8/1/2020

UNIT 2

RELATIONS AND FUNCTIONS

SET THEORY

Set : It is a collection of well defined objects

The objects of a set are called the members or the elements of a set.

For describing a set 2 methods are commonly used

a. **Tabulation Method :**

Here all the elements of set are written down within the brackets.

Eg : $S = \{1, 2, 3, 4, 5\}$

b. **Rule Method :**

Here the set is specified by stating a characteristic property which all the elements of a set prossess.

Eg : $S = \{x \mid x \text{ is a +ve odd integer}\}$

Subset :

If every element of set A is also an element of set B then A is called the subset of B written as $A \subseteq B$

Proper Subset :

If every element of set A is an element of set B and B contains atleast 1 element which does not belong to A, then A is proper subset of B. $A \subset B$

Power Set :

It is the set A is defined as the family consisting of all the subsets of set A and is denoted by P.

Note :

If finite set A has n elements, then $P(A)$ has 2^n elements.

Cartesian Product of Set :

Let A and B be 2 sets, ~~then~~ then the set of all ordered pairs (a, b) where $a \in A$, $b \in B$ is called cartesian product or cross product or product set of A and B, and is denoted by $A \times B$

$$A \times B = \{ (a, b) \mid a \in b \text{ and } b \in B \}$$

Note :

$$A \times B \neq B \times A$$

also $(a, b) \neq (b, a)$.

eg: $A = \{1, 2, 3\}$ $B = \{4, 5, 6, 7, 8\}$

$|A| = 3$ $|B| = 5$

$|A \times B| = |A| \times |B| = 3 \times 5 = 15$

In general

If set A has m elements and set B has n element

then $|A| = m$, $|B| = n$

$|A \times B| = m \times n$.

Also, $|P(A \times B)| = 2^{mn}$

<u>Note</u>:

$|A \times B| = |B \times A|$

$|A \times A| = |A|^2$.

<u>9/1/2020.</u>

So with the ordered pairs $(a_1, a_2, \ldots a_k)$ $(b_1, b_2, \ldots b_k)$, $a_1, a_2, \ldots a_k$ and $b_1, b_2, \ldots b_k$ are k tuples then

$a_i = b_i$ iff $(a_1, a_2, \ldots a_k) = (b_1, b_2, \ldots b_k)$

<u>Results</u>:

i. $A \times (B \cup C) = A \times B \cup A \times C$

ii. $(A \cup B) \times C = A \times C \cup B \times C$

iii. $A \times (B \cap C) = A \times B \cap A \times C$

iv. $(A \times B) \cap C = A \times C \cap B \times C$

v. $A \times (B - C) = A \times B - A \times C$.

Proof :

i. Let $(x,y) \in A \times (B \cup C) \Leftrightarrow x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow (x \in A)$ and $(y \in B$ or $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Leftrightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$

$\Leftrightarrow (x,y) \in \{A \times B \cup A \times C\}$

$\therefore A \times (B \cup C) = A \times B \cup A \times C$

iii. Let $(x,y) \in A \times (B \cap C)$

$\Leftrightarrow x \in A$ and $(y \in B \cap C)$

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Leftrightarrow (x,y) \in A \times B$ and $(x,y) \in A \times C$

$\Leftrightarrow (x,y) \in \{A \times B \cap A \times C\}$

$\therefore A \times (B \cap C) = A \times B \cap A \times C$.

v. Let $(x,y) \in A \times (B - C)$

$\Leftrightarrow x \in A$ and $(y \in (B - C))$

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \notin C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C)$

$\Leftrightarrow (x,y) \in (A \times B)$ and $(x,y) \notin A \times C$

$\Leftrightarrow (x,y) \in \{A \times B - A \times C\}$

$\therefore A \times (B - C) = A \times B - A \times C$

Examples :

1. If $A = \{1, 2, 3\}$ $B = \{2, 3\}$ $C = \{4, 6\}$
Find i. $A \times B$ ii. $A \cup (B \times C)$

Soln :

i. $A \times B = \{(1, 2)(1, 3)(2, 2)(2, 3)(3, 2)(3, 3)\}$

ii. $B \times C = \{(2, 4)(2, 6)(3, 4)(3, 6)\}$

$A \cup (B \times C) = \{(1, 2, 3)(2, 4)(2, 6)(3, 4)(3, 6)\}$

2. Find x and y in the following :

i. $(2x, x+y) = (6, 1)$

ii. $(y-2, 2x+1) = (x-1, y+2)$

Soln

i. $2x = 6$ $\quad$ $x + y = 1$

$x = 3$ $\quad$ $y = -2$

ii. $y - 2 = x - 1$ $\quad$ $2x + 1 = y + 2$

$x - y = -1$ $\quad$ $2x - y = 1$

$$\begin{array}{r} 2x - y = 1 \\ -\; x \underset{+}{-} y = \underset{+}{-} 1 \\ \hline x = 2 \end{array}$$

$x - y = -1$

$x + 1 = y$

$y = 3.$

Suppose $A, B, C \subseteq Z \times Z$ with $A = \{(x,y) \mid y = 5x - 1\}$

$B = \{(x,y) \mid y = 6x\}$ $C = \{(x,y) \mid 3x - y = 7\}$

Find i $A \cap B$ ii. $\overline{\bar{A} \cup \bar{C}}$ iii. $B \cap C$ iv. $\bar{B} \cup \bar{C}$

Sol^n: i. Let $(x,y) \in A \cap B$

$\iff x \in A$ and $y \in B$

$\iff y = 5x - 1$ and $y = 6x$

$5x - 1 = 6x$

$x = -1$ and $y = -6$.

$\therefore A \cap B = (-1, 6)$.

ii. We have $\overline{A \cap C} = \bar{A} \cup \bar{C}$

$\overline{\bar{A} \cup \bar{C}} = \overline{\overline{A \cap C}} = A \cap C$

$(x,y) \in A \cap C$.

$\iff (x,y) \in A$ and $(x,y) \in C$

$\iff y = 5x - 1$ and $3x - y = 7$

$$\begin{array}{r} 5x - y = 1 \\ -3x \mp y = -7 \\ \hline 2x = -6 \end{array}$$

$x = -3$

$y = -15 - 1$

$y = -16$

$\therefore A \cap C = (-3, -16)$

Relations :

Let A and B be 2 sets then a subset of $A \times B$ is called a relation or a binary relation from A to B. Thus if R is a relation from A to B then R is a set of ordered pairs (a,b) where ~~(a,b)~~ $a \in A$ and $b \in B$.

