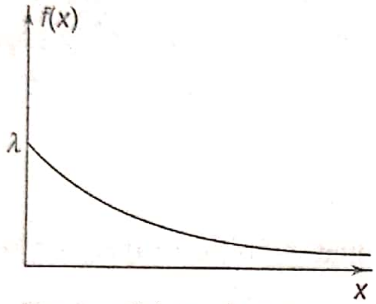
27.12 THE EXPONENTIAL DISTRIBUTION

Many scientific experiments involve the measurement of the duration of time X between an initial point of time and the occurrence of some phenomenon of interest. For example X is the life time of a light bulb which is turned on and left until it burns out. The continuous random variable X having the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Exponential density function

Fig. 27.41

is said to have an *exponential distribution.* Here the only parameter of the distribution is λ which is greater than zero. This distribution, also known as the *negative exponential distribution*, is a special case of the gamma distribution (with $r = 1$). Examples of random variables modeled as exponential are

a. (inter-arrival) time between two successive job arrivals

b. duration of telephone calls

c. life time (or time to failure) of a component or a product

d. service time at a server in a queue

e. time required for repair of a component

The exponential distribution occurs most often in applications of **Reliability Theory** and **Queuing Theory** because of the memoryless property and relation to the (discrete) **Poisson Disribution.** Exponential distribution can be obtained from the Poisson distribution by considering the inter-arrival times rather than the number of arrivals.

Mean and Variance

For any $r \geq 0$,

$$E(X^r) = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \lambda e^{-\lambda x} dx$$

put $\lambda x = t,\ x = \frac{t}{\lambda},\ dx = \frac{1}{\lambda}dt$. Then

$$E(X^r) = \int_0^\infty \left(\frac{t}{\lambda}\right)^r \cdot \lambda \cdot e^{-t} \cdot \frac{1}{\lambda}dt = \frac{1}{\lambda^r}\int_0^\infty e^{-t} t^r dt$$

$$E(X^r) = \frac{\Gamma(r+1)}{\lambda^r}$$

In particular with $r = 0$,

$$\int_0^\infty f(x)dx = \Gamma(1) = 1$$

(i.e., $f(x)$ is a probability density function).
With $r = 1$, mean $= \mu = E(X) = \frac{\Gamma(2)}{\lambda} = \frac{1}{\lambda}$

with $r = 2$, variance$=\sigma^2 = E(X^2) - \{E(X)\}^2 = \frac{\Gamma(3)}{\lambda^2} - \frac{1}{\lambda^2}$

$$\sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Note: Both the mean and standard deviation of the exponential distribution are equal to $\frac{1}{\lambda}$.

Cumulative Distribution Function

$$F(x) = \int_0^x f(t)dt = \int_0^x \lambda e^{-\lambda t} dt = \left.\frac{\lambda e^{-\lambda t}}{-\lambda}\right|_{t=0}^{x}$$

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geqslant 0,$$

and $F(x) = 0$ when $x < 0$

$F(x)$ gives the probability that the "system" will "die" before x units of time have passed.

Probabil...

Worked Out Examples

Example 1: Let the mileage (in thousands of miles) of a particular tyre be a random variable X having the probability density

$$f(x) = \begin{cases} \frac{1}{20}e^{-x/20} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the probability that one of these tyres will last (1) at most 10,000 miles (b) anywhere from 16,000 to 24,000 miles (c) at least 30,000 miles. (d) Find the mean (e) Find the variance of the given probability density function.

Solution: (a) Probability that a tyre will last almost 10,000 miles

$$= P(X \le 10) = \int_0^{10} f(x)dx$$

$$= \int_0^{10} \frac{1}{20}e^{-x/20}dx$$

$$= \frac{1}{20} \cdot e^{-x/20} \cdot \left(\frac{-20}{1}\right)\Big|_0^{10}$$

$$= 1 - e^{-\frac{1}{2}} = 0.3934$$

(b) $$P(16 \le X \le 24) = \int_{16}^{24} f(x)dx$$

$$= \int_{16}^{24} \frac{1}{20}e^{-x/20}dx$$

$$= -e^{-\frac{x}{20}}\Big|_{16}^{24} = e^{-\frac{4}{5}} - e^{-\frac{6}{5}}$$

$$= 0.148$$

(c) $P(X \geq 30) = \int_{30}^{\infty} f(x)dx$

$$= \int_{30}^{\infty} \frac{1}{20} e^{-x/20} dx = -e^{x/20}\Big|_{30}^{\infty} = e^{-\frac{3}{2}}$$

