$K(T)_\bullet := (\ker f_{T,p})_p \subseteq K[\Delta^n]_\bullet,$$

where f_T is the realization map from Section 8.

We define a moduli map Ψ from a Zariski-open locus $\mathcal{U} \subset X_n(\vec{s}; K)$ to a product of Grassmannians by sending T to the collection of subspaces $K(T)_p \subseteq K[\Delta^n]_p$. The image $\mathcal{M}(\vec{s})$ lies within a closed incidence subvariety defined by simplicial compatibility conditions, which we show are linear in the Segre–Plücker coordinates. This reduces the classification of these generated submodules to the geometry of these incidence conditions and of the index collision maps $\mathcal{I}_p$.

Organization

Section 2 establishes notation and the DSTM construction. Section 3 characterizes missing indices and the horn kernel basis. Section 4 develops the Moore filler algorithm and normalization. Section 5 establishes rank formulas and the classification theorem. Section 6 proves the Horn Non-Degeneracy Lemma and the decomposition $X_n = R_{n,j} \oplus D_n$. Section 7 develops the homology dichotomy and the hypergroupoid classification. Section 8 develops the algebraic geometry of generated submodules. Appendix A proves contractibility via an explicit equivariant homotopy.

2. Diagonal simplicial tensor modules

Throughout, let A be a commutative ring and write $\mathbf{Mod}_A$ for the category of A-modules. Denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{Z}^+$ the set of positive integers.

2.1. Simplicial preliminaries

The **simplicial category** Δ has objects $[p] := \{0, \ldots, p\}$ for $p \in \mathbb{N}$; morphisms are nondecreasing maps. It is generated by the **coface maps** $\delta_i^p : [p-1] \to [p]$ (the injection missing i) and the **codegeneracy maps** $\sigma_i^p : [p+1] \to [p]$ (the surjection repeating i).

A **simplicial** A**-module** is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Mod}_A$. We write $X_p := X([p])$. The induced maps are the **face maps** $d_i^p := X(\delta_i^p) : X_p \to X_{p-1}$ and the **degeneracy maps** $s_i^p := X(\sigma_i^p) : X_p \to X_{p+1}$. The associated chain complex $(X_\bullet, \partial_\bullet)$ has boundary maps $\partial_p = \sum_{i=0}^{p} (-1)^i d_i^p$.

2.2. Setup and index sets

We now define the index sets underlying the diagonal simplicial tensor module. Let $k \in \mathbb{Z}^+$. We analyze modules consisting of tensors of order k. A **shape** is a k-tuple $\vec{s} = (n_1, \ldots, n_k) \in (\mathbb{Z}^+)^k$.

We fix the shape $\vec{s}$ throughout and define its **simplicial dimension** as

$$n = n(\vec{s}) := \min(n_1, \ldots, n_k) - 1.$$

For any degree $p \geq 0$ and axis $1 \leq a \leq k$, we define the index bounds:

$$M_a(p) := n_a - 1 - n + p.$$

Since $n_a - 1 \geq n$, we have $M_a(p) \geq p$ for all a. The index set in degree p is defined as

$$I_p := \prod_{a=1}^{k} [\, M_a(p)\,].$$

Since $[p] \subseteq [\, M_a(p)\,]$ for every a, we have the inclusion $[p]^k \subseteq I_p$.

2.3. The Cosimplicial Index Set $I_\bullet$

We define a functor $I_\bullet : \Delta \to \mathbf{Set}$. On objects, $I_\bullet([p]) := I_p$. Note that $M_a(p \pm 1) = M_a(p) \pm 1$.

The structure maps are defined "diagonally" as products of the standard generators.

Definition 2.1 (Cosimplicial Structure Maps)**.** The functor $I_\bullet$ maps the generators of Δ as follows:

For $p \geq 1$ and $0 \leq i \leq p$, the **index coface map** $\Delta_i^p : I_{p-1} \to I_p$ is

$$\Delta_i^p := I_\bullet(\delta_i^p) := \prod_{a=1}^{k} \delta_i^{M_a(p)}.$$

For $p \geq 0$ and $0 \leq i \leq p$, the **index codegeneracy map** $\Sigma_i^p : I_{p+1} \to I_p$ is

$$\Sigma_i^p := I_\bullet(\sigma_i^p) := \prod_{a=1}^{k} \sigma_i^{M_a(p)}.$$

Since $M_a(p) \geq p$, all factors are defined for the common index i. Because the standard maps δ_i, σ_i satisfy the cosimplicial identities, the product maps Δ_i^p, Σ_i^p also satisfy them componentwise, proving that $I_\bullet$ is a cosimplicial set.

2.4. The Diagonal Simplicial Module $X_\bullet(\vec{s}; A)$

We define the contravariant functor $A^{(-)} : \mathbf{Set} \to \mathbf{Mod}_A$. This functor maps a set S to the A-module of functions $S \to A$, and a map of sets $f : S_1 \to S_2$ to the module homomorphism $A^f : A^{S_2} \to A^{S_1}$ defined by pre-composition ($T \mapsto T \circ f$).

Definition 2.2 (Diagonal simplicial tensor module)**.** The **diagonal simplicial A-tensor module** $X_\bullet(\vec{s}; A) : \Delta^{\mathrm{op}} \to \mathbf{Mod}_A$ is the composition $A^{(-)} \circ I_\bullet$.

On objects, $X_p(\vec{s}; A) := A^{I_p}$. Since the index sets I_p are finite, X_p is a free A-module.

The face and degeneracy maps are induced by the corresponding maps in $I_\bullet$ via pre-composition:

$$d_i^p(T) := T \circ \Delta_i^p,$$
$$s_i^p(T) := T \circ \Sigma_i^p.$$

Remark 2.3 (Matrices in the constant shape case). If $k = 2$ and $\vec{s} = (N, N)$ is constant, then $n = N - 1$ and

$$M_a(p) = n_a - 1 - n + p = p \qquad (a = 1, 2).$$

Hence

$$I_p = [p]^2 \qquad \text{and} \qquad X_p(\vec{s}; A) = A^{[p]^2} \cong A^{(p+1)\times(p+1)}.$$

In particular,

$$X_{N-1}(\vec{s}; A) \cong A^{[N-1]^2} \cong A^{N\times N},$$

so an $N\times N$ matrix may be viewed as a simplex in degree $N-1$ of $X_\bullet(\vec{s}; A)$. This identification will be used when we regard adjacency matrices of graphs on N vertices as elements of $X_{N-1}(\vec{s}; A)$.

2.5. Axis Symmetries and Permutations of Tensor Factors

For a shape $\vec{s} = (n_1, \ldots, n_k)$, define its stabilizer

$$\mathrm{Stab}(\vec{s}) := \{\sigma \in S_k : n_{\sigma(a)} = n_a \text{ for all } a\}.$$

Each $\sigma \in \mathrm{Stab}(\vec{s})$ acts (on the left) on the index sets I_p by permuting coordinates:

$$\sigma \cdot (m_1, \ldots, m_k) := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(k)}).$$

This induces a (left) action on the diagonal simplicial module $X_\bullet(\vec{s}; A)$ by precomposition:

$$(\sigma \cdot T)(m) := T(\sigma^{-1} \cdot m), \qquad T \in X_p(\vec{s}; A),\ m \in I_p.$$

Lemma 2.4 (Equivariance under $\mathrm{Stab}(\vec{s})$)**.** *The action of* $\mathrm{Stab}(\vec{s})$ *on* $I_\bullet$ *commutes with the cosimplicial structure maps. Consequently, for every* $\sigma \in \mathrm{Stab}(\vec{s})$ *and all* p, i *we have*

$$d_i^p(\sigma \cdot T) = \sigma \cdot d_i^p(T), \qquad s_i^p(\sigma \cdot T) = \sigma \cdot s_i^p(T).$$

The boundary operators ∂_p *also commute with the* $\mathrm{Stab}(\vec{s})$*-action.*

Proof. Let $\sigma \in \mathrm{Stab}(\vec{s})$. By definition, $n_a = n_{\sigma(a)}$, which implies $n_a = n_{\sigma^{-1}(a)}$. This ensures $M_a(p) = M_{\sigma^{-1}(a)}(p)$ for all a, p.

We first verify the commutation on the index sets. Let $m \in I_{p-1}$. We show $\sigma \cdot \Delta_i^p(m) = \Delta_i^p(\sigma \cdot m)$. The a-th component of the left side is

$$(\sigma \cdot \Delta_i^p(m))_a = (\Delta_i^p(m))_{\sigma^{-1}(a)} = \delta_i^{M_{\sigma^{-1}(a)}(p)}(m_{\sigma^{-1}(a)}).$$

The a-th component of the right side is

$$(\Delta_i^p(\sigma \cdot m))_a = \delta_i^{M_a(p)}((\sigma \cdot m)_a) = \delta_i^{M_a(p)}(m_{\sigma^{-1}(a)}).$$

Since $M_a(p) = M_{\sigma^{-1}(a)}(p)$, the components are equal. Thus $\sigma \circ \Delta_i^p = \Delta_i^p \circ \sigma$. The argument for Σ_i^p is analogous.

Now we verify the equivariance on X_p. Let $T \in X_p$.

$$\begin{aligned} d_i^p(\sigma \cdot T) &= (\sigma \cdot T) \circ \Delta_i^p \\ &= (T \circ \sigma^{-1}) \circ \Delta_i^p \\ &= T \circ (\sigma^{-1} \circ \Delta_i^p) \\ &= T \circ (\Delta_i^p \circ \sigma^{-1}) \quad \text{(by commutativity on } I_\bullet) \\ &= (T \circ \Delta_i^p) \circ \sigma^{-1} \\ &= \sigma \cdot (T \circ \Delta_i^p) = \sigma \cdot d_i^p(T). \end{aligned}$$

The proof for s_i^p is analogous. The statement for ∂_p follows by linearity. □

Remark 2.5 (Constant shape and symmetry subcomplexes). Suppose $\vec{s} = (N, \dots, N)$ is constant. Then $\mathrm{Stab}(\vec{s}) \cong S_k$ acts on $X_\bullet(\vec{s}; A)$ by permuting tensor axes. By Lemma 2.4, any S_k-stable subspace in each degree X_p defines a simplicial A-submodule.

In particular, the symmetric and alternating subspaces

$$X_\bullet^{\mathrm{Sym}} := \{T : \sigma \cdot T = T \ \forall \sigma \in S_k\}, \qquad X_\bullet^{\mathrm{Alt}} := \{T : \sigma \cdot T = \mathrm{sgn}(\sigma)\, T\}$$

form simplicial A-submodules. (If $k!$ is invertible in A, these are direct summands.) In Appendix A we show that the global contracting homotopy H is $\mathrm{Stab}(\vec{s})$-equivariant, so it restricts to chain contractions on $X_\bullet^{\mathrm{Sym}}$ and $X_\bullet^{\mathrm{Alt}}$. Hence these symmetry subcomplexes are also contractible.

Remark 2.6 (Matrix and Hermitian transpose). For $k = 2$ and $\vec{s} = (N, N)$, the nontrivial element $\tau \in S_2$ acts by swapping the tensor axes, corresponding to the transpose: $(\tau \cdot T)(m_1, m_2) = T(m_2, m_1)$. By Lemma 2.4, all structure maps and the boundary operators ∂_p commute with this action.

