UNIT-1

* **Bisection method (Bolzano method)** – (lengthy procedure, may cause error)

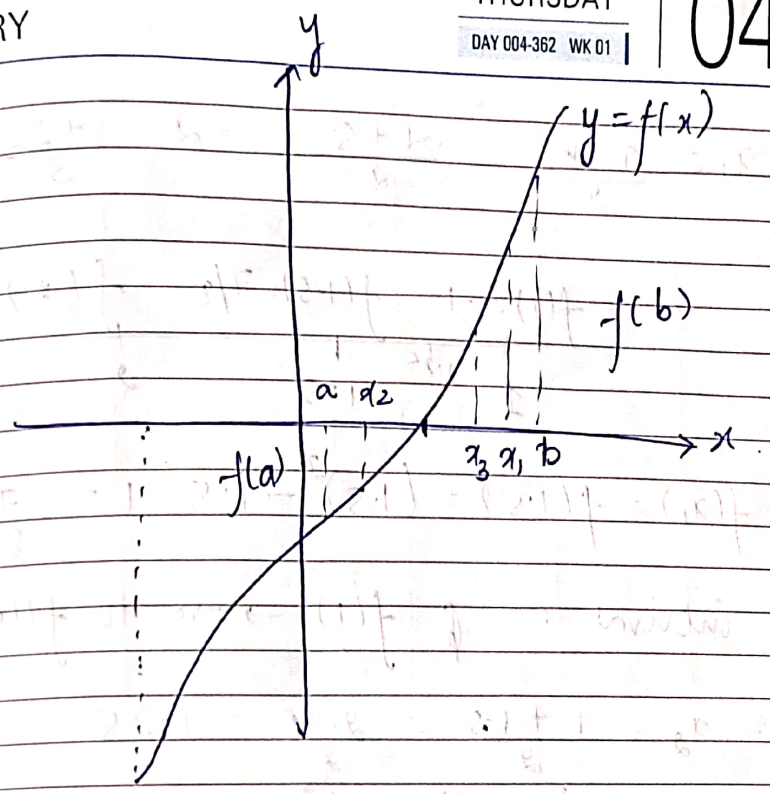
This method is based on the repeated application of intermediate value property.

Let $f(x)$ be continuous b/w a & b & let $f(a)$ be $-ve$, $f(b)$ be $+ve$.

Then, the first approximation of the root is $\boxed{x_1 = \frac{a+b}{2}}$, if $f(x_1)=0$, then, x_1 is the root of $f(x)=0$.

Otherwise, the root lies b/w a & x_1 or x_1 & b, according to $f(x_1)$, if positive or negative.

Then, we bisect the interval as before & continue the process until the root is found accurately.



Q. Find the real root of the equation

$$f(x) = x^3 - x - 1 = 0$$

→ a & b ki aisi values assume krni hai jisse $-ve$ or $+ve$ mile $f(x)$

Let $f(-1) = -1+1-1 = -1 = -ve$

$f(1) = 1-1-1 = -1 = -ve$ } ye lenge

$f(2) = 8-2-1 = 5 = +ve$

$x_1 = \frac{a+b}{2} = \cancel{\frac{1+5}{2}} = 2 \quad \frac{1+2}{3} = 1.5$

$f(1) = -1 \quad f(1.5) = 7/8 \quad f(2) = 5$

1   1.25   1.5   2

$f(x_1) = f(1.5) = (1.5)^3 - 1.5 - 1 = \frac{7}{8} = +ve$

$\Rightarrow$ interval :- $f(1) \to -ve$ to $f(1.5) \to +ve$

$\Rightarrow x_2 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$

$\Rightarrow$ it will lie b/w 1 & 1.5.