$$= 0.2231$$

(d) Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)dx$

$$= \int_{0}^{\infty} x \cdot \frac{1}{20} e^{-\frac{x}{20}} dx$$

$$= -\int_{0}^{\infty} x \cdot d\left(e^{-\frac{x}{20}}\right)$$

$$= -xe^{-\frac{x}{20}} - 20e^{-\frac{x}{20}}\Big|_{0}^{\infty} = 0 - (-20)$$

$$\mu = 20 = \frac{1}{\lambda}$$

(e) Variance $= \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

Consider

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{\infty} x^2 \frac{1}{20} e^{-\frac{x}{20}} dx$$

$$= -x^2 e^{-\frac{x}{20}}\Big|_{0}^{\infty} + 2 \cdot 20 \cdot \int_{0}^{\infty} \frac{1}{20} \cdot xe^{-x/20} dx$$

$$= 0 + 2 \cdot 20 \cdot \mu = 2.20.20 = 2.20^2$$

Then $\sigma^2 = \int_{0}^{\infty} x^2 f(x)dx - \mu^2 = 2.20^2 - 20^2$

$$= 20^2 = \frac{1}{\lambda^2}$$

Example 2: The length of time for one person to be served at a cafeteria is a random variable X having an exponential distribution with a mean of 4 minutes. Find the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days.

Solution: The probability that a person is served at a cafeteria in less than 3 minutes is

$$P(T < 3) = 1 - P(T \geq 3)$$

Since the mean $\mu = \frac{1}{\lambda} = 4$ or $\lambda = \frac{1}{4}$, the exponential distribution is $\frac{1}{4}e^{-\frac{x}{4}}$. Now

$$P(T < 3) = 1 - P(T \geq 3) = 1 - \int_3^\infty \frac{1}{4}e^{-\frac{t}{4}}dt$$

$$P(T < 3) = 1 - \frac{1}{4}e^{-\frac{t}{4}} \cdot \left(\frac{-4}{1}\right)\Big|_3^\infty = 1 - e^{-\frac{3}{4}}$$

Let X represent the number of days on which a person is served in less than 3 minutes. Then using the binomial distribution, the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days is

$$P(X \geq 4) = \sum_{x=4}^{6} {}^6C_x(1 - e^{-3/4})^x(e^{-3/4})^{6-x} = 0.3968$$

EXERCISE

1. Let T be the time (in years) to failure of certain components of a system. The random variable T has exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are in different systems, find the probability that at least 2 are still functioning at the end of 8 years.

Ans. 0.2627

Hint: $P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}dt$

$= e^{-8/5} \simeq 0.2,\ P(X \geq 2) = \sum_{x=2}^{\infty} b(x; 5, 0.2) =$

$1 - \sum_{x=0}^{1} b(x, 5, 0.2) = 1 - 0.7373$

2. If a random variable X has the exponential distribution with mean $\mu = \frac{1}{\lambda} = \frac{1}{2}$ calculate the probabilities that (a) X will lie between 1 and 3 (b) X is greater than 0.5 (c) X is at most 4.

Ans. (a) 0.133 (b) 0.368 (c) 0.98168

Hint: $PDF\ \ f(x) = 2e^{-2x}$ (a) $\int_1^3 2e^{-2x}dx = e^{-2} - e^{-6}$

(b) $\int_{0.5}^\infty 2e^{-2x}dx = e^{-1}$ (c) $\int_0^4 2e^{-2x}dx = 1 - e^{-4}$

3. The life (in years) of a certain electrical switch has an exponential distribution with an average life of $\frac{1}{\lambda} = 2$. If 100 of these switches are installed in

different systems, find the probability that at most 30 fail during the first year.

