UNIT II

Relations and functions

Relation :-

Def^n: Let us Consider the sets $A = \{a_i\}$, $i = 1, 2, 3 . . . m$ and $B = \{b_j\}$, $j = 1, 2, 3, . . . n$ of order m and n respectively. Then R be the relation defined from the set A to B ~~and denoted by aRb~~, $a \in A$ & as the subset of $A \times B$ and it is denoted by aRb, where $a \in A$ and $b \in B$.

Here $A \times B = \{(a_i, b_j)\}$, $\forall\ i = 1, 2, 3, . . . m$ & $j = 1, 2, 3, 4, . . . n$.

Zero - One Matrix

Def^n: A zero-one matrix defined on the relation R is denoted by M_R or $M(R)$ where

$$M_R = M(R) = [m_{ij}]_{m \times n} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & & & \\ m_{m1} & \cdots & \cdots & m_{mn} \end{bmatrix}$$

Here m_{ij} is a dual function defined by

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

In particular, if $A = B$, then zero-one matrix $M(R)$ on a relation aRa is a matrix of order $n \times n$ and whose elements are as follows

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, a_j) \in R \\ 0, & \text{if } (a_i, a_j) \in R \end{cases}$$

Ex For ~~simple~~ example :-
Let us consider the sets $A = \{1, 2\}$ and $B = \{a, b\}$ and the relation R defined as

$$R = \{(1, a), (1, b), (2, b)\}$$

Here $A = \{a_1, a_2\} = \cancel{\{a, b\}}, \{1, 2\}$

$B = \{b_1, b_2\} = \{a, b\}$

$m_{11} = 1$ as $(a_1, b_1) = (1, a) \in R$

$m_{12} = 1$ as $(a_1, b_2) = (1, b) \in R$

$m_{21} = 0$ as $(a_2, b_1) = (2, a) \notin R$

$m_{22} = 1$ as $(a_2, b_2) = (2, b) \in R$

∴ Zero - one matrix of M(R) or M_R is

$$M_R = M(R) = [m_{ij}]_{2\times 2} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$M(R) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Diagraph or Directed Graph

R on set $A = \{a, b, c\} \longrightarrow A \times A$

$R = \{(a, a), (a, b), (b, c)\}$

edge (arrows)

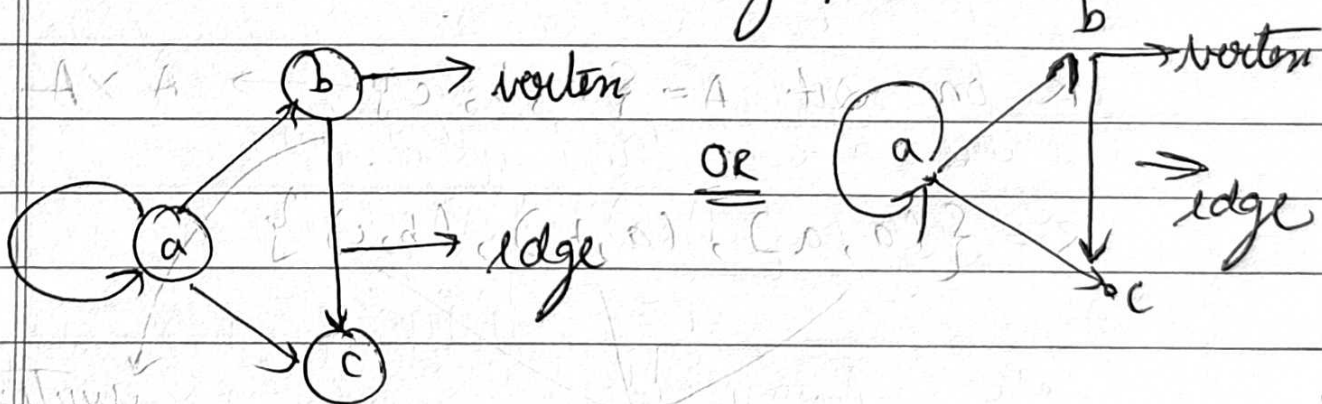
vertex (Bullet, Dot, circles)

Let R be the relation defined on a finite set A. Then R can be pictorially represented as small circles for each element of A called vertices or nodes. And draw an arrow from one vertex 'a' to another vertex b if $(a, b) \in R$, is called an edge.

