Time Series Analysis - Taching Session 01

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1 Assignment 12

Exercise: Let x_t be a zero-mean, unit-variance stationary process with autocorrelation function ρ_h. Suppose that μ_t is a nonconstant function and that σ_t is a positive-valued nonconstant function. The observed series is formed as $y_t = \mu_t + \sigma_t x_t$

a) Find the mean and covariance function for the y_t process.

b) Show that the autocorrelation function for the y_t process depends only on the time lag. Is the y_t process stationary?

c) Is it possible to have a time series with a constant mean and with $\text{corr}(y_t, y_{t+h})$ free of t but with y_t *not* stationary?

1.1 Mean and Covariance Function

1.1.1 Mean

$$E[y_t] = E[\mu_t] + E[\sigma_t x_t] = E[\mu_t] + E[\sigma_t]E[x_t]$$

As $E[x_t]$ is 0, we obtain

$$= E[\mu_t] = \mu_t$$

As μ_t is a non-constant function by definition.

1.1.2 Covariance Function

$$\gamma(s,t) = \text{cov}(y_s, y_t) = E[(y_s - E[y_s])(y_t - E[y_t])]$$

We substitude $y_s = \mu_t + \sigma_t x_t$ (by definition) and we already know that $E[s_t] = \mu_t$.

$$= E[\mu_s + \sigma_s x_s - \mu_s)(\mu_t + \sigma_t x_t - \mu_t)] = E[\sigma_s x_s \sigma_t x_t]$$

As σ is a function by definition, there is no randomness and we can put it out of the expectation.

$$= \sigma_s \sigma_t E[x_s x_t]$$

We know that $E[x_t] = 0$, so we can easily add it to get a known shape.

$$= \sigma_s \sigma_t E[(x_s - E[x_s])(x_t - E[x_t])]$$

The known shape is the definition of the covariance, so we obtain the following.

$$= \sigma_s \sigma_t \text{cov}(x_s, x_t) = \sigma_s \sigma_t \gamma(x_s, x_t)$$

The ACF is defined by

$$\rho(s,t) = \frac{\gamma(x_s, x_t)}{\sqrt{\gamma(x_s, x_s)\gamma(x_t, x_t)}}$$

Solving for $\gamma(s,t)$ and then plugging in in the equation above we obtain

$$= \sigma_s \sigma_t \rho(x_s, x_t) \sqrt{\gamma(x_s, x_s)\gamma(x_t, x_t)}$$

We know that $\gamma(x_s, x_s) = \text{var}(x_s) = 1$ as that is given by definition. Therefor the ACF is finally given by

$$= \sigma_s \sigma_t \rho(x_s, x_t) = \sigma_s \sigma_t \rho_x(s,t)$$

2 Assignment 18

Exercise: For each of the following ARMA models, find the roots of the AR and MA polynomials, identify the values of p and q for which they are ARMA(p,q) (be careful of parameter redundancy), determine whether they are causal, and determine whether they are invertible. In each case, $w_t \sim \mathcal{N}(0,1)$.

c) $x_t - 3x_{t-1} = w_t + 2w_{t-1} - 8w_{t-2}$

d) $x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9}w_{t-1}$

e) $x_t - 4x_{t-2} = w_t - w_{t-1} + 0.5w_{t-2}$

f) $x_t - \frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t$

2.1 c) Solution

2.1.1 Roots of AR and MA Polynomials

Applying the autoregressive operator for ϕ and θ gives us:

AR: $\phi(B) = 1 - 3B$, thus $Z_\phi = (1, -3)$

MA: $\theta(B) = 1 + 2B - 8B^2$, thus $Z_\theta = (1, 2, -8)$

For both parts we calculate the zero points. We start with AR.

$$p_\phi(z) = 1 - 3z$$

setting $p_\phi(z) = 0$

$$1 - 3z = 0$$

From that we get $z_{\theta 1} = \frac{1}{3}$. Now we do the MA part.