And conversely, if R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$ then R is a relation from A to B

- If $(a,b) \in R$ we say that "A is related to B by R" and is denoted as aRb.

Note :

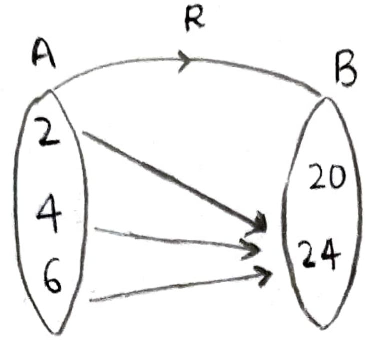
If R is a relation from $A \to A$ i.e. R is a subset of $A \times A$ then we say that R is a binary relation on A.

10/1/19

Eg: $A = \{2, 4, 6\}$ $B = \{20, 24\}$

$A \times B = \{(2,20)(4,20)(6,20)$
$(2,24)(4,24)(6,24)\}$

$R = \{(2,24)(4,24)(6,24)\}$



1. If $A = \{1, 2, 3\}$ then the subsets of A are

$P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$
$\{A\}\}$

$= 2^3$

2. Let $A = \{1, 2, 3\}$ $B = \{2, 4, 5\}$ then determine the following

i. $|A \times B|$

ii. no. of relations from $A \rightarrow B$

iii. no. of binary relations on A

iv no. of relation from $A \rightarrow B$ that contains exactly 5 ordered pairs

v. no. of binary relation on A that contain atleast 7 ordered pair.

vi. No. of relations from $A \rightarrow B$ that contain $(1,2)$ $(1,5)$

Soln: i. $|A \times B| = |A| \times |B| = 3 \times 3 = 9$

ii. $2^9 = 512$

iii. $2^9 = 512$

iv. ${}^9C_5 = 126$

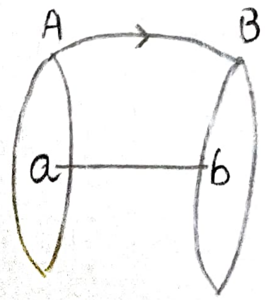
v. ${}^9C_7 + {}^9C_8 + {}^9C_9 = 36+9+1 = 46$

vi.

FUNCTIONS :

Let A and B be 2 non empty sets then a function (mapping) f from $A \to B$ is a relation from $A \to B$ such that for each $a \in A$ and there is a unique $b \in B$ such that $(a,b) \in f$ and is written as $b = f(a)$. Here b is called the image of a and a is called the pre-image of b under f.

A function from A to B is written as $f : A \to B$.



Note :

Every function is a relation, but relation need not be a function,
because if R is a relation from $A \to B$ then, an element of A can be related to 2 different elements of B under R.
but under a function an element of A can be related to only one element / image in B

For the function $f: A \to B$ A is called domain of f and B is co-domain of f. The subset of B consisting of the images of all the elements of A under f is called the range and is denoted by $f(A)$.

11/1/2020

Type of functions :

i. <u>identity function</u> : A function $f: A \to A$ such that $f(a) = a \ \forall \ a \in A$ is called identity function. and is denoted by I_A

ii. <u>Constant function</u> : A function $f: A \to B$ such that $f(a) = c \ \forall \ a \in A$ is a constant function.
Here all the elements of A have the same image in B

iii. <u>onto function</u>: A function $f: A \to B$ is onto if every element of B has a pre-image in A

iv. <u>one one function</u>. Here different elements of A have different images in B under f
i.e. every element of A has a unique image in B and every element of f(a) has unique pre-image in A

<u>One to one correspondance</u>:

A function which is both one one and onto is called one to one correspondance.
Here every element of A has unique image in B and every element of B has a unique image in A.

<u>Zero-one matrices and diagraphs</u>

Consider the sets $A = \{a_1, a_2, \ldots a_m\}$ and $B = \{b_1, b_2, \ldots b_n\}$ of orders m and n respt.
the $A \times B$ consists of all ordered pairs of the form (a_i, b_j) where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ which are mn in numbers

Let R be a relation from $A \rightarrow B$ so that R is a subset of $A \times B$.

Let $m_{ij} = (a_i, b_j)$ and let us assign 0 or 1 accordingly as $(a_i, b_j) \notin R$ and $(a_i, b_j) \in R$.

The $m \times n$ matrix formed by these m_{ij}'s is called the matrix of the relation R or the relation matrix and is denoted by M_R or $M(R)$

Since $M(R)$ contains only 0's and 1's as its elements, $M(R)$ is called zero-one matrix of R

<u>Note</u>:

- In $M(R)$ rows corresponds to elements of set A and column to elements of set B
- $M(R)$ is a zero matrix, if $R = \phi$
- Every element of $M(R)$ is 1 if $R = A \times B$

eg: $A = \{0, 1, 2\}$ $B = \{p, q\}$

$A \times B = \{(0,p)\ (1,p)\ (2,p)\ (0,q)\ (1,q)\ (2,q)\}$

~~$R = \{(0,p)\}$~~

$R = \{(0,p)\ (1,q)\ (2,p)\}$

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

16/1/2020

Digraph :

Let R be a relation on finite set A, draw a small circle for each element of A and label it these circles are called the vertices or nodes, draw an arrow called an edge from a vertex 'x' to a vertex 'y' iff $(x, y) \in R$. The resulting representation is called digraph / directed graph of R.

Here a vertex from which an edge leaves is called the origin or the source of that edge and the vertex where the edge ends is called terminus for that edge.

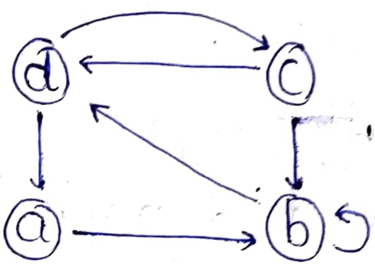
The vertex which is neither a source nor a terminus is called an isolated vertex.

An edge for which the source and the terminus are one and the same is called a loop.

The no of edges terminating at the vertex is called in-degree of that vertex, and The no. of edges leaving the vertex is called out-degree of that vertex.

Eg: $A = \{a, b, c, d\}$

Let $R = \{(a,b)\ (b,b)\ (b,d)\ (c,b)\ (c,d)\ (d,a)\ (d,c)\}$.