Hint: $P(T > 1) = \int_1^{\infty} \frac{1}{2}e^{-\frac{t}{2}} \; dt = +e^{-\frac{1}{2}} = 0.606$

Ans. $P(X \leq 30) = \sum_{x=0}^{30} b(x, 2, 0.606) = \sum_{x=0}^{30} {}^{100}C_x(0.606)^x \; (0.39346)^{100-x}$

4. Suppose the life length X (in hours) of a fuse has exponential distribution with mean $\frac{1}{\lambda}$. Fuses are manufactured by two different processes. Process I yields an expected life length of 100 hours and process II yields an expected life length of 150 hours. Cost of production of a fuse by process I is Rs. C while by the Process II it is Rs 2C. A fine of Rs K is levied if a fuse lasts less than 200 hours. Determine which process should be preferred?

Ans. Prefer Process I if $C > 0.13K$

Hint: $c_1 = c$ if $X > 200$
$= c + k$ if $X \leq 200$

$$E(c_1) = c \cdot P(X > 200) + (c + k)P(X \leq 200)$$
$$= c \cdot e^{-\frac{1}{100} \cdot 200} + (c + k)(1 - e^{-\frac{1}{100} \cdot 200})$$
$$= k(1 - e^{-2}) + c$$

$E(c_2) = k(1 - e^{-4/3}) + 2c, \quad E(c_2) - E(c_1) = c - 0.13k$

5. Suppose N_t be a discrete random variable denoting the number of arrivals in time interval $(0, t]$. Let X be the time of the next arrival, so X is the elapsed time between the occurrences of two successive events. Assuming that N_t is Poisson distributed with parameter λt, show that X is exponentially distributed.

Here λ is the expected numbers of events occurring in one unit of time.

Ans. $P(X > t) = P(N_t = 0) = \dfrac{e^{-\lambda t}(\lambda t)^0}{0} = e^{-\lambda t}$

6. If the average rate of job submission is $\lambda = 0.1$ jobs/second, find the probability that an interval of 10 seconds elapses without job submission.

Ans. $P(X \geq 10) = \int_{10}^{\infty} 0.1e^{0.1t}dt = e^{-1} = 0.368$

Hint: Assume that the number of arrivals/unit time is poisson distributed and the inter arrival time X is exponentially distributed with parameter λ.

7. Let the mileage (in thousands of miles) of a certain radial tyre is a random variable with exponential distribution with mean 40,000 miles. Determine the probability that the tyre will last (a) at least 20,000 km (b) at most 30,000 km.

Ans. (a) $P(X \geq 20,000) = e^{-0.5} = 0.6065$
(b) $P(X \leq 30,0001 = 1 - e^{-0.75} = 0.5270$

8. The amount of time (in hours) required to repair a T.V. is exponentially distributed with mean $\frac{1}{2}$. Find the (a) probability that the repair time exceeds 2 hours (b) the conditional probability that repair takes at least 10 hours given that already 9 hours have been spent repairing the TV.

Ans. (a) $P(X > 2) = e^{-1} = 0.3679$
(b) $P(X \geq 10|X > 9) = P(X > 1) = e^{-0.5} = 0.6065$
(because of the memoryless property).

9. The duration of time X in seconds between presses of the white rat on a bar, which are periodically conditioned, has an exponential distribution with parameter $\lambda = 0.20$. Find the probability that the duration is more than one second but less than 3 seconds (b) more than 3 seconds.

Ans. (a) $P(1 \leq X \leq 3) = e^{-0.2(1)} - e^{-(0.2)3} = 0.819 - 0.549 = 0.270$
(b) $P(X > 3) = e^{-0.2(3)} = 0.549$

10. The time X (seconds) that it takes a certain online computer terminal (the elapsed time between the end of user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected time 20 seconds. Compute the probabilities (a) $P(X \leq 30)$ (b) $P(X \geq 20)$ (c) $P(20 \leq X \leq 30)$ (d) For what value of t is $P(X \leq t) = 0.5$ (i.e., t is the fiftieth percentile of the distribution)

Ans. (a) 0.777 (b) 0.368 (c) 0.145 (d) 13.863

27.14 THE WEIBULL DISTRIBUTION

Lifetimes, waiting times, learning times, travelling times, duration of epidemics are some of the important examples of non-negative random variables whose variability can be explained in many cases by exponential and gamma distributions. However in certain cases Weibull distribution provides good probability model for describing "*length of life*" of objects having the '***weakest link***' property. An object, composed of a large number of separate parts, put under stress is said to have the *weakest link property* if the lifetime of the object is equal to the minimum lifetime of any of its parts.

Example: A chain is as strong as its weakest link.