The resulting pictorial representation of relation R on a finite set A is called digraph or directed graph of R

For Example:— Let us consider a set
$A = \{a, b, c\}$ and a relation
$R = \{(a, a), (a, b), (a, c), (b, c)\}$

Then pictorial representation of relation R on set A is (diagraph)



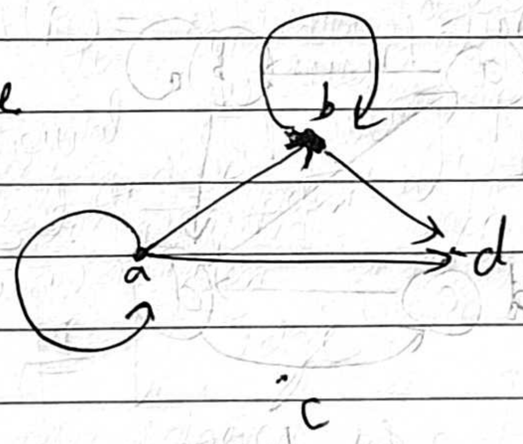
Origin / source of edge and Terminus of edge

In a diagraph, a vertex from which an edge leaves is called "Origin / source of edge and a vertex where an edge ends is called terminus of edge

Isolated Vertex :-

A vertex in a diagraph which is neither a source nor a terminus is called isolated vertex

For example



Here c is isolated vertex.

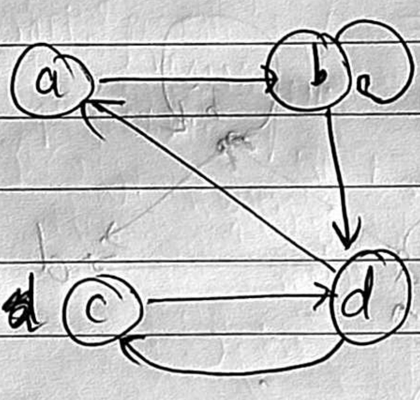
Loop :

An edge for which the source and Terminus is same vertex is called loop

In-degree and out degree

The number of edges terminating at a vertex is called in-degree of that vertex and the number of edges leaving a vertex is called out degree of that vertex

For example: Consider a set $A = \{a, b, c, d\}$ and the relation $R = \{(a,b), (b,b), (b,d), (c,d), (d,a), (d,c)\}$ defined on A. Then digraph of R is



Vertex	In-degree	out-degree
a	1	1
b	2	2
c	1	1
d	2	2

Examples

1) Let $A = \{1, 2\}$, $B = \{p, q, r, s\}$ and let the relation R from A to B defined by

$$R = \{(1, q), (1, r), (2, p), (2, s)\}$$

write down zero - one matrix of R

Sol: Let $A = \{a_1, a_2\} = \{1, 2\}$
$B = \{b_1, b_2, b_3, b_4\} = \{p, q, r, s\}$
$R = \{(1, q), (1, r), (2, p), (2, s)\}$

we ~~kw~~ K.T

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

$m_{11} = 0$ as $(a_1, b_1) \equiv (1, p) \notin R$
$m_{12} = 1$ as $(a_1, b_2) \equiv (1, q) \in R$
$m_{13} = 1$ as $(a_1, b_3) \equiv (1, r) \in R$
$m_{14} = 0$ as $(a_1, b_4) \equiv (1, s) \notin R$
$m_{21} = 1$ as $(a_2, b_1) \equiv (2, p) \in R$
$m_{22} = 0$ as $(a_2, b_2) \equiv (2, q) \notin R$
$m_{23} = 0$ as $(a_2, b_3) \equiv (2, r) \notin R$
$m_{24} = 1$ as $(a_2, b_4) \equiv (2, s) \in R$

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∴ zero-one matrix of relation R is

$$M_R = M(R) = [m_{ij}]_{2\times4} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(2) Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $(a,b) \in R$ if and only if $a < b$, write down R as a set of ordered pairs. Also write down zero-one ~~order~~ matrix of the relation

$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

Sol^n Let $A = \{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 4\}$
the relation R is defined as ~~(a_i, a_j)~~
$(a, b) \in R$
if and only if $a < b$ OR $(a_i, a_j) \in R$
if and only if $a_i < a_j$

$(a_1, a_1) = (1,1) \notin R$ as $1 \not< 1$
$(a_1, a_2) = (1, 2) \in R$, as $1 < 2$
$(a_1, a_3) = (1,3) \in R$, as $1 < 3$

