27.4 BINOMIAL DISTRIBUTION

Binomial distribution (B.D.) due to James Bernoulli (1700) is a discrete probability distribution. The Bernoulli process has the following properties:

i. An experiment is repeated n number of times, called n trials where n is a fixed integer.

ii. The outcome of each trial is classified into two mutually exclusive (dichotomus) categories arbitrarily called a "success" and a "failure".

iii. Probability of success, denoted by p, remains constant for all trials.

iv. The outcomes are independent (of the outcomes of the previous trials).

Each trial in the Bernoulli process is known as Bernoulli trial.

The binomial random variable X is the *number* of successes in n Bernoulli trials. X is discrete since X takes only integer values (we 'count' the number of successes).

Binomial distribution is thus the probability distribution of this discrete random variable X, and is given by

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \ldots, n \quad (1)$$

where n is the number of trials and p is the probability of success in any trial. The probability of x successes is p^x and remaining failures is q^{n-x}. This can happen in n_{C_x} ways. By multiplication rule, the probability of x successes in n trials is $\binom{n}{x} p^x q^{n-x}$.

Note that the $(n+1)$ terms of the binomial expansion

$$(q+p)^n = \binom{n}{0} q^n + \binom{n}{1} pq^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots + \binom{n}{n} p^n$$

$$= b(0; n, p) + b(1; n, p) + b(2; n, p) + \cdots + b(n; n, p)$$

$$= \sum_{x=0}^{n} b(x; n, p)$$

correspond to various values of $b(x; n, p)$ for $x = 0, 1, 2, \ldots, n$.

Since $p + q = 1$, it follows that

$$\sum_{x=0}^{n} b(x; n, p) = 1.$$

B.D. is characterized by the parameter p and the number of trials n.

The mean μ of B.D. is np and the variance σ^2 of B.D. is npq. (see Worked Out Example 1 on page 27.11)

The binomial sums

$$B(r; n, p) = \sum_{x=0}^{r} b(x; n, p)$$

are tabulated (see A2 to A7)
since

$$\frac{{}^nC_{x+1}}{{}^nC_x} = \frac{n!}{(x+1)!(n-x-1)!} \frac{x!(n-x)!}{n!} = \left(\frac{n-x}{x+1}\right)$$

The recurrence relation for B.D. is

$$b(x+1; n, p) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right) b(x; n, p).$$

Example 4: Determine the probability distribution of the number of bad eggs in a box of 6 chosen at random if 10% of eggs are bad, in a large consignment.

Solution: Probability of a bad egg $= p = \frac{10}{100} = 0.1$. Let $X =$ number of bad eggs, $n = 6$. The required B.D. $= b(x; 6, 0.1) = 6_{C_x}(.1)^x(.9)^{6-x}$, for $x = 0, 1, 2, 3, 4, 5, 6$.

X:	0	1	2	3	4	5	6
$P(X)$:	.5311	.35429	0.098	0.015	0.001215	0.000054	0

Example 6: The probability of a man hitting a target is $\frac{1}{3}$. (a) If he fires 5 times, what is the probability of his hitting the target at least twice ? (b) How many times must he fire so that the probability of his hitting the target at least once is more than 90%?

Solution: Probability of hitting $= p = \frac{1}{3}$

probability of no hit (or failure) $= q = \frac{2}{3}$

a. X = number of hits (successes), $n = 5$

$$P(X \geq 2) = \sum_{x=2}^{5} 5_{C_x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x}$$

$$= 1 - \sum_{x=0}^{1} 5_{C_x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x}$$

$$= 1 - \left(\frac{2}{3}\right)^5 - 5_{C_1} \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^4 = \frac{131}{243}.$$

b. The probability of not hitting the target is q^n in n trials (fires). Thus to find the smallest n for which the probability of hitting at least once $1 - q^n$ is more than 90%.

i.e., $\quad 1 - q^n > 0.9$

or $\quad 1 - \left(\frac{2}{3}\right)^n > 0.9 \quad$ i.e., $\left(\frac{2}{3}\right)^n < 0.1$

For $n = 6$, $2^6 = 64 < (0.1)3^6 = 72.9$ this is true.

In other words, he must fire 6 times.