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Useful Formulas for the Analysis of Algorithms

→ **Properties of Logarithms.**

All logarithm bases are assumed to be greater than 1 in the formula below; $\lg x$ denotes the logarithm base 2, $\ln x$ denotes the logarithm base $e = 2.71828$; x, y are arbitrary positive nos.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^y = y \log_a x$
4. $\log_a xy = \log_a x + \log_a y$
5. $\log_a \frac{x}{y} = \log_a x - \log_a y$
6. $a^{\log_b x} = x^{\log_b a}$
7. $\log_a x = \frac{\log_b x}{\log_b a} = \log_a b \log_b x$

Combinatorics → branch of mathematics, dealing with Combinations of objects belonging to a finite set in accordance with Certain Constraints, such as those of graph theory.

1. Number of permutations of an n-element set : $P(n) = n!$
2. Number of k-combinations of an n-element set : $C(n,k) = \frac{n!}{k!(n-k)!}$
3. Number of subsets of an n-element set : 2^n.

Important Summation Formulas

1. $\sum_{i=l}^{u} 1 = \underbrace{1+1+\cdots+1}_{u-l+1 \text{ times}} = u-l+1$ (l, u are integer limits, $l \le u$); $\sum_{i=1}^{n} 1 = n-1+1 = n$

2. $\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$

3. $\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$

4. $\sum_{i=1}^{n} i^k = 1^k+2^k+\cdots+n^k \approx \frac{1}{k+1}n^{k+1}$

5. $\sum_{i=0}^{n} a^i = 1+a+\cdots+a^n = \frac{a^{n+1}-1}{a-1}$ $(a \ne 1)$; $\sum_{i=0}^{n} 2^i = 2^{n+1}-1$

6. $\sum_{i=1}^{n} i2^i = 1\cdot 2+2\cdot 2^2+\cdots+n2^n = (n-1)2^{n+1}+2$

7. $\sum_{i=1}^{n} \frac{1}{i} = 1+\frac{1}{2}+\cdots+\frac{1}{n} \approx \ln n+\gamma$, where $\gamma \approx 0.5772$ (Eulers Constant)

8. $\sum_{i=1}^{n} \lg i \approx n \lg n$

Sum Manipulation Rules

1. $\sum_{i=l}^{u} ca_i = c\sum_{i=l}^{u} a_i$

2. $\sum_{i=l}^{u} (a_i \pm b_i) = \sum_{i=l}^{u} a_i \pm \sum_{i=l}^{u} b_i$

3. $\sum_{i=l}^{u} a_i = \sum_{i=l}^{m} a_i + \sum_{i=m+1}^{u} a_i$, where $l \le m < u$

4. $\sum_{i=l}^{u} (a_i - a_{i-1}) = a_u - a_{l-1}$

Approximation of a Sum by a Definite Integral

$$\int_{l-1}^{u} f(x)\,dx \le \sum_{i=l}^{u} f(i) \le \int_{l}^{u+1} f(x)\,dx \text{ for a nondecreasing } f(x)$$

$$\int_{l}^{u+1} f(x)\,dx \le \sum_{i=l}^{u} f(i) \le \int_{l-1}^{u} f(x)\,dx \text{ for a nonincreasing } f(x)$$

Floor and Ceiling formulas

The floor of a real number x, denoted $\lfloor x \rfloor$, is defined as the greatest integer not larger than x (eg. $\lfloor 3.8 \rfloor = 3$, $\lfloor -3.8 \rfloor = -4$, $\lfloor 3 \rfloor = 3$). The ceiling of a real number x, denoted $\lceil x \rceil$, is defined as the smallest integer not smaller than x (eg., $\lceil 3.8 \rceil = 4$, $\lceil -3.8 \rceil = -3$, $\lceil 3 \rceil = 3$).

1. $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
2. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ and $\lceil x + n \rceil = \lceil x \rceil + n$ for real x and integer n
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$
4. $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$

Miscellaneous

1. $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \longrightarrow \infty$ (Stirling's formula)
2. Modular arithmetic (n, m are integers, p is a positive integer)

$$(n+m) \bmod p = (n \bmod p + m \bmod p) \bmod p$$

$$(nm) \bmod p = ((n \bmod p)(m \bmod p)) \bmod p$$

Basics

→ A.P → $a_n = a + (n-1)d$ d - common difference

a = 1st term, l = last term, d ↗

n = no. of terms.

S_n = Sum to n terms of A.P.

Let $a, a+d, a+2d, \ldots, a+(n-1)d$ be an A.P Then

$$l = a + (n-1)d.$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

We can also write, $S_n = \frac{n}{2}[a + l]$

→ G.P. (Sum to n terms of a G.P).

1st term a, Common ratio be r.

S_n → Sum to first n terms of G.P. Then.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \text{--- (1)}$$

Case 1   if $r = 1$, we have $S_n = a + a + a + \cdots + a$ (n term) $= na$.

Case 2.   if $r \neq 1$ multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \text{--- (2)}$$

~~$rS_n = ar$~~. Sub (2) from (1) we get

Thus given, $(1-r)S_n = a - ar^n = a(1-r^n)$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

→. Sum to n Terms of Special Series.

→ (i) $1+2+3+\cdots+n$ (Sum of 1st n natural nos.).

(ii) $1^2+2^2+3^2+\cdots+n^2$ (Sum of Squares of the 1st n natural nos.).

(iii) $1^3+2^3+3^3+\cdots+n^3$ (Sum of Cubes of the 1st n natural numbers).

Let us take them one by one.

(i) $S_n = 1+2+3+\cdots+n$, then $S_n = \frac{n(n+1)}{2}$

(ii) Here $S_n = 1^2+2^2+3^2+\cdots+n^2$

We Consider the identity $k^3-(k-1)^3 = 3k^2-3k+1$

Putting $k=1,2..,n$ Successively, we obtain

$$1^3-0^3 = 3(1)^2-3(1)+1$$
$$2^3-1^3 = 3(2)^2-3(2)+1$$
$$3^3-2^3 = 3(3)^2-3(3)+1$$
$$\vdots$$
$$n^3-(n-1)^3 = 3(n)^2-3(n)+1$$

Adding both Sides, we get.

$$n^3-0^3 = 3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+n$$
$$n^3 = 3\cdot\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+n$$

W.K.T $\sum_{i=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$, $S_n = \sum_{k=1}^{n}k^2 = \frac{1}{3}\left[n^3+\frac{3n(n+1)}{2}-n\right]$

$= \frac{1}{6}(2n^3+3n^2+n)$

$= \frac{n(n+1)(2n+1)}{6}$.

Solve recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by make a change of Variables.

1st

$$T(n) = 3T(\sqrt{n}) + \underline{\lg n} \quad \text{let } m = \lg n$$

$$T(2^m) = 3T(2^{m/2}) + m \qquad n = 2^m$$

$$S(m) = 3S(m/2) + m.$$

Now we guess $S(m) \leq cm^{\lg 3} + dm$,

$$S(m) \leq 3\left(c(m/2)^{\lg 3} + d(m/2)\right) + m$$

$$\leq cm^{\lg 3} + \left(\tfrac{3}{2}d + 1\right) m \qquad (d \leq -2)$$

$$\leq cm^{\lg 3} + dm.$$

Then we guess $S(m) \geq cm^{\lg 3} + dm$,

$$S(m) \geq 3\left(c(m/2)^{\lg 3} + d(m/2)\right) + m$$

$$\geq cm^{\lg 3} + \left(\tfrac{3}{2}d + 1\right) m \qquad (d \geq -2)$$

$$\geq cm^{\lg 3} + dm.$$

Thus,

$$S(m) = \Theta(m^{\lg 3})$$

$$\underline{T(n) = \Theta(\lg^{\lg 3} n)}.$$

Formulas

Sum of ~~Squares~~ 1st n natural nos $\rightarrow \frac{n(n+1)}{2}$ $\frac{1}{2}n^2$

Sum of Squares of 1st " " $\rightarrow \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}(n^3)$

" " " " odd " $\rightarrow \frac{n(4n^2-1)}{3}$ $1^2+3^2+5^2+\dots n^2$

" " " " Even " $\rightarrow \frac{2n(n+1)(2n+1)}{3}$

Sum of cubes of 1st n natural nos $\rightarrow \frac{n(n+1)^2}{2}$ $1^3+2^3 \dots + n^3$

Sum of " " n odd nos $\rightarrow n^2(2n^2-1)$ $1^3+3^3+5^3 \dots n^3$

Sum of " " n Even nos $\rightarrow 2n^2(n+1)^2$ $2^3+4^3+6^3 \dots + n^3$

$\rightarrow \quad T(n) \le n\lg n + n$

$$T(n) \le 2\left(c\lfloor n/2\rfloor \lg \lfloor n/2\rfloor + \lfloor n/2\rfloor\right) + n$$
$$\le 2c(n/2)\lg(n/2) + 2(n/2) + n$$
$$= cn\lg(n/2) + 2n$$
$$= cn\lg n - cn\lg 2 + 2n$$
$$= cn\lg n + (2-c)n$$
$$\le cn\lg n + n,$$

where the last step holds for $c \geq 1$.