$(a_1, a_4) = (1,4) \in R$, as $1 < 4$
$(a_2, a_1) = (2,1) \notin R$, as $2 \nless 1$
$(a_2, a_2) = (2,2) \notin$, as $2 \nless 2$
$(a_2, a_3) = (2,3) \in R$, as $2 < 3$
$(a_2, a_4) = (2,4) \in R$, as $2 < 4$
$(a_3, a_1) = (3,1) \notin R$, as $3 \nless 1$
$(a_3, a_2) = (3,2) \notin R$, as $3 \nless 2$
$(a_3, a_3) = (3,3) \notin R$, as $3 \nless 3$
$(a_3, a_4) = (3,4) \in R$, as $3 < 4$
$(a_4, a_1) = (4,1) \notin R$, as $4 \nless 1$
$(a_4, a_2) = (4,2) \notin R$, as $4 \nless 2$
$(a_4, a_3) = (4,3) \notin R$, as $4 \nless 3$
$(a_4, a_4) = (4,4) \notin R$, as $4 \nless 4$

$\therefore$ The relation R on a set A is

$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

W.K.T

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, a_j) \in R \\ 0, & \text{if } (a_i, a_j) \in R \end{cases}$$

$m_{11} = 0$, as $(a_1, a_1) = (1,1) \notin R$
$m_{12} = 1$, as $(a_1, a_2) = (1,2) \in R$
$m_{13} = 1$, as $(a_1, a_3) = (1,3) \in R$
$m_{14} = 1$, as $(a_1, a_4) = (1,4) \in R$

$m_{21} = 0$, as $(a_2, a_1) = (2,1) \notin R$

$m_{22} = 0$, as $(a_2, a_2) = (2,2) \notin R$

$m_{23} = 1$, as $(a_2, a_3) = (2,3) \in R$

$m_{24} = 1$, as $(a_2, a_4) = (2,4) \in R$

$m_{31} = 0$, as $(a_3, a_1) = (3,1) \notin R$

$m_{32} = 0$, as $(a_3, a_2) = (3,2) \notin R$

$m_{33} = 0$, as $(a_3, a_3) = (3,3) \notin R$

$m_{34} = 1$, as $(a_3, a_4) = \cancel{(4,1) \notin R}\ (3,4) \in R$

$m_{41} = 0$, as $(a_4, a_1) = (4,1) \notin R$

$m_{42} = 0$, as $(a_4, a_2) = (4,2) \notin R$

$m_{43} = 0$, as $(a_4, a_3) = (4,3) \notin R$

$m_{44} = 0$, as $(a_4, a_4) = (4,4) \notin R$

$\therefore$ The zero-one matrix of R is

$$M_R = M(R) = [m_{ij}]_{4\times 4} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 Determine the relation R from set A to set B as represented by the following matrix

$$M(R) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol^n: Given that

$$M(R) = [m_{ij}]_{m\times n} = [m_{ij}]_{3\times 4} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

It is clear that set A consists of 3 elements and set B consists of 4.

$\therefore\ n(A) = 3,\quad n(B) = 4$

Let us take

$A = \{a_1, a_2, a_3\}$; $B = \{b_1, b_2, b_3, b_4\}$

$m_{11} = 1 \Rightarrow (a_1, b_1) \in R$

$m_{22} = 1 \Rightarrow (a_2, b_2) \in R$

$m_{32} = 1 \Rightarrow (a_3, b_2) \in R$

$m_{33} = 1 \Longrightarrow (a_3, b_3) \in R$

Remaining ordered pairs does not belongs to the relation R

$\therefore R = \{(a_1, b_1), (a_2, b_2), (a_3, b_2), (a_3, b_3)\}$

(4) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by $x R y$ if and only if "x divides y" written as x / y

(i) write down R as a set of ordered pair
(ii) Draw the diagraph of R
(iii) Determine the in-degree and out-degree of the vertices of in the diagraph

sol: Let $A = \{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 4\}$

The relation R on set A is defined as follows $x R y$ iff only if x/y

OR

$(x, y) \in R$ if and only if "x divides y"