$\Rightarrow f(1.25) = (1.25)^3 - (1.25) - 1$

$= \frac{-19}{64} = -ve.$

interval :- 1.25 to 1.5

$\Rightarrow x_3 = \frac{1.25+1.5}{2} = 1.375$

$f(1.375) = (1.375)^3 - (1.375) - 1$

$= -ve.$

interval :- 1.375 to 1.5

$x_4 = \frac{1.375+1.5}{2} = 1.3125$

and soon.

Q. Find the real roots of the eqn

$f(x) = x^3 - 2x - 5 = 0$

$\rightarrow f(0) = 0 - 0 - 5 = -5 = -ve$

$f(1) = 1 - 2 - 5 = -6 = -ve$

$f(2) = 8 - 4 - 5 = -1 = -ve$

$f(3) = 27 - 6 - 5 = 16 = +ve.$ } intervals.

Let $a = 2 \Rightarrow f(a) = -ve$

$b = 3 \Rightarrow f(b) = +ve$

-ve +ve

x	a	b	$x = \frac{a+b}{2}$	$f(x)$
1	2	3	$x_1 = \frac{2+3}{2} = 2.5$	$f(2.5) = 5.6250$
2	2	2.5	2.25	+1.8906
3	2	2.25	2.125	+0.3457
4	2	2.125	2.0625	-0.3513
5	2.0625	2.125	2.09375	-0.0087
6	2.09375	2.125	2.10938	0.1668

$x_2 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$

$f(2.25) = 1.8906 = +ve.$

$x_3 = \frac{2+2.25}{2} = \frac{4.25}{2} = 2.125$

$f(2.125) = 0.3457 > 0.$

$x_4 = \frac{2+2.125}{2} = \frac{4.125}{2} = 2.0625$

$f(2.0625) = -0.3513$

$x_5 = \frac{2.0625+2.125}{2} = 2.09375$

$f(x_5) = -0.0089$

$x_6 = \frac{2.09375+2.125}{2} = 2.10938$

$f(x_6) = 0.1668.$

$a = 2.09375, \quad b = 2.10938$

$x_7 =$

sb table me ayega.

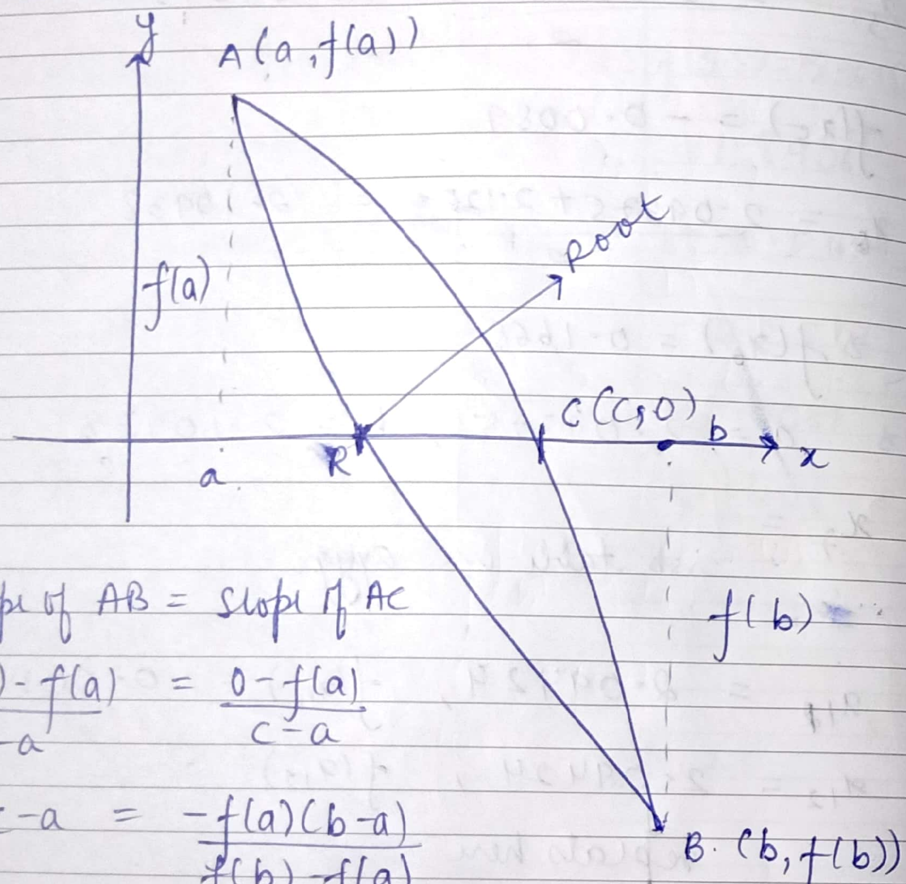
$x_{11} = 2.09424, \quad f(x_{11}) = -0.0034 < 0$

