Implementation of a finite difference 1st order upwind scheme to solve the one-dimensional ideal hydrodynamic equations

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1 Introduction

In this report the one-dimensional hydrodynamic equations will be numerically solved using a finite difference 1st order upwind scheme. The implementation will be checked with two test cases. First, the sod shock problem, where the initial state has a discontinuity and second the isothermal acoustic wave, where the initial conditions are smooth. For the sod shock problem we will perform a convergence analysis.

2 Analytic investigation

The ideal hydrodynamic equations in one dimension are conservation laws for ρ, the mass density, ρv, momentum and e total energy. They can be written in matrix form as:

$$\partial_t (\mathbf{U}) + \partial_x \mathbf{F} (\mathbf{U}) = 0 \tag{1}$$

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ e \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + P \\ ((e+P)v) \end{pmatrix} = 0 \tag{2}$$

Equation (2) can be rewritten in the form $\partial_t \mathbf{U} + \mathbf{A}\left(\mathbf{U}\right)\partial_x \mathbf{U}$. With:

$$\mathbf{A}\left(\mathbf{U}\right) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix} 0 & 1 & 0 \\ -\left(\frac{3-\gamma}{2}\right)v^2 & \left(3-\gamma\right)v & \gamma - 1 \\ -\gamma\frac{ve}{\rho} + \left(\gamma-1\right)v^3 & \gamma\frac{e}{\rho} - \frac{3}{2}\left(\gamma-1\right)\rho v^2 & \gamma\frac{\rho v}{e} \end{pmatrix} \tag{3}$$

To numerically solve this equation we want to find the eigenvalues and eigenvectors of matrix $\mathbf{A}$.

The eigenvalues can be computed by solving a similar system of equations, namely the ideal hydrodynamic equations for primitive variables:

$$\partial_t \begin{pmatrix} \rho \\ v \\ P \end{pmatrix} + \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & P\gamma & v \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ v \\ P \end{pmatrix} = 0$$

With these eigenvalues the eigenvectors of $\mathbf{A}$ can be computed as:

3 Numerical implementation

3.1 Results

3.1.1 Sod shock problem

3.2 Convergence Analysis