1.10: Conversion Factors and Functions

Earlier we showed how unity factors can be used to express quantities in different units of the same parameter. For example, a density can be expressed in g/cm^3 or lb/ft^3. Now we will see how *conversion factors* representing mathematical functions, like D = m/v, can be used to transform quantities into different parameters. For example, what is the volume of a given *mass* of gold? Unity factors and conversion factors are conceptually different, and we'll see that the "dimensional analysis" we develop for unit conversion problems must be used with care in the case of functions.

When we are referring to the same object or sample of material, it is often useful to be able to convert one ***parameter*** into another. For example, in our discussion of fossil-fuel reserves we find that 318 Pg (3.18×10^{17} g) of coal, 28.6 km^3 (2.68×10^{10} m^3) of petroleum, and 2.83×10^3 km^3 (2.83×10^{13} m^3) of natural gas (measured at normal atmospheric pressure and 15°C) are available. But none of these quantities tells us what we really want to know — how much *heat energy* could be released by burning each of these reserves? Only by converting the mass of coal and the volumes of petroleum and natural gas into their equivalent energies can we make a valid comparison. When this is done, we find that the coal could release 7.2×10^{21} J, , the petroleum 1.1×10^{21} J, and the gas 1.1×10^{21} J of heat energy. Thus the reserves of coal are more than three times those of the other two fuels combined. It is for this reason that more attention is being paid to the development of new ways for using coal resources than to oil or gas. Conversion of one kind of quantity into another is usually done with what can be called a **conversion factor**, but the conversion factor is based on a mathematical **function** (D = m / V) or mathematical equation that relates parameters. Since we have not yet discussed energy or the units (joules) in which it is measured, an example involving the more familiar quantities mass and volume will be used to illustrate the way conversion factors are employed. The same principles apply to finding how much energy would be released by burning a fuel, and that problem will be encountered later.

For helpful context about the above discussion, check out the following Crash Course Chemistry video:

Unit Conversion & Significant Figures: Crash Course Chemistry #2(opens in new window) [youtu.be]

Suppose we have a rectangular solid sample of gold which measures 3.04 cm × 8.14 cm × 17.3 cm. We can easily calculate that its volume is 428 cm^3 but how much is it worth? The price of gold is about 5 dollars per gram, and so we need to know the mass rather than the volume. It is unlikely that we would have available a scale or balance which could weigh accurately such a large, heavy sample, and so we would have to determine the mass of gold equivalent to a volume of 428 cm^3. This can be done by manipulating the equation which defines density, ρ = m / V. If we multiply both sides by V, we obtain

$$V \times \rho = \frac{m}{V} \times V = m \tag{1.10.1}$$

$$m = V \times \rho$$

or

$$mass = \text{volume} \times \text{ density}$$

Taking the density of gold from a reference table(opens in new window), we can now calculate

$$\text{Mass} = m = V\rho = 428\ \text{cm}^3 \times \frac{100.32\ \text{g}}{1\ \text{cm}^3} = 8.27 \times 10^3 \text{g} = 8.27\ \text{kg}$$

This is more than 18 lb of gold. At the price quoted above, it would be worth over 40 000 dollars!

The formula which defines density can also be used to convert the mass of a sample to the corresponding volume. If both sides of Eq. 1.10.1 are multiplied by 1/ρ, we have

$$\frac{1}{\rho} \times m = V\rho \times \frac{1}{\rho} = V$$

$$V = m \times \frac{1}{\rho} \tag{1.10.2}$$

Notice that we used the mathematical function D = m/V to convert parameters from mass to volume or vice versa in these examples. How does this differ from the use of unity factors to change units of one parameter?

An Important Caveat

A mistake sometimes made by beginning students is to confuse density with ***concentration***, which also may have units of g/cm^3. By dimensional analysis, this looks perfectly fine. To see the error, we must understand the meaning of the function

$$C = \frac{m}{V}$$

In this case, V refers to the volume of a solution, which contains both a solute and solvent.

Given a concentration of an alloy is 10 g gold in 100 cm^3 of alloy, we see that it is wrong (although dimensionally correct as far as conversion factors go) to ***incorrectly*** calculate the volume of gold in 20 g of the alloy as follows:

$$20\text{g} \times \frac{100\ \text{cm\^{}3}}{10\ \text{g}} = 200\ \text{cm}^3$$

It is only possible to calculate the volume of gold if the density of the alloy is known, so that the volume of alloy represented by the 20 g could be calculated. This volume multiplied by the concentration gives the mass of gold, which then can be converted to a volume with the density function.

The bottom line is that using a simple unit cancellation method does not always lead to the expected results, unless the mathematical function on which the conversion factor is based is fully understood.

✔ Example 1.10.1: Volume of Ethanol

A solution of ethanol with a concentration of 0.1754 g / cm^3 has a density of 0.96923 g / cm^3 and a freezing point of -9 ° F [1]. What is the volume of ethanol (D = 0.78522 g / cm^3 at 25 °C) in 100 g of the solution?

Solution

The volume of 100 g of solution is

$$V = m \div D = 100\ \text{g} \div 0.96923\ \text{g cm}^3 = 103.17\ \text{cm}^3$$

The mass of ethanol in this volume is

$$m = V \times C = 103.17\ \text{cm}^3 \times 0.1754\ \text{g / cm}^3 = 18.097\ \text{g}$$

$$\text{The volume of ethanol } = m \div D = 18.097\ \text{g} \div 0.78522\ \text{g / cm}^3 = 23.05\text{cm}^3$$

Note that we cannot calculate the volume of ethanol by

$$\frac{\dfrac{0.96923g}{cm^3} \times 100cm^3}{\dfrac{0.78522g}{cm^3}} = 123.4\text{cm}^3$$

even though this equation is dimensionally correct.

