Application of Double and Triple Integrals

1) **Area** In the double integral $\iint_A f(x,y)\,dx\,dy$, if $f(x,y) = 1$, then $\iint_A dx\,dy$ is the given area under the ~~curve~~. region

1) Find the area by double integration between the parabola $y = 4x - x^2$ and the line $y = x$

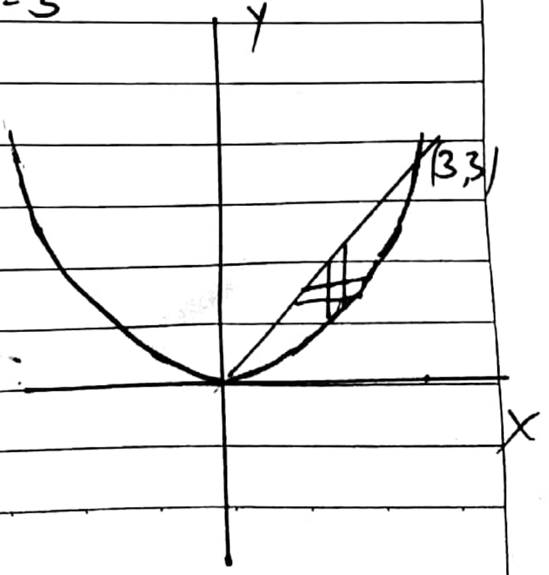
Sol^n^ given eq^n^ $y = 4x - x^2$ —— ①

$y = x$ —— ②

$\therefore$ on solving eq^n^ ① and ② we get

$x = 4x - x^2$

$x^2 - 3x = 0 \Rightarrow x = 0$ or $x = 3$



$$\text{Area} = \iint_A dA$$

$$= \int_0^1 \int_x^{4x-x^2} dy\,dx$$

$$= \int_0^3 [y]_x^{4x-x^2}\,dx$$

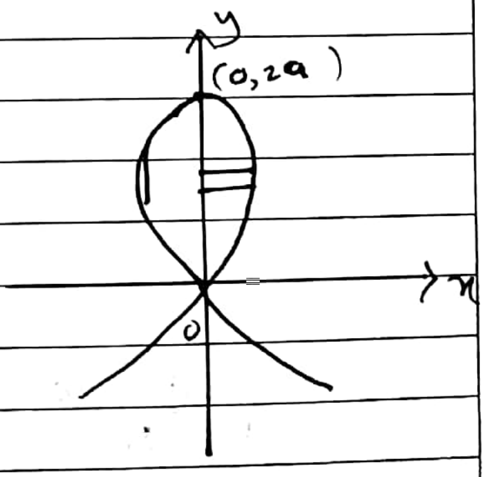
$$= \int_0^3 (3x - x^2)\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= 27\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{9}{2}$$

2) Find the area of the loop of the curve $a^2x^2 = y^3(2a-y)$

Sol^n given $a^2x^2 = y^3(2a-y)$



Since the curve is symmetrical about y-axis

$\therefore$ Required area = 2 times the area of 1st quadrant

Now integrate along horizontal strip for which x varies from 0 to $\frac{y}{a}\sqrt{y(2a-y)}$

y varies from 0 to $2a$

$$\therefore \text{Required Area} = 2\int_0^{2a}\int_0^{\frac{y}{a}\sqrt{y(2a-y)}} dx\,dy$$

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$$= 2\int_0^{2a}\left[x\right]_0^{\frac{y}{a}\sqrt{y(2a-y)}} dy$$

$$= \frac{2}{a}\int_0^{2a} y\sqrt{y(2a-y)}\, dy$$

put $y = 2a\sin^2\theta$

$dy = 4a\sin\theta\cos\theta\, d\theta$

when

$y \to 0 \quad \theta = 0$

$y \to 2a \quad \theta \to \frac{\pi}{2}$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{2a\sin^2\theta\,(2a - \cancel{2a\sin^2\theta})}\,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{2a\sin^2\theta\,.\,2a(1-\sin^2\theta)}\,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= \frac{2}{\cancel{a}}\int_0^{\pi/2} 2\cancel{a}\sin^2\theta\sqrt{4a^2\sin^2\theta\,.\cos^2\theta}\,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= 2\times 2\times 2a\times 4a\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta$$

$$= 32a^2\ \frac{(4-1)(4-3)}{6(6-2)}\times\frac{(2-1)}{2}\times\frac{\pi}{2} = \pi a^2$$