Vertices	a	b	c	d
in-degree	1	3	1	2
out-degree	1	2	2	2.

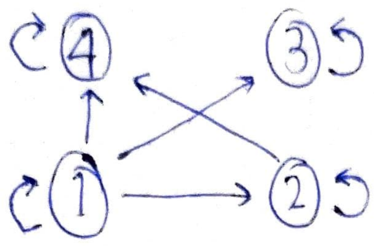
1. Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy iff x divides y

a. write down R as set of orderedpairs

b. draw the graph of R

c. determine the in-degree and out-degrees of the vertices in the diagram

sol^n a. $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)$
$(2,2)\ (2,4)\ (3,3)\ (4,4)\}$

b. 

c

vertices	1	2	3	4
in-degree	1	2	2	3
Out-degree	4	2	1	1

2. Determine the relation R from set A to set B as described by the following.

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

sol^n: Since $|A| = 4$ and $|B| = 3$

$\therefore A = \{a_1, a_2, a_3, a_4\}$ $B = \{b_1, b_2, b_3\}$

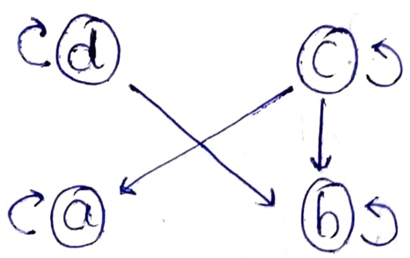
$R = \{(a_1, b_1)\ (a_1, b_3)\ (a_2, b_1)\ (a_2, b_2)\ (a_3, b_3)$
$(a_4, b_1)\}$

3. Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

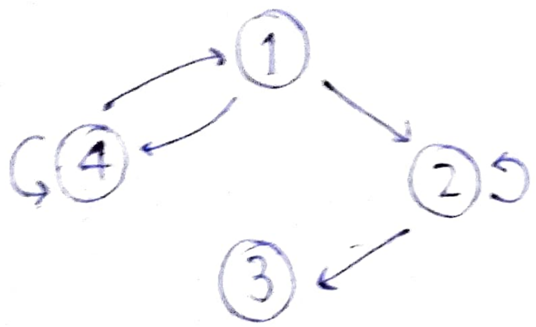
Construct a digraph of R and list the in-degrees and out-degrees of all the vertices.

$R = \{(a,a)\ (b,b)\ (c,a)\ (c,b)\ (c,c)\ (d,b)\ (d,d)\}$



vertices	in-degree	out-degree
a	2	1
b	3	1
c	1	3
d	1	2

4 Find the relation represented by the digraph also write down its matrix



soln: $R = \{(1,2)\ (1,4)\ (2,2)\ (2,3)\ (4,1)\ (4,4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

5. Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy iff $y = 2x$

a. write down R as set of ordered pair

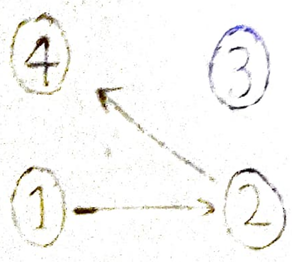
b. draw the diagraph of R

c. determine the in-degrees and out-degrees of the vertices in the digraph

soln:

a. $R = \{(1,2)\ (2,4)\}$

b.

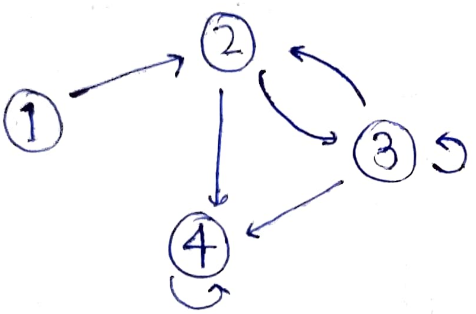


c.

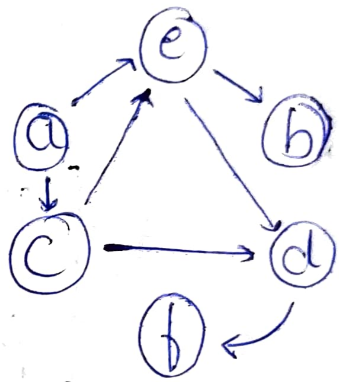
vertices	1	2	3	4
in degree	0	1	0	1
Out degree	1	1	0	0

6 Find the relation R determined by the digraphs below, also write down the matrix of the relation.

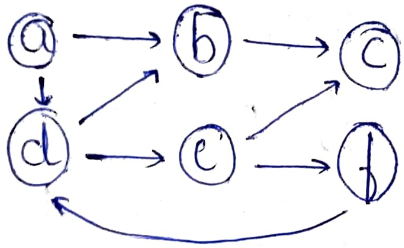
i.



ii.



iii.



Soln: i. R = { (1,2) (2,3) (2,4) (3,2) (3,3) (3,4) (4,4) }

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii. $R = \{ (a,c)\ (a,e)\ (c,d)\ (c,e)\ (d,f)\ (e,b)\ (e,d) \}$

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

iii. $R = \{ (a,b)\ (a,d)\ (b,c)\ (d,b)\ (d,e)\ (e,c)\ (e,f)\ (f,d) \}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

17/1/2020

Properties of Relation

1 Reflexive Relation:

A relation R on a set A is said to be reflexive if $\forall$ $a \in A$, ordered pair $(a,a) \in R$

R is not reflexive, if there is some $a \in A$ such that $(a,a) \notin R$

Note:

i. The matrix of reflexive relation must have 1's on its main diagonal

ii. At every vertex of the digraph of a reflexive relation there must be a cycle of length 1 or loop.

2. Irreflexive Relation:

A relation R on a set A is said to be irreflexive if $(a,a) \notin R$ for any $a \in A$ i.e. R is irreflexive if no element of A is related to itself by R

Note:

i. The matrix of an irreflexive relation must have 0's on its main diagonal

ii. The digraph of an irreflexive relation has no cycle of length 1 at any vertex

Eg : Let $A = \{1, 2, 3\}$
$R_1 = \{(1,1)(1,2)\}$
R_1 is not reflexive because $(2,2)$ and $(3,3) \notin R_1$
R_1 is not irreflexive because $(1,1) \in R_1$

3. <u>Symmetric Relation</u> :

A relation R on a set is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ $\forall a, b \in A$

A relation which is not symmetric is called Asymmetric relation

Eg : $A = \{1, 2, 3\}$

$R_1 = \{(1,2)(2,1)\}$

$R_2 = \{(1,1)(1,3)(1,2)(2,1)\}$

R_1 is symmetric

R_2 is Asymmetric because $(3,1) \notin R_2$

<u>Note</u> :

- Matrix of symmetric relation is symmetric matrix $(a_{ij} = a_{ji})$
- In digraph if there is an edge from vertex a to b, then there must be an edge b to a.