The Weibull distribution was introduced in 1939 by the Swedish physicist *Waloddi Weibull* and is given by probability density function

$$f(x)=\begin{cases}\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right] & \text{if } x\geq\nu\\ 0 & \text{if } x<\nu\end{cases} \tag{3}$$

The three constants $\beta>0$, $\alpha>0$ and $\nu\geq 0$ are the parameters of the distribution. The smallest possible value of X is given by ν. The constant β determines the shape of the density function (1). When $\beta=1$ and $\nu=0$, the Weibull distribution (1) reduces to the exponential distribution with the parameter $\lambda=\frac{1}{\alpha}$. If X has a Weibull distribution with parameters α,β,ν then $Y=\left[\frac{(X-\nu)}{\alpha}\right]^{\beta}$ has an exponential distribution with the parameter $\lambda=1$.

Since $X=\alpha Y^{1/\beta}+\nu$ so $\frac{dX}{dY}=\frac{\alpha}{\beta}Y^{\frac{1}{\beta}-1}$ and $f(X)=\frac{\beta}{\alpha}Y^{(\beta-1)/\beta}e^{-Y}$. Then the probability density function of Y is

$$f(y)=f(x)\frac{dx}{dy}=\frac{\beta}{\alpha}y^{\frac{\beta-1}{\beta}}\cdot e^{-y}\cdot\frac{\alpha}{\beta}y^{\frac{1}{\beta}-1}$$

$$=\begin{cases}e^{-y} & \text{if } y\geq 0\\ 0 & \text{if } y<0\end{cases}$$

which is an exponential distribution with parameter $\lambda=1$.

The cumulative distribution of X is

$$F(x)=P(X\leq x)=\begin{cases}1-\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right] & \text{if } x\geq\nu\\ 0 \text{ if } x<\nu\end{cases} \tag{2}$$

Probabilites are calculated using $F(x)$.

Mean and Variance

$$\text{Mean}=\mu=E(X)=E(\alpha Y^{\frac{1}{\beta}}+\nu)=\alpha E(Y^{\frac{1}{\beta}})+\nu$$

Since $E(X^r)=\frac{\Gamma(r+1)}{\lambda^r}$, with $r=\frac{1}{\beta}$ and $\lambda=1$,

$$\text{Mean}=\mu=\frac{\alpha\Gamma\left(\frac{1}{\beta}+1\right)+\nu}{1} \tag{3}$$

Consider

$$E(X^2)=E\left(\left(\alpha Y^{\frac{1}{\beta}}+\nu\right)^2\right)=E(\alpha^2Y^{2/\beta}+\nu^2+2\alpha\nu Y^{\frac{1}{\beta}})$$

$$=\alpha^2E(Y^{2/\beta})+2\nu\alpha E(Y^{1/\beta})+\nu^2$$

Now with $r=\frac{2}{\beta}$ and $\lambda=1$, we get

Variance $=\sigma^2=E(X^2)-\{E(X)\}^2$

$$=\left[\alpha^2\frac{\Gamma\left(\frac{2}{\beta}+1\right)}{1^2}+2\nu\alpha\frac{\Gamma\left(\frac{1}{\beta}+1\right)}{1}+\nu^2\right]-$$

$$-\left[\alpha\Gamma\left(\frac{1}{\beta}\cdot 1\right)+\nu\right]^2$$

$$\sigma^2=\alpha^2\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\} \tag{4}$$

Median: $P(X\leq x)=F(x)=\frac{1}{2}$ or $1-\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right]=\frac{1}{2}$. Solving $x=\alpha(.693)^{\frac{1}{\beta}}+\nu$

Observe that variance depends upon α and β but independent of ν.

The *survival function* is given by

$$1-F(x)=\begin{cases}1 & \text{if } x<\nu\\ \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right] & \text{if } x\geq\nu\end{cases} \tag{5}$$

Exponential distribution is a special case of both the gamma and Weibull distributions.

Note that gamma distribution with λ and $r=1$ is an exponential distribution with parameter λ. Similarly the Weibull distribution with $\alpha=\frac{1}{\lambda}$, $\beta=1$ and $\nu=0$ is an exponential distribution with parameter λ. Thus the gamma and Weibull distributions are generalization of the exponential distribution. However gamma distribution with $r\neq 1$ is *not* a Weibull distribution. Also Weibull distributions with $\beta\neq 1$ or $\nu\neq 0$ is not gamma distribution.

