1.8. A pair of dice are tossed and the sum of the faces are recorded. Find the smallest set S which includes all possible outcomes.

The faces of the die are the numbers 1 to 6. Thus, no sum can be less than 2 nor greater than 12. Also, every number between 2 and 12 could occur. Thus

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

SET OPERATIONS

1.9. Let $\mathbf{U} = \{1, 2, \ldots, 9\}$ be the universal set, and let

$A = \{1, 2, 3, 4, 5\}$, $C = \{4, 5, 6, 7, 8, 9\}$, $E = \{2, 4, 6, 8\}$,
$B = \{4, 5, 6, 7\}$, $D = \{1, 3, 5, 7, 9\}$, $F = \{1, 5, 9\}$

Find:

(*a*) $A \cup B$ and $A \cap B$ (*c*) $A \cup C$ and $A \cap C$ (*e*) $E \cup E$ and $E \cap E$
(*b*) $B \cup D$ and $B \cap D$ (*d*) $D \cup E$ and $D \cap E$ (*f*) $D \cup F$ and $D \cap F$

Recall that the union $X \cup Y$ consists of those elements in either X or in Y (or both), and the intersection $X \cap Y$ consists of those elements in both X and Y.

(*a*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$, $A \cap B = \{4, 5\}$
(*b*) $B \cup D = \{1, 3, 4, 5, 6, 7, 9\}$, $B \cap D = \{5, 7\}$
(*c*) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathbf{U}$, $A \cap C = \{4, 5\}$
(*d*) $D \cup E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathbf{U}$, $D \cap E = \varnothing$
(*e*) $E \cup E = \{2, 4, 6, 8\} = E$, $E \cap E = \{2, 4, 6, 8\} = E$
(*f*) $D \cup F = \{1, 3, 5, 7, 9\} = D$, $D \cap F = \{1, 5, 9\} = F$

(Observe that $F \subseteq D$; hence, by Theorem 1.4, we must have $D \cup F = D$ and $D \cap F = F$.)

1.10. Consider the sets in the preceding Problem 1.9. Find:

(*a*) A^c, B^c, D^c, E^c; (*b*) $A \setminus B$, $B \setminus A$, $D \setminus E$, $F \setminus D$; (*c*) $A \oplus B$, $C \oplus D$, $E \oplus F$.

(*a*) The complement X^c consists of those elements in the universal set $\mathbf{U}$ which do not belong to X. Hence:

$$A^c = \{6, 7, 8, 9\}, \quad B^c = \{1, 2, 3, 8, 9\}, \quad D^c = \{2, 4, 6, 8\} = E, \quad E^c = \{1, 3, 5, 7, 9\} = D$$

(Note D and E are complements; that is, $D \cup E = \mathbf{U}$ and $D \cap E = \varnothing$.)

(*b*) The difference $X \setminus Y$ consists of the elements in X which do not belong to Y. Therefore

$$A \setminus B = \{1, 2, 3\}, \quad B \setminus A = \{6, 7\}, \quad D \setminus E = \{1, 3, 5, 7, 9\} = D, \quad F \setminus D = \varnothing$$

(Since D and E are disjoint, we must have $D \setminus E = D$; and since $F \subseteq D$, we must have $F \setminus D = \varnothing$.)

(*c*) The symmetric difference $X \oplus Y$ consists of the elements in X or in Y but not in both X and Y. In other words, $X \oplus Y = (X \setminus Y) \cup (Y \setminus X)$. Hence:

$$A \oplus B = \{1, 2, 3, 6, 7\}, \quad C \oplus D = \{1, 3, 8, 9\}, \quad E \oplus F = \{2, 4, 6, 8, 1, 5, 9\} = E \cup F$$

(Since E and F are disjoint, we must have $E \oplus F = E \cup F$.)

1.19. Describe in words: (*a*) $(A \cup B) \setminus (A \cap B)$ and (*b*) $(A \setminus B) \cup (B \setminus A)$. Then prove they are the same set. (Thus, either one may be used to define the symmetric difference $A \oplus B$.)