4. <u>Anti-Symmetric Relation</u>

A Relation R on a set A is anti-symmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

- A relation R is said to be not anti-symmetric if for $(a,b) \in R$ and $(b,a) \in R$ then $a \neq b$

- $M_R = [m_{ij}]$ is matrix of anti-symmetric relation then $\forall\ i \neq j$ we have either $m_{ij} = 0$ or $m_{ji} = 0$
- In the digraph of an anti-symmetric relation for 2 different vertices a and b there cannot be a bi-directional edge betn a and b

Eg: $A = \{1, 2, 3\}$

$R_1 = \{(1,1)\ (2,2)\}$ - both symmetric & anti-symmetric

$R_2 = \{(1,2)\ (2,1)\ (2,3)\}$ - not symmetric and not anti-symmetric

5. <u>Transitive</u>

A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall\ a,b,c \in A$

A relation R is not transitive if for every $a, b, c \in A$ $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \notin R$

Eg : $A = \{1, 2, 3\}$

$R_1 = \{(1,1)(1,2)(2,3)(1,3)(3,1)(3,2)\}$

$R_2 = \{(1,2)(2,3)(1,3)(3,1)\}$

Both are not transitive.

21/1/2020 :

1. $A = \{1, 2, 3, 4\}$. Detemine the nature of R

$R = \{(1,1)(1,2)(2,1)(2,2)(3,3)(3,4)(4,3)(4,4)\}$

Soln : R is reflexive, symmetric, transitive

2. $A = \{1, 2, 3\}$. Detemine the nature of the following relations.

$R_1 = \{(1,2)(2,1)(1,3)(3,1)\}$

$R_2 = \{(1,1)(2,2)(3,3)(2,3)\}$

$R_3 = \{(1,1)(2,2)(3,3)\}$

$R_4 = \{(1,1)(2,2)(3,3)(3,2)\}$

Soln : R_1 is symmetric, not reflexive, not transitive. irreflexive.

R_2 is reflexive, transitive

R_3 is reflexive, anti-symmetric, transitive.

R_4 is reflexive, transitive.

3. Find the nature of the relations

i. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

symmetric

ii. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Reflexive and Symmetric

$R_1 = \{ (a,b)\ (a,c)\ (b,a)\ (b,b)\ (c,a)\ (c,c)\ (c,d)$
$(d,c)\ (d,d) \}$

4. ST the relation R represented by the matrix is transitive

$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

22/1/2020

(Theorem)

5 On the set Z^+, relation R is defined by aRb iff 'a divides b'. PT R is reflexive, transitive, anti-symmetric, but not symmetric.

soln: Proof:

$\forall a \in Z^+$, the statement a divides b is true

i.e., aRa is true

Hence R is reflexive

for any $(a,b) \in Z^+$ a divides b need not imply b divides a (eg: 3 divides 6 but 6 does not divide 3)

$\therefore$ R is not symmetric.

'a divides b' and 'b divides c' implies 'a divides c'

$\therefore$ R is transitive

Also 'a divides b' and 'b divides a' implies b = a $\therefore$ R is anti-symmetric.

(Theorem)

6 Let R and S be 2 relations on set A then PT a] if R and S are reflexive so are $R \cap S$ and $R \cup S$ is also

b. if R and S are symmetric so are $R \cap S$ and $R \cup S$

c. if R and S are anti-symmetric so is $R \cap S$

d. if R and S are transitive so is $R \cap S$

soln. Proof :

a. for $a \in A$

$(a,a) \in R$ and $(a,a) \in S$

$\because$ R and S are reflexive.

$\Rightarrow (a,a) \in R \cap S$

$\Rightarrow$ ~~R~~ $R \cap S$ is also reflexive

IIIly $(a,a) \in R$ or $(a,a) \in S$

$\Rightarrow (a,a) \in R \cup S$

$\therefore R \cup S$ is also reflexive

b. Let $(a,b) \in R$ and $(a,b) \in S$

then $(b,a) \in R$ the $(b,a) \in S$.

$\Rightarrow (a,b) \in R \cap S$ also $(b,a) \in R \cap S$

$\therefore (R \cap S)$ is symmetric.

Let $(x,y) \in R$ or $(x,y) \in S$

then $(y,x) \in R$ or $(y,x) \in S$

$\Rightarrow (x,y) \in R \cup S$

$\therefore$ $R \cup S$ is symmetric.

c. Suppose $(a,b)\,(b,a) \in R \cap S$

$\Rightarrow (a,b)\,(b,a) \in R$ and $(a,b)\,(b,a) \in S$.

$\because$ R and S are anti-symmetric, it follows that $a = b$.

$\Rightarrow R \cap S$ is anti symmetric.

d. Let $(a,b)\,(b,c) \in R \cap S$

then $\exists\,(a,c) \in R$ and $(a,c) \in S$

$(a,c) \in R \cap S$

Equivalence Relation:

A relation R on set A is said to be an equivalence relation on A, if

i. R is reflexive

ii. R is symmetric

iii. R is transitive on A.

Eg: "is equal to" is example for equivalence relation

"is less than" is not an equivalence relation

1. Let $A = \{1, 2, 3, 4\}$, then

$R = \{(1,1)(1,2)(2,1)(2,2)(3,1)(3,3)(1,3)(4,1)(4,4)\}$

be a relation on A. IS R an equivalence relation

Soln: R is reflexive.

R is not symmetric because for $(4,1)$ $(1,4) \notin R$

R is not transitive $\because$ for $(4,1)$ and $(1,2)$

R is not equivalence relation $(4,2) \notin R$

Equivalence classes

Let R be an equivalence relation on a set A and $a \in A$, then the set of all those elements x of A which are related to A by R is called the equivalence class of A wrt R. This equivalence class ~~id~~ is denoted by $R(a)$ or $[a]$ or $\bar{a}$

$$[a] = \{ x \in A \mid (x, a) \in R \}$$

eg: $A = \{1, 2, 3\}$

$R = \{(1,1)\ (1,3)\ (2,2)\ (3,1)\ (3,3)\}$ is an equivalence relation.

