Suppose we have a collection of n objects. A *combination* of these n objects taken r at a time is any selection of r of the objects where order doesn't count. In other words, an *r combination* of a set of n objects is any subset of r elements. For example, the combinations of the letters a, b, c, d taken three at a time are

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \quad \text{or simply} \quad abc, abd, acd, bcd$$

Observe that the following combinations are equal:

$$abc, acb, bac, bca, cab, cba$$

That is, each denotes the same set $\{a, b, c\}$.

The number of combinations of n objects taken r at a time will be denoted by

$$C(n, r)$$

Before we derive the general formula for $C(n, r)$, we consider a particular case.

EXAMPLE 2.8 Find the number of combinations of four objects, a, b, c, d, taken three at a time.

Each combination consisting of three objects determines $3! = 6$ permutations of the objects in the combination as pictured in Fig. 2-2. Thus, the number of combinations multiplied by $3!$ equals the number of permutations. That is,

$$C(4, 3) \cdot 3! = P(4, 3) \quad \text{or} \quad C(4, 3) = \frac{P(4, 3)}{3!}$$

But $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ and $3! = 6$. Thus, $C(4, 3) = 4$, which is noted in Fig. 2-2.

Combinations	Permutations
abc	*abc, acb, bac, bca, cab, cba*
abd	*abd, adb, bad, bda, dab, dba*
acd	*acd, adc, cad, cda, dac, dca*
bcd	*bcd, bdc, cbd, cdb, dbc, dcb*

Fig. 2-2

Formula for $C(n, r)$

Since any combination of n objects taken r at a time determines $r!$ permutations of the objects in the combination, we can conclude that

$$P(n, r) = r!\, C(n, r)$$

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Thus, we obtain the following formula for $C(n, r)$:

Theorem 2.7: $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$.

Recall that the binomial coefficient $\binom{n}{r}$ was defined to be $\dfrac{n!}{r!(n-r)!}$. Accordingly,

$$\boxed{C(n,r) = \binom{n}{r}}$$

We shall use $C(n, r)$ and $\binom{n}{r}$ interchangeably.

EXAMPLE 2.9

(*a*) Find the number m of committees of 3 that can be formed from 8 people.

Each committee is, essentially, a combination of the 8 people taken 3 at a time. Thus

$$m = C(8,3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

(*b*) A farmer buys 3 cows, 2 pigs, and 4 hens from a person who has 6 cows, 5 pigs, and 8 hens. How many choices does the farmer have?

The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways. Accordingly, altogether the farmer can choose the animals in

$$\binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 20 \cdot 10 \cdot 70 = 14{,}000 \text{ ways}$$

EXAMPLE 2.10 Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

There are $C(9, 3) = 84$ ways to first choose 3 toys for the youngest. Then there are $C(6, 2) = 15$ ways to choose 2 of the remaining 6 toys for the oldest. Next, there are $C(4, 2) = 6$ ways to choose 2 of the remaining 4 toys for the second oldest. The third oldest receives the remaining 2 toys. Thus, by the product rule,

$$m = 84(15)(6) = 7560$$

Alternately, by Problem 2.37,

$$m = \frac{9!}{3!2!2!2!} = 7560$$

EXAMPLE 2.11 Find the number m of ways that 12 students can be partitioned into 3 teams, T_1, T_2, T_3, so that each team contains 4 students.

Method 1: Let A be one of the students. Then there are $C(11, 3)$ ways to choose 3 other students to be on the same team as A. Now let B denote a student who is not on the same team as A; then there are $C(7, 3)$ ways to choose 3 students out of the remaining students to be on the same team as B. The remaining 4 students constitute the third team. Thus, altogether, the number m of ways to partition the students is as follows:

$$m = C(11,3) \cdot C(7,3) = \binom{11}{3} \cdot \binom{7}{3} = 165 \cdot 35 = 5775$$

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Method 2: Each partition $[T_1, T_2, T_3]$ of the students can be arranged in $3! = 6$ ways as an ordered partition. By Problem 2.37 (or using the method in Example 2.10), there are

$$\frac{12!}{4!4!4!} = 34{,}650$$

such ordered partitions. Thus, there are $m = 34{,}650/6 = 5775$ (unordered) partitions.

2.7 TREE DIAGRAMS

A *tree diagram* is a device used to enumerate all the possible outcomes of a sequence of experiments or events where each event can occur in a finite number of ways. The construction of tree diagrams is illustrated in the following examples.

EXAMPLE 2.12 Find the product set $A \times B \times C$ where

$$A = \{1, 2\},\ B = \{a, b, c\},\ C = \{3, 4\}$$

The tree diagram for the $A \times B \times C$ appears in Fig. 2-3. Observe that the tree is constructed from left to right and that the number of branches at each point corresponds to the number of possible outcomes of the next event. Each endpoint of the tree is labeled by the corresponding element of $A \times B \times C$. As expected from Theorem 1.11, $A \times B \times C$ contains $n = 2 \cdot 3 \cdot 2 = 12$ elements.