(*a*) $(A \cup B) \setminus (A \cap B)$ consists of the elements in A or B but not in both A and B.

(*b*) $(A \setminus B) \cup (B \setminus A)$ consists of the elements in A which are not in B, or the elements in B which are not in A.

Using $X \setminus Y = X \cap Y^c$ and the laws in Table 1-1, including DeMorgan's law, we obtain:

$$\begin{aligned}(A \cup B) \setminus (A \cap B) &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cap B^c) \\ &= (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c) \\ &= \varnothing \cup (A \cap B^c) \cup (B \cap A^c) \cup \varnothing \\ &= (A \cap B^c) \cap (B \cap A^c) = (A \setminus B) \cup (B \setminus A)\end{aligned}$$

FINITE SETS AND COUNTING PRINCIPLE, COUNTABLE SETS

1.21. Determine which of the following sets are finite:

(*a*) $A = \{\text{seasons in the year}\}$
(*b*) $B = \{\text{states in the United States}\}$
(*c*) $C = \{\text{positive integers less than 1}\}$
(*d*) $D = \{\text{odd integers}\}$
(*e*) $E = \{\text{positive integral divisors of 12}\}$
(*f*) $F = \{\text{cats living in the United States}\}$

(*a*) A is finite since there are four seasons in the year, that is, $n(A) = 4$.
(*b*) B is finite because there are 50 states in the United States, that is, $n(B) = 50$.
(*c*) There are no positive integers less than 1; hence C is empty. Thus, C is finite and $n(C) = \varnothing$.
(*d*) D is infinite.
(*e*) The positive integer divisors of 12 are 1, 2, 3, 4, 6, 12. Hence E is finite and $n(E) = 6$.
(*f*) Although it may be difficult to find the number of cats living in the United States, there is still a finite number of them at any point in time. Hence F is finite.

1.22. Suppose 50 science students are polled to see whether or not they have studied French (F) or German (G), yielding the following data:

25 studied French, 20 studied German, 5 studied both

Find the number of students who: (*a*) studied only French, (*b*) did not study German, (*c*) studied French or German, (*d*) studied neither language.

(*a*) Here 25 studied French, and 5 of them also studied German; hence $25 - 5 = 20$ students only studied French. That is, by Corollary 1.7,

$$n(F \setminus G) = n(F) - N(F \cap G) = 25 - 5 = 20$$

(*b*) There are 50 students of whom 20 studied German; hence $50 - 20 = 30$ did not study German. That is, by Corollary 1.8,

$$n(G^c) = n(\mathbf{U}) - n(G) = 50 - 20 = 30$$

(*c*) By the inclusion-exclusion principle in Theorem 1.9,

$$n(F \cup G) = n(F) + n(G) - n(F \cap G) = 25 + 20 - 5 = 40$$

That is, 40 students studied French or German.

(*d*) The set $F^c \cap G^c$ consists of the students who studied neither language. By DeMorgan's law, $F^c \cap G^c = (F \cup G)^c$. By (*c*), 40 studied at least one of the languages; hence

$$n(F^c \cap G^c) = n(\mathbf{U}) - n(F \cup G) = 50 - 40 = 10$$

That is, 10 students studied neither language.

1.23. Each student at some college has a mathematics requirement M (to take at least one mathematics course) and a science requirement S (to take at least one science course). A poll of 140 sophomore students shows that:

60 completed M, 45 completed S, 20 completed both M and S

Use a Venn diagram to find the number of students who had completed:

(*a*) At least one of the two requirements
(*b*) Exactly one of the two requirements
(*c*) Neither requirement

Translating the above data into set notation yields:

$$n(M) = 60, \qquad n(S) = 45, \qquad n(M \cap S) = 20, \qquad n(\mathbf{U}) = 140$$

Draw a Venn diagram of sets M and S with four regions, as in Fig. 1-9(a). Then, as in Fig. 1-9(b), assign numbers to the four regions as follows:

