timeseries-to-egt (ts2eg): Mathematical Background and Construction

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1 Overview

This note collects the mathematics underlying the *timeseries-to-egt* pipeline. The procedure

1. constructs per-time *payoff vectors* $v(t) \in \mathbb{R}^N$ from an $N \times T$ multivariate time series X,
2. decomposes $v(t)$ into *common-interest* and *zero-sum* parts via orthogonal projections,
3. maps player space to a k-strategy space using a data-driven dictionary $S \in \mathbb{R}^{N\times k}$ and mixtures $x(t) \in \Delta_k$,
4. estimates a strategy-level operator $A \in \mathbb{R}^{k\times k}$ compatible with replicator dynamics, and
5. computes (and tests stability of) mixed equilibria, using IAAFT surrogates to assess significance.

Finding an ESS is interpreted as evidence for *strategic interaction* in the data.

2 Notation

- $X = [X_{\cdot,1} \ \cdots \ X_{\cdot,T}] \in \mathbb{R}^{N\times T}$: N player signals over T times.
- $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^N$.
- $\Delta_k = \{x \in \mathbb{R}^k_{\geq 0} : \ \mathbf{1}^\top x = 1\}$: simplex.
- M_I, M_Z: common-interest and zero-sum projectors in player space.
- $M_Z^{(k)}$: zero-sum projector in strategy space.
- $S \in \mathbb{R}^{N\times k}$: strategy archetypes (columns).
- $x(t) \in \Delta_k$: mixture at time t; $X_k = [x(1) \cdots x(T)] \in \mathbb{R}^{k\times T}$.
- $g(t) \in \mathbb{R}^k$: strategy-level signal at time t; $G = [g(1) \cdots g(T)]$.

3 Common-interest and Zero-sum Subspaces

3.1 Unweighted projectors

Let $\bar{v} = \frac{1}{N}\mathbf{1}^\top v$ denote the arithmetic mean. Define

$$M_I \;=\; \frac{1}{N}\,\mathbf{1}\mathbf{1}^\top, \qquad M_Z \;=\; I - M_I. \tag{1}$$

For any $v \in \mathbb{R}^N$, $v_I = M_I v = \bar{v}\,\mathbf{1}$ is the *common-interest* component and $v_Z = M_Z v = v - \bar{v}\,\mathbf{1}$ is the *zero-sum* (mean-centered) component. Both M_I and M_Z are symmetric idempotent matrices and satisfy $M_I M_Z = 0$, with $\mathrm{im}(M_I) = \mathrm{span}\{\mathbf{1}\}$ and $\mathrm{im}(M_Z) = \{\mathbf{1}\}^\perp$.

3.2 Weighted projectors

Let $w \in \mathbb{R}^N_{>0}$ and $D_w = \mathrm{diag}(w)$; define the weighted inner product $\langle u, v\rangle_w = u^\top D_w v$. The orthogonal projector onto $\mathrm{span}\{\mathbf{1}\}$ *with respect to* $\langle \cdot, \cdot\rangle_w$ is

$$M_I^{(w)} \;=\; \mathbf{1}\,\pi_w^\top, \qquad \pi_w^\top \;=\; \frac{\mathbf{1}^\top D_w}{\mathbf{1}^\top D_w \mathbf{1}} \;=\; \frac{w^\top}{\sum_i w_i}, \tag{2}$$

so that $M_I^{(w)} v = (\pi_w^\top v)\,\mathbf{1}$ equals the vector with each entry the w-weighted mean of v. The weighted zero-sum projector is $M_Z^{(w)} = I - M_I^{(w)}$. One checks $D_w M_I^{(w)} = (M_I^{(w)})^\top D_w$ and likewise for $M_Z^{(w)}$, i.e., self-adjointness under $\langle \cdot, \cdot\rangle_w$.