Over $K = \mathbb{C}$, viewing $X_\bullet(\vec{s}; K)$ as a simplicial $\mathbb{R}$-module, complex conjugation commutes with all structure maps. The Hermitian conjugate action (transpose combined with conjugation) also commutes with d_i^p, s_i^p and ∂_p (as $\mathbb{R}$-linear operators). The real subspaces of symmetric, skew-symmetric, Hermitian, and skew-Hermitian tensors form simplicial $\mathbb{R}$-submodules, and the global contracting homotopy H restricts to each of them.

2.6. Base Change and Realization Maps

Proposition 2.7 (Base change for $X_\bullet(\vec{s};-)$). *For any ring homomorphism $\varphi : A \to B$ and shape $\vec{s}$, there is a canonical isomorphism of simplicial B-modules*

$$X_\bullet(\vec{s};A) \otimes_A B \xrightarrow[\cong]{\theta_\bullet} X_\bullet(\vec{s};B).$$

Proof. Each $X_p(\vec{s};A) = A^{I_p}$ is a finite free A-module with basis $\{E_m\}_{m\in I_p}$ (indicator functions). Define

$$\theta_p : X_p(\vec{s};A) \otimes_A B \longrightarrow X_p(\vec{s};B), \qquad \theta_p(E_m \otimes b) = b \cdot E_m.$$

Since I_p is finite, $(A^{I_p}) \otimes_A B \cong B^{I_p}$, so each θ_p is a B-module isomorphism.

Simplicial compatibility follows because the face and degeneracy maps are given by precomposition with the same maps Δ_i^p, Σ_i^p on $I_\bullet$ in both $X_\bullet(\vec{s};A)$ and $X_\bullet(\vec{s};B)$, so each θ_p commutes with all d_i^p and s_i^p. □

Corollary 2.8 (Naturality and flat base change). *Let $T \in X_n(\vec{s};A)$, and let $f_T : A[\Delta^n] \to X_\bullet(\vec{s};A)$ be the corresponding realization map (defined in Section 8). Let $\varphi : A \to B$ be a ring homomorphism, and let $T_B := \theta_n(T \otimes 1) \in X_n(\vec{s};B)$. Let $\varphi_p : A[\Delta^n]_p \otimes_A B \xrightarrow{\cong} B[\Delta^n]_p$ denote the canonical isomorphism.*

1. *(Naturality) The following diagram commutes for all p:*

$$\begin{tikzcd} A[\Delta^n]_p \otimes_A B \arrow[r, "f_{T,p}\otimes 1_B"] \arrow[d, "\varphi_p"', "\cong"] & X_p(\vec{s};A) \otimes_A B \arrow[d, "\theta_p"', "\cong"] \\ B[\Delta^n]_p \arrow[r, "f_{T_B,p}"'] & X_p(\vec{s};B) \end{tikzcd}$$

2. *(Kernel preservation) If B is a flat A-module, the kernel sequence is preserved:*

$$(\ker f_{T,p}) \otimes_A B \;\cong\; \ker(f_{T_B,p}) \qquad (\forall p).$$

Proof. (1) Let $\alpha \in \Delta([p],[n])$ be a basis element of $A[\Delta^n]_p$. Both paths in the diagram map the element $\alpha \otimes b$ to $b \cdot f_{T_B,p}(\alpha)$, establishing commutativity.

(2) Consider the exact sequence $0 \to \ker f_{T,p} \to A[\Delta^n]_p \xrightarrow{f_{T,p}} X_p(\vec{s};A)$. If B is a flat A-module, the functor $(-) \otimes_A B$ is exact. Applying it yields the exact sequence

$$0 \to (\ker f_{T,p}) \otimes_A B \to A[\Delta^n]_p \otimes_A B \xrightarrow{f_{T,p}\otimes 1_B} X_p(\vec{s};A) \otimes_A B.$$

This implies $(\ker f_{T,p}) \otimes_A B \cong \ker(f_{T,p} \otimes 1_B)$. By the commutativity of the diagram in (1) and the fact that φ_p and θ_p are isomorphisms, $\ker(f_{T,p} \otimes 1_B)$ is isomorphic to $\ker(f_{T_B,p})$. □

3. Horns, Kernels, and Missing Indices

We investigate the structure of face kernels and the uniqueness of fillers for horns in $X_\bullet(\vec{s};A)$.

3.1. Horns and the Horn Kernel

Let $p \geq 1$ and $0 \leq j \leq p$. We denote $F_j = [p] \setminus \{j\}$.

Definition 3.1 ((p, j)-Horn and Filler)**.** A (p, j)-**horn** in $X_\bullet$ is a tuple $H = (h_i)_{i \in F_j}$ of elements $h_i \in X_{p-1}$ satisfying the **horn compatibility condition**:

$$d_i(h_\ell) = d_{\ell-1}(h_i) \quad \text{for all } i < \ell \text{ with } i, \ell \in F_j.$$

A tensor $T \in X_p$ is a **filler** of H if $d_i(T) = h_i$ for all $i \in F_j$.

In the category of simplicial modules, every horn has at least one filler. The uniqueness of fillers is measured by the horn kernel.

Definition 3.2 (Horn Kernel)**.** The (p, j)-**horn kernel** $R_{p,j}$ is the submodule of X_p defined by

$$R_{p,j} := \bigcap_{i \in F_j} \ker(d_i^p).$$

If T and T' are two fillers for the same horn H, their difference $T - T' \in R_{p,j}$. The set of fillers $\mathcal{F}(H)$ is an affine space (torsor) modeled on $R_{p,j}$. Fillers are unique if and only if $R_{p,j} = \{0\}$.

3.2. Structure of the Face Kernels

We characterize the kernels of the face maps using the standard basis $\{E_m\}_{m \in I_p}$ of X_p.

Lemma 3.3 (Action on basis elements)**.** *Let $m \in I_p$. The action of the face map d_i^p on E_m is*

$$d_i^p(E_m) = \begin{cases} E_{m'} & \text{if } m \in \operatorname{im}(\Delta_i^p), \text{ where } m' \text{ is the unique preimage,} \\ 0 & \text{if } m \notin \operatorname{im}(\Delta_i^p). \end{cases}$$

Proof. Evaluate at $m'' \in I_{p-1}$: $(d_i^p(E_m))(m'') = E_m(\Delta_i^p(m'')) = \delta_{m,\Delta_i^p(m'')}$. If $m \notin \operatorname{im}(\Delta_i^p)$, this is zero. If $m \in \operatorname{im}(\Delta_i^p)$, since Δ_i^p is injective, there is a unique m' such that $\Delta_i^p(m') = m$. Then $d_i^p(E_m) = E_{m'}$. □

Lemma 3.4 (Injectivity Lemma)**.** *If $d_i^p(E_{m_1}) = d_i^p(E_{m_2}) \neq 0$, then $m_1 = m_2$.*

Proof. By Lemma 3.3, the common value is some $E_{m^*} \in X_{p-1}$. Then $m_1 = \Delta_i^p(m^*)$ and $m_2 = \Delta_i^p(m^*)$. □

Theorem 3.5 (Basis of the Face Kernel)**.** *The kernel of $d_i^p : X_p \to X_{p-1}$ is the free A-submodule*

$$\ker(d_i^p) = \operatorname{span}_A\{E_m \mid m \in I_p \setminus \operatorname{im}(\Delta_i^p)\}.$$

Proof. Let $T = \sum_{m \in I_p} a_m E_m \in X_p$. Applying d_i^p gives

$$d_i^p(T) = \sum_{m \in \operatorname{im}(\Delta_i^p)} a_m d_i^p(E_m),$$

since $d_i^p(E_m) = 0$ if $m \notin \operatorname{im}(\Delta_i^p)$ (Lemma 3.3). By the Injectivity Lemma 3.4, the set $\{d_i^p(E_m) \mid m \in \operatorname{im}(\Delta_i^p)\}$ consists of distinct standard basis elements in X_{p-1}, hence is A-linearly independent.

Thus $d_i^p(T) = 0$ if and only if $a_m = 0$ for all $m \in \operatorname{im}(\Delta_i^p)$. □

3.3. *Missing Indices and the Support Characterization*

Definition 3.6 (Missing Index)**.** An index $m \in I_p$ is **missing** from the (p, j)-horn if

$$\forall i \in F_j : m \notin \mathrm{im}(\Delta_i^p).$$

Let $M_{p,j} \subset I_p$ denote the set of missing indices.

Theorem 3.7 (Support Characterization / Horn Kernel Basis)**.** *The horn kernel $R_{p,j}$ is a free A-module with basis $\{E_m\}_{m \in M_{p,j}}$. Consequently, if T_1 and T_2 are two fillers of the same horn, their difference is supported on the missing indices:* $\mathrm{supp}(T_1 - T_2) \subseteq M_{p,j}$.

Proof. By Theorem 3.5, $\ker(d_i^p)$ is the coordinate subspace spanned by the basis $B_i = \{E_m \mid m \notin \mathrm{im}(\Delta_i^p)\}$. The horn kernel $R_{p,j}$ is the intersection of these coordinate subspaces for $i \in F_j$. The intersection of coordinate subspaces is spanned by the intersection of their bases, $\bigcap_{i \in F_j} B_i$. An element E_m lies in this intersection if and only if $m \notin \mathrm{im}(\Delta_i^p)$ for all $i \in F_j$, which defines $M_{p,j}$. The consequence regarding the support of the difference follows because $T_1 - T_2 \in R_{p,j}$. □

3.4. *Combinatorial Characterization*

We provide a combinatorial interpretation of missing indices. We denote the image of a multi-index $m = (m_1, \dots, m_k)$ by $\mathrm{im}(m) = \{m_1, \dots, m_k\}$.

Proposition 3.8 (Characterization of Missing Indices)**.** *An index $m \in I_p$ is missing from the (p, j)-horn ($m \in M_{p,j}$) if and only if* $\mathrm{im}(m) \supseteq F_j$.

Proof. We first establish the equivalence: $m \in \mathrm{im}(\Delta_i^p) \iff i \notin \mathrm{im}(m)$. Recall $\Delta_i^p = \prod_{a=1}^k \delta_i^{M_a(p)}$. The image of δ_i^q is $[q] \setminus \{i\}$.

$$\begin{aligned} m \in \mathrm{im}(\Delta_i^p) &\iff \forall a : m_a \in \mathrm{im}(\delta_i^{M_a(p)}) \\ &\iff \forall a : m_a \neq i \quad (\text{since } i \in [p] \subseteq [M_a(p)]) \\ &\iff i \notin \mathrm{im}(m). \end{aligned}$$

Taking the contrapositive, $m \notin \mathrm{im}(\Delta_i^p) \iff i \in \mathrm{im}(m)$.

By definition, $m \in M_{p,j}$ if and only if $\forall i \in F_j : m \notin \mathrm{im}(\Delta_i^p)$. Therefore,

$$m \in M_{p,j} \iff \forall i \in F_j : i \in \mathrm{im}(m) \iff F_j \subseteq \mathrm{im}(m).$$

□

Corollary 3.9. *The horn kernel is non-trivial ($R_{p,j} \neq \{0\}$) if and only if $k \geq p$.*

Proof. By Theorem 3.7, $R_{p,j} \neq \{0\}$ iff $M_{p,j} \neq \emptyset$.

($\Rightarrow$) If $m \in M_{p,j}$, then $k \geq |\mathrm{im}(m)| \geq |F_j| = p$.

($\Leftarrow$) Suppose $k \geq p$. Let $F_j = \{f_1, \dots, f_p\}$. Define $m = (f_1, \dots, f_p, 0, \dots, 0) \in \mathbb{N}^k$. Since $M_a(p) \geq p$, we have $[p]^k \subseteq I_p$, so $m \in I_p$. Since $\mathrm{im}(m) \supseteq F_j$, $m \in M_{p,j}$. □

3.5. The ℓ-free Subspace

The combinatorial characterization of $\mathrm{im}(\Delta_\ell^p)$ yields a structural property of the face maps.