$x_{12} = 2.09424, \quad f(x_{12})$ -

repeats here

root = 2.09424.

* False Position method (Regula Falsi)



Slope of AB = Slope of AC

$$\frac{f(b)-f(a)}{b-a} = \frac{0-f(a)}{c-a}$$

$$\Rightarrow c-a = \frac{-f(a)(b-a)}{f(b)-f(a)}$$

$\cancel{c-a}$ $c = a - \dfrac{f(a)(b-a)}{f(b)-f(a)}$

$$\Rightarrow \boxed{c = \frac{a f(b) - b f(a)}{f(b) - f(a)}} \quad *$$

Working procedure -

1. Find the interval $[a, b]$ st $f(a)f(b) < 0$
2. find $c = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$
3. $f(a)f(c) < 0 \Rightarrow$ roots lie in $[a, c]$

   $f(b)f(c) < 0 \Rightarrow$ roots lie in $[c, b]$
4. Repeat steps 1 & 2.

Q. Find a real root of $x^3 - 2x - 5 = 0$, using the False posn method upto four iterations.

$\rightarrow$ $\left.\begin{array}{l} f(2) = 8-4-5 = -1 = -ve \\ f(3) = 27-6-5 = 16 = +ve \end{array}\right\} \begin{array}{l} \rightarrow a \\ \rightarrow b. \end{array}$

<u>1st iterat^n -</u>

$\Rightarrow a = 2, \quad b = 3.$

$\Rightarrow f(a) = -1, \quad f(b) = 16$

$$\Rightarrow c = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{2(16) - 3(-1)}{16 - (-1)}$$

$$= \frac{32 + 3}{17} = \frac{35}{17} = 2.0588$$

$$f(c) = (2.0588)^3 - 2(2.0588) - 5 = -0.3908 < 0$$

<u>2nd iterat^n -</u>

Now, $a = 2.0588 \Rightarrow f(a) = -0.3908$
$b = 3 \Rightarrow f(b) = 16.$

$$c = \frac{2.0588(16) + 3(0.3908)}{16 + 0.3908}$$

$$= 2.0812$$

$$f(c) = (2.0812)^3 - 2(2.0812) - 5 = -0.1479 < 0$$

<u>3rd iteration.</u>

Now, $a = 2.0812 \Rightarrow f(a) = -0.1479$
$b = 3 \Rightarrow f(b) = 16.$

$$c = \frac{2.0812(16) + 3(0.1479)}{16 + 0.1479} = 2.0896$$

$$\Rightarrow f(c) = -0.0551 < 0$$

<u>4th iterat^n -</u>

Now, $a = 2.0896 \Rightarrow f(a) = -0.0551$
$b = 3 \Rightarrow f(b) = 16$

$$c = \frac{2.0896(16) + 3(0.0551)}{16 + 0.0551} = 2.0927$$

Hence, the root is <u>2.0927</u>.