3.3 Explicit Helmert (contrast) basis

A particularly convenient orthonormal change of variables is the *Helmert* matrix $Q \in \mathbb{R}^{N\times N}$:

$$\text{First column: } q_1 \;=\; \frac{1}{\sqrt{N}}\,\mathbf{1}. \tag{3}$$

$$\text{For } k = 2, \dots, N: \;\; q_k \text{ has entries } \big(q_k\big)_i \;=\; \begin{cases} \frac{1}{\sqrt{k(k-1)}} & \text{if } i < k, \\ -\frac{k-1}{\sqrt{k(k-1)}} & \text{if } i = k, \\ 0 & \text{if } i > k. \end{cases} \tag{4}$$

Then $Q^\top Q = I$, $Q^\top \mathbf{1} = \sqrt{N}\,e_1$, and

$$M_I \;=\; q_1 q_1^\top \;=\; \frac{1}{N}\mathbf{1}\mathbf{1}^\top, \qquad M_Z \;=\; I - q_1 q_1^\top \;=\; \sum_{k=2}^{N} q_k q_k^\top. \tag{5}$$

A weighted analogue is obtained by Gram–Schmidt using the D_w inner product, with first vector $q_1^{(w)} = \mathbf{1}/\|\mathbf{1}\|_w$ where $\|\mathbf{1}\|_w^2 = \mathbf{1}^\top D_w \mathbf{1}$.

Worked example ($N = 3$). Using (3)–(4),

$$Q_3 \;=\; \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \qquad M_I \;=\; \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad M_Z \;=\; I - M_I. \tag{6}$$

For $v = (p_1, p_2, p_3)^\top$,

$$\tilde{v} \;=\; Q_3^\top v = \begin{bmatrix} \frac{p_1+p_2+p_3}{\sqrt{3}} \\ \frac{p_1-p_2}{\sqrt{2}} \\ \frac{p_1+p_2-2p_3}{\sqrt{6}} \end{bmatrix}, \qquad v_I \;=\; \frac{p_1+p_2+p_3}{3}\,\mathbf{1}, \qquad v_Z \;=\; v - v_I. \tag{7}$$

4 From Time Series to Payoffs

4.1 Static (profile) game

Let $Z = M_Z X$ (or $M_Z^{(w)} X$). Discretize each row of Z into b bins to form a joint *profile* $a(t) \in \{1, \ldots, b\}^N$. For each profile a, define a payoff vector

$$v(a) \;=\; \mathbb{E}\big[X_{\cdot,t+1} \mid a(t) = a\big], \tag{8}$$

estimated empirically. At time t set $v(t) = v\big(a(t)\big)$ and take $v_Z(t) = M_Z v(t)$ (or weighted).

4.2 Information-sharing (VAR) game

For each player i, consider two linear predictors of $X_{i,t}$ from past lags up to order p:

$$\text{Self model:} \quad \hat{X}_{i,t} = \sum_{\ell=1}^{p} \alpha_{i,\ell} X_{i,t-\ell} + u_{i,t}, \tag{9}$$

$$\text{Full model:} \quad \hat{X}_{i,t} = \sum_{j=1}^{N} \sum_{\ell=1}^{p} \beta_{i,j,\ell} X_{j,t-\ell} \;+\; \sum_{m \in \mathcal{S}} \gamma_{i,m} s_m(t) \;+\; e_{i,t}, \tag{10}$$

where $s_m(t)$ are optional seasonal regressors (e.g., $\sin / \cos$ at multiples of a base period). Let $\sigma_{i,\text{self}}^2 = \mathbb{E}[u_{i,t}^2]$ and $\sigma_{i,\text{full}}^2 = \mathbb{E}[e_{i,t}^2]$. Define the *information-sharing payoff* for player i as the reduction in mean squared error

$$v_i \;:=\; \sigma_{i,\text{self}}^2 - \sigma_{i,\text{full}}^2 \quad \text{(optionally normalized)}. \tag{11}$$

Finally set $v_Z = M_Z v$ (or $M_Z^{(w)} v$).

