13.7: Successive Approximation

An approximation is often useful even when it is not a very good one, because we can use the initial inaccurate approximation to calculate a better one. A good example of this occurs in Example 2 from Calculating the Extent of Reactions where it was necessary to solve the equation

$$0.0200 = \frac{(1+x)x}{(1-2x)^2}$$

The conditions of the problem suggest that x is so small that it can be ignored.

1. Accordingly we can approximate

$$1 + x \approx 1 \approx 1 - 2x$$

from which we obtain the approximate result (called the first approximation), x_1:

$$0.0200 \approx \frac{1x_1}{1^2} \quad \text{or} \quad x_1 \approx 0.0200$$

Although for some purposes this is a sufficiently accurate result, a much better approximation can be obtained by feeding this one back into the formula. If we write the formula as

$$x = \frac{0.0200\,(1-2x)^2}{1+x}$$

we can now substitute $x_1 = 0.0200$ on the right-hand side, giving the second approximation:

$$x_2 = \frac{0.0200\,(1 - 2 \times 0.0200)^2}{1 + 0.0200} = \frac{0.0200 \times 0.96^2}{1.0200} = 0.0181$$

If we repeat this process, a third approximation is obtained:

$$x_3 \approx 0.0182$$

in exact agreement with the accurate result obtained from the quadratic formula in the example.

With practice, using this method of successive approximations is much faster than using the quadratic formula. It also has the advantage of being self-checking. A mistake in any of the calculations almost always leads to an obviously worse approximation. In general, if the last approximation for x differs from the next to last by less than 5 percent, it can be assumed to be accurate, and the successive-approximation procedure can be stopped. In the example just given,

$$\frac{x_2 - x_1}{x_1} = \frac{0.0181 - 0.0200}{0.0200} = \frac{-0.0019}{0.0200} \approx -0.10 = -10 \text{ percent}$$

so a third approximation was calculated. This third approximation was almost identical to the second and so was taken as the final result.