Q. Using False-Position method, find a real root of the eq^n

$$x \log_{10} x - 1.2 = 0 \text{ in 3 steps}$$

$f(1) = 1\log_{10} 1 - 1.2 = -1.2 < 0$

$f(2) = 2\log_{10} 2 - 1.2 = -0.6 < 0$

$f(3) = 3\log_{10} 3 - 1.2 = 0.23 > 0.$

$\therefore$ $a = 2 \Rightarrow f(a) = -0.6$
$b = 3 \Rightarrow f(b) = 0.23.$

<u>1st iteratn-</u>

$a = 2 \Rightarrow f(a) = -0.6$
$b = 3 \Rightarrow f(b) = 0.23$

$$c = \frac{2(0.23) + 3(0.6)}{0.23 + 0.6} = \frac{0.46 + 1.8}{0.83}$$

$$= 2.722$$

$f(c) = -0.0186 < 0.$

<u>2nd itern-</u>

$a = 2.722 \Rightarrow f(a) = -0.0186$
$b = 3 \Rightarrow f(b) = 0.23$

0.0024×10
0.0024

0.00009×10^{-7}
0.009

$$c = \frac{2.722(0.23) + 3(0.0186)}{0.23 + 0.0186} = \frac{0.626 + 0.0558}{0.2486}$$

$= 2.742$

$f(c) = 0.0009 > 0$

<u>3rd iteratn-</u>

$a = 2.722 \Rightarrow f(a) = -0.0186$
$b = 2.742 \Rightarrow f(b) = 0.0009$

$$\therefore c = \frac{2.722(0.0009) + 2.742(0.0186)}{0.0009 + 0.0186}$$

$$= \frac{0.0024 + 0.051}{0.0195} = 2.738$$

$\therefore$ Root of the given eqn is $\underline{\underline{2.738}}$

★ ## Simple Fixed Point iteration method -

Suppose we have eq^n $f(x)=0$, then eq^n can be expressed as $\boxed{x=\phi(x)}$.

$$\boxed{|\phi'(x)|_{\text{at } x=x_0} < 1}$$

Then iterative method is applied.

The successive approximation is given by

$$\boxed{x_n = \phi(x_{n-1})}$$

1) $x_1 = \phi(x_0)$

$x_2 = \phi(x_1)$

$x_3 = \phi(x_4)$

$x_n = \phi(x_{n-1})$

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Q. Find a real root of the eq^n :-

$$f(x) = x^3 + x^2 - 1 = 0.$$

$\rightarrow$ $f(0) = 0 + 0 - 1 = -1.$

$f(1) = 1 + 1 - 1 = 1$

Let $x_0 = 0.5$ $\left(\frac{0+1}{2} = 0.5\right)$. ~~near to~~ 0

$\phi(x) = x$ can be written in 3 ways-

$x^3 = 1 - x^2$	$x^2 = 1 - x^3$	$x^2(x+1) - 1 = 0$
$x = (1-x^2)^{1/3}$	$x = (1-x^3)^{1/2}$	$x = \frac{1}{\sqrt{1+x}}$.

$\phi(x) = (1-x^2)^{1/3}$

$\phi'(x) = \frac{1}{3}\frac{(-2x)}{(1-x^2)^{2/3}}$

$|\phi'(x)|_{x_0 = 0.5}$

$= \frac{1}{3}\frac{\left(2\left(\frac{1}{2}\right)\right)}{\left(1-\frac{1}{4}\right)^{2/3}}$

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$x^3 = 1-x^2 \Rightarrow x = (1-x^2)^{1/3}$

$\phi(x) = x = (1-x^2)^{1/3}$

$\phi'(x) = \frac{1}{3}\frac{(-2x)}{(1-x^2)^{2/3}}$

$\Rightarrow \left|\phi'(x)\right|_{x=x_0} = \left|\phi'(x)\right|_{x=0.5} = \frac{\frac{1}{3}\left(2\left(\frac{1}{2}\right)\right)}{\left(\frac{3}{4}\right)^{2/3}}$

$= \frac{1}{3}\frac{1}{(0.75)^{0.66}} = \frac{1}{3}\left(\frac{1}{0.827}\right)$

$= \frac{1}{2.481} = \underline{0.403} < 1.$

lly $x = (1-x^3)^{1/2}$ & $x = \frac{1}{\sqrt{1+x}}$ be lige

$\left|\phi'(x)\right|_{x=0.5} < 1$

But, we'll consider $\phi(x) = \frac{1}{\sqrt{1+x}}$ because it leads to a value close to 0.