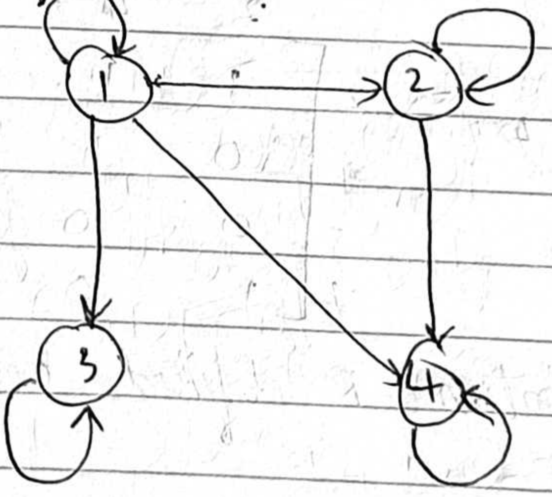
$(1, 1) \in R$ as $1/1$
$(1, 2) \in R$ as $1/2$
$(1, 3) \in R$ as $1/3$
$(1, 4) \in R$ as $1/4$

$(2,1) \notin R$, as $2 \nmid 1$
$(2,2) \in R$, as $2 | 2$
$(2,3) \notin R$, as $2 \nmid 3$
$(2,4) \in R$, as $2 | 4$
$(3,1) \notin R$, as $3 \nmid 1$
$(3,2) \notin R$, as $3 \nmid 2$
$(3,3) \in R$, as $3 | 3$
$(3,4) \notin R$, as $3 \nmid 4$
$(4,1) \notin R$, as $4 \nmid 1$
$(4,2) \notin R$, as $4 \nmid 2$
$(4,3) \notin R$, as $4 \nmid 3$
$(4,4) \in R$, as $4 | 4$

(i) The relation R defined on set A is

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(ii) Diagraph of the relation R is



(iii)

Vertex	In-Degree	Out-Degree
1	1	4
2	2	2
3	2	1
4	3	1

(5) Let $A = \{2, 4, 5, 7\}$ and R be the relation on A having the matrix

$$M(R) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Construct the diagraph of R

Sol^n Given that $A = \{2, 4, 5, 7\} = \{a_1, a_2, a_3, a_4\}$

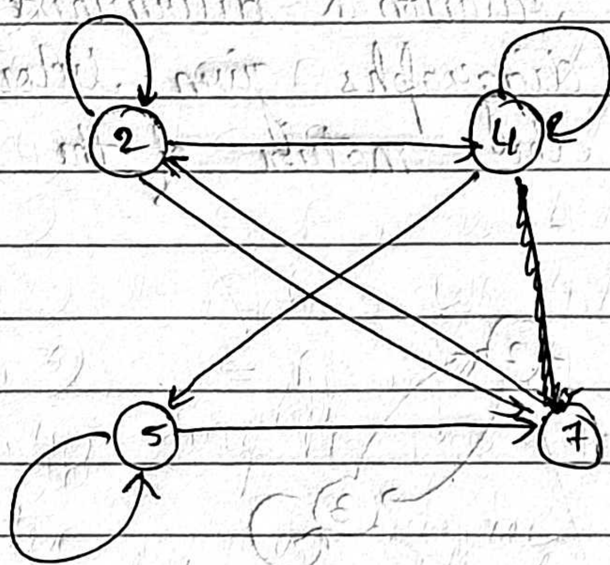
$$M(R) = [m_{ij}]_{4\times4} = \begin{array}{c|cccc} & 2 & 4 & 5 & 7 \\ \hline 2 & 1 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \\ 5 & 0 & 0 & 1 & 1 \\ 7 & 1 & 0 & 0 & 0 \end{array}$$

∴ The relation R defined on set A is

(1, 4)