This time, the boundary cond^n is

$$T(1) = 1 \le cn\lg n + n = 0 + 1 = 1.$$

$T(n) = 4T(n/2) + n$ is $T(n) = \Theta(n^2)$. Show subst proof with assumption fails $T(n) \le cn^2$. Then show how to subtract off a lower-order term to make the subst proof work.

Guess $T(n) \le cn^2$ we have

$$T(n) = 4T(n/2) + n$$
$$\le 4c(n/2)^2 + n$$
$$= cn^2 + n$$

Lets try guess which subtracts off a lower-order term

Now we have $T(n) \le cn^2 - cn$.

$$T(n) = 4T(n/2) + n$$
$$= 4\left(c(n/2)^2 - c(n/2)\right) + n$$
$$= 4c(n/2)^2 - 4c(n/2) + n$$
$$= cn^2 + (1-2c)n \qquad c \geq 1/2.$$
$$\le cn^2.$$

(1)

→ $T(n) = \begin{cases} T(n/2) + c & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$ — base Eqⁿ (initial Condition boundary Condition.

$T(n) = T(n/2) + c$ — (1)

$T(n/2) = T\left(\frac{n/2}{2}\right) + c$

$= T\left(\frac{n}{2\times2}\right) + c$

$= T(n/4) + c$ — (2) in (1)

$T(n/4) = T(n/8) + c$ — (3) in (1)

$T(n) = T(n/4) + c + c$ — (2) in (1)

$= T(n/4) + 2c$

$= T(n/2^2) + 2c$

$= T(n/8) + c + 2c$

$= T(n/2^3) + 3c$

$= T(n/2^4) + 4c \dots$

$= T(n/2^5) + 5c \dots$

'c' there is a pattern appearing thus we need to Generalize it and Eliminate the T(n) fⁿ using base Condition.

$T(n/2^i) + ic$

i steps

To terminate we need (1).

So n should be 2^i

$n = 2^i$

take log on both sides.

$\log n = \log 2^i$

$\log n = i(\log 2) - 1.$

$\log n = i$

$T(n/2^i) + ic$ sub interms of n.

$T\left(\frac{n}{n}\right) + ic$

$T(1) + ic$

$1 + ic$ we need to write time complexity in terms of 1. So Calculate i value

→ $1 + \log n \cdot c$ — constant.

$\therefore O(\log_2 n)$

Recurrence Relations.

$$T(n) = \begin{cases} T(n/2) + C & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

What is a recurrence relation?
How to write it & How do we solve it.

For Competitive Exams
Placements.

Let us take an Example of Binary Search algm.

- Recurrence – name itself tells us, calling itself again & again.

array should be sorted.

0	1	2	3	4	5	6
10	20	30	40	50	60	70

$i = 0$, $j = 6$

$40 = 30$? ✗

BS (a, i, j, x) — $x = 30$

mid = $(i+j)/2$ — $\frac{0+6}{2} = 3$

if ($a[mid] == x$) — $O(c)$

return (mid);

else if ($a[mid] > x$) — $40 > 30$? ✓

→ BS($a, i, mid-1, x$)

else — $40 < 30$

→ BS($a, mid+1, j, x$)

if $x = 40$ then Search is Complete so constant amount of time is used (1)

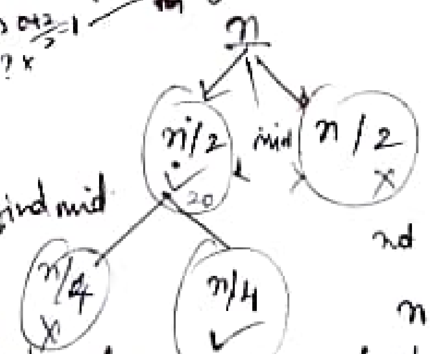
Based on behavior of the algm we will write the recurrence relation.

10 20 30 < mid > 50 60 70

0 1 2

mid → $\frac{0+2}{2} = 1$

$m = 30$? ✗



So what will be the recurrence relation

(coz Everytime we are looking towards only one subproblem)

$$\boxed{T(n) = T(n/2) + C}$$

– Some Constant (1) time, in finding mid or Comparison.

not interested going only towards one $n/2$

$n \rightarrow n/2 \rightarrow n/4 \rightarrow n/4$

again interested in only 1 sub problem

④ Substitution Method

② $T(n) = \begin{cases} 1 & \text{if } n=1 \\ n * T(n-1) & \text{if } n>1 \end{cases}$

1. Can solve all the recurrence relation using substit method.
2. which is not possible by Master theorem as it can solve only a Specific kind of problem.
3. Always gives correct answer, but takes longer time as it involves lot of mathematical f^n

$T(n) = n * T(n-1)$ — ① ← this fn will go on decreasing in the next f^n call. from (n-1) to (n-2) (n-3) .. decrease by subtracting by 1.

$T(n-1) = (n-1) * T((n-1)-1)$
$= (n-1) * T(n-2)$ — ②

$T(n-2) = (n-2) * T((n-2)-1)$
$(n-2) * T(n-3)$ — ③ } → we are finding, (Computing) iteration.

Now Substitute →

② in ① $T(n) = n * \{(n-1) * T(n-2)\}$ ④

③ in ④ $= n * (n-1) * \cancel{(\ldots)} * T\{(n-2) * T(n-3)$

$= n * (n-1) * (n-2) * T(n-3)$ — trend is keeps on decrease approaching

=

(n-i) steps

$n * (n-1) * (n-2) * (n-3) \ldots T(n-(n-1))$ we need to terminate (Eliminate) steps at some point using base Condition, initial Condit

$n * (n-1) * (n-2) * (n-3) \ldots * 1$

$n * (n-1) * (n-2) * \ldots * 3 * 2 * 1$ $\quad T(n-(n-1)$

$n * n(1-\frac{1}{n}) * n(1-\frac{2}{n}) * \ldots n(\frac{3}{n}) * n(\frac{2}{n}) * n(\frac{1}{n})$ $\quad (n-n+1)$

does matter in many

$T(1) = 1$

be $\frac{1}{n^3} \frac{1}{n^4}$ incoming n times

which is $< n^n$

$n! \to$

$n \cdot n \cdot n = n^3$

$O(n!)$ Exponential.

$T(n) = n * T(n-1)$ $\quad T(1) = 1$

$= T(n-2) * (n-1) * n$ $\quad T(n) = T(n-1) * n$

$= T(n-3) * (n-2) * (n-1) * n$ $\quad T(n-1) = T(n-2) * (n-1)$

$=$ $\quad T(n-2) = T(n-3) * (n-2)$

i times

$n - i = 1$

$n - 1 = i$

$= T(n-i) * (n-(i-1)) * (n-(i-2)) * \ldots n$

$= T(n-(n-1)) * (n-(n-1)-1) * (n-(n-1)-2) \ldots n$

$= T(n-n+1) * (n-n+2) * (n-n+3) \ldots n$

$= T(1) * 2 * 3 * \ldots n$

$= 1 * 2 * 3 * \ldots n$

$= n!$ $\quad O(n!)$.

→ 3

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

Now let use these eqn ② & ③ by substituting them in ①.

$T(n) = 2T(n/2) + n$ — ①

$T(n/2) = 2T(n/4) + n/2$ — ②

$T(n/4) = 2T(n/8) + n/4$ — ③

$T(n) = 2T(n/2) + n$

$= 2\left[2T(n/4) + n/2\right] + n = 2^2 T(n/2^2) + n + n$

$= 2^2 T(n/2^2) + 2n$

$= 2^2 T(n/4) + 2n$

$= 2^2 T(n/4) + 2n$

$= 2^2\left[2T(n/8) + n/4\right] + 2n = 2^3 (T(n/8) + n) + 2n$

$= 2^3 T(n/2^3) + 3n$ $= 2^3 T(n/2^3) + 3n$

$2^4 T(n/2^4) + 4n$, $2^5 T(n/2^5) + 5n$ ... pattern approaching thus Generalize.

i times

$2^i T(n/2^i) + in$ — we need to terminate it $T(1)$.

$2^i T(n/n) + in$

$2^i T(1) + in$

$n \cdot (1) + n\log n$

$\boxed{O(n\log n)}$.