Definition 3.10 (ℓ-free subspace)**.** Define the ℓ-free subspace

$$X_p^{(\ell\text{-free})} := \mathrm{span}_A\{\, E_m :\ \ell \notin \mathrm{im}(m)\,\} \ \subseteq X_p.$$

Corollary 3.11 (Isomorphism on the ℓ-free subspace)**.** *The face map d_ℓ^p induces a linear isomorphism*

$$d_\ell^p :\ X_p^{(\ell\text{-free})} \xrightarrow{\ \cong\ } X_{p-1}.$$

Proof. By the proof of Proposition 3.8, $m \in \mathrm{im}(\Delta_\ell^p) \iff \ell \notin \mathrm{im}(m)$. Thus $X_p^{(\ell\text{-free})}$ is spanned by $\{E_m \mid m \in \mathrm{im}(\Delta_\ell^p)\}$. By the Injectivity Lemma 3.4, d_ℓ^p is injective on this subspace. Since $\Delta_\ell^p : I_{p-1} \to \mathrm{im}(\Delta_\ell^p)$ is a bijection, Lemma 3.3 shows that d_ℓ^p maps the basis of $X_p^{(\ell\text{-free})}$ bijectively onto the basis of X_{p-1}. $\square$

3.6. Genericity

Let $T' = \mu_j(\Phi_j(T))$ be the Moore filler of T (see Section 4). The difference $T - T'$ lies in $R_{p,j}$. By Theorem 3.7, $\mathrm{supp}(T - T') \subseteq M_{p,j}$.

Definition 3.12 (Generic Tensor)**.** Assume A is an integral domain. A tensor $T \in X_p$ is called (p,j)-**generic** if the difference $T - T'$ has maximal support: $\mathrm{supp}(T - T') = M_{p,j}$. (This requires $k \geq p$).

Proposition 3.13 (Generic Locus)**.** *Assume A is an integral domain and $k \geq p$. The set of (p,j)-generic tensors forms a Zariski-open subset $\mathcal{U}_{p,j} \subset X_p$. If moreover A is infinite, then $\mathcal{U}_{p,j}$ is non-empty.*

Proof. Let $R(T) = T - T'$. The map $T \mapsto T'$ is A-linear (as μ_j and Φ_j are linear), hence $R(T)$ is A-linear. Write $R(T) = \sum_{m \in M_{p,j}} c_m(T) E_m$. Each coefficient $c_m(T)$ is a linear form in the entries of T.

We show that $c_m(T)$ is not identically zero. Consider $T = E_m$ for $m \in M_{p,j}$. Since $E_m \in R_{p,j}$, its horn is $\Phi_j(E_m) = 0$. The Moore filler of the zero horn is $T' = \mu_j(0) = 0$. Then $R(E_m) = E_m$, so $c_m(E_m) = 1$.

The generic locus is defined by the non-vanishing of these forms:

$$\mathcal{U}_{p,j} \ := \ \{\, T \in X_p :\ c_m(T) \neq 0 \text{ for all } m \in M_{p,j}\,\}.$$

The complement is the union of the zero loci of the non-zero forms c_m, hence a finite union of proper linear subspaces. Therefore $\mathcal{U}_{p,j}$ is Zariski-open. If A is infinite, a finite union of proper linear subspaces cannot equal X_p, so $\mathcal{U}_{p,j}$ is non-empty. $\square$

4. Normalization and the Moore Filler

We review the normalization of simplicial modules. Let $X_\bullet$ be a simplicial A-module.

4.1. The Normalization Theorem

Definition 4.1 (Normalized and degenerate submodules)**.** The **normalized submodule** in degree p is $N_p(X) := \bigcap_{i=1}^{p} \ker d_i \subseteq X_p$. The **degenerate submodule** in degree p is $D_p(X) := \sum_{r=0}^{p-1} \mathrm{im}(s_r) \subseteq X_p$.

By the normalization theorem, the homology of $X_\bullet$ is isomorphic to the homology of its normalized (Moore) complex $(N_\bullet(X), d_0)$.

Theorem 4.2 (Normalization theorem / Dold-Kan Correspondence)**.** *For every $p \geq 0$ there is a functorial direct sum decomposition*

$$X_p \cong N_p(X) \oplus D_p(X).$$

There is a functorial projection $\pi_p : X_p \to N_p(X)$ with $\ker(\pi_p) = D_p(X)$. Moreover, X_p decomposes as a direct sum of images of $N_q(X)$ under iterated degeneracies (Eilenberg–Zilber decomposition):

$$X_p \;=\; \bigoplus_{q=0}^{p} \; \bigoplus_{0 \leq i_1 < \cdots < i_{p-q} \leq p-1} s_{i_{p-q}} \cdots s_{i_1} \, N_q(X).$$

Refs.: *Dold [2]; Kan [3]; May [4]; Weibel [5, Thm 8.3.8, Cor 8.4.2]; Goerss–Jardine [6, III.2].*

Remark 4.3 (Eilenberg–Mac Lane idempotents)*.* The decomposition is established by the **Eilenberg–Mac Lane idempotents**. The projection onto $N_p(X)$ corresponding to the convention $(N_\bullet, d_0)$ is given explicitly by $\pi_p = (\mathrm{id} - s_{p-1} d_p) \cdots (\mathrm{id} - s_0 d_1)$. These operators rely only on the simplicial identities and are thus functorial for any simplicial module. *Refs.:* Mac Lane [10, Ch. VIII, §6]; Weibel [5, Thm 8.3.8].

4.2. The Moore Horn Filler Algorithm

Definition 4.4 (Horn space)**.** Fix $p \geq 1$ and $j \in \{0, \dots, p\}$. Let $F_j = [p] \setminus \{j\}$. The space of compatible (p, j)-horns, $\mathrm{Horns}(p, j)$, is the submodule of $\prod_{i \in F_j} X_{p-1}$ defined by the compatibility conditions $d_i x_\ell = d_{\ell-1} x_i$ for $i < \ell$ in F_j.

The horn map is $\Phi_j : X_p \to \prod_{i \in F_j} X_{p-1}$, defined by $\Phi_j(T) = (d_i T)_{i \in F_j}$. Its image lies in $\mathrm{Horns}(p, j)$, and its kernel is the horn kernel $R_{p,j}(X) = \bigcap_{i \in F_j} \ker d_i$.

Proposition 4.5 (Exactness of the Horn Sequence)**.** *The sequence*

$$0 \longrightarrow R_{p,j}(X) \longrightarrow X_p \xrightarrow{\Phi_j} \mathrm{Horns}(p, j) \longrightarrow 0$$

is exact. The surjectivity of Φ_j (the Kan condition in the abelian setting) is established by an explicit iterative construction (the Moore filler algorithm) which yields a filler in the degenerate submodule $D_p(X)$.

Proof. Exactness at $R_{p,j}(X)$ and X_p is by definition. The surjectivity of Φ_j and the construction of a degenerate filler is standard; see e.g., Weibel [5, Lemma 8.2.6] or Duskin [7, Lemma 3.1]. ∎

We detail the algorithm used in the proof of Proposition 4.5.

Definition 4.6 (Moore Filler Map)**.** The **Moore filler map** $\mu_j : \mathrm{Horns}(p, j) \to D_p(X)$ is defined iteratively for a horn $H = (x_i)_{i \in F_j}$.

Phase I (Ascending indices $i < j$). Initialize $T^{(-1)} := 0$. For $i = 0, \dots, j-1$:

$$T^{(i)} := T^{(i-1)} + s_i\big(x_i - d_i T^{(i-1)}\big).$$

Phase II (Descending indices $i > j$). Initialize $U^{(p+1)} := T^{(j-1)}$. For $i = p, \dots, j+1$:

$$U^{(i)} := U^{(i+1)} + s_{i-1}\big(x_i - d_i U^{(i+1)}\big).$$

Output. $\mu_j(H) := U^{(j+1)}$.

By construction, $\mu_j(H) \in D_p(X)$. The map μ_j provides an explicit right inverse (section) to Φ_j.

Corollary 4.7. *The Moore filler map μ_j induces a direct sum decomposition*

$$X_p = R_{p,j}(X) \oplus \mathrm{Im}(\mu_j).$$

Furthermore, $\mathrm{Im}(\mu_j) \subseteq D_p(X)$.

Proof. Since μ_j is a section of Φ_j ($\Phi_j \circ \mu_j = \mathrm{id}$), the exact sequence in Proposition 4.5 splits, yielding the decomposition. The inclusion follows from Definition 4.6. □

5. Combinatorics and Classification

We apply the combinatorial characterization of the horn kernel (Section 3) to classify the DSTM $X_\bullet(\vec{s}; A)$ and derive exact rank formulas.

5.1. Classification Theorem

We establish the main classification result based on the tensor order k and the simplicial dimension $n = \min(\vec{s}) - 1$.

Definition 5.1 (Strict Algebraic n-Hypergroupoid)**.** A simplicial module $X_\bullet$ is a **strict algebraic n-hypergroupoid** if fillers are unique in dimensions strictly greater than n, and not unique in dimension n. That is:

1. $R_{p,j}(X) = 0$ for all $p > n$ and all j.
2. $R_{n,j}(X) \neq 0$ for at least one j.

Remark 5.2 (Relation to Duskin–Glenn). Condition *(1)* in Definition 5.1 is exactly the n-hypergroupoid condition of Duskin–Glenn (as in the nLab definition): all horns in dimensions $p > n$ admit unique fillers. Condition *(2)* adds a non-uniqueness requirement in dimension n: there exists j with $R_{n,j}(X) \neq 0$. Thus a strict algebraic n-hypergroupoid in our sense is an n-hypergroupoid in which horn fillers are non-unique in dimension n and unique in all dimensions $p > n$.

Theorem 5.3 (Hypergroupoid Classification)**.** *$X_\bullet(\vec{s};A)$ is a strict algebraic n-hypergroupoid if and only if $k = n$.*

Proof. We use the criterion established in Corollary 3.9: $R_{p,j} \neq 0 \iff k \geq p$.

Condition (1) requires $R_{p,j} = 0$ for all $p > n$. This is equivalent to $k < p$ for all $p \geq n+1$. This holds if and only if $k < n+1$, i.e., $k \leq n$.

Condition (2) requires $R_{n,j} \neq 0$. This is equivalent to $k \geq n$.

$X_\bullet$ is a strict algebraic n-hypergroupoid if and only if both conditions hold: $(k \leq n) \wedge (k \geq n)$, which is equivalent to $k = n$. □

5.2. Rank Formulas via Inclusion-Exclusion

We analyze the ranks of the components of the DSTM. Since $X_p(\vec{s};A)$ is a free A-module and the normalization projections are defined over $\mathbb{Z}$ (Remark 4.3), the ranks of the submodules $R_{p,j}, N_p, D_p$ are independent of the ring A.

Let $N_a = n_a - 1$. Recall $n = \min_a N_a$.

Proposition 5.4 (Rank of $R_{p,j}$)**.** *Let $F_j = [p] \setminus \{j\}$. The rank of $R_{p,j}(X)$ is*

$$\operatorname{rank} R_{p,j} = \sum_{t=0}^{p} (-1)^t \binom{p}{t} \prod_{a=1}^{k} (M_a(p) + 1 - t).$$

Proof. By Theorem 3.7 and Proposition 3.8, the rank is the count of $m \in I_p$ such that $\operatorname{im}(m) \supseteq F_j$. We use inclusion–exclusion on $I_p = \prod [M_a(p)]$.