$R = \{(2,2), (2,4), (2,7), (4,4), (4,5), (5,5), (5,7), (7,2)\}$

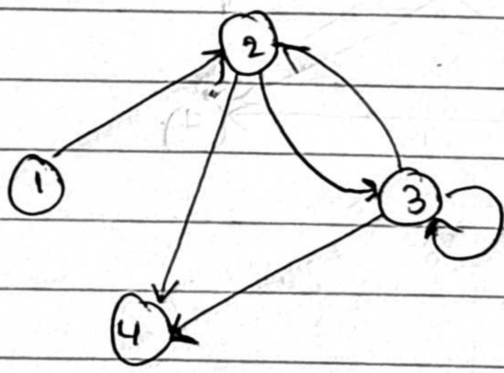
The diagraph of R is



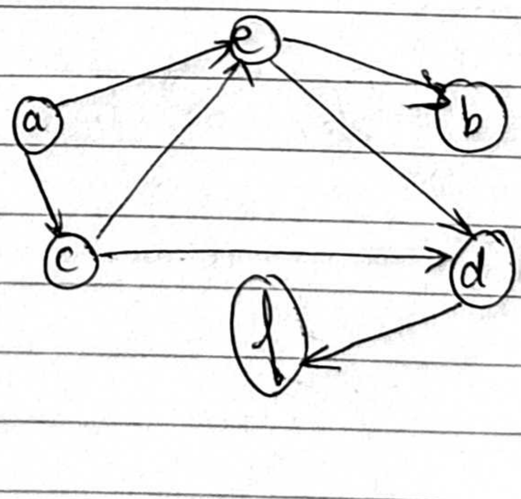
(6) ~~Let $A = \{2, 4, 5, 7\}$ and R be the relation on A having the matrix M~~

(8) Find the relation R determined by each of the diagraphs given below. Also write the matrix of the relation.

(i)



(ii)



Sol: Let A be the set defined on diagraph (i)

$A = \{1, 2, 3, 4\}$

∴ Relation R of the diagraph (i) is

$R = \{(1,2), (2,3), (2,4), (3,3), (3,2)$
$(3,4)\}$

∴ Zero-one matrix associated with above relation

$$M(R) = [m_{ij}]_{4\times4} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & m_{11} & m_{12} & m_{13} & m_{14} \\ 2 & m_{21} & m_{22} & m_{23} & m_{24} \\ 3 & m_{31} & m_{32} & m_{33} & m_{34} \\ 4 & m_{41} & m_{42} & m_{43} & m_{44} \end{array}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) Let B be the set of digraph (ii)

$B = \{a, b, c, d, e, f\}, \ n(B) = 6$

∴ Relation R associated with digraph (ii) is

$R = \{(a,c), (a,e), (c,d), (c,e),$
$(d,f), (e,b), (e,d)\}$

∴ Zero-one matrix of the relation R is

$$M(R) = [m_{ij}]_{6\times 6} = \begin{array}{c|c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \hline \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} m_{11} & \cdots & \cdots & \cdots & \cdots & m_{16} \\ m_{21} & \cdots & \cdots & \cdots & \cdots & m_{26} \\ \vdots & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ m_{61} & \cdots & \cdots & \cdots & \cdots & m_{66} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note

Reflexive : $a R a \Rightarrow (a, a) \in R, \ \forall \ a \in A$

Symmetric : if $(a, b) \in R$, then $(b, a) \in R$

Transitive : if $(a, b) \in R$ & $(b, c) \in R$, then $(a, c) \in R$

Equivalence : Reflexive, Symmetric & Transitive

Properties of Relations

1) **Reflexive Relation** : A relation R on set A is said to reflexive if any element a of A is related to itself.

i.e $(a,a) \in R$, $\forall a \in A$

2) **Symmetric Relation** : A relation R on set A is said to be symmetric if $(a,b) \in R$, then $(b,a) \in R$, $\forall a,b \in A$

3) **Transistive Relation** :- A relation R on set A is said to transistive if $(a,b) \in R$, and $(b,c) \in R$, then $(a,c) \in R$, $\forall a,b,c \in A$

Equivalence Relation

A relation R on a set A is said to be equivalence on A if R is reflexive, symmetric and transitive on A.

Examples

(1) Let $A = \{1, 2, 3, 4\}$ and the relation
$R = \{(1,1), (1,2), (2,1), (3,4), (4,3)$
$(2,2), (4,4)\}$

be the relation on A, verify that R is an equivalence relation

Sol^n Let $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (3,3), (2,2), (3,4), (4,3), (4,4)\}$

(i) Reflexive Relation : R is reflexive relation

$\because (a, a) \in R, \forall a \in A$ i.e $((1,1), (2,2), (3,3), (4,4)) \in R$

(ii) Symmetric Relation : R is symmetric relation

$\because (a, b) \in R$, then $(b, a) \in R$

i.e $(2,1)\ (1,2)\ (3,4)\ (4,3) \in R$

~~(2,1) and (1,2)~~

$(1,2)$ and $(2,1) \in R$

$(3,4)$ and $(4,3) \in R$

(iii) Transitive Relation :- R is transitive relation

$\because$ ~~$(a,b) \in R$~~ if (a, b) & $(b, c) \in R$, then $(a, c) \in R$

i.e. $(3,3)$ & $(3,4) \in R \Rightarrow (3,4) \in R$

∴ R is reflexive, symmetric and transitive on A

∴ R is equivalence relation.