20 completed both M and S, so $n(M \cap S) = 20$
$60 - 20 = 40$ completed M but not S, so $n(M \setminus S) = 40$
$45 - 20 = 25$ completed S but not M, so $n(S \setminus M) = 25$
$140 - 20 - 40 - 25 = 55$ completed neither M nor S

By the Venn diagram:

(a) $20 + 40 + 25 = 85$ completed M or S. Alternately, we can find $n(M \cup S)$ without the Venn diagram by using the inclusion-exclusion principle:

$$n(M \cup S) = n(M) + n(S) - n(M \cap S) = 60 + 45 - 20 = 85$$

(b) $40 + 25 = 65$ completed exactly one of the requirements. That is, $n(M \oplus S) = 65$.
(c) 55 completed neither requirement. That is, $n(M^c \cap S^c) = 55$.

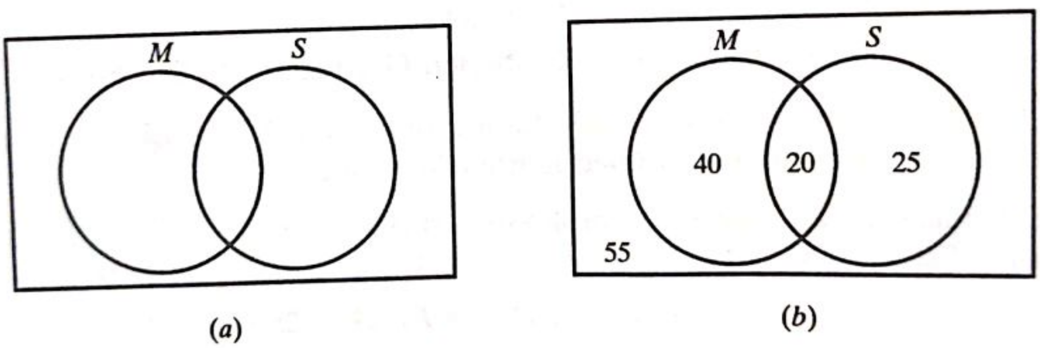

Fig. 1-9

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ORDERED PAIRS AND PRODUCT SETS

1.26. Find x and y given that $(2x, x - 3y) = (6, -9)$.

Two ordered pairs are equal if and only if the corresponding entries are equal. This leads to the equations

$$2x = 6 \qquad \text{and} \qquad x - 3y = -9$$

Solving the equations yields $x = 3$, $y = 4$.

1.27. Given: $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find: (a) $A \times B$, (b) $B \times A$, (c) $B \times B$.

(a) $A \times B$ consists of all ordered pairs (x, y) where $x \in A$ and $y \in B$. Thus

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

(*b*) $B \times A$ consists of all ordered pairs (x, y) where $x \in B$ and $y \in A$. Thus

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(*c*) $B \times B$ consists of all ordered pairs (x, y) where $x, y \in B$. Thus

$$B \times B = \{(a, a), (a, b), (b, a), (b, b)\}$$

Note that, as expected from Theorem 1.11, $n(A \times B) = 6$, $n(B \times A) = 6$, $n(B \times B) = 4$; that is, the number of elements in a product set is equal to the product of the numbers of elements in the factor sets.

1.28. Given $A = \{1, 2\}$, $B = \{x, y, z\}$, $C = \{3, 4\}$. Find $A \times B \times C$.

$A \times B \times C$ consists of all ordered triples (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called "tree diagram" as in Fig. 1-11. The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the diagram.