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Now, $x_n = \frac{1}{\sqrt{1+x_{n-1}}}$

$x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = \frac{1}{\sqrt{1.5}} = 0.81649$

$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.81649}} = \frac{1}{\sqrt{1.81649}}$

$= 0.74196$

$x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.74196}} = \frac{1}{\sqrt{1.74196}}$

$= 0.75767$

$x_4 = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1.75767}} = 0.75427$

$x_5 = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1.75427}} = 0.75500$

$x_6 = \frac{1}{\sqrt{1+x_5}} = 0.\underline{7548}8$

$x_7 = 0.\underline{7548}7$ same $\underline{4 \text{ digit}}$ the stop.

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Q. Find the roots of $\cos x = 3x - 1$, correct to four decimal places by using iteration method.

$\rightarrow$ $f(x) = \cos x - 3x + 1$

$f(0) = 1 - 0 + 1 = 2 \quad > 0$

$f\left(\frac{\pi}{2}\right) = 0 - \frac{3\pi}{2} + 1 \quad < 0.$

is case me 0 as compare to $\frac{\pi}{2}$.

Let $x_0 = 0 \rightarrow$ zero se near.

$3x = 1 + \cos x$

$x = \frac{1 + \cos x}{3} \Rightarrow \phi(x) = \frac{1 + \cos x}{3}.$

$\Rightarrow \phi'(x) = \frac{1}{3}(-\sin x) = \frac{-\sin x}{3}$

$\Rightarrow \left|\phi'(x)\right|_{x=0} = \frac{\sin 0}{3} = 0. < 1$

x_0 ki value $\frac{0 + \frac{\pi}{2}}{2} \Rightarrow \frac{\pi}{4}$ bhi le skte the bs $\left|\phi'(x)\right|_{x=x_0} < 1$ aana chaiye

$\Rightarrow \phi(x) = \frac{1 + \cos x}{3}.$

$\Rightarrow x_n = \phi(x_{n-1})$

$\Rightarrow x_1 = \frac{1 + \cos x_0}{3} = \frac{1 + \cos 0}{3} = \frac{2}{3} = \cancel{0.6667}$

$= 0.66667$

$x_2 = \frac{1 + \cos(0.66667)}{3} = \frac{1 + 0.99993}{3} = 0.66664$

$x_3 = \frac{1 + \cos(0.66664)}{3} = \frac{1 + 0.999932}{3}$

$= 0.666644$

$\Rightarrow$ Root is 0.66664.

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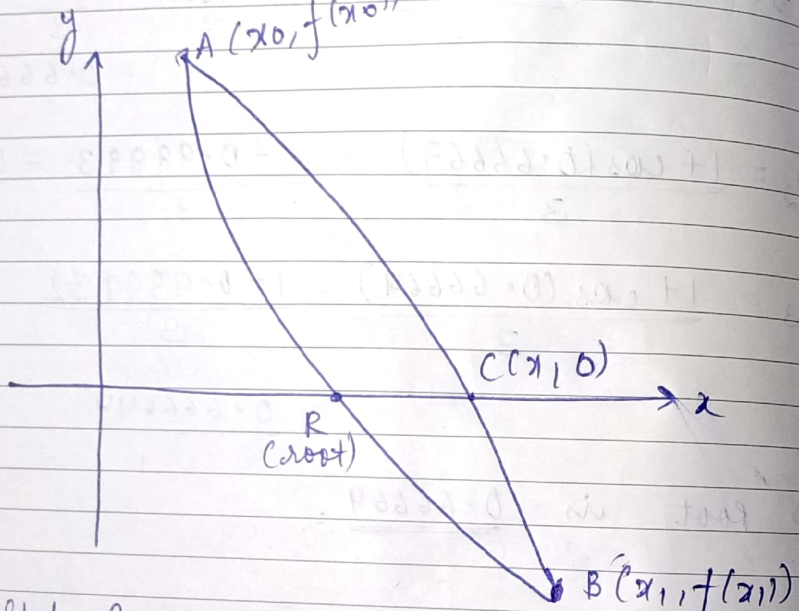
* **Secant method (Chord method)-**