For a subset $S \subseteq F_j$ with $|S| = t$, we count the indices $m \in I_p$ such that $\operatorname{im}(m) \cap S = \emptyset$. Since $M_a(p) \geq p$, $S \subseteq [M_a(p)]$. The number of choices for the a-th component is $|[M_a(p)] \setminus S| = M_a(p) + 1 - t$. The total count of such indices is $\prod_{a=1}^{k} (M_a(p) + 1 - t)$.

Since $|F_j| = p$, the inclusion–exclusion principle yields the formula for the count of indices covering F_j. □

We analyze the normalized complex $(N_\bullet, d_0)$. The cycles are $Z_p(N_\bullet) = \ker(d_0 : N_p \to N_{p-1}) = \bigcap_{i=0}^{p} \ker d_i$.

Corollary 5.5 (Rank of $Z_p(N_\bullet)$)**.** *The rank of the normalized cycles $Z_p(N_\bullet)$ is*

$$\operatorname{rank} Z_p(N_\bullet) = \sum_{t=0}^{p+1} (-1)^t \binom{p+1}{t} \prod_{a=1}^{k} (M_a(p) + 1 - t).$$

Proof. $Z_p(N_\bullet)$ is spanned by basis elements E_m such that $\operatorname{im}(m) \supseteq [p]$. The formula follows by applying inclusion–exclusion with the covered set $[p]$ (size $p+1$). □

5.3. Constant Shape and Stirling Numbers

If the shape is constant, then $n_a = n + 1$ for all a, where n is the simplicial dimension. In this case $N_a = n$ and hence $M_a(p) = p$ for all a.

The formulas simplify using the Stirling numbers of the second kind $\left\{ {k \atop m} \right\}$ (see, e.g., Stanley [9]).

Theorem 5.6 (Ranks for Constant Shape)**.** *If $M_a(p) = p$ for all a (i.e., $I_p = [p]^k$), the ranks of the components of the Moore complex in degree p are:*

$$\operatorname{Rank} Z_p(N_\bullet) = (p+1)! \left\{ {k \atop p+1} \right\},$$
$$\operatorname{Rank} N_p(X) = p! \left\{ {k \atop p} \right\} + (p+1)! \left\{ {k \atop p+1} \right\}.$$

Proof. We interpret these ranks in terms of finite differences of the polynomial $P(x) = x^k$.

The rank of $Z_p(N_\bullet)$ specializes to

$$\sum_{t=0}^{p+1} (-1)^t \binom{p+1}{t} (p+1-t)^k = \Delta^{p+1}(x^k)\big|_{x=0}.$$

Using the expansion $x^k = \sum_m \left\{ {k \atop m} \right\} x^{\underline{m}}$. The operator acts by $\Delta^{p+1}(x^{\underline{m}}) = m^{\underline{p+1}} x^{\underline{m-p-1}}$. Evaluating at $x = 0$, only the term $m = p+1$ contributes (since $0^{\underline{0}} = 1$ and $0^{\underline{j}} = 0$ for $j > 0$), yielding $(p+1)! \left\{ {k \atop p+1} \right\}$.

The rank of $N_p(X) = R_{p,0}$ specializes to

$$\sum_{t=0}^{p} (-1)^t \binom{p}{t} (p+1-t)^k = \Delta^{p}(x^k)\big|_{x=1}.$$

Evaluating $\Delta^p(x^{\underline{m}}) = m^{\underline{p}} x^{\underline{m-p}}$ at $x = 1$. The term $1^{\underline{j}}$ is 1 if $j = 0$ or $j = 1$, and 0 if $j \geq 2$. If $m = p$, the contribution is $p! \left\{ {k \atop p} \right\}$. If $m = p+1$, the contribution is $(p+1)! \left\{ {k \atop p+1} \right\}$. ☐

5.4. Boundaries and Contractibility

We analyze the boundaries $B_p(N_\bullet) = \operatorname{im}(d_0 : N_{p+1} \to N_p)$.

Proposition 5.7 (Non-triviality of Boundaries)**.** *$B_p(N_\bullet) \neq 0$ if and only if $k \geq p+1$.*

Proof. If $k < p+1$, then $N_{p+1}(X) = 0$ by Corollary 3.9 (since $N_{p+1} = R_{p+1,0}$), so $B_p(N_\bullet) = 0$.

If $k \geq p+1$, we construct a non-zero boundary. We seek $T \in N_{p+1}(X)$ such that $d_0(T) \neq 0$. We require an index $m \in I_{p+1}$ such that $E_m \in N_{p+1}(X)$ and $d_0(E_m) \neq 0$. $E_m \in N_{p+1}(X)$ requires $\operatorname{im}(m) \supseteq \{1, \dots, p+1\}$. $d_0(E_m) \neq 0$ requires $m \in \operatorname{im}(\Delta_0^{p+1})$, which is equivalent to $0 \notin \operatorname{im}(m)$. Thus we seek m such that $\operatorname{im}(m) = \{1, \dots, p+1\}$. Since $k \geq p+1$, such an m exists. Since $M_a(p+1) \geq p+1$, $m \in I_{p+1}$. Therefore $B_p(N_\bullet) \neq 0$. ☐

The DSTM $X_\bullet$ is contractible (Theorem Appendix A.4). Consequently, $H_*(X_\bullet) = 0$, implying $Z_p(N_\bullet) = B_p(N_\bullet)$ for all p. The differential $d_0 : N_{p+1} \to N_p$ maps surjectively onto $Z_p(N_\bullet)$ with kernel $Z_{p+1}(N_\bullet)$.

Proposition 5.8 (Rank Consistency Check)**.** *The rank formulas derived satisfy the consistency condition required by contractibility (Rank-Nullity):*

$$\operatorname{Rank} N_{p+1}(X) = \operatorname{Rank} Z_{p+1}(N_\bullet) + \operatorname{Rank} Z_p(N_\bullet).$$

Proof. We verify the identity using the formulas from Theorem 5.6 in the constant shape case.

$$\operatorname{Rank} Z_{p+1} + \operatorname{Rank} Z_p = (p+2)! \left\{ {k \atop p+2} \right\} + (p+1)! \left\{ {k \atop p+1} \right\}.$$

This matches the formula for $\operatorname{Rank} N_{p+1}(X)$ derived in Theorem 5.6 (by replacing p with $p+1$). The consistency for the general shape case follows because the complex is defined and contractible over $\mathbb{Z}$. □

6. The Horn Non-Degeneracy Lemma and Decomposition

We establish that the horn kernel and the degenerate submodule intersect trivially. This ensures the Horn decomposition is compatible with the standard decomposition of simplicial modules.

Lemma 6.1 (Horn Non-Degeneracy)**.** *For any simplicial module $X_\bullet$, any degree $p \geq 1$, and any $j \in [p]$,*

$$R_{p,j}(X) \cap D_p(X) = \{0\}.$$

Proof. We filter the degenerate submodule $D_p(X)$ by the maximum index of the degeneracy operators. For $0 \leq r \leq p-1$, set

$$H_r := \sum_{i=0}^{r} \operatorname{im}(s_i) \subseteq X_p, \quad H'_r := \sum_{i=0}^{r} \operatorname{im}(s_i) \subseteq X_{p-1},$$

and set $H_{-1} = H'_{-1} = \{0\}$. Note that $D_p(X) = H_{p-1}$.

Let $x \in R_{p,j}(X) \cap D_p(X)$ and assume $x \neq 0$. Choose $m \geq 0$ minimal such that $x \in H_m$. Then we can write

$$x = s_m(z) + y, \qquad z \in X_{p-1}, \quad y \in H_{m-1}.$$

If we can show $z \in H'_{m-1}$, then $z = \sum_{i=0}^{m-1} s_i(u_i)$. Using the identity $s_m s_i = s_i s_{m-1}$ for $i < m$, we obtain

$$s_m(z) = \sum_{i=0}^{m-1} s_m s_i(u_i) = \sum_{i=0}^{m-1} s_i s_{m-1}(u_i) \in H_{m-1}.$$

This implies $x \in H_{m-1}$, contradicting the minimality of m. Thus, it suffices to prove $z \in H'_{m-1}$.

We split the proof into cases based on the index j.

Case 1: $j \neq m+1$. Since $m+1 \leq p$ and $m+1 \neq j$, we have $d_{m+1}(x) = 0$.

$$0 = d_{m+1}(x) = d_{m+1}s_m(z) + d_{m+1}(y) = z + d_{m+1}(y),$$

where we used $d_{m+1}s_m = \mathrm{id}$. Since $y \in H_{m-1}$, we write $y = \sum_{i=0}^{m-1} s_i(w_i)$. For $i < m$, the identity $d_{m+1}s_i = s_i d_m$ holds, so

$$d_{m+1}(y) = \sum_{i=0}^{m-1} s_i d_m(w_i) \in H'_{m-1}.$$

Hence $z = -d_{m+1}(y) \in H'_{m-1}$, as required.

Case 2: $j = m+1$. In this case, the face maps $d_0, \ldots, d_m$ all annihilate x (since $j = m+1$). Since $x \in H_m$, we may write x as a sum $x = \sum_{i=0}^{m} s_i(w_i)$ for some coefficients $w_i \in X_{p-1}$. (Note that z from the setup corresponds to w_m).

We prove by induction on k (for $0 \leq k \leq m$) that $w_k \in H'_{m-1}$.

Base step ($k = 0$): Consider $d_0(x) = 0$.

$$0 = \sum_{i=0}^{m} d_0 s_i(w_i) = d_0 s_0(w_0) + \sum_{i=1}^{m} d_0 s_i(w_i).$$

Using $d_0 s_0 = \mathrm{id}$ and $d_0 s_i = s_{i-1} d_0$ for $i \geq 1$:

$$0 = w_0 + \sum_{i=1}^{m} s_{i-1} d_0(w_i).$$

The sum consists of terms in $\mathrm{im}(s)$, so $w_0 \in H'_{m-1}$.

Inductive step: Assume $w_0, \ldots, w_{k-1} \in H'_{m-1}$ for some $1 \leq k \leq m$. Consider $d_k(x) = 0$.

$$0 = \sum_{i=0}^{m} d_k s_i(w_i).$$

We split the sum into four parts based on the index i:

1. $i < k-1$: $d_k s_i(w_i) = s_i d_{k-1}(w_i) \in H'_{m-1}$.
2. $i = k-1$: $d_k s_{k-1}(w_{k-1}) = w_{k-1}$ (using $d_k s_{k-1} = \mathrm{id}$). By the inductive hypothesis, $w_{k-1} \in H'_{m-1}$.
3. $i = k$: $d_k s_k(w_k) = w_k$ (using $d_k s_k = \mathrm{id}$).
4. $i > k$: $d_k s_i(w_i) = s_{i-1} d_k(w_i) \in H'_{m-1}$.

Substituting these into the equation yields

$$0 = (\text{terms in } H'_{m-1}) + w_{k-1} + w_k + (\text{terms in } H'_{m-1}).$$

Solving for w_k, we see that w_k is a sum of elements in H'_{m-1}. Thus $w_k \in H'_{m-1}$.

By induction, $w_m \in H'_{m-1}$. Since $z = w_m$, we have $z \in H'_{m-1}$.

