The Clever Machine

Topics in Computational Neuroscience & Machine Learning

Blog Archives

Rejection Sampling

SEP 10
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Suppose that we want to sample from a distribution $f(x)$ that is difficult or impossible to sample from directly, but instead have a simpler distribution $q(x)$ from which sampling is easy. The idea behind Rejection sampling (aka Acceptance-rejection sampling) is to sample from $q(x)$ and apply some rejection/acceptance criterion such that the samples that are accepted are distributed according to $f(x)$.

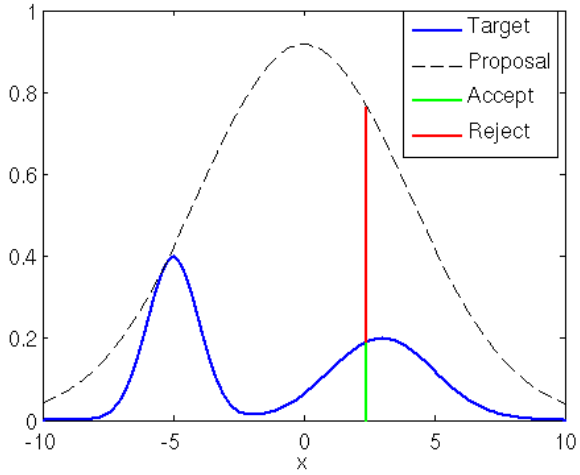
Envelope distribution and rejection criterion

In order to be able to reject samples from $q(x)$ such that they are sampled from $f(x)$, $q(x)$ must "cover" or envelop the distribution $f(x)$. This is generally done by choosing a constant $c > 1$ such that $cq(x) > f(x)$ for all x. For this reason $cq(x)$ is often called the *envelope distribution*. A common criterion for accepting samples from $x \sim q(x)$ is based on the ratio of the target distribution to that of the envelope distribution. The samples are accepted if

$$\frac{f(x)}{cq(x)} > u$$

where $u \sim Unif(0,1)$, and rejected otherwise. If the ratio is close to one, then $f(x)$ must have a large amount of probability mass around x and that sample should be more likely accepted. If the ratio is small, then it means that $f(x)$ has low probability mass around x and we should be less likely to accept the sample. This criterion is demonstrated in the chunk of MATLAB code and the resulting figure below:


```
rand('seed',12345);
x = -10:.1:10;
% CREATE A "COMPLEX DISTRIBUTION" f(x) AS A MIXTURE OF TWO NORMAL
% DISTRIBUTIONS
f = inline('normpdf(x,3,2) + normpdf(x,-5,1)','x');
t = plot(x,f(x),'b','linewidth',2); hold on;

% PROPOSAL IS A CENTERED NORMAL DISTRIBUTION
q = inline('normpdf(x,0,4)','x');

% DETERMINE SCALING CONSTANT
c = max(f(x)./q(x))

%PLOT SCALED PROPOSAL/ENVELOP DISTRIBUTION
p = plot(x,c*q(x),'k--');

% DRAW A SAMPLE FROM q(x);
qx = normrnd(0,4);
fx = f(qx);

% PLOT THE RATIO OF f(q(x)) to cq(x)
a = plot([qx,qx],[0 fx],'g','Linewidth',2);
r = plot([qx,qx],[fx,c*q(qx)],'r','Linewidth',2);
legend([t,p,a,r],{'Target','Proposal','Accept','Reject'});
xlabel('x');
```



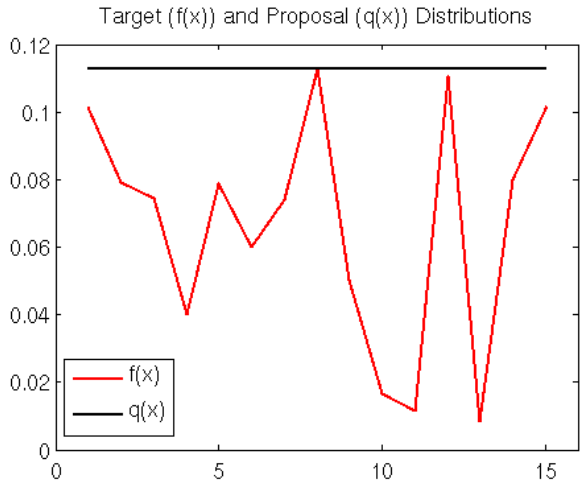
(https://theclevermachine.files.wordpress.com/2012/09/rejectionsamplingcriterion.png)
Rejection Sampling with a Normal proposal distribution

