SETS

INTRODUCTION

- Discrete structures are used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects.
- Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines.

Cont….

- Sets are a fundamental discrete structure on which all other discrete structures are built.
- Sets are used to group objects together.
- Often, but not always, the objects in a set have similar properties. For instance, all the students who are currently enrolled in Discrete structures make up a set.

Definition

- A *set* is an unordered collection of objects, called *elements* or *members* of the set.
- A set is said to *contain* its elements.
- We write $a \in A$ to denote that a is an element of the set A.
- The notation $a \notin A$ denotes that a is not an element of the set A.

Representation of Sets

- It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.
- There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces.
- For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d.
- This way of describing a set is known as the **roster method**.
- For example the set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

Cont....

- Another way to describe a set characterize all those elements is to use set in the set by statin g the property builder notation. We or properties they must have to be members.

- For instance, the set O of all odd positive integers less than 10 can be written as; O = {x | x is an odd positive integer less than 10},

- or, specifying the universe as the set of positive integers, as O = {x ∈ Z+ | x is odd and x < 10}.

Cont….

- These sets, each denoted using a boldface letter, play an important role in computational mathematics:
  - **N** = {0, 1, 2, 3, . . .}, the set of **natural numbers**
  - **Z** = {. . . , −2, −1, 0, 1, 2, . . .}, the set of **integers**
  - $\mathbf{Z}^+$ = {1, 2, 3, . . .}, the set of **positive integers**
  - **Q** = $\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
  - **R**, the set of **real numbers**
  - $\mathbf{R}^+$, the set of **positive real numbers**
  - **C**, the set of **complex numbers**.

Sets as members of other sets

- Sets can have other sets as members.
- For instance the set {**N**,**Z**,**Q**,**R**} is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers.

Please Note

- Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set.
- For example, *boolean* is the name of the set {0, 1} together with operators

on one or more elements of this set, such as AND, OR, and NOT.

Equal Sets

- Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

Definition:

- Two sets are *equal* if and only if they have the same elements.

Therefore, if

A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

- We write $A = B$ if A and B are equal sets.

Example

- The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.
- Note that the order in which the elements of a set are listed does not matter.
- Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

Empty and Singleton Sets

- There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅.
- The empty set can also be denoted by { } (that is, we represent the empty set with a pair of braces that encloses all the elements in this set).
- Often, a set of elements with certain properties turns out to be the null set.
- For instance, the set of all positive integers that are greater than their squares is the null set.

Cont….

- A set with one element is called a **singleton set**.
- A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set.
- The single element of the set $\{\emptyset\}$ is the empty set itself!
- A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Subsets

- It is common to encounter situations where the elements of one set are also the elements of a second set.

Definition:

- The set A is a *subset* of B if and only if every element of A is also an element of B.
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ is true.

Examples

- The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10.
- The set of all Information technology students at Ankole Western University is a subset of the set of all students at Ankole Western University .

The Size of a Set (Cardinality)

- Sets are used extensively in counting problems, and for such applications we need to know the sizes of sets.

Definition:

- Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

- **Example:** What is the cardinality of a set A of all positive odd integers less than 10?

Power Sets

- Many problems involve testing all combinations of elements of a set to see if they satisfy some property.
- To consider all such combinations of elements of a set *S*, we build a new set that has as its members all the subsets of *S*.

Definition:

- Given a set *S*, the *power set* of *S* is the set of all subsets of the set *S*. The power set of *S* is denoted by *P(S)*.

Example

Question: What is the power set of the set {0, 1, 2}?

Solution:

The power set *P({0, 1, 2})* is the set of all subsets of {0, 1, 2}.

Hence, *P({0, 1, 2})* = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.

Question:

What is the power set of the empty set? What is the power set of the set {∅}?

Solution: ????

Note: If a set has *n* elements, then its power set has 2^n elements.

Cartesian Products

- The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

<u>**Definition:**</u>

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its *n*th element.

Cont…

- We say that two ordered *n*-tuples are equal if and only if each corresponding pair of their elements is equal.
- In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.
- In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs

(a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product Definition

- Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

Hence, $A \times B = \{(a, b) \mid a \in A$ A $b \in B\}$.