This method is quite similar to regula falsi method except for the condⁿ

$f(x_1)f(x_2)<0$



Slope of AB = slope of AC

$\Rightarrow \dfrac{f(x_1)-f(x_0)}{x_1-x_0} = \dfrac{0-f(x_0)}{x-x_0}$

On solving, we get,

$$\boxed{x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1)-f(x_0)}} \star$$

General form,

$$\boxed{x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1})-f(x_n)}}$$

This method fails when $\boxed{f(x_n) = f(x_{n+1})}$.

Q. A real root of the eqⁿ $x^3-5x+1=0$ lies in the interval $(0,1)$. Perform four iterations of the secant method.

→ Let $f(x) = x^3-5x+1$

$f(0) = 0-0+1 = 1$
$f(1) = 1-5+1 = -3$

$\Rightarrow x_0 = 0, x_1 = 1$

$\Rightarrow f(0) = 1, f(1) = -3$

~~Put x~~ $x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

Put $n = 1$ (kyuki x_1 already hai)

$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0(-3) - 1(1)}{-3-1}$

$= \frac{1}{4} = 0.25$

$\therefore f(0.25) = -0.234375$

Put $n = 2$

$\therefore x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$= \frac{1(-0.234375) - 0.25(-3)}{-0.234375 + 3} = 0.18644$

$f(x_3) = 0.07428$

Put $n = 3$

$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$= \frac{0.25(0.07428) - 0.18644(-0.234375)}{0.07428 + 0.234375}$

$= 0.20174$

$\therefore f(x_4) = -0.00048$

Put $n = 4$

$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$

$= \frac{0.18644(-0.00048) - 0.20174(0.07428)}{-0.00048 - 0.07428}$

$= 0.20081$

$\therefore$ Root is $\underline{0.20081}$

Q. Estimate the root of the eq^n $\cos x - xe^x = 0$ using the secant method with the initial estimate of $x_1 = 0.5$, $x_2 = 1$

$\rightarrow f(x_1) = f(0.5) = \cos 0.5 - 0.5e^{0.5}$
$= 0.0532$.

$f(x_2) = f(1) = \cos 1 - 1e^1 = -e$
$= -2.17798$

We know,

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})},$$

Put $n = 2$

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

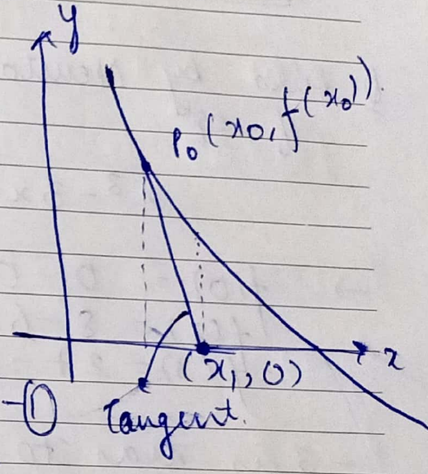
$$= \frac{0.5(-2.17798) - 1(0.0532)}{-2.17798 - 0.0532}$$

$= 0.5119$

$\Rightarrow f(x_3) = 0.01773$

$\|$ly on solving, we get root at $n=5 \Rightarrow x_6 = 0.5178$.