Now $[1] = \{1, 3\}$

$[2] = \{2\}$

$[3] = \{1, 3\}$

13-01-2020

Partition of a Set :

Let A be a non-empty set, suppose if there exist non-empty subsets $A_1, A_2, \ldots\ldots A_k$ of set A such that the following conditions hold

i. A is the union of $A_1, A_2, \ldots\ldots, A_k$ i.e.

$A = A_1 \cup A_2 \ldots\ldots \cup A_k$

ii. Any 2 of the subsets $A_1, A_2, A_3 \ldots\ldots A_k$ are disjoint

i.e. $A_i \cap A_j = \phi$

Then the set $P = \{A_1, A_2, A_3, \ldots\ldots, A_k\}$ is called partition of set A also $A_1, A_2, \ldots\ldots, A_k$ are known as the blocks

Eg $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A_1 = \{1, 3, 5, 7\}$ $A_2 = \{2, 4\}$ $A_3 = \{6, 8\}$

$P = \{A_1, A_2, A_3\}$

1 For an equivalence relation $R = \{(1,1)(1,2)(2,1)(2,2)$ $(3,4)(4,3)(3,3)(4,4)\}$ defined on a set $A = \{1, 2, 3, 4\}$. Determine the partition induced

Soln. The equivalence classes are

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3, 4\}$

$[4] = \{3, 4\}$

$P = \{[1], [3]\}$

$P = \{\{1, 2\}, \{3, 4\}\}$

2. $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1)(2,2)(2,3)(3,2)(3,3)(4,4)(4,5)$ $(5,4)(5,5)\}$ is an equivalence relation on A

Find the partition of A induced by R.

Sol^n: $[1] = \{1\}$

$[2] = \{2, 3\}$

$[3] = \{2, 3\}$

$[4] = \{4, 5\}$

$[5] = \{4, 5\}$

$P = \{[1]\,[2]\,[5]\}$

$= \{\{1\}\{2, 3\}\{4, 5\}\}$

Note: Let A and B be 2 sets. Let R_1 and R_2 be relations on A and B then $R_1 \cup R_2$ is a relation from $A \to B$ such that $(a, b) \in R_1 \cup R_2$ iff $(a, b) \in R_1$ or $(a, b) \in R_2$

Ill^{ly}, $R_1 \cap R_2$ defined on $A \to B$ such that $(a, b) \in R_1$ and $(a, b) \in R_2$

Compliment of a relation:

The compliment of R, is denoted by $\bar{R}$, defined as the relation from $A \to B$ such that $(a, b) \in \bar{R}$ iff $(a, b) \notin R$

Converse of a relation:

Converse of R denoted R^c, is a relation from $B \to A$ with the property that if $(a,b) \in R^c$ iff $(b,a) \in R$

- If M_R is a matrix of R, then transpose of M_R is a matrix of $\bar{R}$
- $(R^c)^c = R$

Prove the following:

1. If R is an equivalence relation on a set A then so is R^c
2. If R and S are equivalence relations on set A then so is $R \cap S$
3. If R and S are equivalence relations on set A then $R \cup S$ need not be a equivalence relation.

1 Proof:

for any $(a,b) \in R^c \Rightarrow (b,a) \in R$

$\Rightarrow (a,b) \in R \quad \because$ R is symmetric

$\Rightarrow (b,a) \in R^c$

$\therefore R^c$ is symmetric.

for any $(a,a) \in R^c \Rightarrow (a,a) \in R$

$\therefore$ R^c is reflexive.

for any $(a,b)(b,c) \in R^c$

$\Rightarrow (b,a)(c,b) \in R$

$\Rightarrow (c,a) \in R$ $\because$ R is transitive

$\Rightarrow (a,b) \in R^c$

$\therefore$ R^c is transitive.

$\therefore$ R^c is equivalence relation.

2. Since R and S are equivalence relations they are reflexive, symmetric and transitive. Consequenty $R \cap S$ is reflexive, symmetric and transitive

3. Here R and S are equivalence relation and hence they are transitive

But $R \cup S$ need not be transitive, hence $R \cup S$ need not be an equivalence relation

24/1/2020

Eg: For a fixed integer $n>1$. the relation "Congruent modulo n" is an equivalence relation on the set of all +ve integer set

Note: Two integers a and b are congruent to mod n i.e. $a \equiv b \pmod n$ if

$a-b$ is divisble by n or a and b has the same remainder when divided by n

Soln. Proof

~~$\forall (a,b) \in$~~ $\cdot\ a, b \in Z$ "a is congruent to b modulo n" if $a-b$ is multiple of n

Let us denote $a \equiv b \pmod n$ by aRb, then to prove R is an equivalence relation.

$\forall\ a \in Z \quad a-a=0$, is multiple of n

i.e. $a \equiv a \pmod n$

$\Rightarrow aRa$

$\therefore$ R is reflexive

$\forall\ a, b \in Z$

$aRb \Rightarrow a \equiv b \pmod{n}$

$\Rightarrow (a-b)$ is multiple of n

$\Rightarrow (b-a)$ is multiple of n

$\Rightarrow b \equiv a \pmod{n}$

$\Rightarrow bRa$

$\therefore$ R is symmetric.

$\forall\ a, b, c \in Z$

aRb & bRc $\Rightarrow a \equiv b \pmod{n}$ and & $b \equiv c \pmod{n}$

$\Rightarrow (a-b)$ is multiple of n and $(b-c)$ is multiple of n.

$\Rightarrow (a-\cancel{b}+\cancel{b}-c)$ is multiple of n

$\Rightarrow a \equiv c \pmod{n}$

$\Rightarrow aRc$

$\therefore$ R is transitive

$\therefore$ R is equivalence relation.

* **Partial Orders:**

A relation R on a set A is said to be partial ordering relation or a partial order on A if
i. R is reflexive
ii. R is anti-symmetric
iii. R is transitive

The set A with partial order R defined on it is called a partially ordered set or an ordered set or a 'poset' and denoted by the pair (A, R)

Total Order:

Let R be a partial order on A, then R is called total order on A if $\forall x, y \in A$ either xRy or yRx

For eg: a partial order relation less than or equal to is total order on the set $\mathbb{R}$
$(\mathbb{R}, \leq)$ is totally ordered set.

