1.5 FINITE AND COUNTABLE SETS

Sets can be finite or infinite. A set S is *finite* if S is empty or if S consists of exactly m elements where m is a positive integer; otherwise S is infinite.

EXAMPLE 1.4

(*a*) Let A denote the letters in the English alphabet, and let D denote the days of the week, that is, let

$$A = \{a, b, c, \ldots, y, z\} \quad \text{and} \quad D = \{\text{Monday, Tuesday, \ldots, Sunday}\}$$

Then A and D are finite sets. Specifically, A has 26 elements and D has 7 elements.

(*b*) Let $R = \{x : x \text{ is a river on the earth}\}$. Although it may be difficult to count the number of rivers on the earth, R is still a finite set.

(*c*) Let E be the set of even positive integers, and let $\mathbf{I}$ be the *unit interval*; that is, let

$$E = \{2, 4, 6, \ldots\} \quad \text{and} \quad \mathbf{I} = [0, 1] = \{x : 0 \leq x \leq 1\}$$

Then both E and $\mathbf{I}$ are infinite sets.

Countable Sets

A set S is *countable* if S is finite or if the elements of S can be arranged in the form of a sequence, in which case S is said to be *countably infinite*. A set is *uncountable* if it is not countable. The above set E of even integers is countably infinite, whereas it can be proven that the unit interval $\mathbf{I} = [0, 1]$ is uncountable.

1.6 COUNTING ELEMENTS IN FINITE SETS, INCLUSION-EXCLUSION PRINCIPLE

The notation $n(S)$ or $|S|$ will denote the number of elements in a set S. Thus $n(A) = 26$ where A consists of the letters in the English alphabet, and $n(D) = 7$ where D consists of the days of the week. Also $n(\varnothing) = 0$, since the empty set has no elements.

The following lemma applies.

Lemma 1.6: Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

This lemma may be restated as follows:

Lemma 1.6: Suppose S is the disjoint union of finite sets A and B. Then S is finite and

$$n(S) = n(A) + n(B)$$

Proof: In counting the elements of $A \cup B$, first count the elements of A. There are $n(A)$ of these. The only other elements in $A \cup B$ are those that are in B but not in A. Since A and B are disjoint, no element of B is in A. Thus, there are $n(B)$ elements which are in B but not in A. Accordingly, $n(A \cup B) = n(A) + n(B)$.

For any sets A and B, the set A is the disjoint union of $A \setminus B$ and $A \cap B$ (Problem 1.45). Thus, Lemma 1.6 gives us the following useful result.

Corollary 1.7: Let A and B be finite sets. Then

$$n(A \setminus B) = n(A) - n(A \cap B)$$

That is, the number of elements in A but not in B is the number of elements in A minus the number of elements in both A and B. For example, suppose an art class A has 20 students and 8 of the students are also taking a biology class B. Then there are

$$20 - 8 = 12$$

students in the class A which are not in the class B.

Given any set A, we note that the universal set $\mathbf{U}$ is the disjoint union of A and A^c. Accordingly, Lemma 1.6 also gives us the following result.

Corollary 1.8: Suppose A is a subset of a finite universal set $\mathbf{U}$. Then

$$n(A^c) = n(\mathbf{U}) - n(A)$$

For example, suppose a class $\mathbf{U}$ of 30 students has 18 full-time students. Then there are

$$30 - 18 = 12$$

part-time students in the class.

Inclusion-Exclusion Principle

There is also a formula for $n(A \cup B)$, even when they are not disjoint, called the *inclusion-exclusion principle*. Namely,

Theorem (Inclusion-Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cap B$ and $A \cup B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to get a similar result for three sets.

Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.9) may be used to further generalize this result to any finite number of finite sets.

EXAMPLE 1.5 Suppose list A contains the 30 students in a mathematics class and list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:

(*a*) Only on list A (*c*) On list A or B (or both)

(*b*) Only on list B (*d*) On exactly one of the two lists (*a*) List A contains 30 names and 20 of them are on list B; hence $30 - 20 = 10$ names are only on list A. That is, by Corollary 1.7,

$$n(A \setminus B) = n(A) - n(A \cap B) = 30 - 20 = 10$$

(*b*) Similarly, there are $35 - 20 = 15$ names only on list B. That is,

$$n(B \setminus A) = n(B) - n(A \cap B) = 35 - 20 = 15$$

(*c*) We seek $n(A \cup B)$. Note we are given that $n(A \cap B) = 20$.

One way is to use the fact that $A \cup B$ is the disjoint union of $A \setminus B$, $A \cap B$, and $B \setminus A$ (Problem 1.54), which is pictured in Fig. 1-5 where we have also inserted the number of elements in each of the three sets $A \setminus B$, $A \cap B$, $B \setminus A$. Thus

$$n(A \cup B) = 10 + 20 + 15 = 45$$

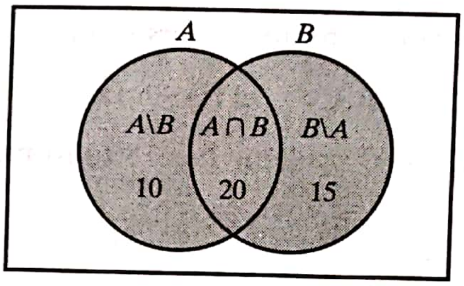
Alternately, by Theorem 1.8,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

(*d*) By (*a*) and (*b*), there are $10 + 15 = 25$ names on exactly one of the two lists; so $n(A \oplus B) = 25$. Alternately, by the Venn diagram in Fig. 1-5, there are 10 elements in $A \setminus B$, and 15 elements in $B \setminus A$; hence

$$n(A \oplus B) = 10 + 15 = 25$$



$A \cup B$ is shaded.