Take $n = 2^i$

$n = 2^i$

$\log n = \log 2^i$

$\log n = K(\log 2)$ 1.

$\boxed{\log n = K}$.

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n-1) + \log n, & \text{if } n > 1 \end{cases}$$

$$T(n) = T(n-1) + \log n \quad \text{--- (1)}$$

$$T(n-1) = T(n-2) + \log(n-1) \quad (2)$$

$$T(n-2) = T(n-3) + \log(n-2) \quad (3)$$

$$T(n) = T(n-1) + \log n$$

$$= T(n-2) + \log(n-1) + \log n$$

$$= T(n-3) + \log(n-2) + \log(n-1) + \log n$$

pattern approach.

$$= T(n-4) + \log(n-3) + \log(n-2) + \log(n-1) + \log n$$

run i times

$$= \underbrace{T(n-i)}_{\text{Solve (1)}} + \log(n-(i-1)) + \log(n-(i-2)) + \log(n-(i-3)) + \dots + \log n$$

put $i = n$

$$= \frac{T(n-n)}{T(1)}$$

$$\log(\cancel{n-((n-1)+1)})$$

$$\log(\cancel{n-n+2})$$

$$\log(n-(n-1))$$

$$T(n-n+1) = 1$$

$\log m + \log n$
$= \log(m \cdot n)$

$\log(n-n+1) = \log 1 \quad \log 2 \quad \log 3 \dots \log n$

$$= 1 + \log 1 + \log 2 + \log 3 + \dots \log n$$

$$= 1 + \log(\underbrace{1 \cdot 2 \cdot 3 \cdot 4 \dots n}_{n!})$$

$$= 1 + \log(n!)$$

$n! = n(n-1)!$

$= n(n-1) \times (n-2) \times (n-3)$

$$= 1 + \boxed{\log n^n} = 1 + n \log n$$

may go up to 'n' time thus you can write it as n^n go upto over 'n' time

$$= \cancel{O \log n} \quad \boxed{O(n \log n)}$$

(5) $\to$ $T(n) = 4T(n/4) + 4$, $T(1) = 1$

$= 4[4T(n/4^2) + 4] + 4$

$= 4^2 T(n/4^2) + 8$ $\cancel{+4 = 20}$ $\cancel{4 \cdot 2} = 4 \times 5$

$4^2[4T(n/4^3) + 4] + 20$ (4^2+4)

$= 4^3 T(n/4^3) + 4^3 + 4^2 + 4$

$\vdots$

i times

$= 4^i T(n/4^i) + 4^i + 4^{(i-1)} + \dots + 4^{(i-i)}$ Summation

$= nT(n/n) + \cancel{n + \frac{n}{4}} \frac{n}{n} + \sum_{i=1}^{k} 4^i$ $\Rightarrow 4^k -$

$= \cancel{nT(1) +} \dots$

$= 4^i T(n/4^i) + \frac{1-4^i}{1-4}$ $\to \cancel{n + n} + \frac{n}{n}$

$\therefore$ $= nT(n/n) + \frac{1-n}{1-4}$

$= n + \frac{1-n}{1-4}$

$O(n)$.

$T(n) = 4T(n/4) + 4$

$T(n/4) = 4T(n/4^2) + 4$

$T(n/4^2) = 4T(n/4^3) + 4$

$\sum_{i=1}^{k} a \cdot r^i = a\left(\frac{1-r^i}{1-r}\right)$

Put. $n/4^i = 1$

$n = 4^i$

$\log n = \log 4^i$

$\log n = i \log_2 4$

$\log n = i \lg 4 - \log 2$

$\log_4 n = i$.

$4^{\log_4 n} = n$

⑥ $T(n) = 3T(n-1) + 1, \quad T(1)=1$

$T(n) = 3T(n-1) + 1$
$T(n-1) = 3T(n-2) + 1$
$T(n-2) = 3T(n-3) + 1$

Note:
$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$$
Summation formula

$= 3[3T(n-2)+1]+1$

$= 3^2 \cdot T(n-2) + 3 + 1 = 4 = 2 \cdot 2$

$= 3^2[3T(n-3)+1] + 3 + 1$

$= 3^3 \cdot T(n-3) + \underbrace{3^2 + 3 + 1}_{= 3^2 + 3^1 + 3^0}$

General form

i times.

$= 3^i \cdot T(n-i) + 3^{(i-1)} + 3^{(i-2)} + \dots + 3^0 \longrightarrow \frac{3^i}{3}$

$= 3^i T(n-i) + \sum_{i=0}^{k-1} 3^i \longrightarrow \sum_{i=0}^{3-1} 3^i = \sum_{i=0}^{2} 3^i = 3^0 + 3^1 + 3^2$

$T(1)=1$

$n-i=1$

$(n-1)=i$

$= 3^{(n-1)} T(n-n+1) + \sum_{i=0}^{n-1-1} 3^i = 3^{n-1} \cdot \underbrace{T(1)}_{1} + \sum_{i=0}^{n-2} 3^i$

$= 3^{n-1} + \sum_{i=0}^{n-2} 3^i$

$= 3^{n-1} + \frac{3^{n-1}-1}{3-1} = 3^{n-1} + \frac{3^{n-1}-1}{2} = \frac{2 \cdot 3^{n-1} + 3^{n-1} - 1}{2} = \frac{3^{n}-1}{2}$

$O(3^n)$.

(7) $T(n) = 8T(n/2) + n^2$ $\qquad T(1) = 1.$

$\frac{64}{16}$

$= 8\left[8T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$

$= 8^2 T\left(\frac{n}{2^2}\right) + 2n^2 + n^2$

$= 8^2\left[8T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + 2n^2 + n^2$

$= 8^3 T\left(\frac{n}{2^3}\right) + \underset{2^2 n^2}{4n^2} + 2n^2 + n^2$

$\longrightarrow 2^4 n^2 + 2^3 n^2 + 2^2 n^2 + 2n^2 + n^2$

$T(n) = 8T(n/2) + n^2$

$T(n/2) = 8T(n/2^2) + \left(\frac{n}{2}\right)^2$

$T(n/2^2) = 8T(n/2^3) + \left(\frac{n}{2^2}\right)^2$

$\vdots$

....

i times

$= 8^i T\left(\frac{n}{2^i}\right) + 2^3 n^2 (2^3 + 2^2 + 2^1 + 2^0) \longrightarrow 8^i T(n/2^i) + \dots kn^2 + \dots 4n^2 + 2n^2 + n^2$

$n^2(1+2+4\dots k)$

G.P $= 8$

$\frac{n}{2^i} = 1$

$n = 2^i$

$\log n = \log_2 {}^i$

$\log_2 n = i \log_2 2$

$i = \log_2 n$

$= 8^{\log_2 n} T(n/n) + 2n^2(2^3 + 2^2 + 2^1 + 2^0)$

$\sum_{k=0}^{\log n - 1} 2^k n^2$

$= 8^i \cdot 1 + n^2(1+2+4+\dots i)$ $\longrightarrow 2^i = n$

$= 8^i \cdot 1 + n^2(2^k - 1) = 2^{3(i)} + n^2(2^i - 1)$

$= \cancel{\theta(n)} = 2^{3(i)} + n^2(n-1)$

$8^{\log_2 n}$

$n^{\log_2 8}$

let $n/2^k$

$2^{3 \times \log_2 n} \rightarrow n^{3 \cdot \log_2 2 = 1} = 2^{3i} + n^3 - n =$

$n^3 + n^3 - n \longrightarrow O(n^3)$

$\frac{9}{8}\left(n^3 - \frac{n^3}{n^2}\right) = \dots \frac{9}{8}\left(\frac{n^3 \times n^2 - n^3}{n^2}\right)$

$= 9/8\left(\frac{n^5 - n^3}{n^2}\right) = 9/8\left(\frac{n^2 - 1}{\cancel{n^2}}\right) \times n^2 \times n$

$= 9/8(n^3 - n)$

(3):

$T(n) = 3T(n/3) + n^3$ $\qquad T(1) = 0.$

$= 3\left[3T(n/3^2) + \left(\frac{n}{3}\right)^3\right] + n^3$ $\qquad T(n) = 3T(n/3) + n^3$

$= 3^2 \cdot T\left(\frac{n}{3^2}\right) + \frac{n^3}{3^2} + n^3$ $\qquad T(n/3) = 3T(n/3^2) + \left(\frac{n}{3}\right)^3$

$T(n/3^2) = 3T\left(\frac{n}{3^3}\right) + \left(\frac{n}{3^2}\right)^3$

$= 3^2\left[3T\left(\frac{n}{3^3}\right) + \left(\frac{n}{3^2}\right)^3\right] + \frac{n^3}{3^2} + n^3$

$= 3^3 \cdot T\left(\frac{n}{3^3}\right) + \frac{n^3}{3^4} + \frac{n^3}{3^2} + n^3$ $\qquad 3^2 \times 3^2 = 3^2$

$= 27 \cdot T\left(\frac{n}{27}\right) + \frac{n^3}{9^2} + \frac{n^3}{9} + n^3$ ... Pattern approaching $\qquad \frac{3\times3\times3\times3}{9\times9} = 81$

i times

General form

$T(n) = 3^i \cdot T\left(\frac{n}{3^i}\right) + \cancel{n^3}\left(\frac{1}{9^{(i-1)}} + \frac{1}{9^{(i-2)}} + \dots + \frac{1}{9^{(i-i)}}\right) n^3$

Summation.