Example 1

Question: Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution:

- The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

Example 2

Question: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution:

The Cartesian product $A \times B$ is,

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Note: The Cartesian products $A \times B$ and $B \times A$ are not equal unless

$A = \emptyset$ or $B = \emptyset$

Definition 2

- The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by

  $A_1 \times A_2 \times \cdot \cdot \cdot \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where

  a_i belongs to A_i for $i = 1, 2, \ldots, n$.

- In other words,

  $$A_1 \times A_2 \times \cdot \cdot \cdot \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Example

Question: What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution:

- The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$

Set Operations

- Two, or more, sets can be combined in many different ways.
- Such combinations are performed using set operations like;

i. Union

ii. Intersection

iii. Difference among others.

Union

Definition:

- Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.
- An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that;

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

- The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, {1, 3, 5} ∪ {1, 2, 3} = {1, 2, 3, 5}.

Cardinality of a Union

- We are often interested in finding the cardinality of a union of two finite sets

  A and *B*.

- Note that $|A| + |B|$ counts each element that is in *A* but not in *B* or

  in *B* but not in *A* exactly once, and each element that is in both *A* and

  B exactly twice.

- Thus, if the number of elements that are in both *A* and *B* is subtracted from

  $|A| + |B|$, elements in $A \cap B$ will be counted only once.

- Hence, $|A \cup B| = |A| + |B| - |A \cap B|$

Intersection

Definition:

- Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

- An element x belongs to the intersection of the sets A and B if and only if x

belongs to A and x belongs to B. This tells us that;

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

- The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is,

{1, 3, 5} ∩ {1, 2, 3} = {1, 3}.

Differenc
e

Definition:

- Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by $A - B$, is the set containing those elements that are in *A* but not in *B*.
- The difference of *A* and *B* is also called the *complement of B with respect to A*.
- An element x belongs to the difference of *A* and *B* if and only if $x \in A$ and

$x \in B$.

- This tells us that $A - B = \{x \mid x \in A \text{ A } x \in B\}$.

Example

- The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is,

  $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$.

- This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set

  {2}.

Complement

Definition:

- Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.
- An element belongs to A if and only if $x \notin A$. This tells us that;

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

- Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$	Absorption laws

Membership Tables

- Set identities can be proved using **membership tables**. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity.
- To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Example

Question: Use a membership table to show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution:

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0

Computer Representation of Sets

- There are various ways to represent sets using a computer.
- One method is to store the elements of the set in an unordered fashion.
- However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements.
- A better method is to store elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Cont….

- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).
- First, specify an arbitrary ordering of the elements of U,

for instance $a_1, a_2, \ldots, a_n$.

- Represent a subset A of U with the bit string of length n, where the i^{th} bit in

this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example

Question

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Cont…

Solution:

- The bit string that represents the set of odd integers in *U*, namely,

  {1, 3, 5, 7, 9}, has a 1 bit in the first, third, fifth, seventh, and ninth

  positions, and a zero elsewhere. Hence the bit string is; **10 1010**

  1010.

- The subset of all even integers in *U*, namely, {2, 4, 6, 8, 10}, is represented

  by the bit string **01 0101 0101**.

- The set of all integers in *U* that do not exceed 5, namely, {1, 2, 3, 4, 5}, is

Representing Complements

- To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1.
- This is because $x \in A$ if and only if $x \in A$.

Example:

Question: Using the previous example, what is the complement of the subset of the odd integers in U?

Solution:

- The bit string for the set {1, 3, 5, 7, 9} is 10 1010 1010.
- The bit string for the complement of this set is obtained by replacing 0_s with 1_s and vice versa. This yields the string **01 0101 0101**, which corresponds to the set {2, 4, 6, 8, 10}.

Representation of Union and Intersection

Union

- To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets.
- The bit in the i^{th} position of the bit string of the union is 1 if either of the bits in the i^{th} position in the two strings is 1 (or both are 1), and is 0 when both bits are 0.
- Hence, the bit string for the union is the bitwise *OR* of the bit strings for the

Cont….

Intersection

- The bit in the *i* th position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are).
- Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

Example

Question:

The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Cont….

Solution:

- The bit string for the union of these sets is

  11 1110 0000 ∨ 10 1010 1010 = 11 1110 1010,

  which corresponds to the set {1, 2, 3, 4, 5, 7, 9}.

- The bit string for the intersection of these

  sets is 11 1110 0000 A 10 1010 1010 = 10

  1010 0000,

which corresponds to the set {1, 3, 5}.

END