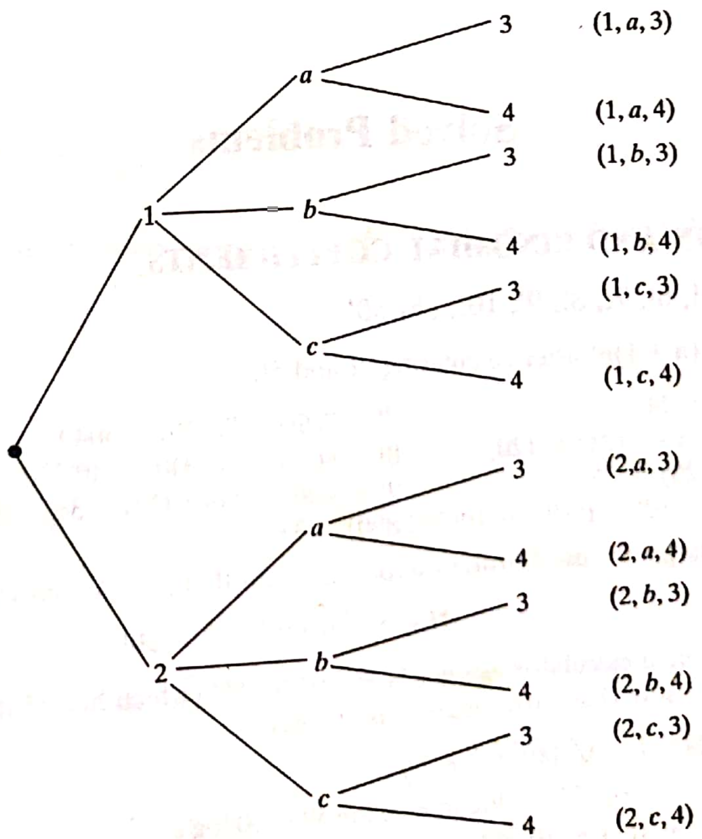

Fig. 2-3

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EXAMPLE 2.13 Marc and Erik are to play a tennis tournament. The first person to win 2 games in a row or who wins a total of 3 games wins the tournament. Find the number of ways the tournament can occur.

The tree diagram showing the possible outcomes of the tournament appears in Fig. 2-4. Specifically, there are 10 endpoints which correspond to the following 10 ways that the tournament can occur:

$$MM,\ MEMM,\ MEMEM,\ MEMEE,\ MEE,\ EMM,\ EMEMM,\ EMEME,\ EMEE,\ EE$$

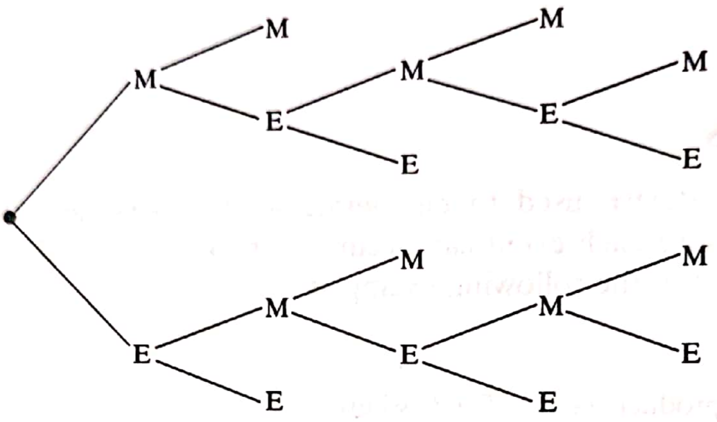

Fig. 2-4

The path from the beginning of the tree to the endpoint describes who won which game in the individual tournament.

Solved Problems

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

2.1. Compute: (*a*) 4!, 5!, 6!, 7!, 8!, 9!, 10!; (*b*) 50!

(*a*) Use $(n+1)! = (n+1)n!$ after calculating 4! and 5!:

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24, \qquad 7! = 7(6!) = 7(720) = 5040$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5(24) = 120, \qquad 8! = 8(7!) = 8(5040) = 40{,}320$$
$$6! = 6(5!) = 6(120) = 720, \qquad 9! = 9(8!) = 9(40{,}320) = 362{,}880$$
$$10! = 10(9!) = 10(362{,}880) = 3{,}628{,}800$$

(*b*) Since n is very large, we use Stirling's approximation that $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ (where $e = 2.718$). Let

$$N = \sqrt{100\pi}\, 50^{50} e^{-50} \sim 50!$$

Evaluating N using a calculator, we get $N = 3.04 \times 10^{64}$ (which has 65 digits).

Alternately, using (base 10) logarithms, we get

$$\begin{aligned} \log N &= \log(\sqrt{100\pi}\, 50^{50} e^{-50}) \\ &= \tfrac{1}{2}\log 100 + \tfrac{1}{2}\log \pi + 50 \log 50 - 50 \log e \\ &= \tfrac{1}{2}(2) + \tfrac{1}{2}(0.497\,2) + 50(1.699\,0) - 50(0.434\,3) \\ &= 64.483\,6 \end{aligned}$$

The antilog yields $N = 3.04 \times 10^{64}$

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