In all cases, we conclude $z \in H'_{m-1}$, which contradicts the minimality of m unless $x = 0$. $\square$

Theorem 6.2 (Horn decomposition)**.** *For any simplicial module $X_\bullet$, any $p \geq 1$, and any $j \in [p]$, the Moore filler map μ_j (Definition 4.6) is a section of the horn map Φ_j, and*

$$X_p = R_{p,j}(X) \oplus \mathrm{im}(\mu_j), \qquad \mathrm{im}(\mu_j) \subseteq D_p(X).$$

In particular, the horn kernel $R_{p,j}(X)$ is a direct summand of X_p, complementary to a degenerate summand.

Proof. By construction (Definition 4.6) and Proposition 4.5, μ_j is a section of Φ_j, i.e. $\Phi_j \circ \mu_j = \mathrm{id}_{\mathrm{Horns}(p,j)}$. Thus the short exact sequence in Proposition 4.5 splits, and we obtain

$$X_p = R_{p,j}(X) \oplus \mathrm{im}(\mu_j).$$

Each step in the definition of μ_j adds a degeneracy, so $\mathrm{im}(\mu_j) \subseteq D_p(X)$. Lemma 6.1 ensures that the intersection of these summands is trivial ($R_{p,j}(X) \cap \mathrm{im}(\mu_j) = \{0\}$), confirming that the decomposition is compatible with the standard degenerate filtration. □

Remark 6.3 (Normalization Conventions). Theorem 6.2 generalizes the classical Normalization Theorem (Theorem 4.2). Standard conventions for the Moore complex (or normalized chain complex) vary:

1. $N_p^{(0)}(X) = \bigcap_{i=0}^{p-1} \ker d_i = R_{p,p}(X)$.
2. $N_p^{(1)}(X) = \bigcap_{i=1}^{p} \ker d_i = R_{p,0}(X)$.

In this paper we adopt the second convention $N_p(X) = N_p^{(1)}(X) = R_{p,0}(X)$. Our horn decomposition encompasses both as the cases $j = p$ and $j = 0$ respectively.

7. Homology Dichotomy and Classification

7.1. The Horn Complex

Fix $j \in [n]$. Define the linear maps

$$\Phi_j : X_n \longrightarrow \bigoplus_{i \neq j} X_{n-1}, \qquad \Phi_j(T) = (d_i T)_{i \neq j},$$

$$\Psi : \bigoplus_{i \neq j} X_{n-1} \longrightarrow \bigoplus_{\substack{i<m \\ i,m \neq j}} X_{n-2}, \qquad \Psi((x_i)) = (d_i x_m - d_{m-1} x_i)_{i<m}.$$

Proposition 7.1. *The sequence*

$$C_j^{\mathrm{horn}} : \qquad 0 \longrightarrow X_n \xrightarrow{\partial_2 = \Phi_j} \bigoplus_{i \neq j} X_{n-1} \xrightarrow{\partial_1 = \Psi} \bigoplus_{i<m,\, i,m \neq j} X_{n-2} \longrightarrow 0$$

is a chain complex (indexed homologically with X_n in degree 2). Moreover,

$$H_2\big(C_j^{\mathrm{horn}}\big) \;\cong\; R_{n,j}, \qquad H_1\big(C_j^{\mathrm{horn}}\big) = 0.$$

Proof. $\partial_1 \circ \partial_2 = 0$ by the simplicial identities $d_i d_m = d_{m-1} d_i$ for $i < m$. Since the complex starts in degree 2, $H_2 = \ker \partial_2 = R_{n,j}$. The kernel of ∂_1 is the space of compatible horns $\mathrm{Horns}(n,j)$; by the existence of fillers (Proposition 4.5), $\mathrm{im}\, \partial_2 = \mathrm{Horns}(n,j)$. Thus $H_1 = \ker \partial_1 / \mathrm{im}\, \partial_2 = 0$. □

Theorem 7.2 (Short exact sequence in the horn kernel)**.** *The face map $d_j : R_{n,j} \to X_{n-1}$ induces a natural short exact sequence*

$$0 \longrightarrow Z_n \longrightarrow R_{n,j} \xrightarrow{d_j} d_j(R_{n,j}) \longrightarrow 0.$$

Proof. The kernel of $d_j|_{R_{n,j}}$ is $R_{n,j} \cap \ker d_j = \bigcap_{i \neq j} \ker d_i \cap \ker d_j = Z_n$. □

7.2. The Classification Dichotomy

We use the ranks established in Section 5.

Corollary 7.3 (Filler dichotomy)**.** *The ranks of $R_{n,j}$ and Z_n (Proposition 5.4, Corollary 5.5) determine the following classification based on the tensor order k:*

1. *If $k < n$, then $R_{n,j} = 0$ (fillers are unique in dimension n).*
2. *If $k = n$, then $R_{n,j} \neq 0$ and $Z_n = 0$. Hence $R_{n,j} \cong d_j(R_{n,j})$.*
3. *If $k \geq n+1$, then $Z_n \neq 0$ and Z_n injects canonically into $R_{n,j}$.*

Corollary 7.4 (Constant shape ranks)**.** *For the constant shape $\vec{s} = (n+1, \ldots, n+1)$, the ranks are given by Theorem 5.6:*

$$\operatorname{rank} Z_n = (n+1)! \left\{ {k \atop n+1} \right\},$$

$$\operatorname{rank} R_{n,j} = n! \left\{ {k \atop n} \right\} + (n+1)! \left\{ {k \atop n+1} \right\}.$$

In particular:

- *$Z_n \neq 0$ if and only if $k \geq n+1$.*
- *$R_{n,j} \neq 0$ if and only if $k \geq n$.*

Remark 7.5 (Moore filler vs. T). For the horn $\Lambda^n_j(T)$, Moore's filler T' satisfies $T' = T$ if and only if $T \in D_n$: the "only if" direction holds because T' is always degenerate (Definition 4.6), and the "if" direction follows from Lemma 6.1, since in that case $T - T'$ lies in $R_{n,j} \cap D_n = \{0\}$. If $k < n$, then $R_{n,j} = 0$, so $T' = T$ always holds (and $D_n = X_n$). If $k \geq n$ and $T \notin D_n$, then $T - T' \in R_{n,j} \setminus \{0\}$ and is supported on the missing indices (Theorem 3.7).

7.3. Interpretation as Algebraic n-Hypergroupoids

We adapt the definition of Duskin (1979) and Glenn (1982).

Definition 7.6 (Algebraic n-hypergroupoid)**.** A simplicial module $X_\bullet$ is an **algebraic n-hypergroupoid** if horn fillers are unique in dimensions $p > n$. It is **strict** if it is not an $(n-1)$-hypergroupoid (i.e., fillers in dimension n are not unique).

Proposition 7.7. *A simplicial module $X_\bullet$ is an algebraic n-hypergroupoid if and only if $R_{p,j}(X) = 0$ for all $p > n$ and all j.*

Proof. Simplicial modules are Kan complexes (fillers always exist, e.g., Proposition 4.5). As observed in Section 3, the set of fillers of a given horn is an affine torsor modeled on $R_{p,j}$, hence a singleton if and only if $R_{p,j} = 0$. □

Applying the criterion of Corollary 3.9 to the DSTM $X_\bullet(\vec{s}; A)$ with simplicial dimension n yields the hypergroupoid classification of Theorem 5.3: $X_\bullet(\vec{s}; A)$ is a strict algebraic n-hypergroupoid if and only if the tensor order k equals n.

7.4. The Subcomplex of Normalized Cycles

We analyze the subcomplex formed by the normalized cycles $Z_r(N_\bullet)$. In the DSTM, $Z_r(N_\bullet)$ is spanned by basis elements E_m such that $\mathrm{im}(m) \supseteq [r]$.

Lemma 7.8. *The family $(Z_r(N_\bullet))_{r\geq 0}$ forms a chain subcomplex of $(X_\bullet, \partial)$. Furthermore, the differential on this subcomplex is zero:*

$$\partial_r\big(Z_r(N_\bullet)\big) = 0, \qquad H_r\big(Z_\bullet(N_\bullet)\big) \cong Z_r(N_\bullet).$$

There is a short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet(N_\bullet) \longrightarrow X_\bullet \longrightarrow X_\bullet/Z_\bullet(N_\bullet) \longrightarrow 0. \tag{1}$$

Proof. By definition, $Z_r(N_\bullet) = \bigcap_{i=0}^{r} \ker d_i$. If $x \in Z_r(N_\bullet)$, then $d_i(x) = 0$ for all i. Therefore the boundary $\partial_r(x) = \sum(-1)^i d_i(x) = 0$. This confirms that $Z_\bullet(N_\bullet)$ is a chain subcomplex with zero differential. The homology of a complex with zero differential is the complex itself. □

Corollary 7.9 (Quotient homology)**.** *For the DSTM $X_\bullet(\vec{s}; A)$, which is contractible by Theorem Appendix A.4, the short exact sequence of chain complexes* (1) *induces isomorphisms*

$$H_r\big(X_\bullet/Z_\bullet(N_\bullet)\big) \;\cong\; H_{r-1}\big(Z_\bullet(N_\bullet)\big) \cong Z_{r-1}(N_\bullet)$$

for all $r \geq 1$.

Proof. By Theorem Appendix A.4, the DSTM $X_\bullet(\vec{s}; A)$ is contractible, hence $H_r(X_\bullet) = 0$ for all $r \geq 0$. Taking homology of the short exact sequence (1) gives a long exact sequence

$$\cdots \to H_r(Z_\bullet(N_\bullet)) \to H_r(X_\bullet) \to H_r(X_\bullet/Z_\bullet(N_\bullet)) \to H_{r-1}(Z_\bullet(N_\bullet)) \to \cdots .$$

The differential on $Z_\bullet(N_\bullet)$ is zero, so $H_r(Z_\bullet(N_\bullet)) \cong Z_r(N_\bullet)$ for all r. Using $H_r(X_\bullet) = 0$ and exactness, we obtain

$$H_r\big(X_\bullet/Z_\bullet(N_\bullet)\big) \;\cong\; H_{r-1}\big(Z_\bullet(N_\bullet)\big) \cong Z_{r-1}(N_\bullet).$$

□

Remark 7.10 (Homology spheres). For a normalized cycle $C \in Z_n(N_\bullet)$, the generated simplicial subobject $\langle C\rangle \subseteq X_\bullet$ (the degeneracy-closure) provides an algebraic model of the n-sphere. If C is primitive (e.g., a basis element E_m over a PID A), $\langle C\rangle$ is isomorphic to the simplicial sphere $A[\Delta^n]/\operatorname{Sk}_{n-1}(A[\Delta^n])$. It satisfies $H_n(\langle C\rangle) \cong A$ and $H_m(\langle C\rangle) = 0$ for $m \neq n$. (See Weibel [5, Exercise 8.3.4].)

8. The Geometry of Generated Subobjects and Moduli Spaces

We analyze the isomorphism classes of the simplicial submodules $\langle T\rangle \subseteq X_\bullet(\vec{s}; K)$ generated by a tensor $T \in X_n$. We assume the ground ring K is an infinite field.

8.1. Realization Maps and Kernel Sequences

Let $C_\bullet := K[\Delta^n]$ be the standard simplicial K-module representing the n-simplex. In degree p, $C_p = K[\Delta^n]_p$ is the free K-vector space with basis the set of morphisms $\Delta([p],[n])$. The dimension is $S_p := \dim(C_p) = \binom{n+p+1}{p+1}$. Let $\iota_n \in C_n$ be the generator corresponding to the identity map.