With $\nu = 0$ the above results takes the following form:

The Weibull distribution is

$$f(x) = \begin{cases} \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\frac{x^\beta}{\alpha^\beta}}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

put $\alpha^* = \alpha^{-\beta}$ ($or\ \alpha = (\alpha^*)^{-\frac{1}{\beta}}$) then

$$f(x) = \begin{cases} \alpha^* \beta e^{-\alpha^* x^\beta} \cdot x^{\beta-1}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (6)$$

The cumulative distribution is

$$F(x) = \begin{cases} 1 - e^{-\alpha^* x^\beta}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (7)$$

$$\text{Mean} : \mu = (\alpha^*)^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) \quad (8)$$

$$\text{Variance} : \sigma^2 = (\alpha^*)^{\frac{-2}{\beta}} \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\}$$

$$\text{Median} : \hat{x} = \alpha^{*^{-\frac{1}{\beta}}} (0.693)^{\frac{1}{\beta}} = \left(\frac{0.693}{\alpha^*}\right)^{\frac{1}{\beta}} \quad (10)$$

Example: Suppose the life time X (in hours) of a semiconductor is a random variable having the Weibull distribution with parameters, $\alpha^* = (200)^{-2.5}$, $\beta = 2.5$, $\nu = 0$. Determine the probability that the lifetime of the semiconductor is (a) at most 200, (b) less than 200, (c) more than 300, (d) between 100 and 200 hours. Find (e) the mean (f) variance (g) median of life time X.

Solution: Recall that the cumulative distribution with $\alpha^* = (200)^{-2.5}$, $\beta = 2.5$ is

$$F(x) = P(X \le x) = 1 - e^{-\alpha^* x^\beta} = 1 - e^{-\left(\frac{x}{200}\right)^{2.5}}$$

(a) Probability that life time X of the semiconductor is at most 200 hours

$$= P(X \le 200) = F(200)$$

$$= 1 - e^{-\left(\frac{200}{200}\right)^{2.5}} = 1 - e^{-1} = 1 - 0.36787 = 0.632$$

(b) $P(X < 200) = P(X \le 200) = 0.632$

(c) $P(X > 300) = 1 - F(300) = e^{-\left(\frac{200}{200}\right)^{2.5}} = e^{-\left(\frac{3}{2}\right)^{2.5}} = 0.0635$

(d) $P(100 < X < 200) = F(200) - F(100)$

$$= \left[1 - e^{-\left(\frac{200}{200}\right)^{2.5}}\right] - \left[1 - e^{-\left(\frac{100}{200}\right)^{2.5}}\right]$$

$$= e^{-\left(\frac{1}{2}\right)^{2.5}} - e^{-1} = 0.83796 - 0.36789$$

$$= 0.4700$$

(e) Mean: $\mu = (\alpha^*)^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta} + 1\right) = \dfrac{\Gamma\left(\frac{1}{2.5}+1\right)}{200} = 200 \cdot \Gamma\left(\frac{2}{5} + 1\right) = \frac{400}{5}\Gamma\left(\frac{2}{5}\right) = 80 \cdot \Gamma(2/5)$

(f) Variance $= (\alpha^*)^{-2/\beta}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right] = \left(\frac{1}{200}\right)^{-2}\left[\Gamma\left(1+\frac{4}{5}\right) - \left\{\Gamma\left(1+\frac{2}{5}\right)\right\}^2\right]$

$= 40000\left[\frac{4}{5}\cdot\Gamma\left(\frac{4}{5}\right) - \left\{\frac{2}{5}\Gamma\left(\frac{2}{5}\right)\right\}^2\right]$

(g) Median: $\hat{x}$ middlemost value such that $P(X \leq \hat{x}) = \frac{1}{2}$. Then

$$F(\hat{x}) = 1 - e^{-\alpha^* \hat{x}^\beta} = \frac{1}{2}$$

So

$$\hat{x} = \text{median} = (\alpha^*)^{-\frac{1}{\beta}}(.693)^{\frac{1}{\beta}}$$

$$= 200(.693)^{\frac{1}{2.5}} = 200(.693)^{\frac{2}{5}}$$

$$= 200(.86356) = 172.7123$$