Hasse diagram

It is the graphical representation of the relation of elements of partially ordered set with an implied orientation

Note : If there is an edge from A ← B and B ← C then because of transitivity there should be an edge from A → C But in Hasse diagram we need not execute

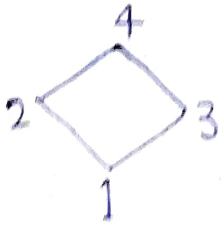
- $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (2,2)\ (2,4)\ (1,3)\ (3,3)\ (3,4)\ (1,4)$
$(4,4)\}$

R is reflexive, anti symmetric, transitive

So, R is poset on A

Hasse diagram :

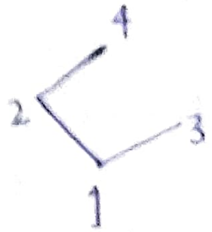


- $A = \{1, 2, 3, 4\}$ defined by xRy iff x divides y

Soln. $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4), (2,2)\ (2,4), (3,3)$
$(4,4)$

R is reflexive, anti symmetric, transitive

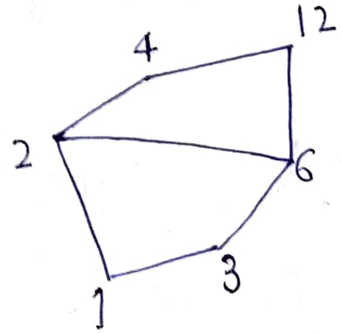
Hasse diagram :



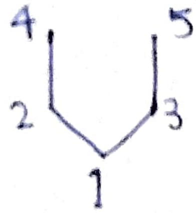
- A = {1, 2, 3, 4, 6, 12} aRb a divides b

R {(1,1)(1,2)(1,3)(1,4)(1,6)(1,12)
(2,4) (2,6) (2,12) (3,3) (2,2)
(3,6) (3,12) (4,4) (4,12) (6,6)
(6,12) (12,12)}

Hasse diagram :



Determine the matrix whose Hasse diagram is given below.

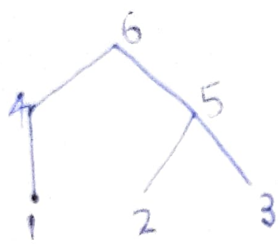


R = {(1,2) (1,3) (1,4) (1,5), (4,4)
(2,4) (3,5) (1,1) (2,2) (3,3) (5,5)}

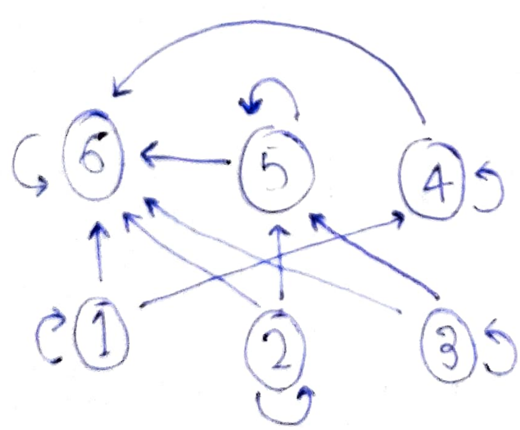
$$
M = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}
$$

31/1/2020

Hasse diagram of partial order R on a set A = {1, 2, 3, 4, 5, 6} is as given below. Write down ~~as~~ R as a subset of AXA Construct its ~~diagram~~ digraph



Soln: R = {(1,1) (1,4) (1,6) (2,2) (2,5) (2,6) (3,3) (3,5) (3,6) (4,4) (4,6) (5,5) (5,6) (6,6)}

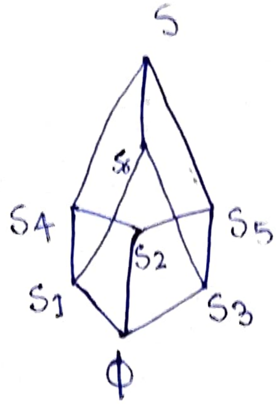


Let S = {1, 2, 3} & P(S) be the power set of S which defines the relation R by xRy iff $x \subseteq y$.
ST this relation is a partial order on P(S)
Draw the hasse diagram

Sol^n ϕ. S_1 = {1}, S_2 {2}, S_3 {3} S_4 {1,2},
S_5 {2,3}, S_6 {1,3}

ϕ is the subset of $S_1, S_2, S_3, S_4, S_5, S_6, S$

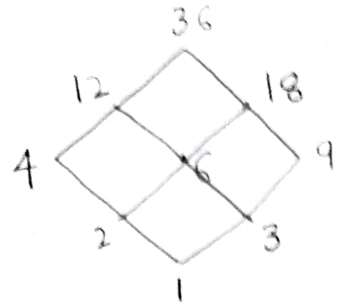
$S_1 \subseteq S_4, S_6, S$
$S_2 \subseteq S_4, S_5, S$
$S_3 \subseteq S_5, S_6, S$
$\phi \subseteq S_1, S_2, S_3, S_4, S_5, S_6$



ST the positive divisors of 36 is a poset
Draw Hasse diagram

Sol^n: A = {1, 2, 3, 4, 6, 9, 12, 18, 36}
R = { a divides b }
R = {(1,1) (1,2) (1,3) (1,4) (1,6) (1,9) (1,12)
(1,18) (1,36) (2,2) (2,4) (2,6) (2,12)
(2,18) (2,36) (3,3) (3,6) (3,9) (3,12)
(3,18) (3,36) (4,12) (4,18) (6,12)

(6,18) (6,36) (9,18) (9,36) (12,12) (12,36)
(6,6) (9,9)



Extremal elements of poset

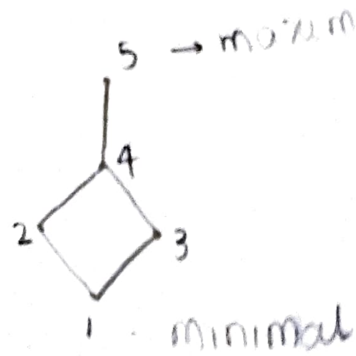
Consider a poset (A,R) we can define some special elements :

1. maximal element :

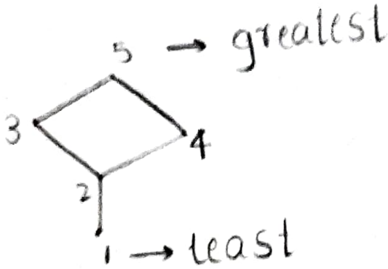
An element 'a' is a maximal element of A iff in the Hasse diagram of R no edge starts ~~terminates~~ at 'a'