Observe that $n(A) = 2$, $n(B) = 3$, $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B \cdot n(C)$$

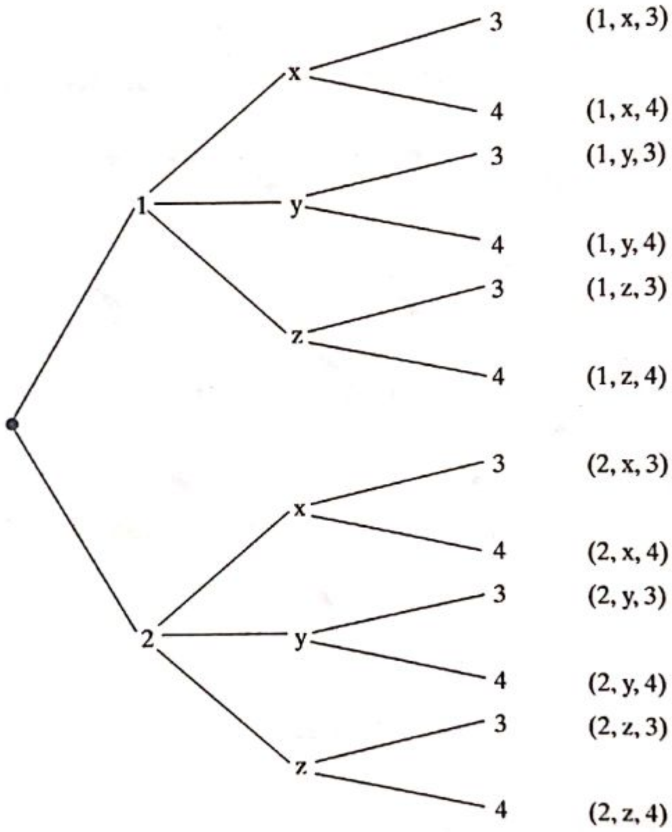

Fig. 1-11

1.29. Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of outcomes. Find C^3, $n(C^3)$, and explain what C^3 represents.

Since $n(C) = 2$, we have $n(C^3) = 2^3 = 8$. Omitting certain commas and parenthesis for notational convenience,

$$C^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

C^3 represents all possible sequences of outcomes of three tosses of the coin.

1.30. Prove: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$\begin{aligned} A \times (B \cap C) &= \{(x, y) : x \in A,\ y \in B \cap C\} \\ &= \{(x, y) : x \in A,\ y \in B,\ y \in C\} \\ &= \{(x, y) : x \in A,\ y \in B,\ x \in A,\ y \in C\} \\ &= \{(x, y) : (x, y) \in A \times B,\ (x, y) \in A \times C\} \\ &= (A \times B) \cap (A \times C) \end{aligned}$$

CLASSES OF SETS AND PARTITIONS

1.31. Consider the set $A = [\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}]$. (*a*) Find the elements of A. (*b*) Find $n(A)$.

(*a*) A is a collection of sets; its elements are the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.

(*b*) A has only three elements; hence $n(A) = 3$.

1.32. Consider the class A of sets in Problem 1.31. Determine whether or not each of the following is true or false:

(*a*) $1 \in A$ (*c*) $\{6, 7, 8\} \in A$ (*e*) $\varnothing \in A$

(*b*) $\{1, 2, 3\} \subseteq A$ (*d*) $\{\{4, 5\}\} \subseteq A$ (*f*) $\varnothing \subseteq A$

(*a*) False. 1 is not one of the three elements of A.

(*b*) False. $\{1, 2, 3\}$ is not a subset of A; it is one of the elements of A.

(*c*) True. $\{6, 7, 8\}$ is one of the elements of A.

(*d*) True. $\{\{4, 5\}\}$, the set consisting of the element $\{4, 5\}$, is a subset of A.

(*e*) False. The empty set $\varnothing$ is not an element of A, that is, it is not one of the three sets listed as elements of A.

(*f*) True. The empty set $\varnothing$ is a subset of every set; even a class of sets.

1.33. List the elements of the power set $\mathscr{P}(A)$ of $A = \{a, b, c, d\}$.

The elements of $\mathscr{P}(A)$ are the subsets of A. Hence:

$$\begin{aligned} \mathscr{P}(A) = [A,\ & \{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\},\ \{a, b\},\ \{a, c\}, \\ & \{a, d\},\ \{b, c\},\ \{b, d\},\ \{c, d\},\ \{a\},\ \{b\},\ \{c\},\ \{d\},\ \varnothing] \end{aligned}$$

As expected, $\mathscr{P}(A)$ has $2^4 = 16$ elements.