$= 3^i \cdot T(n/3^i) + \left(\sum_{i=0}^{k-1} \frac{1}{9^i}\right) n^3$

$\boxed{\text{NOTE: } \sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}}$ $\qquad = 3^i \cdot T(n/3^i) + \left(\frac{1-\left(\frac{1}{9}\right)^i}{1-\left(\frac{1}{9}\right)}\right) n^3$

$n/3^i = 1$ $\qquad = 3^i \cdot T(n/3^i) + \left(\frac{1-\left(\frac{1}{9^i}\right)}{\frac{9-1}{9}}\right) n^3 = 3^i \cdot T(n/3^i) + \left(\frac{1-\frac{1}{9^i}}{8/9}\right)$

$n = 3^i$ $\qquad T(n) = 3^{\log_3 n} T(n/3^{\log_3 n}) + \left(\frac{1-\frac{1}{9}^{\log_3 n}}{8/9}\right) n^3$

$\log n = \log 3^i$ $\qquad n^{\log_3 3} = 1$ $\qquad = n \cdot T(n/n) + \left(\frac{1-\frac{1}{n^2}}{8/9}\right) \times n^3$ $\qquad \frac{1}{n^{\log_3 9}}$

$\log_3 n = i \log_3 3$

$\log_3 n = i$ $\qquad = n \cdot T(1) + \frac{9}{8}\left(1 - \frac{1}{n^2}\right) \times n^3$ $\qquad n^{\frac{\log 9}{\log 3}}$

$= n(0) + \frac{9}{8}\left(n^3 - \frac{n^3}{n^2}\right)$ $\qquad n^2$

$= 0 + 9/8\,(n^3 - n)$ $\qquad \frac{n^2 n^3 - n^3}{n^2}$

$\therefore O(n^3)$

$2n^3 - 2n^3$

$n^3\left(1 - \frac{1}{n^2}\right) = \left(\frac{n^2 - n}{n^2}\right) \times n^3 \to n^3 - n$

Basics.

(iii) Here $S_n = 1^3 + 2^3 + \dots + n^3$

We consider, $(K+1)^4 - K^4 = 4K^3 + 6K^2 + 4K + 1$

Putting $K = 1, 2, 3 \dots n$, we get

$2^4 - 1^4 = 4(1)^3 + 6(1)^2 + 4(1) + 1$

$3^4 - 2^4 = 4(2)^3 + 6(2)^2 + 4(2) + 1$

$4^4 - 3^4 = 4(3)^3 + 6(3)^2 + 4(3) + 1$

$\vdots$

$$S_n = \frac{n^2(n+1)^2}{4} = \frac{[n(n+1)]^2}{4}$$

---

(9) $T(n) = 2T(n/2) + 1 \qquad T(1) = 1.$

$T(n) = 2T(n/2) + 1$

$T(n/2) = 2T(n/2^2) + 1$

$T(n/2^2) = 2T(n/2^3) + 1$

$T(n/2^3) + 2T(n/2^4) + 1$

$n = 2^i$

$\log_2 n = i \log_2 2$

$\log n = i$

$= 2\left[2T(n/2^2) + 1\right] + 1$

$= 2^2 \cdot T(n/2^2) + 2 + 1$

$= 2^2\left[2 \cdot T(n/2^3) + 1\right] 2 + 1$

$= 2^3 \cdot T(n/2^3) + 4 + 2 + 1$

$= 2^3 \cdot \left[2T(n/2^4) + 1\right] + 4 + 2 + 1$

$= 2^4 \cdot T(n/2^4) + 2^3 + 4 + 2 + 1$

$2^3 + 2^2 + 2^1 + 2^0$

$8 + 4 + 2 + 1 = 15 \quad 2^4 - 1$

$2^4 - 1 = 16 - 1 = 15$

i times

$= 2^i \cdot T(n/2^i) + 2^i - 1$

$= 2^{\log_2 n} \cdot T(n/2^{\log_2 n}) + 2^{\log_2 n} - 1 = n^{\log_2 2} T(n/n) + n^{\log_2 2} - 1 \quad = 1$

$= n + T(1) + n - 1 = n + n = 2n - O(n)$ //

(10)

$T(1) = 1$.

$\rightarrow T(n) = 3T(n/3) + n$

$= 3\left[3 \cdot T(n/3^2) + n/3\right] + n$

$= 3^2 \cdot T(n/3^2) + n + n$

$= 3^2 \cdot \left[3T(n/3^3) + \frac{n}{3^2}\right] + n + n$

$= 3^3 \cdot T(n/3^3) + n + n + n$

$= 3^3\left[3 \cdot T(n/3^4) + \frac{n}{3^3}\right] + \underbrace{n + n + n}_{3n}$

$= 3^4 \cdot T(n/3^4) + \underbrace{n + 3n}_{4n}$

$\vdots$

i times

$= 3^i \cdot T(n/3^i) + in$

$T(n) = 3T(n/3) + n$

$T(n/3) = 3T(n/3^2) + \frac{n}{3}$

$T(n/3^2) = 3 \cdot T(n/3^3) + \frac{n}{3^2}$

$T(n/3^3) = 3 \cdot T(n/3^4) + \frac{n}{3^3}$

$n/3^i = 1$

$n = 3^i$

$\log_3 n = i \log_3 3$

$\log_3 n = i$

$= 3^{\log_3 n} T(n/n) + n\log_3 n$

$= n^{\log_3 3 \to 1} T(1) + n\log_3 n$

$= n + n\log_3 n$

$O(n\log_3 n)$..

(11) $T(n) = T(n-1) + n^4$

$\left\{ T(n) = \begin{cases} 2 & \text{for } n=0 \\ 2+T(n-1) & \text{for } n>0 \end{cases} \right.$

$= T(n-2) + (n-1)^4 + n^4$

$\{ T(n) = T(n-1)+1 \}$ $T(1) = 1.$

$= T(n-3) + (n-2)^4 + (n-1)^4 + n^4$

$= T(n-4) + (n-3)^4 + (n-2)^4 + (n-1)^4 + n^4$ ... pattern appearing..

.... i times.

$= T(n-i) + (n-(i-1))^4 + (n-(i-2))^4 + (n-(i-3))^4 + \cdots + (n^4)$

Put $n=i$ $= T(n-n) + (n-(n-1))^4 + (n-(n-2))^4 + (n-(n-3))^4 + n^4$

$T(0) = 0$ $= T(0) + 2^4 + 2^4 + \ldots\ 3^4 \ldots\ldots n^4.$

$= 0 + 1^4 + 2^4 + 3^4 + \cdots\ (n-1)^4 + n^4$

Sum of 1st n natural nos.

$\frac{n(n+1)}{2} = O(n^2)$

2nd power. Sum of 1st sq. n. natural nos.

$\frac{n(n+1)(2n+1)}{6} = O(n^3).$

Sum of 1st 4 power of natural nos.

$= O(n^5)$

(12) $\rightarrow$ $T(n) = T(n-1) + \log(n)$, $T(0) = 0$

Logarithm Product Rule : $\log(mn) = \log(m) + \log(n)$

$$T(n) = T(n-1) + \log(n)$$

$$T(n) = T(n-1) + \log(n)$$
$$T(n-1) = T(n-2) + \log(n-1)$$
$$T(n-2) = T(n-3) + \log(n-2)$$

$$= T(n-2) + \log(n-1) + \log(n)$$
$$= T(n-3) + \log(n-2) + \log(n-1) + \log n$$
$$= T(n-3) + \log\{(n-2)(n-1).n\}$$
$$= T(n-3) + \log(\underbrace{n.(n-1).(n-2)}_{n!}) \ldots$$ pattern approaches!