2. minimal element

An element 'a' is a minimal element of A iff in the Hasse diagram of R no edges terminates at 'a'



3 An element $a \in A$ is called a <u>greatest element</u> of A if $xRa \ \forall x \in A$



4 An element $a \in A$ is called a <u>least element</u> of A if $aRx \ \forall x \in A$

4/2/2020

5 An element $a \in A$ is called an <u>upper bound</u> of a subset B of A if xRa for $x \in B$

6. An element $a \in A$ is called a <u>lower bound</u> of a subset B of A if aRx for all $x \in B$

7. An element $a \in A$ is the <u>least upper bound</u> (LUB) of a subset B of A if

i. a is an upper bound of B

ii. a' is an upper bound of B then aRa'

LUB is also called <u>Supremum</u>

8. An element $a \in A$ is called greatest lower bound (GLB) of a subset B of A if

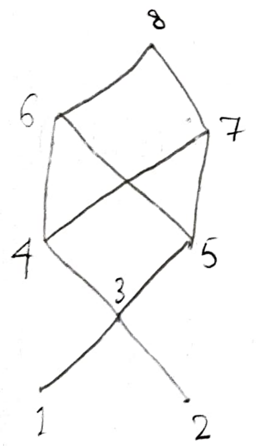
i. a is the lower bound of B

ii. a' is a lower bound of B.

then $a' R a$

GLB is also called infimum.

Eg 1: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and a partial order on A whose Hasse diagram is as below and let us consider subsets $B_1 = \{1, 2\}$ $B_2 = \{3, 4, 5\}$ of A.



i. $1R3$, $2R3$ $\therefore$ 3 is the upper bound for B_1
III'ly 4, 5, 6, 7, 8 are also the upper bounds of B_1

Since 3 is the lowest of all these upper bounds, therefore LUB of B_1 is 3

ii. For B_2.

Since for each $x \in B_2$. 6 is an upper bound of B_2 III'ly 7 and 8 are also upper bounds of B_2.

Since 6 is the least upper bound of B_2 but it is not related to the upper bound 7

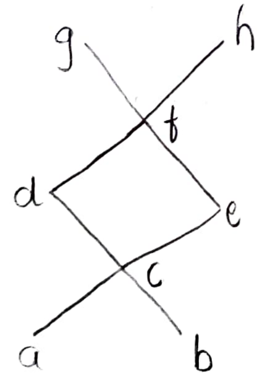
$\therefore$ B_2 has no LUB

Among 1, 2 and 3, 1R3, 2R3, 3R3. therefore 3 is the lower bound of B_2

Also 3 is the GLB of B_2

2. The Hasse diagram for the poset (A, R) is as given below.

If $B = \{c, d, e\}$ then find the following if they exist.



i. all upper bounds of B

ii. all lower bounds of B

iii. LUB of B

iv. GLB of B.

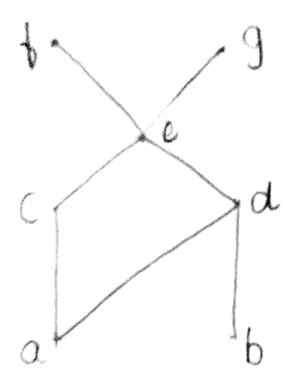
Sol^n i. f, g, h

ii. a, b, c

iii. f

iv. c

3. Consider the poset whose Hasse diagram is as below. Find LUB and GLB of $B = \{c, d, e\}$



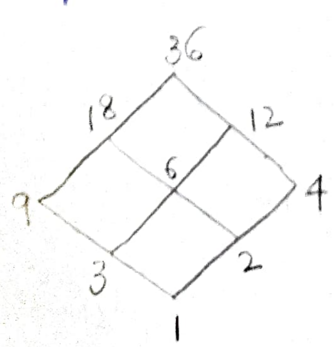
Sol^n: LUB(B) → e

GLB(B) → a

LATTICE:

Let (A, R) be a poset. This poset is called a lattice if $\forall x, y \in A$, the element LUB[x, y] and GLB[x, y] exists in A

1. ST the poset D(36) is a lattice.



5-2-2020

Sol^n Since $\forall\, x, y \in A$ there exist LUB and GLB

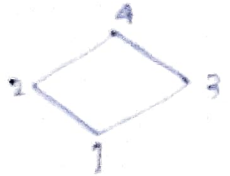
i.e for $\{9, 4\}$

LUB - 36

GLB - ~~4~~ 1

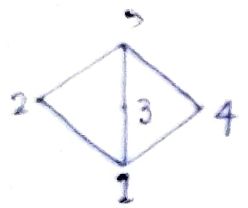
D36 is a lattice

2



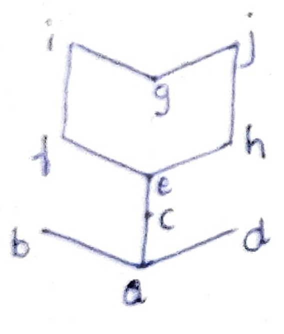
→ It is a lattice

3



- Lattice

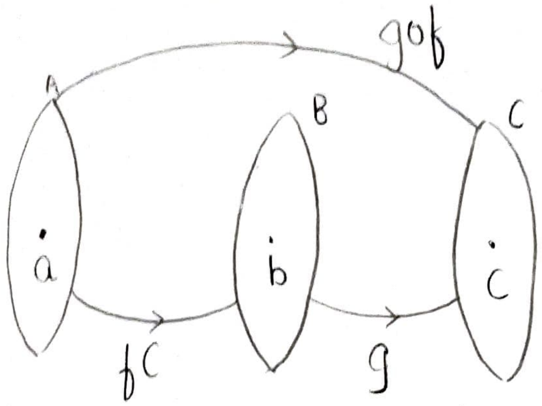
4



Compositions of functions.