$$p_\theta(z) = 1 + 2z - 8z^2$$

setting $p_\theta(z) = 0$

$$0 = 1 + 2z - 8z^2$$

First we divide by -8 and swap sides:

$$z^2 - \frac{1}{4}z - \frac{1}{8} = 0$$

We add $(\frac{1}{4}/2)^2$ on both sides

$$z^2 - \frac{1}{4}z - \frac{1}{8} + \frac{1}{64} - \frac{8}{64} = \frac{1}{64}$$

Next we create the polynomial form and add $\frac{8}{64}$ on both sides

$$(z - \frac{1}{8})^2 = \frac{9}{64}$$

Taking the square root

$$z - \frac{1}{8}2 = \pm\frac{3}{8}$$

Thus is follows that $z_{\theta 1 = \frac{1}{2}}$ and $z_{\theta 2 = -\frac{1}{4}}$.

2.1.2 Finding p and q

To check for redundancy, we have to identify any common roots. As $z_\phi \cap z_\theta = \{\}$, $\phi(B)$ and $\theta(B)$ share no common roots and no redundancy is given.

Therefore p and q are simply given by the highest order terms, so the model is ARMA$(p = 1, q = 2)$.

2.1.3 Causality and Invertibility

Causality: As $\forall Z_\theta : |Z_\theta| > 1$ is **not** given, the model is not causal.

Invertibility: As $\forall Z_\phi : |Z_\phi| > 1$ is **not** given, the model is not invertible.

2.2 d) Solution

2.2.1 Roots of AR and MA Polynomials

Applying the autoregressive operator for ϕ and θ gives us:

AR: $\phi(B) = 1 - 2 + 2B^2$, thus $Z_\phi = (1, -2, 2)$

MA: $\theta(B) = 1 - \frac{8}{9}B$, thus $Z_\theta = (1, -\frac{8}{9})$

For both parts we calculate the zero points. We start with AR.

$$p_\phi(z) = 1 - 2z + 2z^2$$

Setting $p_\phi(z) = 0$ and swapping sides

$$1 - 2z + 2z^2 = 0$$

Divide by 2 and order

$$z^2 - z + \frac{1}{2} = 0$$

We add $(1/2)^2$ on both sides

$$z^2 - z + \frac{1}{2} + \frac{1}{4} = \frac{1}{4}$$

Next we create the polynomial form and substract $\frac{1}{2}$ on both sides:

$$(z - \frac{1}{2})^2 = -\frac{1}{4}$$

Taking the square root:

$$z - \frac{1}{2} = \pm\frac{1}{2}i$$

From that we get $z_{\theta 1} = \frac{1}{2} + \frac{1}{2}i$ and $z_{\theta 2} = \frac{1}{2} - \frac{1}{2}i$. Now we do the MA part.

$$p_\theta(z) = 1 - \frac{8}{9}z$$

setting $p_\theta(z) = 0$

$$0 = 1 - \frac{8}{9}z$$

So $z_{\theta 1 = \frac{8}{9}}$

2.2.2 Finding p and q

To check for redundancy, we have to identify any common roots. As $z_\phi \cap z_\theta = \{\}$, $\phi(B)$ and $\theta(B)$ share no common roots and no redundancy is given.

Therefore p and q are simply given by the highest order terms, so the model is ARMA$(p = 1, q = 2)$.

2.2.3 Causality and Invertibility

Causality: As $\forall Z_\theta : |Z_\theta| > 1$ is **not** given, the model is not causal.

Invertibility: As $\forall Z_\phi : |Z_\phi| > 1$ is **not** given, the model is not invertible.

