1.7 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B". By definition,

$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

We note that ordered pairs (a, b) and (c, d) are equal if and only if their *first* elements, a and c, are equal and their *second* elements, b and d, are equal. That is,

$$\boxed{(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d}$$

EXAMPLE 1.6 $\mathbf{R}$ denotes the set of real numbers, and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane, as in Fig. 1-6. Here each point P represetns an ordered pair (a, b) of real numbers, and vice versa; the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

EXAMPLE 1.7 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

Also,

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

There are two things worth noting in the above Example 1.7. First of all, $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important.

Secondly, using $n(S)$ for the number of elements in a set S, we have:

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

In fact, $n(A \times B) = n(A) \cdot n(B)$ for any finite sets A and B. This follows from the observation that, for each $a \in A$, there will be $n(B)$ ordered pairs in $A \times B$ beginning with a. Hence, altogether there will be $n(A)$ times $n(B)$ ordered pairs in $A \times B$.

We state the above result formally.

Theorem 1.11: Suppose A and B are finite. Then $A \times B$ is finite and

$$n(A \times B) = n(A) \cdot n(B)$$

The concept of a product of sets can be extended to any finite number of sets in a natural way. That is, for any sets $A_1, A_2, \ldots, A_m$, the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$, where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_m \in A_m$, is called the *product* of the sets $A_1, A_2, \ldots, A_m$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_m \qquad \text{or} \qquad \prod_{i=1}^{m} A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^m for $A \times A \times \cdots \times A$, where there are m factors.

Furthermore, for finite sets $A_1, A_2, \ldots, A_m$, we have

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1)n(A_2) \cdots n(A_m)$$

That is, Theorem 1.11 may be easily extended, by induction, to the product of m sets.

1.8 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S, we may wish to talk about some of its subsets. Thus, we would be considering a "set of sets". Whenever such a situation arises, to avoid confusion, we will speak of a *class* of sets or a *collection* of sets. The words "subclass" and "subcollection" have meanings analogous to subset.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let $\mathscr{A}$ be the class of subsets of S which contains exactly three elements of S. Then

$$\mathscr{A} = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

The elements of $\mathscr{A}$ are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.

Let $\mathscr{B}$ be the class of subsets of S which contains the numeral 2 and two other elements of S. Then

$$\mathscr{B} = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of $\mathscr{B}$ are $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus $\mathscr{B}$ is a subclass of $\mathscr{A}$. (To avoid confusion, we will usually enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S, we may consider the class of all subsets of S. This class is called the *power set* of S, and it will be denoted by $\mathscr{P}(S)$. If S is finite, then so is $\mathscr{P}(S)$. In fact, the number of elements in $\mathscr{P}(S)$ is 2 raised to the power of S; that is,

$$n(\mathscr{P}(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.) We emphasize that S and the empty set $\varnothing$ belong to $\mathscr{P}(S)$ since they are subsets of S.

EXAMPLE 1.9 Suppose $S = \{1, 2, 3\}$. Then

$$\mathscr{P}(S) = [\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$$

As expected from the above remark, $\mathscr{P}(S)$ has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A *partition* of S is a *subdivision* of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that

(i) Each a in S belongs to one of the A_i.

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j, \text{ the } A_i \cap A_j = \varnothing.$$

The subsets in a partition are called *cells*. Figure 1-7 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, A_5.

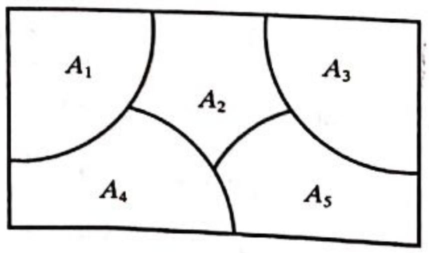

Fig. 1-7

1.9 MATHEMATICAL INDUCTION

An essential property of the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers which is used in many proofs follows:

Principle of Mathematical Induction I: Let $A(n)$ be an assertion about the set $\mathbf{N}$ of positive integers, that is, $A(n)$ is true or false for each integer $n \geq 1$. Suppose $A(n)$ has the following two properties:

(i) $A(1)$ is true.

(ii) $A(n + 1)$ is true whenever $A(n)$ is true.

Then $A(n)$ is true for every positive integer.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when $\mathbf{N}$ is developed axiomatically.

EXAMPLE 1.11 Let $A(n)$ be the assertion that the sum of the first n odd numbers is n^2; that is,

$$A(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The nth odd number is $2n - 1$ and the next odd number is $2n + 1$.)

Observe that $A(n)$ is true for $n = 1$ since

$$A(1): 1 = 1^2$$

Assuming $A(n)$ is true, we add $2n + 1$ to both sides of $A(n)$, obtaining

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

However, this is $A(n + 1)$. That is, $A(n + 1)$ is true assuming $A(n)$ is true. By the principle of mathematical induction, $A(n)$ is true for all $n \geq 1$.

There is another form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the above principle of induction.

Principle of Mathematical Induction II: Let $A(n)$ be an assertion about the set $\mathbf{N}$ of positive integers with the following two properties:

(i) $A(1)$ is true.

(ii) $A(n)$ is true whenever $A(k)$ is true for $1 \leq k \leq n$.

Then $A(n)$ is true for every positive integer.

Remark: Sometimes one wants to prove that an assertion A is true for a set of integers of the form

$$\{a, a + 1, a + 2, \ldots\}$$

where a is any integer, possibly 0. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.