Here a zero-mean Normal distribution is used as the proposal distribution. This distribution is scaled by a factor $c = 9.2$, determined from $f(x)$ and $q(x)$ to ensure that the proposal distribution covers $f(x)$. We then sample from $q(x)$, and compare the proportion of $cq(x)$ occupied by $f(x)$. If we compare this proportion to a random number sampled from $Unif(0,1)$ (i.e. the criterion outlined above), then we would accept this sample with probability proportional to the length of the green line segment and reject the sample with probability proportional to the length of the red line

segment.

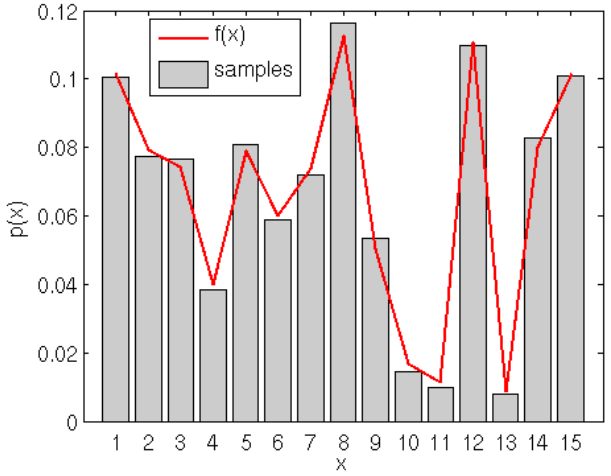
Rejection sampling of a random discrete distribution

This next example shows how rejection sampling can be used to sample from any arbitrary distribution, continuous or not, and with or without an analytic probability density function.



(https://theclevermachine.files.wordpress.com/2012/09/rejectionsamplingtargetproposal.png)
Random Discrete Target Distribution and Proposal that Bounds It.

The figure above shows a random ***discrete*** probability density function $f(x)$ generated on the interval (0,15). We will use rejection sampling as described above to sample from $f(x)$. Our proposal/envelope distribution is the uniform discrete distribution on the same interval (i.e. any of the integers from 1-15 are equally probable) multiplied by a constant c that is determined such that the maximum value of $f(x)$ lies under (or equal to) $cq(x)$.



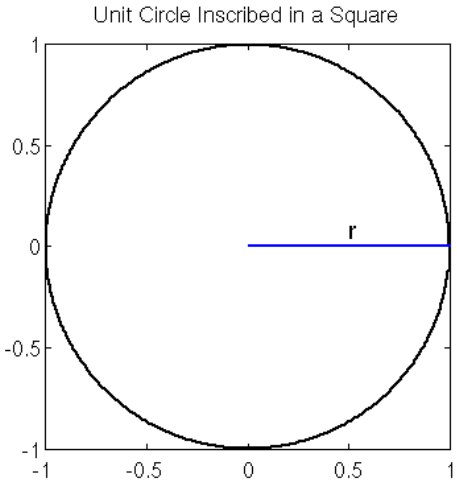
(https://theclevermachine.files.wordpress.com/2012/09/rejectionsamplingdiscrete.png)
Rejection Samples For Discrete Distribution on interval [1 15]

Plotted above is the target distribution (in red) along with the discrete samples obtained using the rejection sampling. The MATLAB code used to sample from the target distribution and display the plot above is here:

```
rand('seed',12345)
randn('seed',12345)

fLength = 15;
% CREATE A RANDOM DISTRIBUTION ON THE INTERVAL [1 fLength]
f = rand(1,fLength); f = f/sum(f);

figure; h = plot(f,'r','Linewidth',2);
hold on;
l = plot([1 fLength],[max(f) max(f)],'k','Linewidth',2);

legend([h,l],{'f(x)','q(x)'},'Location','Southwest');
xlim([0 fLength + 1])
xlabel('x');
ylabel('p(x)');
title('Target (f(x)) and Proposal (q(x)) Distributions');

% OUR PROPOSAL IS THE DISCRETE UNIFORM ON THE INTERVAL [1 fLength]
% SO OUR CONSTANT IS
c = max(f/(1/fLength));

nSamples = 10000;
i = 1;
while i < nSamples
   proposal = unidrnd(fLength);
   q = c*1/fLength; % ENVELOPE DISTRIBUTION
   if rand < f(proposal)/q
      samps(i) = proposal;
      i = i + 1;
```