* Newton Raphson method -



We know, eq^n of tangent:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

here, $x_1 = x_0$ & $y_1 = f(x_0)$

$$\Rightarrow \boxed{y - f(x_0) = f'(x_0)(x - x_0)} \quad -(1)$$

Pt. $(x_1, 0)$ satisfies (1)

$\Rightarrow 0 - f(x_0) = f'(x_0)(x_1 - x_0)$

$$\Rightarrow x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$ → Formula of Newton Raphson

$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ — general formula & $f'(x_n) \neq 0$.

Q. Find by Newton Raphson method a root of eq^n

$$x^3 - 3x - 5 = 0$$

$\rightarrow$ $f(0) = 0 - 0 - 5 = -ve$

$f(2) = 8 - 6 - 5 = -ve = -3$

$f(3) = 27 - 9 - 5 = 16 = +ve.$

-3 is near to 0 $\Rightarrow$ $\boxed{x_0 = 2}$

Now, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Put $n = 0$

$\Rightarrow$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $= \cancel{6}\ 2$

$f(x_0) = -3$, $f'(x_0) = 3x^2 - 3$
$= 3(4) - 3 = 9.$

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$\Rightarrow$ $x_1 = 2 - \frac{(-3)}{9\ 3}$ $\Rightarrow 2 + \frac{1}{3} = \frac{7}{3} = 2.33334$

$f(x_1) =$ ______ .

Put $n = 1$.

$\Rightarrow$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = 3x^2 - 3$ $\Rightarrow$ $f'(x_1) = 3(2.33334)^2 - 3$
$=$ ______ .

$x_2 =$ ______ .

so on. until value of x repeats

Q. Use Newton Raphson method to find a real root of
$$\cos x - xe^x = 0$$
corrected to four decimal places.

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→ $\cos x - x e^{x} = f(x)$

$f(1) = \cos 1 - 1e^{1} = -2.7182$
$f(0) = 1.$

$\boxed{\cancel{x_0 = 0}}$ transcendtal eqⁿ we take mid pt

$\therefore x_0 = \frac{1+0}{2} = 0.5$

$f'(x) = -\sin x - e^{x} - x e^{x}$
$= -\sin x - e^{x}(x+1)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 0.5 - \frac{\cos(0.5) - 0.5 e^{0.5}}{-\sin(0.5) - e^{0.5}(0.5+1)}$
$= 0.51802$

$x_2 = 0.51802 - \frac{\cos(0.51802) - 0.51802 e^{0.51802}}{-\sin(0.51802) - e^{0.51802}(1.51802)}$
$= 0.5180 \rightarrow$ repeats.

$\therefore$ Root $= \underline{0.5180}$.

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Q. Apply Newton Raphson method to solve the eqⁿ:

$2(x-3) = \log_{10} x \Rightarrow 2(x-3) - \log_{10} x = 0$

→ $f(1) = 2(1-3) - 0 = 2(-2) = -4$
$f(2) = 2(-1) - 0.301 = -2.301$
$f(3) = 2(0) - 0.4771 = -0.4771$
$f(4) = 2 - 0.602 = 1.398$

$\Rightarrow x_0 = \frac{3+4}{2} = \frac{7}{2} = 3.5$

$f'(x) = 2(1-0) - \frac{1}{x} = 2 - \frac{1}{x} =$.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 3.5 - \frac{2(3.5-3) - \log(3.5)}{2 - \frac{1}{3.5}} = 3.25696$

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WK	M	T	W	T	F	S	S
5				1	2	3	4
6	5	6	7	8	9	10	11
7	12	13	14	15	16	17	18
8	19	20	21	22	23	24	25
9	26	27	28	29			

On solving,

09 $x_2 = 3.256366$.

10 $x_3 = \underline{\underline{3.256}}$ repeats.

11 Root is $\underline{\underline{3.256}}$.

12

1

2

3

4

5

6

7