5 Strategies and Mixtures

Let $S = [s_1 \cdots s_k] \in \mathbb{R}^{N \times k}$ be a strategy dictionary (e.g. via nonnegative matrix factorization). For each time t, infer $x(t) \in \Delta_k$ by projecting the observed signal onto the cone spanned by S with a simplex constraint, e.g.

$$x(t) \;\in\; \operatorname*{arg\,min}_{x \in \Delta_k} \; \|X_{\cdot,t} - Sx\|_2^2 \quad \text{(optionally with nonnegativity on } S\text{)}. \tag{12}$$

Define the strategy-level signal $g(t) = S^\top v_Z(t)$ and collect $X_k = [x(1) \cdots x(T)] \in \mathbb{R}^{k \times T}$, $G = [g(1) \cdots g(T)] \in \mathbb{R}^{k \times T}$.

6 Replicator-compatible Operator

Let $M_Z^{(k)} = I_k - \frac{1}{k}\mathbf{1}\mathbf{1}^\top$ be the strategy-space centering matrix. We fit $A \in \mathbb{R}^{k\times k}$ via ridge regression:

$$\min_A \left\|M_Z^{(k)} G - A X_k\right\|_F^2 + \lambda \|A\|_F^2 \qquad \Longrightarrow \qquad A \;=\; \left(M_Z^{(k)} G\right) X_k^\top \left(X_k X_k^\top + \lambda I\right)^{-1}. \tag{13}$$

Row-centering $A\mathbf{1} = 0$ (and optionally $\mathbf{1}^\top A = 0$) enforces invariance of replicator dynamics under constant payoff shifts.

7 Replicator Dynamics, Jacobian, Nash, and ESS

The continuous-time replicator dynamics on Δ_k are

$$\dot{x}_i \;=\; x_i\Big((Ax)_i - \bar{f}(x)\Big), \qquad \bar{f}(x) = x^\top A x, \quad i = 1, \ldots, k. \tag{14}$$

The Jacobian of the vector field $F_i(x) = x_i\big((Ax)_i - x^\top A x\big)$ at an interior point x is

$$\frac{\partial F_i}{\partial x_j}(x) \;=\; \delta_{ij}\left((Ax)_i - x^\top A x\right) \;+\; x_i\Big(A_{ij} - (A^\top x)_j - (Ax)_j\Big). \tag{15}$$

A mixed rest point $x^\star$ with support $J = \{i : x_i^\star > 0\}$ satisfies

$$A_{JJ} x_J^\star = \alpha\,\mathbf{1}, \quad \mathbf{1}^\top x_J^\star = 1, \qquad (Ax^\star)_\ell \le \alpha \;\; \forall \ell \notin J, \tag{16}$$

i.e., $x^\star$ is a (symmetric) mixed Nash equilibrium. For (local) evolutionary stability one may use the following sufficient test.

Proposition 1 (Tangent-space test)**.** *If $x^\star$ is a mixed Nash equilibrium with support J and the quadratic form $y^\top \frac{A_{JJ} + A_{JJ}^\top}{2}\, y$ is negative definite on $\{y \in \mathbb{R}^{|J|} : \;\mathbf{1}^\top y = 0\}$, then $x^\star$ is a (locally) evolutionary stable strategy for the replicator dynamics.*

8 Seasonal VAR Details

Let $S(t)$ collect seasonal regressors such as $\{\sin(2\pi m t/P), \cos(2\pi m t/P)\}_{m\in\mathcal{M}}$. For player i the full regression is

$$X_{i,t} \;=\; \sum_{j=1}^{N}\sum_{\ell=1}^{p} \beta_{i,j,\ell} X_{j,t-\ell} \;+\; \gamma_i^\top S(t) \;+\; e_{i,t}, \tag{17}$$

with ridge penalty on β, γ. The information-sharing payoff v_i is computed from the MSE reduction relative to the self-only regression. Center v across players with M_Z (or $M_Z^{(w)}$). See [5] for VAR practice.

9 IAAFT Surrogates and Significance

To assess whether an observed ESS arises from cross-player coupling rather than marginal autocorrelation, we generate IAAFT surrogates for each player series:

- Each surrogate preserves the *empirical amplitude distribution* and the *power spectrum* of the original player series but destroys cross-player dependence.

- For each surrogate dataset, recompute payoffs, re-estimate A via (13), and tally whether an ESS is found. The surrogate ESS rate provides a null benchmark.