Rejection sampling from the unit circle to estimate π

Though the ratio-based acceptance-rejection criterion introduced above is a common choice for drawing samples from complex distributions, it is not the only criterion we could use. For instance we could use a different set of criteria to generate some geometrically-bounded distribution. If we wanted to generate points uniformly within the unit circle (i.e. a circle centered at $(y, x) = 0$ and with radius $r = 1$), we could do so by sampling Cartesian spatial coordinates x and y uniformly from the interval (-1,1)–which samples form a square centered at (0,0)–and reject those points that lie outside of the radius $r = \sqrt{x^2 + y^2} = 1$



(https://theclevermachine.files.wordpress.com/2012/09/unitcircleinsquare2.png)
Unit Circle Inscribed in Square

Something clever that we can do with such a set of samples is to approximate the value π: Because a square that inscribes the unit circle has area:
$A_{square} = (2r)^2 = 4r^2$

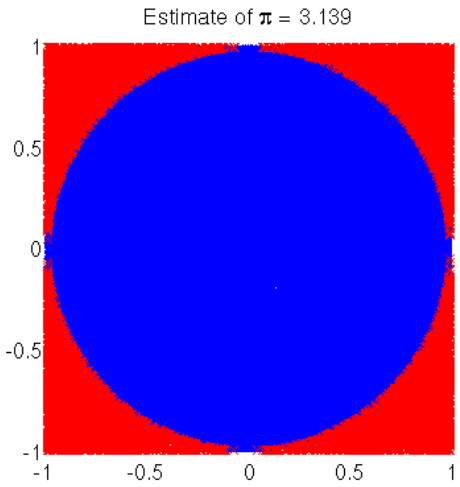
and the unit circle has the area:

$A_{circle} = \pi r^2$

We can use the ratio of their areas to approximate π:

$\pi = 4\frac{A_{circle}}{A_{square}}$

The figure below shows the rejection sampling process and the resulting estimate of π from the samples. One-hundred thousand 2D points are sampled uniformly from the interval (-1,1). Those points that lie within the unit circle are plotted as blue dots. Those points that lie outside of the unit circle are plotted as red x's. If we take four times the ratio of the area in blue to the entire area, we get a very close approximation to 3.14 for π.



(https://theclevermachine.files.wordpress.com/2012/09/rejectionsamplingpi.png)
Rejection Criterion

The MATLAB code used to generate the example figures is below:


```
% DISPLAY A CIRCLE INSCRIBED IN A SQUARE

figure;
a = 0:.01:2*pi;
x = cos(a); y = sin(a);
hold on
plot(x,y,'k','Linewidth',2)

t = text(0.5, 0.05,'r');
l = line([0 1],[0 0],'Linewidth',2);
axis equal
box on
xlim([-1 1])
ylim([-1 1])
title('Unit Circle Inscribed in a Square')

pause;
rand('seed',12345)
randn('seed',12345)
delete(l); delete(t);

% DRAW SAMPLES FROM PROPOSAL DISTRIBUTION
samples = 2*rand(2,100000) - 1;

% REJECTION
reject = sum(samples.^2) > 1;

% DISPLAY REJECTION CRITERION
scatter(samples(1,~reject),samples(2,~reject),'b.')
scatter(samples(1,reject),samples(2,reject),'rx')
hold off
```

Wrapping Up

Rejection sampling is a simple way to generate samples from complex distributions. However, Rejection sampling also has a number of weaknesses:

- Finding a proposal distribution that can cover the support of the target distribution is a non-trivial task.
- Additionally, as the dimensionality of the target distribution increases, the proportion of points that are rejected also increases. This curse of dimensionality makes rejection sampling an inefficient technique for sampling multi-dimensional distributions, as the majority of the points proposed are not accepted as valid samples.
- Some of these problems are solved by changing the form of the proposal distribution to "hug" the target distribution as we gain knowledge of the target from observing accepted samples. Such a process is called *Adaptive Rejection Sampling*, which will be covered in another post.

Posted in Density Estimation, Sampling Methods, Statistics

Tags: Curse of Dimensionality, Envelope Distribution, Proposal Distribution, Rejection Sampling, Sampling Methods, Target Distribution 4 Comments