Note: Note that this result required when to use the function C = m/V, and when to use the function D=m/V as conversion factors. Pure dimensional analysis could not reliably give the answer, since both functions have the same dimensions.

Example 1.10.2: Volume of Benzene

Find the volume occupied by a 4.73-g sample of benzene.

Solution

The density of benzene is 0.880 g cm^{-3}. Using Eq. (2),

$$\text{Volume} = V = m \times \frac{1}{\rho} = 4.73 \text{ g} \times \frac{1 \text{ cm}^3}{0.880 \text{ g}} = 5.38 \text{ cm}^3$$

Note: Note that taking the reciprocal of $\frac{0.880 \text{ g}}{1 \text{ cm}^3}$ simply inverts the fraction — 1 cm^3 goes on top, and 0.880 g goes on the bottom.

The two calculations just done show that density is a conversion factor which changes volume to mass, and the reciprocal of density is a conversion factor changing mass into volume. This can be done because the mathematical formula defining density relates it to mass and volume. Algebraic manipulation of this formula gave us expressions for mass and for volume [Eq. 1.10.1 and 1.10.2], and we used them to solve our problems. If we understand the function D = m/V and heed the caveat above, we can devise appropriate conversion factors by unit cancellation, as the following example shows:

Example 1.10.3: Volume of Mercury

A student weighs 98.0 g of mercury. If the density of mercury is 13.6 g/cm^3, what volume does the sample occupy?

Solution

We know that volume is related to mass through density.

Therefore

$$V = m \times \text{ conversion factor}$$

Since the mass is in grams, we need to get rid of these units and replace them with volume units. This can be done if the reciprocal of the density is used as a conversion factor. This puts grams in the denominator so that these units cancel:

$$V = m \times \frac{1}{\rho} = 98.0 \text{ g} \times \frac{1 \text{ cm}^3}{13.6 \text{ g}} = 7.21 \text{ cm}^3$$

If we had multiplied by the density instead of its reciprocal, the units of the result would immediately show our error:

$V = 98.0 \text{ g} \times \frac{13.6\ g}{1 \text{ cm}^3} = 1.333 \text{g}^2/\text{cm}^3$ (no cancellation!)

It is clear that square grams per cubic centimeter are not the units we want.

Using a conversion factor is very similar to using a unity factor — we know the conversion factor is correct when units cancel appropriately. A conversion factor is not unity, however. Rather it is a physical quantity (or the reciprocal of a physical quantity) which is related to the two other quantities we are interconverting. The conversion factor works because of the relationship [ie. the definition of density as defined by Eqs. 1.10.1 and 1.10.2 includes the relationships between density, mass, and volume], *not* because it is has a value of one. Once we have established that a relationship exists, it is no longer necessary to memorize a mathematical formula. The units tell us whether to use the conversion factor or its reciprocal. Without such a relationship, however, mere cancellation of units does not guarantee that we are doing the right thing.

A simple way to remember relationships among quantities and conversion factors is a “road map“of the type shown below:

$$\text{Mass} \overset{density}{\longleftrightarrow} \text{volume or } m \overset{\rho}{\longleftrightarrow} V$$

This indicates that the mass of a particular sample of matter is related to its volume (and the volume to its mass) through the conversion factor, density. The double arrow indicates that a conversion may be made in either direction, provided the units of the conversion factor cancel those of the quantity which was known initially. In general the road map can be written

$$\text{First quantity} \overset{\text{conversion factor}}{\longleftrightarrow} \text{second quantity}$$

As we come to more complicated problems, where several steps are required to obtain a final result, such road maps will become more useful in charting a path to the solution.

Example 1.10.1: Volume to Mass Conversion

Black ironwood has a density of 67.24 lb/ft^3. If you had a sample whose volume was 47.3 ml, how many grams would it weigh? (1 lb = 454 g; 1 ft = 30.5 cm).

Solution

The road map

$$V \overset{\rho}{\rightarrow} m$$

tells us that the mass of the sample may be obtained from its volume using the conversion factor, density. Since milliliters and cubic centimeters are the same, we use the SI units for our calculation:

$$\text{Mass} = m = 47.3 \text{cm}^3 \times \frac{67.24 \text{ lb}}{1 \text{ ft}^3}$$

Since the volume units are different, we need a unity factor to get them to cancel:

$$m = 47.3 \text{ cm}^3 \times \left(\frac{1 \text{ ft}}{30.5 \text{ cm}}\right)^3 \times \frac{67.24 \text{ lb}}{1 \text{ ft}^3} = 47.3 \text{ cm}^3 \times \frac{1 \text{ ft}^3}{30.5^3 \text{ cm}^3} \times \frac{67.24 \text{ lb}}{1 \text{ ft}^3}$$

We now have the mass in pounds, but we want it in grams, so another unity factor is needed:

$$m = 47.3 \text{ cm}^3 \times \frac{1 \text{ ft}^3}{30.5^3 \text{ cm}^3} \times \frac{67.24 \text{ lb}}{1 \text{ ft}^3} \times \frac{454 \text{ g}}{1 \text{ lb}} = 500.9 \text{ g}$$

In subsequent chapters we will establish a number of relationships among physical quantities. Formulas will be given which define these relationships, but we do not advocate slavish memorization and manipulation of those formulas. Instead we recommend that you remember that a relationship exists, perhaps in terms of a road map, and then adjust the quantities involved so that the units cancel appropriately. Such an approach has the advantage that you can solve a wide variety of problems by using the same technique.