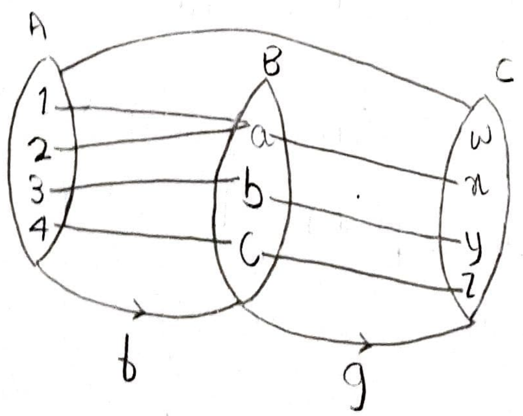
Let us consider 3 non empty sets A, B, .. (not necessarily distinct) and the funct^n $f: A \to B$ and $g: B \to C$. The composition (product

of these 2 funct^n is defined as the function gof from A to C i.e. $gof : A \to C$ with
$gof(a) = g\{f(a)\} \;\; \forall \, a \in A$



For a function $f : A \to A$ fof is denoted by f^2

1 Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c\}$
$C = \{w, x, y, z\}$ with $f : A \to B$ $g : B \to C$
given by $f = \{(1, a)\,(2, a)\,(3, b)\,(4, c)\}$
$g = \{(a, x)(b, y)(c, z)\}$ find gof



Sol^n: $gof(1) = g\{f(1)\} = g(a) = x$
$gof(2) = g\{f(2)\} = g(a) = x$

$gof(3) = g\{f(3)\} = g(b) = y$
$gof(4) = g\{f(4)\} = g(c) = z$
$\therefore gof = \{(1, x)(2, x)(3, y)(4, z)\}$

2. Consider the functions f and g defined by $f(x) = x^3$ $g(x) = x^2 + 1$ $\forall x \in R$. then find gof, fog, f^2, g^2

Soln

$$gof(x) = g\{f(x)\} = g(x^3) = (x^3)^2 + 1 = x^6 + 1$$

$$fog(x) = f\{g(x)\} = f(x^2+1) = (x^2+1)^3$$

$$f^2(x) = f\{f(x)\} = f(x^3) = (x^3)^3 = x^9$$

$$g^2(x) = g\{g(x)\} = g(x^2+1) = (x^2+1)^2 + 1$$

Note: gof and fog are not same.

3. Let f and g be the functions from $R \to R$ defined by $f(x) = ax + b$ $g(x) = 1 - x + x^2$ if $gof(x) = 9x^2 - 9x + 3$ then determine a and b

Soln: $gof(x) = g\{f(x)\} = g\{ax+b\} = 1 - (ax+b) + (ax+b)^2$

$$9x^2 - 9x + 3 = 1 \cdot (ax+b) + (ax+b)^2$$

$$9x^2 - 9x + 3 = 1 - ax - b + a^2x^2 + b^2 + 2abx$$

$$= (1 - b + b^2) + (-a + 2ab)x + a^2x^2$$

Comparing co-efficients we get :

$1 - b + b^2 = 3$ $\qquad$ $9 = a^2$

$b^2 - b - 2 = 0$ $\qquad$ $a = \pm 3$

$b = 2, -1.$

6/2/2020.

4. Let f, g and h be functions from set $Z \to Z$ defined by $f(x) = x - 1$, $g(x) = 3x$ and $h(x) = \begin{cases} 0, & x \text{ even} \\ 1, & x \text{ odd} \end{cases}$

Determine $[fo(goh)](x)$ and $[(fog)oh](x)$ and verify $fo(goh) = (fog)oh$

Sol^n: $[fo(goh)](x) = f[(goh)(x)]$

Consider $goh(x) = g\{h(x)\}$.

$[fo(goh)](x) = f[g\{h(x)\}]$

$= g\{h(x)\} - 1$ $\qquad$ $f(x) = x - 1$

$= 3\{h(x)\} - 1$

for x as even

$= 3(0) - 1$

$= -1$

for x as odd

$= 3(1) - 1$

$= 2.$

$$[fo(goh)](x) = \begin{cases} -1 & , x \text{ is even} \\ 2 & , x \text{ is odd.} \end{cases}$$

$[(fog)oh](x) = [fog(x)]h$

$= [f\{g(x)\}]h$

$= [g(x) - 1]h$

$= [3x - 1]h$

x is even

$= 3(0) - 1 = -1$

x is odd

$= 3(1) - 1 = 2.$

$$[(fog)oh](x) = \begin{cases} -1 & , x \text{ is even} \\ 2 & , x \text{ is odd} \end{cases}$$

Properties of Functions

Theorem 1

Let $f: X \to Y$ be a function and A and B be, non-empty subsets of X then

i. if $A \subseteq B$ then $f(A) \subseteq f(B)$

ii. $f(A \cup B) = f(A) \cup f(B)$

iii $f(A \cap B) \subseteq f(A) \cap f(B)$

Equality holds if f is 1-1, onto, and one to one correspondance.

Theorem 2 :

Let A and B be finite sets and f be a function $f: A \to B$ then the following statements are true,

i. If f is one to one then $|A| \leq |B|$

ii. If f is onto then $|B| \leq |A|$

Invertible Function :

A function $f: A \to B$ is said to be invertible if there exist a function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$ where I_A and I_B is the identity function on A and B respectively.

Then g is called an inverse of f and written as $g = f^{-1}$

1. Let $A = \{1, 2, 3, 4\}$ and f and g be given by

$f = \{(1,4)\ (2,1)\ (3,2)\ (4,3)\}$ and

$g = \{(1,2)\ (2,3)\ (3,4)\ (4,1)\}$

Here f and g are functions from $A \to A$

PT f and g are the inverses of each other

Soln.

$$gof(1) = g\{f(1)\} = g\{4\} = 1 = I_A$$
$$gof(2) = g\{f(2)\} = g\{1\} = 2 = I_A$$
$$gof(3) = g\{f(3)\} = g\{2\} = 3 = I_A$$
$$gof(4) = g\{f(4)\} = g\{3\} = 4 = I_A$$

Also

$$fog(1) = f\{g(1)\} = f(2) = 1 \quad I_1$$
$$fog(2) = f\{g(2)\} = f\{3\} = 2 \quad I_2$$
$$fog(3) = f\{g(3)\} = f\{4\} = 3 \quad I_3$$
$$fog(4) = f\{g(4)\} = f\{1\} = 4 \quad I_4$$

2. Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d\}$ detamine whether the function from $A \to B$ are invertible or not.

$f = \{(1, a)\ (2, a)\ (3, c)\ (4, d)\}$

$g = \{(1, a)\ (2, c)\ (3, d)\ (4, d)\}$

Sol^n: $f \circ g(1) = f\{g(1)\} = f\{a\}$