Definition 8.1 (Realization Map and Kernel Sequence)**.** For $T \in X_n(\vec{s}; K)$, the **realization map** $f_T : C_\bullet \to X_\bullet(\vec{s}; K)$ is the unique morphism of simplicial modules sending ι_n to T. The **kernel sequence** of T is the simplicial submodule

$$K(T)_\bullet := \ker(f_T) \subset C_\bullet,$$

defined degreewise by $K(T)_p := \ker(f_{T,p}) \subseteq C_p$.

The image is $\langle T\rangle_\bullet = \mathrm{im}(f_T)$, and the realization map induces an isomorphism of simplicial K-modules

$$C_\bullet/K(T)_\bullet \xrightarrow{\cong} \langle T\rangle_\bullet.$$

The isomorphism class of $\langle T\rangle_\bullet$ is therefore determined by its kernel sequence $K(T)_\bullet$.

Remark 8.2 (Finite Generation). Since $K[\Delta^n]$ is generated in degree n, any simplicial submodule is determined by its components up to degree n.

Proposition 8.3 (Isomorphism classes and kernel sequences)**.** *For each $T \in X_n(\vec{s}; K)$, the realization map $f_T : C_\bullet \to X_\bullet(\vec{s}; K)$ induces an isomorphism*

$$C_\bullet/K(T)_\bullet \xrightarrow{\cong} \langle T\rangle_\bullet.$$

In particular, all invariants of the generated submodule $\langle T\rangle_\bullet$ can be computed from its kernel sequence $K(T)_\bullet \subseteq C_\bullet$.

The simplicial automorphism group $\mathrm{Aut}_{\mathrm{simp}}(C_\bullet)$ acts on kernel sequences by

$$\varphi \cdot K_\bullet := \varphi^{-1}\big(K_\bullet\big), \qquad \varphi \in \mathrm{Aut}_{\mathrm{simp}}(C_\bullet),$$

and if $K(T_2)_\bullet = \varphi \cdot K(T_1)_\bullet$ for some φ, then $\langle T_1\rangle_\bullet \cong \langle T_2\rangle_\bullet$.

Proof. Since $K(T)_\bullet = \ker(f_T)$, the universal property of quotients gives a simplicial map

$$C_\bullet/K(T)_\bullet \longrightarrow \langle T\rangle_\bullet$$

which is an isomorphism in each degree, because $K(T)_p = \ker(f_{T,p})$ and $\langle T\rangle_p = \mathrm{im}(f_{T,p})$. This proves the first claim.

For the second claim, suppose $K(T_2)_\bullet = \varphi^{-1}(K(T_1)_\bullet)$ for some $\varphi \in \mathrm{Aut}_{\mathrm{simp}}(C_\bullet)$. Then φ induces an isomorphism of quotients

$$C_\bullet/K(T_2)_\bullet \xrightarrow{\cong} C_\bullet/K(T_1)_\bullet,$$

and composing with the degreewise isomorphisms $C_\bullet/K(T_i)_\bullet \xrightarrow{\cong} \langle T_i\rangle_\bullet$ for $i = 1, 2$ yields an isomorphism $\langle T_2\rangle_\bullet \cong \langle T_1\rangle_\bullet$. □

Remark 8.4. We do not need any converse to Proposition 8.3. For our purposes it suffices that each generated submodule $\langle T\rangle_\bullet$ arises as a quotient $C_\bullet/K(T)_\bullet$, and that all invariants we study (homology, Laplacians, spectra, etc.) can be expressed in terms of the kernel sequence $K(T)_\bullet \subseteq C_\bullet$. We make no assertion that isomorphic quotients have isomorphic kernels, nor that every isomorphism between quotients lifts to an automorphism of $C_\bullet$.

8.2. Index Collisions and the Realization Matrix

Let $R_p := \dim(X_p)$.

Definition 8.5 (Index collision map)**.** For each $p \geq 0$, the **index collision map** is

$$\mathcal{I}_p : I_p \times \Delta([p],[n]) \longrightarrow I_n, \qquad \mathcal{I}_p(m,\alpha) := I_\bullet(\alpha)(m).$$

Definition 8.6 (Realization matrix)**.** The **realization matrix** $M_{T,p} \in \mathrm{Mat}_{R_p \times S_p}(K)$ represents $f_{T,p}$. Its entry in row $m \in I_p$ and column $\alpha \in \Delta([p],[n])$ is

$$M_{T,p}[m,\alpha] = (f_{T,p}(\alpha))(m) = T_{\mathcal{I}_p(m,\alpha)}.$$

A **collision** in degree p is a pair $(m,\alpha) \neq (m',\alpha')$ with $\mathcal{I}_p(m,\alpha) = \mathcal{I}_p(m',\alpha')$. Collisions force the corresponding entries of $M_{T,p}$ to agree for every T, and are the source of rank loss in the symbolic matrices $M_{T_{\mathrm{sym}},p}$.

8.3. The Moduli Map and Grassmannians

Let $\mathcal{V} = \{v_u\}_{u \in I_n}$ be indeterminates and $\mathbb{K} := K(\mathcal{V})$ the function field. The symbolic tensor T_{sym} has entries v_u.

Definition 8.7 (Generic Rank and Kernel Dimension)**.** The **generic rank** $R'_p(\vec{s})$ is the rank of the symbolic matrix $M_{T_{\mathrm{sym}},p}$ over $\mathbb{K}$. The **generic kernel dimension** is $K'_p(\vec{s}) := S_p - R'_p(\vec{s})$.

Definition 8.8 (Moduli Map)**.** The **generic locus** $\mathcal{U} \subset X_n(\vec{s};K)$ is the non-empty Zariski open set where $\mathrm{rank}(M_{T,p}) = R'_p(\vec{s})$ for all p.

The **Grassmannian associated to the shape** $\vec{s}$ is

$$\mathrm{Gr}(\vec{s}) := \prod_{p:\ K'_p(\vec{s})>0} \mathrm{Gr}\big(K'_p(\vec{s}), K[\Delta^n]_p\big).$$

By Remark 8.2, this product involves at most $n+1$ factors ($0 \leq p \leq n$).

The **Moduli Map** Ψ is the algebraic map:

$$\Psi : \mathcal{U} \longrightarrow \mathrm{Gr}(\vec{s}), \quad T \mapsto (\ker f_{T,p})_p.$$

The **Moduli Space of Kernel Sequences** $\mathcal{M}(\vec{s})$ is the image $\Psi(\mathcal{U})$.

8.4. Injectivity Analysis and Examples

Lemma 8.9. *The functor $I_\bullet : \Delta \to$ **Set** preserves monomorphisms.*

Proof. Monomorphisms in Δ are generated by coface maps δ_i. $I_\bullet(\delta_i^p) = \Delta_i^p$ is a product of injections, hence injective. □

Theorem 8.10 (Injectivity and Dominance at $p=0$)**.** *In the DSTM:*

1. *The map $\mathcal{I}_0$ is injective. $R'_0 = \min(R_0, S_0)$.*
2. *If $K'_0 > 0$, the projection $\Psi_0 : \mathcal{U} \to \mathrm{Gr}(K'_0, S_0)$ is dominant.*

Proof. (1) $S_0 = n+1$. Let a_0 be an axis where $n_{a_0} = n+1$. Then $M_{a_0}(0) = 0$. For $m \in I_0$, $m_{a_0} = 0$. Let $\alpha_i : [0] \to [n]$ be the vertex map $\alpha_i(0) = i$. The a_0-th coordinate of $\mathcal{I}_0(m, \alpha_i)$ is $\alpha_i(m_{a_0}) = i$. If $\mathcal{I}_0(m, \alpha_i) = \mathcal{I}_0(m', \alpha_{i'})$, then $i = i'$. By Lemma 8.9, $I_\bullet(\alpha_i)$ is injective, so $m = m'$.

(2) Since $\mathcal{I}_0$ is injective, the entries of $M_{T_{\mathrm{sym}},0}$ are distinct variables. The map $L_0 : X_n \to \mathrm{Mat}_{R_0 \times S_0}(K)$ sending T to $M_{T,0}$ is a projection onto these coordinates, hence surjective. The map from the space of matrices of maximal rank R_0' to the Grassmannian of their kernels is dominant. Ψ_0 is the composition of L_0 (restricted to $\mathcal{U}$) and this dominant map. $\square$

Lemma 8.11 (Non-injectivity for $p \geq 1$)**.** *For $p \geq 1$, $\mathcal{I}_p$ is not injective.*

Proof. Let $\alpha_0 : [p] \to [n]$ be the constant map to 0. It factors as $\alpha_0 = \iota_0 \circ \pi_0$. Since $p \geq 1$, $|I_p| > |I_0|$. The map $I_\bullet(\pi_0) : I_p \to I_0$ cannot be injective. There exist $m \neq m'$ such that $I_\bullet(\pi_0)(m) = I_\bullet(\pi_0)(m')$. Applying $I_\bullet(\iota_0)$ yields $\mathcal{I}_p(m, \alpha_0) = \mathcal{I}_p(m', \alpha_0)$. $\square$

Example 8.12 (Rank Drop: Shape (2,2)). $n = 1, k = 2$. $S = (2, 3)$. $R = (1, 4)$. $p = 1$. $S_1 = 3, R_1 = 4$. The morphisms $\Delta([1], [1])$ are $\{\mathrm{id}, (0,0), (1,1)\}$. $I_1 = [1]^2$. The symbolic realization matrix $M_{T_{\mathrm{sym}},1}$ is:

$$M = \begin{pmatrix} v_{00} & v_{00} & v_{11} \\ v_{01} & v_{00} & v_{11} \\ v_{10} & v_{00} & v_{11} \\ v_{11} & v_{00} & v_{11} \end{pmatrix}.$$

All four 3×3 minors vanish identically over $\mathbb{K}$. The generic rank is $R_1' = 2$. $K_1' = 1$.

8.5. The Incidence Variety and Segre–Plücker Embedding

The image $\mathcal{M}(\vec{s})$ lies within the closed incidence subvariety $\mathbf{Gr}^{\mathrm{simp}}(\vec{s}) \subset \mathrm{Gr}(\vec{s})$ defined by the simplicial compatibility conditions.

Theorem 8.13 (Global Structure of the Moduli Space)**.** *$\mathcal{M}(\vec{s})$ is a constructible set contained in $\mathbf{Gr}^{\mathrm{simp}}(\vec{s})$. The projection $\mathcal{M}(\vec{s}) \to \mathrm{Gr}(K_0'(\vec{s}), S_0)$ is dominant (if $K_0'(\vec{s}) > 0$). The moduli space $\mathcal{M}(\vec{s})$ is irreducible and unirational.*

Proof. $\mathcal{U}$ is an open subset of an affine space, hence irreducible. $\mathcal{M}(\vec{s})$ is the image of an irreducible variety under a rational map, hence it is constructible (Chevalley's theorem), irreducible, and unirational. Dominance at $p = 0$ follows from Theorem 8.10. $\square$

Proposition 8.14 (Segre–Plücker linearity)**.** *Embed $\mathrm{Gr}(\vec{s})$ into a projective space via the composition of the Plücker embeddings $\iota_p : \mathrm{Gr}(K_p', S_p) \hookrightarrow \mathbb{P}(\Lambda^{K_p'} K[\Delta^n]_p)$ and the Segre embedding Σ. The image $\Sigma(\mathbf{Gr}^{\mathrm{simp}}(\vec{s}))$ is the intersection of $\Sigma(\mathrm{Gr}(\vec{s}))$ with a linear subspace of the ambient projective space.*

Proof. We analyze the condition $d_i(K_p) \subseteq K_{p-1}$. Let $C_q = K[\Delta^n]_q$. Let ω_p and ω_{p-1} be the Plücker coordinates representing K_p and K_{p-1}. The condition is equivalent to $\Lambda^{K_p'} d_i(\omega_p) \wedge \eta_{p-1} = 0$, where η_{p-1} is a volume element of a complement to K_{p-1}. Under the Segre embedding, these bilinear equations in the Plücker coordinates become linear equations in the homogeneous coordinates of the tensor product space. $\square$

8.6. *Homology of Generated Subobjects*

Fix $T \in X_n(\vec{s}; K)$ and write $C_\bullet := K[\Delta^n]$ and $K_\bullet := K(T)_\bullet = \ker(f_T) \subseteq C_\bullet$. We analyze $H_\bullet(\langle T \rangle_\bullet)$ using the short exact sequence

$$0 \longrightarrow K_\bullet \longrightarrow C_\bullet \longrightarrow \langle T \rangle_\bullet \longrightarrow 0.$$

Recall that $H_p(C_\bullet) = 0$ for $p > 0$ and $H_0(C_\bullet) \cong K$.