2.3 e) Solution

2.3.1 Roots of AR and MA Polynomials

Applying the autoregressive operator for ϕ and θ gives us:

AR: $\phi(B) = 1 - 4B^2$, thus $Z_\phi = (1, 0, -4)$

MA: $\theta(B) = 1 - B + \frac{1}{2}B^2$, thus $Z_\theta = (1, -1, -\frac{1}{2})$

For both parts we calculate the zero points. We start with AR.

$$p_\phi(z) = 1 - 4Z^2$$

setting $p_\phi(z) = 0$

$$1 - 4Z^2 = 0$$

dividing by 4 and putting Z on one side

$$Z^2 = \frac{1}{4}$$

From that we get $z_{\theta 1} = \frac{1}{2}$ and $z_{\theta 2} = -\frac{1}{2}$. Now we do the MA part.

$$p_\theta(z) = 1 - z + \frac{1}{2}z^2$$

setting $p_\theta(z) = 0$

$$0 = 1 - z + \frac{1}{2}z^2$$

First we divide by $\frac{1}{2}$ and swap sides:

$$z^2 - 2z + 2 = 0$$

We add $(\frac{2}{2}/2)^2$ on both sides

$$z^2 - 2z + 1 + 2 = 1$$

Next we create the polynomial form and substract -2 on both sides

$$(z-1)^2 = -1$$

Taking the square root

$$z - 1 = \pm i$$

Thus is follows that $z_{\theta 1} = 1 + 1i$ and $z_{\theta 2} = 1 - 1i$.

2.3.2 Finding p and q

To check for redundancy, we have to identify any common roots. As $z_\phi \cap z_\theta = \{\}$, $\phi(B)$ and $\theta(B)$ share no common roots and no redundancy is given.

Therefore p and q are simply given by the highest order terms, so the model is ARMA$(p = 2, q = 2)$.

2.3.3 Causality and Invertibility

Causality: As $\forall Z_\theta : |Z_\theta| > 1$ is **not** given, the model is not causal.

Invertibility: As $\forall Z_\phi : |Z_\phi| > 1$ is given, the model is invertible. The length of all elements of the set Z_ϕ is given by $|Z_\phi| = \sqrt{1^2 + (\pm 1)^2} \approx 1.414214$.

2.4 f) Solution

2.4.1 Roots of AR and MA Polynomials

Applying the autoregressive operator for ϕ and θ gives us:

AR: $\phi(B) = 1 - \frac{9}{4}B - \frac{9}{4}B^2$, thus $Z_\phi = (1, -\frac{9}{4}, -\frac{9}{4})$

MA: $\theta(B) = 1$, thus $Z_\theta = (1)$

For both parts we calculate the zero points. We start with AR.

$$p_\phi(z) = 1 - \frac{9}{4}z - \frac{9}{4}z^2$$

setting $p_\phi(z) = 0$

$$1 - \frac{9}{4}z - \frac{9}{4}z^2 = 0$$

dividing by $\frac{9}{4}$ and ordering

$$z^2 + z - \frac{4}{9} = 0$$

Adding $(\frac{1}{2})^2$ on both sides

$$z^2 + z + \frac{1}{4} - \frac{4}{9} = \frac{1}{4}$$

Forming the polynimial and adding $\frac{4}{9}$

$$(z + \frac{1}{2})^2 = \frac{25}{36}$$

Taking the square root

$$z + \frac{1}{2} = \pm\frac{5}{6}$$

From that we get $z_{\theta 1} = \frac{1}{3}$ and $z_{\theta 2} = -\frac{4}{3}$. Now we do the MA part.

$$p_\theta(z) = 1$$

As setting this to 0 is an inequality, this ARMA model does not have an MA part.

2.4.2 Finding p and q

To check for redundancy, we have to identify any common roots. As $\theta(B)$ is not given, it does not really make sense to ask this question. There are no common roots.

Therefore p and q are simply given by the highest order terms, so the model is ARMA$(p = 2, q = 0)$ which can also be expressed as AR(2).

2.4.3 Causality and Invertibility

Causality: As $\forall Z_\theta : |Z_\theta| > 1$ is **not** given, the model is not causal.

Invertibility: As the model does not have an MA part, it does not make sense to talk about invertibility as an MA(∞) representation cannot exist for something that is non-existent. Also a (weak) stationary property does not make sense in this case.