⋮

i times

$T(0)=0$

$n = i$

$$= T(n-i) + \log(n^{(i-3)}.(n-(i-2)).(n-(i-1)) \ldots$$
$$= T(\underbrace{\cancel{n}-\cancel{n}}_{0}) + \log(n-(n-3)).(n-(n-2)).(n-(n-1))$$
$$= 0 + \log(\cancel{n}-\cancel{n}+3).(\cancel{n}-\cancel{n}+2).(\cancel{n}-\cancel{n}+1)$$
$$= 0 + \log(\underbrace{(3).(2).(1)}_{})$$ Pattern.

$$= 0 + \log(n!)$$ $3 \times 2 \times 1$

$$= \log(n!)$$ $3!$

$$T(n) \in O(\log n!)$$

→ (13) $T(n) = 3T\lfloor n/4 \rfloor + n$ $\qquad = T(1) = 1$

$= 3(3T\lfloor n/16 \rfloor + n/4) + n = 9T\lfloor n/16 \rfloor + 3n/4 + n$

$= 9[3T\lfloor n/64 \rfloor + n/16] + 3n/4 + n = 27T\lfloor n/64 \rfloor + \frac{9n}{16} + \frac{3n}{4} + n$

$= 3^3 T\lfloor n/4^3 \rfloor + n\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right)$

i times

$= 3^i T\lfloor n/4^i \rfloor + n\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right]$

$= 3^{\log_4 n} + \underset{=1}{(n/n)} + n\underbrace{[1 + 3/4 + (3/4)^2 + \cdots]}$

G.P with common ratio 3/4.

$n = 4^i$

$\log_4 n = \log_4 4^i$

$\log_4 n = i \log_4 4$

$\log_4 n = i$

↓ rearrange

$= n^{\log_4 3} + n\left[\frac{1}{1 - 3/4}\right]$

'i' when G.P common ratio is less than 1

$\frac{1}{\frac{4-3}{4}} = \frac{1}{\frac{1}{4}}$

Simply take sum of infinite terms in a G.P. series.

$= n^{\log_4 3} + \frac{n}{\frac{1}{4}} \rightarrow 4n$

↓

$\text{Sum of } " = \frac{a}{1-r}$ (a → 1st term, r → common ratio.)

$= n^{\log_4 3} + 4n$

$= n^{.75} + 4n^1 = O(n)$

(14) $T(n) = 2T(n/2) + 4n$

$= 2[2T(n/4) + 4(n/2)] + 4n$

$= 2^2 T(n/2^2) + 4n + 4n \rightarrow (4 \cdot 2n)$

$= 2^2 [2T(n/2^3) + 4(n/4)] + 4 \cdot 2n$

$= 2^3 \cdot T(n/2^3) + 4n + 4 \cdot 2n$

$= 2^3 \cdot T(n/2^3) + 3 \cdot 4n$

$\vdots$

i times

$= 2^i \cdot T(n/2^i) + i \cdot 4n$

$= 2^{\log n} \quad 2^i \cdot T(n/n) + i \cdot 4n$

$= nT(1) + i \cdot 4n$

$= n \times 4 + i \cdot 4n$

$= 4n + (\log_2 n) \cdot 4n$

$= 4n + 4n \log n$

$= O(n \log n)$

$T(1) = 4$

$T(n) = 2T(n/2) + 4n$

$T(n/2) = 2T(n/4) + 4(n/2)$

$T(n/4) + 2T(n/2^3) + 4(n/4)$

$2^{\log_2 n}$ swap $= n^{\log_2 2} = n$

$n = 2^i$

$\log n = i \log 2$

$\log n = i$

$2^i = 2^{\log_2 n}$ — same base

$= n$

$T(n) = 8T(n/2) + n^2$ and $T(1)=1$

$T(n) = 8T(n/2) + n^2$

$T(n/2) = 8T(n/4) + (\frac{n}{2})^2$

$T(n/4) = 8T(n/4^2) + (n/4)^2$

$$T(n) = 8T(n/2) + n^2$$

$$= 8[8T(n/2^2) + (n/2)^2] + n^2$$

$$= 8^2 \cdot T(n/2^2) + \frac{2n^2}{8} + n^2 = 8^2 \cdot T(n/2^2) + 2n^2 + n^2$$

$$= 8^2[8T(n/4^2) + (n/4)^2] + 2n^2 + n^2$$

$$= 8^3 \cdot T(n/4^2) + 4n^2 + 2n^2 + n^2$$

4
× 4
16
× 4
64

⋮

i times

=

---

(15) $T(n) = T(n-1) + 2n$ $T(0) = 0$

$$= T(n-2) + 2(n-1) + 2n$$

$$= T(n-3) + 2(n-2) + 2(n-1) + 2n$$

$$= T(n-3) + 2n - 4 + 2n - 2 + 2n$$

$$= T(n-3) + 3(2n) - 6 \quad 3 \times 2$$

$T(n) = T(n-1) + 2n$

$T(n-1) = T(n-2) + 2(n-1)$

$T(n-2) = T(n-3) + 2(n-2)$

⋮

i times

$$= T(n-i) + i(2n) - i \times 2$$

sub. $i = n$.

$$= T(\cancel{n} - \cancel{n}) + n(2n) - 2n$$

$$= 0 + 2n^2 - 2n = 2(n^2 - n)$$

$O(n^2)$.

→ Lower Bound $T(n) \geq 2T(n/2) + cn$

1. Guess : $T(n) = \Omega(n \log n)$ $T(n) \geq dn\lg n$

2. Prove by • Induction for n

$$T(n) \geq 2T(n/2) + cn$$
$$= 2\left(d\tfrac{n}{2} \lg \tfrac{n}{2}\right) + cn$$
$$= dn \lg \frac{n}{2} + cn$$
$$= dn\lg n - dn + cn$$
$$\geq dn\lg n \quad \text{if } -dn + cn \geq 0, \quad d \leq c$$

→ $T(n) = T(\sqrt{n}) + n$

$$= T(n^{1/2}) + n$$
$$= T(n^{1/4}) + n^{1/2} + n$$
$$= T(n^{1/2^3}) + n^{1/2^2} + n^{1/2} + n$$
$$= T(n^{1/2^k}) + n^{1/2^{k-1}} + n^{1/2^{k-2}} + \cdots + n.$$
$$= 1 + n + n^{1/2} + \cdots + n^{2/\log n}$$
$$= O(n).$$

$T(2) = 1$

$n^{1/2^k} =$

$T(n) = T(\underset{T(n^{1/2})}{\sqrt{n}}) + n$

$T(n^{1/2}) = T(n^{1/4}) + n^{1/2}$

$T(n^{1/4}) = T(n^{1/2^3}) + n^{1/4}$

$[n^a + n^{a-1} + n^{a-2} = O(n^a)]$

(16)

By Master → $a=9,\ b=3,\ d=1.$ $b^d\ 3^1=3.$
$a > b^d$ ✓ Case 3 $n^{\log_b a} = n^{\log_3 9}$
$= O(n^2)$.

$T(n) = 9T(n/3)+n$ $\quad T(1)=1$

$= 9[9T(n/3^2)+n/3]+n$

$= 9^2 T(n/3^2)+\underline{3n+n} \rightarrow (4n\ 2.2n)$

$= 9^2[9T(n/3^3)+n/3^2]+3n+n$

$= 9^3 T(n/3^3)+\underline{9n+3n+n} \rightarrow \cancel{13n} \rightarrow$
$(3^2+3^1+3^0)\,n$

$= 9^3[9T(n/3^4)+(n/3^3)]+9n+3n+n$

$= 9^4 T(n/3^4)+27n+9n+3n+n$
$3^3 n+3^2 n+3^1 n+\cdots+n.$

$=\ \vdots$

itimes.

$= 9^i T(n/3^i)+3^{(i-1)}n+3^{(i-2)}n+3^{(i-3)}n+\cdots+n$

$= 9^{\log_3 n}\cdot T(n/n)+\frac{3^i}{3}n+\frac{3^i}{3^2}n+\frac{3^i}{3^3}n+\cdots+n$

$= n^{\log_3 9}\cdot 1+\frac{3^{\log_3 n}}{3}n+\frac{3^{\log_3 n}}{3^2}n+\frac{3^{\log_3 n}}{3^3}+\cdots+n$

$= n^{\log_3 9}+\frac{n^{\log_3 3}}{3}n+\frac{n^{\log_3 3}}{3^2}n+\frac{n^{\log_3 3}}{3^3}n+\cdots+n$ $\quad 3^{i-1}$ $\quad \frac{3^{\log_3 n}}{3^{\log_3 3}}=1$

$= n^{\log_3 9}+\frac{n^2}{3}+\frac{n^2}{3^2}+\frac{n^2}{3^3}+\cdots+n$

$= n^2+\frac{n^2}{2}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)\times 3.$ $= 3/2$

$\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{3-1}{3}}$
$=\frac{1}{\frac{2}{3}}=3/2.$

$= n^2+\frac{n^2}{2}$ $\quad O(n^2)$ //.