Proposition 8.15 (Homology of the generated subobject)**.** *Let $B_p(C_\bullet)$ and $B_p(K_\bullet)$ denote the boundary subspaces in $C_\bullet$ and $K_\bullet$, respectively. Then:*

1. *For $p \geq 2$, there is a natural isomorphism*
$$H_p(\langle T \rangle_\bullet) \cong H_{p-1}(K_\bullet).$$
2. $H_1(\langle T \rangle_\bullet) \cong \big(K_0 \cap B_0(C_\bullet)\big)/B_0(K_\bullet)$.
3. $H_0(\langle T \rangle_\bullet) \cong C_0/\big(K_0 + B_0(C_\bullet)\big)$.

Proof. Applying homology to

$$0 \to K_\bullet \to C_\bullet \to \langle T \rangle_\bullet \to 0$$

gives a long exact sequence. For $p \geq 2$ we have $H_p(C_\bullet) = H_{p-1}(C_\bullet) = 0$, so the connecting homomorphism induces an isomorphism $H_p(\langle T \rangle_\bullet) \cong H_{p-1}(K_\bullet)$, which proves (1).

For $p = 1$ and $p = 0$, the long exact sequence reads

$$0 \to H_1(\langle T \rangle_\bullet) \to H_0(K_\bullet) \to H_0(C_\bullet) \to H_0(\langle T \rangle_\bullet) \to 0,$$

and the standard identification of H_0 as cycles modulo boundaries in degree 0 yields the descriptions in (2) and (3). □

Corollary 8.16 (Generic connectivity)**.** *If $K_0'(\vec{s}) > 0$, then for a generic tensor $T \in \mathcal{U}$, $H_0(\langle T \rangle_\bullet) = 0$.*

Proof. $B_0(C_\bullet)$ is a hyperplane in C_0. By Proposition 8.15, we have $H_0(\langle T \rangle_\bullet) \neq 0$ if and only if $K_0(T) \subseteq B_0(C_\bullet)$. This condition cuts out a proper closed subvariety of $\mathrm{Gr}(K_0'(\vec{s}), S_0)$. Since Ψ_0 is dominant (Theorem 8.10), the generic kernel $K_0(T)$ avoids this locus. □

8.7. *Example: Shape* $(3, 3)$

Let $\vec{s} = (3, 3)$, so $k = 2$ and $n = 2$. Then X_2 consists of 3×3 matrices. We write $T = (v_{ij})_{0 \leq i,j \leq 2}$. The dimensions are

$$S = (3, 6, 10), \qquad R = (1, 4, 9).$$

We fix the following bases:

- For X_2, the basis $I_2 = [2]^2$ in lexicographic order:
$$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).$$
- For $K[\Delta^2]_1$, the basis of monotone maps $[1] \to [2]$ ordered as
$$(0,0),\ (0,1),\ (0,2),\ (1,1),\ (1,2),\ (2,2).$$
- For $K[\Delta^2]_2$, the basis of monotone maps $[2] \to [2]$ ordered as the ten non-decreasing triples (a_0, a_1, a_2) with $0 \leq a_0 \leq a_1 \leq a_2 \leq 2$:
$$(0,0,0), (0,0,1), (0,0,2), (0,1,1), (0,1,2), (0,2,2), (1,1,1), (1,1,2), (1,2,2), (2,2,2).$$

Degree $p = 0$.. Here $R_0 = 1$ and $S_0 = 3$. The realization matrix is

$$M_{T,0} = \begin{pmatrix} v_{00} & v_{11} & v_{22} \end{pmatrix}.$$

For generic T, its rank is 1, so $K_0' = 2$. The kernel is the 2-dimensional subspace of K^3 of solutions to

$$v_{00}x_0 + v_{11}x_1 + v_{22}x_2 = 0.$$

On the open chart $v_{00} \neq 0$, it is spanned by

$$(-v_{11}/v_{00},\ 1,\ 0), \qquad (-v_{22}/v_{00},\ 0,\ 1).$$

Thus $K_0(T)$ depends only on the projective class $[v_{00} : v_{11} : v_{22}] \in \mathbb{P}^2$.

Degree $p = 1$.. Here $R_1 = 4$ and $S_1 = 6$. With the bases above, the realization matrix is

$$M_{T,1} = \begin{pmatrix} v_{00} & v_{00} & v_{00} & v_{11} & v_{11} & v_{22} \\ v_{00} & v_{01} & v_{02} & v_{11} & v_{12} & v_{22} \\ v_{00} & v_{10} & v_{20} & v_{11} & v_{21} & v_{22} \\ v_{00} & v_{11} & v_{22} & v_{11} & v_{22} & v_{22} \end{pmatrix}.$$

Two obvious kernel vectors can be written purely in terms of the diagonal entries. Set

$$c_1^{(1)} := \left(-v_{11}/v_{00},\ 0,\ 0,\ 1,\ 0,\ 0\right)^T, \qquad c_2^{(1)} := \left(-v_{22}/v_{00},\ 0,\ 0,\ 0,\ 0,\ 1\right)^T.$$

A direct calculation shows

$$M_{T,1}\, c_1^{(1)} = 0, \qquad M_{T,1}\, c_2^{(1)} = 0$$

as identities in the polynomial ring $K[v_{ij}]$ (on the open set $v_{00} \neq 0$). Hence $K_1' \geq 2$.

On the other hand, taking e.g.

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix},$$

one obtains

$$M_{T,1} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 1 & 3 \\ 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 3 \end{pmatrix},$$

which has rank 4 (one checks that some 4×4 minor is non-zero). Thus the generic rank is $R_1' = 4$, and therefore $K_1' = S_1 - R_1' = 2$.

In particular, for generic T, the kernel $K_1(T) = \ker f_{T,1}$ is spanned by $c_1^{(1)}$ and $c_2^{(1)}$, so it depends only on (v_{00}, v_{11}, v_{22}).

Degree $p = 2.$. Here $R_2 = 9$ and $S_2 = 10$. With the bases chosen above, the symbolic realization matrix $M_{T,2}$ is

$$M_{T,2} = \begin{pmatrix} v_{00} & v_{00} & v_{00} & v_{11} & v_{11} & v_{22} & v_{11} & v_{11} & v_{22} & v_{22} \\ v_{00} & v_{00} & v_{00} & v_{01} & v_{01} & v_{02} & v_{11} & v_{11} & v_{12} & v_{22} \\ v_{00} & v_{00} & v_{00} & v_{10} & v_{10} & v_{20} & v_{11} & v_{11} & v_{21} & v_{22} \\ v_{00} & v_{00} & v_{01} & v_{11} & v_{12} & v_{22} & v_{11} & v_{12} & v_{22} & v_{22} \\ v_{00} & v_{01} & v_{02} & v_{11} & v_{12} & v_{22} & v_{11} & v_{12} & v_{22} & v_{22} \\ v_{00} & v_{02} & v_{02} & v_{12} & v_{12} & v_{22} & v_{12} & v_{12} & v_{22} & v_{22} \\ v_{00} & v_{00} & v_{10} & v_{11} & v_{21} & v_{22} & v_{11} & v_{21} & v_{22} & v_{22} \\ v_{00} & v_{10} & v_{20} & v_{11} & v_{21} & v_{22} & v_{11} & v_{21} & v_{22} & v_{22} \\ v_{00} & v_{20} & v_{20} & v_{21} & v_{21} & v_{22} & v_{21} & v_{21} & v_{22} & v_{22} \end{pmatrix}.$$

Again there are two explicit kernel vectors depending only on the diagonal. Set

$$c_1^{(2)} := \bigl(-v_{11}/v_{00},\, 0,\, 0,\, 0,\, 0,\, 0,\, 1,\, 0,\, 0,\, 0\bigr)^T, \qquad c_2^{(2)} := \bigl(-v_{22}/v_{00},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 1\bigr)^T.$$

A direct substitution into the matrix above shows

$$M_{T,2}\, c_1^{(2)} = 0, \qquad M_{T,2}\, c_2^{(2)} = 0$$

as polynomial identities (again on the open chart $v_{00} \neq 0$). Thus $K_2' \geq 2$ and $R_2' \leq 10-2 = 8$.

To see that the generic rank is exactly 8, we exhibit a single tensor for which $M_{T,2}$ has rank 8. Take for example

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

Substituting these values into $M_{T,2}$ gives a 9×10 numerical matrix with rank 8. Since rank is upper semicontinuous in families, this shows that the generic rank is $R_2' = 8$, and therefore $K_2' = S_2 - R_2' = 2$.

Moreover, the two kernel generators $c_1^{(2)}, c_2^{(2)}$ depend only on v_{00}, v_{11}, v_{22}, not on the off-diagonal entries.

Collecting the three degrees, we have:

Proposition 8.17 (Generic kernel dimensions for shape $(3,3)$)**.** *For $\vec{s} = (3,3)$, the generic kernel dimensions are*

$$K'(\vec{s}) = (K_0', K_1', K_2') = (2,2,2).$$

On the open chart $v_{00} \neq 0$, the kernels $K_p(T)$ are generated by vectors whose coordinates are rational functions of v_{11}/v_{00} and v_{22}/v_{00} only.

Corollary 8.18 (Moduli space for shape $(3,3)$)**.** *For $\vec{s} = (3,3)$, the moduli space $\mathcal{M}(\vec{s})$ is a surface of dimension 2, birational to $\mathbb{P}^2$.*

Proof. The generic kernel vector in degree 0 is the hyperplane

$$K_0(T) = \{(x_0, x_1, x_2) \mid v_{00}x_0 + v_{11}x_1 + v_{22}x_2 = 0\} \subset K^3.$$

Thus the point $K_0(T) \in \mathrm{Gr}(2,3)$ determines the projective class $[v_{00} : v_{11} : v_{22}] \in \mathbb{P}^2$, and hence determines the ratios v_{11}/v_{00} and v_{22}/v_{00} on the chart $v_{00} \neq 0$. By the explicit formulas above, $K_1(T)$ and $K_2(T)$ are then uniquely determined. Therefore, on this chart, the moduli map Ψ factors through the projection

$$X_2 \dashrightarrow \mathbb{P}^2, \qquad T \longmapsto [v_{00} : v_{11} : v_{22}],$$

and the induced map $\mathbb{P}^2 \dashrightarrow \mathcal{M}(\vec{s})$ is birational onto its image. Hence $\mathcal{M}(\vec{s})$ has dimension 2 and is birational to $\mathbb{P}^2$. □

Appendix A. Contractibility of the Diagonal Simplicial Module

We prove that the diagonal simplicial A-module $X_\bullet(\vec{s}; A)$ is acyclic by constructing an explicit chain contraction.