1.34. Let $S = \{a, b, c, d, e, f, g\}$. Determine which of the following are partitions of S:

(*a*) $P_1 = [\{a, c, e\}, \{b\}, \{d, g\}]$ (*c*) $P_3 = [\{a, b, e, g\}, \{c\}, \{d, f\}]$

(*b*) $P_2 = [\{a, e, g\}, \{c, d\}, \{b, e, f\}]$ (*d*) $P_4 = [\{a, b, c, d, e, f, g\}]$

(*a*) P_1 is not a partition of S since $f \in S$ does not belong to any of the cells.

(*b*) P_2 is not a partition of S since $e \in S$ belongs to two of the cells, $\{a, e, g\}$ and $\{b, e, f\}$.

(*c*) P_3 is a partition of S since each element in S belongs to exactly one cell.

(*d*) P_4 is a partition of S into one cell, S itself.

1.35. Find all partitions of $S = \{a, b, c, d\}$.

Note first that each partition of S contains either one, two, three, or four distinct cells. The partitions are as follows:

(1) $[\{a, b, c, d\}] = [S]$

(2a) $[\{a\}, \{b, c, d\}]$, $[\{b\}, \{a, c, d\}]$, $[\{c\}, \{a, b, d\}]$, $[\{d\}, \{a, b, c\}]$

(2b) $[\{a, b\}, \{c, d\}]$, $[\{a, c\}, \{b, d\}]$, $[\{a, d\}, \{b, c\}]$

(3) $[\{a\}, \{b\}, \{c, d\}]$, $[\{a\}, \{c\}, \{b, d\}]$, $[\{a\}, \{d\}, \{b, c\}]$, $[\{b\}, \{c\}, \{a, d\}]$,
$[\{b\}, \{d\}, \{a, c\}]$, $[\{c\}, \{d\}, \{a, b\}]$

(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$

[Note (2a) refers to partitions with one-element and three-element cells, whereas (2a) refers to partitions with two two-element cells.] There are $1 + 4 + 3 + 6 + 1 = 15$ different partitions of S.

1.41. Prove the assertion $A(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$A(n): 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The assertion holds for $n = 1$ since

$$A(1): \qquad 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $A(n)$ is true, we add $n+1$ to both sides of $A(n)$. This yields

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] \\ &= \tfrac{1}{2}[(n+1)(n+2)] \end{aligned}$$

which is $A(n+1)$. That is, $A(n+1)$ is true whenever $A(n)$ is true. By the principle of induction, $A(n)$ is true for all $n \geq 1$.

1.42. Prove the following assertion (for $n \geq 0$):

$$A(n): 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

$A(0)$ is true since $1 = 2^1 - 1$. Assuming $A(n)$ is true, we add 2^{n+1} to both sides of $A(n)$. This yields:

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2(2^{n+1}) - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

which is $A(n+1)$. Thus, $A(n+1)$ is true whenever $A(n)$ is true. By the principle of induction, $A(n)$ is true for all $n \geq 0$.

1.43. Prove: $n^2 \geq 2n + 1$ for $n \geq 3$.

Since $3^2 = 9$ and $2(3) + 1 = 7$, the formula is true for $n = 3$. Assuming $n^2 \geq 2n + 1$, we have

$$(n + 1)^2 = n^2 + 2n + 1 \geq (2n + 1) + 2n + 1 = 2n + 2 + 2n \geq 2n + 2 + 1 = 2(n + 1) + 1$$

Thus, the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 3$.

1.44. Prove: $n! \geq 2^n$ for $n \geq 4$.

Since $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $2^4 = 16$, the formula is true for $n = 4$. Assuming $n! \geq 2^n$ and $n + 1 \geq 2$, we have

$$(n + 1)! = n!(n + 1) \geq 2^n(n + 1) \geq 2^n(2) = 2^{n+1}$$

Thus, the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 4$.