$T(n)=9T(n/3)+n$
$T(n/3)=9T(n/3^2)+(n/3)$
$T(n/3^2)=9T(n/3^3)+(n/3^2)$
$T(n/3^3)=9T(n/3^4)+(n/3^3)$
$\vdots$

$3^2+3^1+3^0$
$9+3+1=13.$

$3\times3\times3$

$\sum_{i=1}^{n} 3^i$
$= \cancel{3\times3^3}$
$= (n-1)$
$= (3-1)3^{3+1}+2$

$\frac{n}{3^i}=1,\ \log_3 n = \log_3 3^i$
$\log_3 n = i$

⇒ 17

$T(n) = 8 \cdot T(n/2)^2 + n^2$ $\qquad$ $T(1) = 1$.

$= 8\left[8T(n/2^2) + (n/2)^2\right] + n^2$

$= 8^2 \cdot T(n/2^2) + 2n^2 + n^2$

$= 8^2\left[8T(n/2^3) + (n/2^2)^2\right] + 2n^2 + n^2$

$= 8^3 \cdot T(n/2^3) + 4n^2 + 2n^2 + n^2$

$= 8^3 \cdot \left[8T(n/2^4) + (n/2^3)^2\right] + 4n^2 + 2n^2 + n^2$

$= 8^4 \cdot T(n/2^4) + \cancel{8}n^2 + 4n^2 + 2n^2 + n^2$

$T(n) = 8T(n/2) + n^2$

$T(n/2) = 8T(n/2^2) + (n/2)^2$

$T(n/2^2) = 8T(n/2^3) + (n/2^2)^2$

$T(n/2^3) = 8T(n/2^4) + (n/2^3)^2$

$\cancel{16} \times 81$

324

$8 \times 8 \times 8$

$64.$

⋮

i times $\qquad$ General form

$= 8^i \cdot T(n/2^i) + n^2(\cancel{\cdots} + 4 + 2 + 1)$

$2^3 + 2^2 + 2^1 + 2^0 \quad — \sum_{i=1}^{n} 2^i - 1$

$= 8^i \cdot T(n/2^i) + n^2(2^i - 1)$

$n/2^i = 1$

$n = 2^i$, $\log_2 n = i\log_2 2$

$\log_2 n = i$

$= 8^{\log_2 n} \cdot T(n/n) + n^2(2^{\log_2 n} - 1)$

$= n^{\log_2 8} \cdot T(1) + n^2(n^{\log_2 2} - 1)$

$= n^{\log_2 8} \cdot 1 + n^2(n^2 - 1)$

$= n^{\log_2 8} \cdot 1 + n^3 - n^2$

$= n^3 + n^3 - n^2$ $\qquad$ $O(n^3)$ //.

By Masters.

$a = 8$, $b = 2$, $d = 2$

$b^d = 2^2 = 4$

$a > b^d$

Case 3 $\quad O(n^{\log_b a})$

$= O(n^{\log_2 8}) = O(n^3)$

⑰ $\Rightarrow T(n) = T(n-2) + n^2$ $\qquad T(0) = 1$

$$= T(n-4) + (n-2)^2 + n^2$$

$$= T(n-6) + (n-4)^2 + (n-2)^2 + n^2$$

$\vdots$

i times

$$= T(n-i) + (n-(i-2))^2 + (n-(i-4))^2 + \cdots + n^2$$

$n = i$

$$= T(n-n) + (n-n+2)^2 + (n-n+4)^2 + \cdots + n^2$$

$$= \underbrace{1 + 2^2 + 4^2 + \cdots + n^2}_{\text{Sum of Square of Even numbers.}}$$

$$\rightarrow 2n(n+1)(2n+1)/3 \approx \frac{1}{3}(n^3)$$

$$\therefore O(n^3)$$

⑱ $\Rightarrow T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-1) + \frac{1}{n} & \text{otherwise} \end{cases}$

$$T(n) = T(n-1) + \frac{1}{n}$$

$$= T(n-2) + \frac{1}{(n-1)} + \frac{1}{n}$$

$$= T(n-3) + \frac{1}{(n-2)} + \frac{1}{(n-1)} + \frac{1}{n}$$

$$= T(n-4) + \frac{1}{(n-3)} + \frac{1}{(n-2)} + \frac{1}{(n-1)} + \frac{1}{n}$$

$$T(n) = T(n-1) + \frac{1}{n}$$

$$T(n-1) = T(n-2) + \frac{1}{(n-1)}$$

$$T(n-2) = T(n-3) + \frac{1}{(n-2)}$$

$$T(n-3) = T(n-4) + \frac{1}{(n-3)}$$

i times

$$= T(n-i) + \frac{1}{(n(i-1))} + \frac{1}{(n+(i-2))} + \frac{1}{(n-(i-3))} + \cdots + \frac{1}{n}$$

P.T.O

$n-i=1$

$\cancel{n=i}$

$n-1=i$

$$= T(n-n+1) + \frac{1}{(n-n+1+1)} + \frac{1}{(n-n+1+2)} + \frac{1}{(n-n+1+3)} + \cdots + \frac{1}{n}$$

$$= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}}$$

Series (Summation) $\sum_{i=1}^{n} \frac{1}{i} \approx \log n + \gamma$

$\gamma \downarrow$ Eulers 0.557 const.

$T(n) = O(\log_2 n)$

$\Rightarrow$ (19) $T(n) = T(n-1) + n^2$ $\qquad T(0) = 1$

$= T(n-2) + (n-1)^2 + n^2$

$= T(n-3) + (n-2)^2 + (n-1)^2 + n^2$

$= T(n-4) + (n-3)^2 + (n-2)^2 + (n-1)^2 + n^2$

(*)

$= T(n-4) + n^2 + n^2 + n^2 + 9 + 4 + 1 - 6n - 4n - 2n$

$= T(n-4) + 3n^2 + 14 - 12n$

$4+10 - 4 \cdot 3n$

$\vdots$ i times

$= T(n-i) + (i-1)n^2 + i \times 3n + 14$

$\cancel{= T(n-n+1) + ((n-1)-1)n^2 + (n-1) \times 3n + 14}$

$= T(n-n) + (n-1)n^2 + n \times 3n + 14$

$= 1 + n^3 - n^2 + 3n^2 + 14$

$= n^3 + 2n^2 + 15 \qquad O(n^3)$ //

$T(n) = T(n-1) + n^2$

$T(n-1) = T(n-2) + (n-1)^2$

$T(n-2) = T(n-3) + (n-2)^2$

$T(n-3) = T(n-4) + (n-3)^2$

$(n-1)^2 = n^2 - 2n + 1$

$(n-2)^2 = n^2 - 4n + 4$

$(n-3)^2 = n^2 - 6n + 9$

$\cancel{n=i-1}$

$n-1 = i$

$n = i$ $\qquad$ Put $n = i$

(*) $\rightarrow = T(n-i) + (n-(i-1))^2 + (n-(i-2))^2 + (n-(i-3))^2 + \cdots + n^2$

$= T(n-n) + (n-(n-1))^2 + (n-(n-2))^2 + (n-(n-3))^2 + \cdots + n^2$

$= 1 + (n-n+1^2) + (n-n+2^2) + (n-n+3^2) + \cdots + n^2$

$1^2 + 2^2 + 3^2 + \cdots n^2 \approx \frac{1}{3}(n^3)$

(20) $T(n) = 3T(n/2) + cn^2 \qquad T(1)=1$

$= 3\left[3T(n/2^2) + c(n/2)^2\right] + cn^2$

$= 3^2 \cdot T(n/2^2) + \frac{3}{4}n^2 + cn^2$

$= 3^2\left[3T(n/2^3) + c[n/2^2]^2\right] + \frac{3}{4}n^2 + cn^2$

$= 3^3 T(n/2^3) + \frac{9}{4^2}n^2 + \frac{3}{4}n^2 + cn^2$

$= 3^3\left[3T(n/2^4) + c(n/2^3)^2\right] + 9/4^2 n^2 + 3/4 n^2 + cn^2$

$= 3^4 T(n/2^4) + \frac{27}{4^3}cn^2 + 9/4^2 n^2 + 3/4 cn^2 + cn^2$

$T(n) = 3T(n/2) + cn^2$

$T(n/2) = 3T(n/2^2) + c(n/2)^2$

$T(n/2^2) = 3T(n/2^3) + c(n/2^2)^2$

$T(n/2^3) = 3T(n/2^4) + c(n/2^3)^2$

generalize ... i times.