Appendix A.1. The Chain Complex and Homotopy Operator

The DSTM $X_p(\vec{s}; A)$ is the A-module of functions $T : I_p \to A$. The index bounds are $M_a(p) = n_a - 1 - n + p$, where $n = \min_a(n_a) - 1$.

Remark Appendix A.1 (Index Bounds at $p = 0$)*.* By the definition of n, there exists at least one index a_0 such that $n_{a_0} - 1 = n$. For this index, $M_{a_0}(0) = 0$. Consequently, for any index $m \in I_0$, the coordinate m_{a_0} must be 0.

Let $(X_\bullet, \partial_\bullet)$ be the associated unaugmented chain complex. We set $X_p = 0$ for $p < 0$, and $\partial_0 = 0$.

Definition Appendix A.2 (Shift-and-Truncate Homotopy H)**.** We define $H_p : X_p \to X_{p+1}$ for $p \geq -1$. Set $H_{-1} := 0$. For $p \geq 0$, $T \in X_p$, and $m \in I_{p+1}$:

$$H_p(T)(m) := \begin{cases} 0 & \text{if } \exists a : m_a = 0, \\ T(m - \vec{1}) & \text{if } \forall a : m_a > 0. \end{cases}$$

Lemma Appendix A.3. *The operator H satisfies the following identities for $p \geq 0$:*

1. $d_0^{p+1} H_p = \mathrm{id}_{X_p}$.
2. $d_i^{p+1} H_p = H_{p-1} d_{i-1}^p$ *for $i > 0$.*

Proof. Let $T \in X_p$ and $m \in I_p$.

(1) $d_0 H(T)(m) = H(T)(\Delta_0(m)) = T(m + \vec{1} - \vec{1}) = T(m)$.

(2) Let $i > 0$.

Case $p > 0$**:** If some coordinate $m_a = 0$, then $\Delta_i(m)$ also has a zero coordinate (since $i > 0$ and $\delta_i(0) = 0$), so $H_p(T)(\Delta_i(m)) = 0$ and $H_{p-1}(d_{i-1}T)(m) = 0$ by definition of H. Thus $d_i H_p(T)(m) = H_{p-1} d_{i-1}(T)(m) = 0$.

If all coordinates $m_a > 0$, then

$$d_i^{p+1} H_p(T)(m) = H_p(T)(\Delta_i^p(m)) = T(\Delta_i^p(m) - \vec{1}),$$

while

$$H_{p-1}d^p_{i-1}(T)(m) = d^p_{i-1}(T)(m-\vec{1}) = T(\Delta^p_{i-1}(m-\vec{1})).$$

The coface maps satisfy the coordinate identity

$$\delta_i(x) - 1 = \delta_{i-1}(x-1) \qquad (x > 0),$$

hence $\Delta^p_i(m) - \vec{1} = \Delta^p_{i-1}(m-\vec{1})$ when all $m_a > 0$, and the two expressions agree.

Case $p = 0$: We verify $d^1_1 H_0 = H_{-1}d^0_0 = 0$. Let $m \in I_0$. By Remark Appendix A.1, there exists a_0 such that $m_{a_0} = 0$. $d_1 H_0(T)(m) = H_0(T)(\Delta_1(m))$. The a_0-th coordinate of $\Delta_1(m)$ is $\delta_1(m_{a_0}) = \delta_1(0)$. Since $i = 1 > 0$, $\delta_1(0) = 0$. By the definition of H, since $\Delta_1(m)$ has a zero coordinate, $H_0(T)(\Delta_1(m)) = 0$. Thus $d^1_1 H_0 = 0$. □

Theorem Appendix A.4. *The diagonal simplicial module $X_\bullet(\vec{s}; A)$ is contractible.*

Proof. We verify the chain contraction identity $\partial_{p+1}H_p + H_{p-1}\partial_p = \mathrm{id}$. Using Lemma Appendix A.3 (valid for all $p \geq 0$):

$$\begin{aligned}\partial_{p+1}H_p &= d_0 H_p + \sum_{i=1}^{p+1}(-1)^i d_i H_p \\ &= \mathrm{id} + \sum_{i=1}^{p+1}(-1)^i H_{p-1}d^p_{i-1}.\end{aligned}$$

Re-indexing the sum ($j = i-1$):

$$\sum_{i=1}^{p+1}(-1)^i H_{p-1}d^p_{i-1} = \sum_{j=0}^{p}(-1)^{j+1}H_{p-1}d^p_j = -\,H_{p-1}\partial_p.$$

(This holds even for $p = 0$, as $\partial_0 = 0$ and $H_{-1} = 0$). Substituting back, we obtain

$$\partial_{p+1}H_p + H_{p-1}\partial_p \;=\; \mathrm{id}_{X_p},$$

so H is a chain contraction of $X_\bullet(\vec{s}; A)$. □

Appendix A.2. Equivariance Properties of the Homotopy

Recall the action of the stabilizer group $\mathrm{Stab}(\vec{s})$ on the diagonal simplicial module $X_\bullet(\vec{s}; A)$ defined in Section 2. The face and degeneracy maps commute with this action (Lemma 2.4).

Lemma Appendix A.5 (Equivariance under $\mathrm{Stab}(\vec{s})$)**.** *For every $p \geq -1$ and $\sigma \in \mathrm{Stab}(\vec{s})$, the shift-and-truncate homotopy H_p is equivariant with respect to the action of* $\mathrm{Stab}(\vec{s})$*:*

$$H_p(\sigma \cdot T) = \sigma \cdot H_p(T).$$

Proof. Let $T \in X_p$, $\sigma \in \mathrm{Stab}(\vec{s})$, and $m \in I_{p+1}$. We compare $H_p(\sigma \cdot T)(m)$ and $(\sigma \cdot H_p(T))(m)$. By definition of the action, $(\sigma \cdot H_p(T))(m) = H_p(T)(\sigma^{-1}m)$.

The definition of H_p depends on the presence of zero coordinates. Since permutation only changes the positions of the coordinates, the condition "$\exists a : m_a = 0$" is equivalent to "$\exists b : (\sigma^{-1}m)_b = 0$".

Case 1: $\exists a : m_a = 0$. In this case, $H_p(\sigma \cdot T)(m) = 0$. Since $\sigma^{-1}m$ also contains a zero coordinate, $H_p(T)(\sigma^{-1}m) = 0$.

Case 2: $\forall a : m_a > 0$. In this case, $\sigma^{-1}m$ is also strictly positive.

$$\begin{aligned} H_p(\sigma \cdot T)(m) &= (\sigma \cdot T)(m - \vec{1}) \\ &= T(\sigma^{-1}(m - \vec{1})). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\sigma \cdot H_p(T))(m) &= H_p(T)(\sigma^{-1}m) \\ &= T((\sigma^{-1}m) - \vec{1}). \end{aligned}$$

The equality holds because the shift operation commutes with permutation: $\sigma^{-1}(m - \vec{1}) = (\sigma^{-1}m) - \vec{1}$. Therefore, H_p is equivariant under the action of $\mathrm{Stab}(\vec{s})$. □

Appendix A.3. Shifted Depth Filtration and Spectral Sequence

We introduce a filtration on the complex $(X_\bullet, \partial)$ that is compatible with the differential and the homotopy operator H, leading to the collapse of the associated spectral sequence.

Definition Appendix A.6 (Shifted Depth Filtration)**.** For $p \geq 0$ and $t \in \mathbb{Z}$, define a decreasing filtration $\mathcal{F}^\bullet X_p$. Write $\min(m) := \min_a m_a$ for $m \in I_p$. Set

$$\mathcal{F}^t X_p \ := \ \{\, T \in X_p \mid T(m) = 0 \text{ unless } \min(m) \geq t + p \,\}.$$

This filtration is bounded: $\mathcal{F}^{-p} X_p = X_p$ (since $\min(m) \geq 0$). Furthermore, since $\min(m) \leq \min_a M_a(p)$ for any $m \in I_p$, we have $\mathcal{F}^t X_p = 0$ if $t + p > \min_a M_a(p)$.

Lemma Appendix A.7 (Behavior of ∂ and H w.r.t. $\mathcal{F}^\bullet$)**.** *The differential ∂ and the homotopy H preserve the shifted depth filtration (filtration degree 0). For all $p \geq 0$ and $t \in \mathbb{Z}$:*

1. $d_i^p(\mathcal{F}^t X_p) \subseteq \mathcal{F}^t X_{p-1}$ *for all* $i \in \{0, \dots, p\}$.
2. $H_p(\mathcal{F}^t X_p) \subseteq \mathcal{F}^t X_{p+1}$.

Proof. (1) Let $T \in \mathcal{F}^t X_p$. We want to show $d_i^p(T) \in \mathcal{F}^t X_{p-1}$. Let $m \in I_{p-1}$. We must show that if $\min(m) < t + p - 1$, then $d_i^p(T)(m) = 0$. $d_i^p(T)(m) = T(\Delta_i^p(m))$. The coface maps satisfy $\delta_i(x) \leq x + 1$ for all i, x. Thus $\min(\Delta_i^p(m)) \leq \min(m) + 1$. If $\min(m) < t + p - 1$, then $\min(\Delta_i^p(m)) < t + p$. Since $T \in \mathcal{F}^t X_p$, $T(\Delta_i^p(m)) = 0$. Thus $d_i^p(T) \in \mathcal{F}^t X_{p-1}$.

(2) Let $T \in \mathcal{F}^t X_p$. We want to show $H_p(T) \in \mathcal{F}^t X_{p+1}$. Let $m \in I_{p+1}$. We must show that if $\min(m) < t + p + 1$, then $H_p(T)(m) = 0$.

If $\min(m) = 0$, $H_p(T)(m) = 0$ by definition. If $\min(m) > 0$, $H_p(T)(m) = T(m - \vec{1})$. If $\min(m) < t + p + 1$, then $\min(m - \vec{1}) = \min(m) - 1 < t + p$. Since $T \in \mathcal{F}^t X_p$, $T(m - \vec{1}) = 0$. Thus $H_p(T) \in \mathcal{F}^t X_{p+1}$. □

Proposition Appendix A.8 (Spectral sequence collapses at E_1)**.** *Let $\mathcal{F}^\bullet$ be the shifted depth filtration. The spectral sequence of the filtered complex $(X_\bullet, \partial, \mathcal{F}^\bullet)$ satisfies*

$$E_1^{t,q} = 0 \qquad \text{for all } t, q.$$

Hence it collapses at the E_1—page.

Proof. By Lemma Appendix A.7, $(X_\bullet, \partial, \mathcal{F}^\bullet)$ is a filtered complex, as ∂ has filtration degree 0. Let $E_0 = \mathrm{gr}_{\mathcal{F}}(X_\bullet)$ be the associated graded complex, with differential d^0 induced by ∂.

By Lemma Appendix A.7, the homotopy operator H also has filtration degree 0. It induces a map $[H]$ on E_0. The chain contraction identity $\partial H + H\partial = \mathrm{id}$ holds in the filtered complex. Since all operators involved preserve the filtration, the identity descends to the associated graded complex:

$$d^0[H] + [H]d^0 = [\mathrm{id}] = \mathrm{id}_{E_0}.$$

This shows that the complex (E_0, d^0) is contractible via the homotopy $[H]$. Therefore, its homology is trivial. The E_1 term is defined as the homology of (E_0, d^0). We conclude $E_1 = H(E_0) = 0$. □

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