IAAFT alternates phase randomization in the Fourier domain (matching the original magnitudes) with rank-order remapping to the original amplitudes until convergence [6].

10 Edge Cases and Identifiability

- If $X_{\cdot,t} = c_t \mathbf{1}$ (pure common mode), then $v_Z \equiv 0$ and the EGT signal is null by design.
- Replicator dynamics are invariant under adding the same constant to all payoffs; centering A enforces this.
- Choice of S matters; NMF/archetypes produce parts-based, interpretable strategies. PCA is acceptable but mixes signs; rectification may be used.

11 Relation to Jessie & Saari's Coordinate Systems

Jessie and Saari advocate analyzing *classes* of games by decomposing each game into coordinates that separate individual incentives (*"Nash"*), group/behavioral components, and kernel (payoff inflation/deflation), yielding a uniform methodology across formats [7]. In their words, a coordinate system "explicitly separates each payoff into contributions being made to each feature," replacing ad hoc analysis with a systematic decomposition.

How ts2eg differs. Our construction is **time-series first**. Rather than starting from a static payoff table and decomposing it, we:

1. derive per-time payoff vectors from multivariate signals (static or predictive-gain definitions);
2. isolate common-interest vs. zero-sum directions *in player space* via ANOVA/Helmert at each time;
3. map payoffs to a learned strategy dictionary S to obtain strategy-level signals; and
4. estimate a replicator-compatible operator A by ridge, enabling ESS analysis with surrogate significance (IAAFT).

Thus, while philosophically aligned with "coordinate systems for games," ts2eg couples signal processing (VAR with seasonal regressors; IAAFT) and evolutionary dynamics through an explicit change of variables in player space.

12 Algorithms (summary)

1. Compute M_I/M_Z (or weighted) and optionally a (weighted) Helmert basis using (3)–(4).
2. Build per-time payoffs $v(t)$ via static profiles or information-sharing; take $v_Z(t) = M_Z v(t)$.
3. Learn strategies S and mixtures $x(t) \in \Delta_k$.
4. Form $g(t) = S^\top v_Z(t)$ and fit A by ridge: $A = (M_Z^{(k)} G) X_k^\top (X_k X_k^\top + \lambda I)^{-1}$; enforce $A\mathbf{1} = 0$.
5. Enumerate supports to find mixed Nash equilibria; apply the tangent-space test for ESS.
6. Generate IAAFT surrogates; re-estimate A; report ESS frequency.

13 Symbols (quick reference)

$X \in \mathbb{R}^{N \times T}$	player signals (rows = players, cols = time)
M_I, M_Z	common-interest and zero-sum projectors
$M_I^{(w)}, M_Z^{(w)}$	weighted projectors (inner product $\langle \cdot, \cdot \rangle_w$)
Q	(weighted) Helmert matrix (orthonormal change of basis)
$S \in \mathbb{R}^{N \times k}$	strategy archetypes (columns)
$x(t) \in \Delta_k$	mixture on the simplex at time t
$g(t) = S^\top v_Z(t)$	strategy-level signal
$A \in \mathbb{R}^{k \times k}$	replicator-compatible operator

References

[1] D. Monderer and L. S. Shapley (1996). Potential Games. *Games and Economic Behavior* 14(1):124–143.

[2] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo (2011). Flows and Decompositions of Games: Harmonic and Potential Games. *Mathematics of Operations Research* 36(3):474–503.

[3] J. Hofbauer and K. Sigmund (1998). *Evolutionary Games and Population Dynamics*. Cambridge University Press.

[4] W. H. Sandholm (2010). *Population Games and Evolutionary Dynamics*. MIT Press.

[5] H. Lütkepohl (2005). *New Introduction to Multiple Time Series Analysis*. Springer.

[6] T. Schreiber and A. Schmitz (2000). Surrogate time series. *Physica D* 142(3–4):346–382.

[7] D. T. Jessie and D. G. Saari (2019). *Coordinate Systems for Games: Simplifying the "me" and "we" Interactions*. Springer, in *Static & Dynamic Game Theory: Foundations & Applications*.