$= 3^i T(n/2^i) + 27/4^3 cn^2 + 9/4^2 cn^2 + 3/4 cn^2 + cn^2$

$= 3^i \cdot \underbrace{T(n/n)}_{1} + cn^2\left(1 + 3/4 + (3/4)^2 + (3/4)^3 \ldots\right)$

$\rightarrow$ G P with common ratio of $3/4$ $\therefore$ we can represent it as $\frac{a}{1-r} = \frac{1}{1-3/4}$

$= \frac{1}{4-3/4} = \frac{1}{1/4} = 4$

$n/2^i = 1$

$n = 2^i$

$i \log_2 2 = \log_2 n$

$\log n = i$

$= 3^{\log_2 n} \cdot 1 + 4cn^2$

$n^{\log_2 3 \rightarrow 0.1761} + 4cn^2$

$= n^0 + 4cn^2 \qquad 1 + 4cn^2 \rightarrow O(n)^2$

Solve By Master's Theorem. Compute $n^{\log_b a}$ $\qquad n^{\log_2 3}$

$T(n) = 3T(n/2) + cn^2, \; a=3, b=2 \quad f(n) = n^2$ $\qquad n^{0.17}$

$d=2, \; b^d = 2^2 = 4$

$a < b^d$ - Case 1.

$O(n^d) \rightarrow O(n^2)$

$\Rightarrow$ 2.1.

$x(n) = x(n/2) + n$ $\qquad x(1) = 1$

$x(n) = x(n/2) + n$

$x(n/2) = x(n/2^2) + n/2$

$x(n/2^2) = x(n/2^3) + n/2^2$

$x(n/2^3) = x(n/2^4) + n/2^3$

$\vdots$

$= x(n/2^2) + n/2 + n$

$= x(n/2^3) + n/2^2 + n/2 + n$

$= x(n/2^4) + n/2^3 + n/2^2 + n/2 + n \ldots$ ~~$+n$~~

$\vdots$ Generalize

i times

$= x(n/2^i) + n/2^{(i-1)} + n/2^{(i-2)} + n/2^{(i-i)} \ldots - + n.$

$n/2^i = 1$

$n = 2^i$

$\log n = i \log_2 2$

$\log_2 n = i$

$= x\left(n/\underset{\text{swap}}{2^{\log_2 n}}\right) + \dfrac{n}{\frac{2^i}{2}} + \dfrac{n}{\frac{2^i}{2^2}} + \ldots + n$

$= x\left(\dfrac{n}{n^{\log_2 2}}\right) + \dfrac{2n}{2^i_n} + \dfrac{2^2 n}{2^i} + \ldots + n$

$= x(n/n) + \dfrac{2n}{n} + \dfrac{4n}{n} + \ldots + n$

$= x\left(\dfrac{1}{1}\right) + 2 + 4 + \ldots + n$

$= 1 + 2 + 4 + \ldots + n$

$= 2^0 + 2^1 + 2^2 + \ldots + n$

Sum of 1^{st} even nos. $\Rightarrow \sum_{i=1}^{n} 2^i - 1$

$\therefore 2^i - 1 = 2^{\log_2 n} - 1 = n^{\log_2 2} - 1 = n - 1$

$O(n)$ //.

By M.T $x(n) = x(n/2) + n^1$

$a = 1, b = 2, d = 1.$ $b^d = 2^1 = 2$

$\therefore a < b^d$ – Case 1. $O(n^d) = O(n^1)$

$\therefore O(n)$ //.

22)

$T(n) = 2T(n/4) + n/2$ $\quad T(1) = 1$

$= 2[2T(n/4^2) + n/8] + n/2$

$= 2^2 T(n/4^2) + n/4 + \frac{n}{2}$

$= 2^2[2\cdot T(n/4^3) + \frac{n}{16\times 2}] + \frac{n}{4} + \frac{n}{2}$

$= 2^3 T\left(\frac{n}{4^3}\right) + \frac{n}{8} + \frac{n}{4} + \frac{n}{2}$

$= 2^3\left[2\cdot T(n/4^4) + \frac{n}{4^3\times 2}\right] + \frac{n}{8} + \frac{n}{4} + \frac{n}{2}$

$= 2^4 T\left(\frac{n}{4^4}\right) + \frac{n}{16} + \frac{n}{8} + \frac{n}{4} + \frac{n}{2}$

$T(n) = 2T(n/4) + n/2$

$T(n/4) = 2T(n/4^2) + n/8$

$T(n/4^2) = 2T(n/4^3) + \frac{n}{4^2\times 2} \to \frac{n}{16\times 2}$

$T(n/4^3) = 2T(n/4^4) + \frac{n}{4^3\times 2}$

$n/4^i = 1$

$n = 4^i$

$\log_4 n = \log 4^i$

$\log_4 n = i \log_4 4$

$\log_4 n = i$

Generalize ... i times.

$= 2^i \cdot T\left(\frac{n}{4^i}\right) + n\left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right)$ ∴ G.P with common ratio $1/2$ ∴ $\frac{a}{1-r}$

$= \frac{1/2}{1-1/2} = \frac{1/2}{\frac{2-1}{2}} = \frac{1/2}{1/2} = 1.$

$= 2^i \cdot T(n/4^i) + n$

$= 2^{\log_4 n} \cdot T(n/4^{\log_4 n}) + n = n^{\log_4 2} \cdot T(n/n^{\log_4 4}) + n$

$= n^{0.5} \cdot T(n/n) + n$

$= n^{0.5} \cdot 1 + n \implies O(n)$

By M.T $T(n) = 2T(n/4) + \frac{1}{2}n$

$a = 2, b = 4, n^1, d = 1, b^d = 4^1 = 4$

∴ $a < b^d = 2 < 4$ Case 1.

$O(n^d) = O(n^1) = \underline{O(n)}$

(23) $T(n) = 3T(n/3) + n \quad T(1) = 1$

$= 3\left[3T\left(n/3^2\right) + \frac{n}{3}\right] + n$

$= 3^2 T\left(n/3^2\right) + n + n$

$= 3^2\left[3T\left(n/3^3\right) + \frac{n}{3^2}\right] + 2n$

$= 3^3 T\left(n/3^3\right) + n + 2n$

$= 3^3\left[3T\left(n/3^4\right) + n/3^3\right] + 3n$

$= 3^4 \cdot T\left(n/3^4\right) + \underbrace{n + 3n}_{4n}$

$\Rightarrow$ Generalize ... i times

$= 3^i \cdot T\left(n/3^i\right) + in$

$= n \cdot T\left(n/n\right) + in$

$= n \cdot 1 + in = n + n\log_3 n$

$\therefore \Theta\left(n\log_3 n\right)$

$T(n) = 3T(n/3) + n$

$T(n/3) = 3T\left(n/3^2\right) + \frac{n}{3}$

$T\left(n/3^2\right) = 3T\left(n/3^3\right) + n/3^2$

$T\left(n/3^3\right) = 3T\left(n/3^4\right) + n/3^3$

...

$n = 3^i = \log_3 n = \log_3 3^i$

$\log_3 n = i\log_3 3$

$\log_3 n = i$

$3^i = 3^{\log_3 n} = n^{\log_3 3}$

$= n^1 = n \quad \therefore 3^i = n$

Substitute in general eq.

By M.T. $T(n) = 3T(n/3) + n^1$

$a = 3,\ b = 3,\ d = 1 \quad b^d = 3^1 = 3$

$a = b^d, \quad 3 = 3, \quad \therefore$ Case 2 $\rightarrow \Theta(n^d \log n) = \Theta(n\log_3 n)$

$\therefore \cancel{\Theta(n^d \log n)} = \cancel{n}^{\log_3 3} = n^1 = n \quad \therefore n^d\ n^1\ n$

$\Theta(n\log_3 n)$

⇒ (24) $T(n) = 9T(n/3) + n \qquad T(1) = 1.$

$= 9[9T(n/3^2) + n/3] + n$

$= 9^2 T(n/3^2) + 3n + n$

$= 9^2 [9T(n/3^3) + \frac{n}{3^2}] + 3n + n$

$= 9^3 T(n/3^3) + 9n + 3n + n$

$= 9^3 [9T(n/3^4) + \frac{n}{3^3}] + 9n + 3n + n$

$= 9^4 T(n/3^4) + 27n + 9n + 3n + n$

Generalize ... i terms

$= 9^i T(n/3^i) + 27n + 9n + 3n + n$

$= 9^{\log_3 n} T(n/n) + n(\underline{1 + 3 + 3^2 + 3^3})$ → G.P with C.R = 3

$= n^{\log_3 9} T(1) - \frac{1}{2} n \qquad \therefore \frac{a}{1-r} = \frac{1}{1-3} = \frac{1}{-2}$

$= n^2 - \frac{1}{2} n \qquad \therefore \Theta(n^2)$

$T(n) = 9T(n/3) + n$

$T(n/3) = 9T(n/3^2) + \frac{n}{3}$

$T(n/3^2) = 9T(n/3^3) + \frac{n}{3^2}$

$T(n/3^3) = 9T(n/3^4) + \frac{n}{3^3}$

⋮

$\frac{\cancel{3} \times \cancel{3} \times 9}{\cancel{3} \times \cancel{3} \times 3}$

$3^i = n$

$\log_3 n = \log_3 3^i$

$\log_3 n = i \log_3 3$

$\log_3 n = i$

By M.T. $T(n) = 9T(n/3) + n$

$a = 9,\ b = 3,\ d = 1,\ b^d = 3^1 = 3$

$\therefore a > b^d \quad 9 > 3 \quad \therefore$ Case 3 $\quad \Theta(n^{\log_b 9})$

$= \Theta(n^{\log_3 9})$

$= \Theta(n^2)$

(25)

$T(n) = 2T(n-1) + c \quad T(1) = 1$

$2[2T(n-2) + c] + c$

$= 2^2 T(n-2) + 2c$

$= 2^2[2 \cdot T(n-3) + c] + 2c$

$= 2^3 T(n-3) + 3c$

$= 2^3[2T(n-4) + c] + 3c$

$= 2^4 T(n-4) + 4c$

$T(n) = 2T(n-1) + c$

$T(n-1) = 2T(n-2) + c$

$T(n-2) = 2T(n-3) + c$

$T(n-3) = 2T(n-4) + c$

i items — General form

$= 2^i \cdot T(n-i) + ic$

$n - i = 1$

$n - 1 = i$

$= 2^i \cdot T(n-n+1) + ic = 2^i \cdot T(1) + (n-1)c$

$= 2^{n-1} + (n-1)c \Rightarrow O(2^n)$

---

(26) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + \frac{1}{n} & \text{otherwise} \end{cases}$

$T(n) = T(n-1) + \frac{1}{n}$

$T(n-1) = T(n-2) + \frac{1}{(n-1)}$

$T(n-2) = T(n-3) + \frac{1}{(n-2)}$

$T(n-3) = T(n-4) + \frac{1}{(n-3)}$

$T(n) = T(n-1) + \frac{1}{n}$

$= T(n-2) + \frac{1}{(n-1)} + \frac{1}{n}$

$= T(n-3) + \frac{1}{(n-2)} + \frac{1}{(n-1)} + \frac{1}{n}$

$= T(n-4) + \frac{1}{(n-3)} + \frac{1}{(n-2)} + \frac{1}{(n-1)} + \frac{1}{n}$

i items General form.

$= T(n-i) + \frac{1}{(n-(i-1))} + \frac{1}{(n-(i-2))} + \frac{1}{(n-(i-3))} + \dots + \frac{1}{n}$

$= T(n-i) + \frac{1}{(n-(i-1))} + \frac{1}{(n-(i-2))} + \frac{1}{(n-(i-3))} + \cdots + \frac{1}{n}$

$n-i=1$

$n-1=i$

$= T(\cancel{n}-\cancel{n}+1) + \frac{1}{(n-(n-1-1))} + \frac{1}{(n-(n-1-2))} + \frac{1}{(n-(n-1-3))} + \cdots + \frac{1}{n}$

$= 1 + \frac{1}{(\cancel{n}-\cancel{n}+1+1)} + \frac{1}{(\cancel{n}-n+1+2)} + \frac{1}{(n-n+1+3)} + \cdots + \frac{1}{n}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

H.P $\rightarrow \sum_{i=1}^{n} \frac{1}{i} \approx \log n + r \rightarrow r =$ Euler Constant

$r = 0.557$

$\therefore T(n) = O(\log_2 n)$

$\Rightarrow$ (27) $T(n) = 8T(n/2) + n^2 \quad T(1) = 1$

$= 8[8 \cdot T(n/2^2) + (n/2)^2] + n^2$

$= 8^2 \cdot T\left(\frac{n}{2^2}\right) + 2n^2 + n^2$

$= 8^2 [8T(n/2^3) + (n/2^2)^2] + 2n^2 + n^2$

$= 8^3 \cdot T(n/2^3) + 4n^2 + 2n^2 + n^2$

$= 8^3 [8T(n/2^4) + (n/2^3)^2] + 4n^2 + 2n^2 + n^2$

$= 8^4 T(n/2^4) + \underline{8n^2 + 4n^2 + 2n^2 + n^2} \rightarrow n^2(1+2+4+8)$

$2^0 + 2^1 + 2^2 + 2^3 \rightarrow \sum_{i=0}^{n} 2^i - 1$

Generalize .. i times

$= 8^i \cdot T(n/2^i) + n^2(2^i - 1)$

$= 8^{\log_2 n} T(n/n) + n^2 \cdot 2^{\log_2 n} - 1 = n^{\log_2 8} \cdot T(1) + n^2 \cdot n^{\log_2 2} - 1$

$= n^3 + n^2 \cdot n - 1 \Rightarrow n^3 + n^3 - 1 \longrightarrow O(n^3)$

$T(n) = 8T(n/2) + n^2$

$T(n/2) = 8T(n/2^2) + (n/2)^2$

$T(n/2^2) = 8T(n/2^3) + (n/2^2)^2$

$T(n/2^3) = 8T(n/2^4) + (n/2^3)^2$

$\frac{n}{2^i} = 1 \rightarrow \log_2 n = i$

By M.T $a=8, b=2, d=2, b^d = 2^2 = 4 \therefore a > b^d \quad 8>4$ Case 3: $O(n^{\log_b a}) = O(n^3)$

→ (28) $T(n) = T(\sqrt{n}) + 1$ $T(1) = 1$.

$T(n^{1/2}) + 1$

$= T(n^{1/4}) + 1 + 1$

$= T(n^{1/8}) + 1 + 1 + 1$

$= T(n^{1/2^3}) + 3$

⋮

Generalize i terms

$= T(n^{\frac{1}{2^i}}) + i$

$= T(2) + \log\log_2 n$

$O(\log\log_2 n)$.

for instance.

$n^{\frac{1}{2^i}} = 2$

$\log n^{\frac{1}{2^i}} = \log 2$

$\frac{1}{2^i} \log n = \log 2$

$\frac{1}{2^i} = \frac{\log 2}{\log n}$

$2^i = \frac{\log n}{\log 2}$

$2^i = \log_2 n$

apply log again

$\log 2^i = \log\log_2 n$

$i \log_2 2 = \log\log_2 n$

$i = \log\log_2 n$

Another Way.

$n = 2^i$

$T(n) = T(\sqrt{n}) + 1$

$T(2^i) = T(2^{i/2}) + 1$

$\sqrt{n} = n^{1/2}$

$2^i = 2^{i/2}$

$2^i = k$ if $S(k) = S(k/2) + 1$

Now it is in the form where we can apply Master method.

$S(k) = S(k/2) + 1$

$n = 2^k$

$k = \log_2 n$

$a = 1,\ b = 2,\ f(n) = 1$. $d = 0$. b^d 2^0 $1 = n^d$

$a = b^d$ $= 1 = 1$ ∴ Case 2 → $\Theta(n^d \log k)$

k

$i = \log_2 n$

$\Theta(1 \log\log_2 n) = \Theta(\log\log_2 n)$

$n = 2^i$

$\log_2 n = i \log_2 2$

$\log_2 n = i$

(29) $T(n) = 2T(\sqrt{n}) + \log n$ $\longrightarrow$ Change in Variable problem

$n = 2^m$

$\log 2^m$

$m \log_2 2$

$= m.$

$T(2^m) = 2T(n^{1/2}) + \log 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

Let $m \to 2^m$ $S(m) = 2T(m/2) + m$

$a = 2,\ b = 2,\ d = 1$

$b^d = 2$, $a = b$ $\therefore$ Case (2).

$2^k = n$

$\log n = k \lg 2$

$\log_2 n = k$

$\Theta(k^d \log k) = \Theta(k \log k)$ $\log_2 n$

$= \Theta(\log_2 \log \log_2 n)$

$\Theta(m \log m)$

(30) $T(n) = 3T(\sqrt{n}) + \log n$

$\to n = 2^m$

$T(2^m) = 3T(2^{m/2}) + m$

$S(m) = 3T(m/2) + m$

$a = 3,\ b = 2,\ d = 1,\ b^d = 2^1 = 2$

$a > b^d$ Case (3).

$\Theta(m^{\log_2 3}) \longrightarrow \Theta(\cancel{\log}^{\log_2 3})$.