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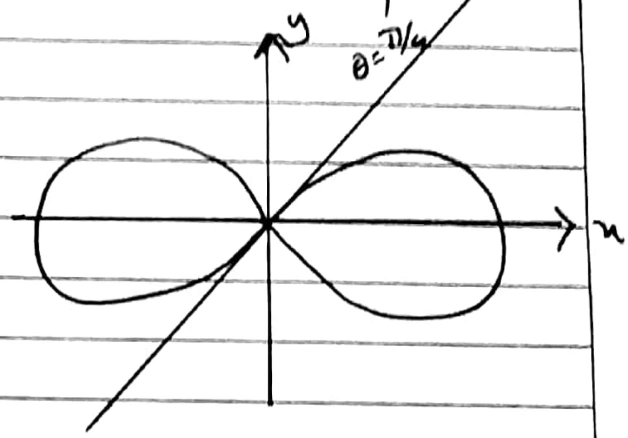
3) Find the area of (Leminscate Bernoulli) the curve $r^2 = a^2\cos 2\theta$

Sol^n Since the curve is symmetrical about the pole.

$\therefore$ Required Area = 4 times area above the line $\theta = 0$ for which θ varies from 0 to $\frac{\pi}{4}$



and r varies from 0 to $a\sqrt{\cos 2\theta}$

$\therefore$ Required Area $= 4 \iint_A dx\,dy = \iint_B r\,dr\,d\theta$

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

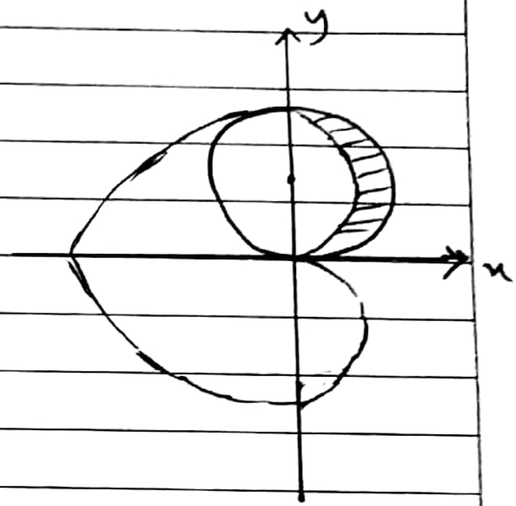
$$= 4 \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = 2a^2 \int_0^{\pi/2} \cos 2\theta\, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= a^2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= a^2$$

4) Find the area lying inside the circle $r = a\sin\theta$ and outside the cardiode $r = a(1-\cos\theta)$



Sol^n Area $= \iint_A r\,dr\,d\theta$

$$= \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

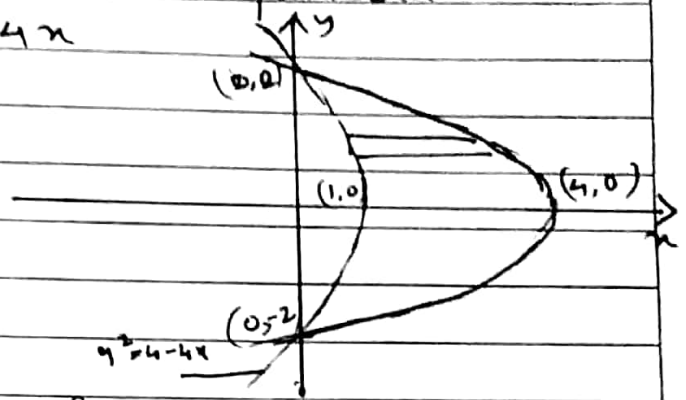
$$= \frac{a^2}{2} \left[\int_0^{\pi/2} \sin^2\theta\,d\theta - \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \right]$$

$$= \frac{a^2}{2} \left[\frac{(2-1)}{2} \times \frac{\pi}{2} - \left\{ [\theta]_0^{\pi/2} - 2[\sin\theta]_0^{\pi/2} + \frac{(2-1)}{2} \times \frac{\pi}{2} \right\} \right]$$

$$= \frac{a^2}{2} \left[\frac{\pi}{4} - \frac{\pi}{2} + 2 - \frac{\pi}{4} \right]$$

$$= \frac{a^2}{2} \left[2 - \frac{\pi}{2} \right] = \frac{a^2}{4} (4 - \pi)$$

7) Find the area bounded between the parabola $y^2 = 4 - x$ and $y^2 = 4 - 4x$



Sol^n given $y^2 = 4 - x$ — (1)

$y^2 = 4 - 4x$ — (2)

By Symmetry

Required area = 2 times area of 1^{st} quadrant

$\therefore$ Integrating along horizontal strip we ~~have~~

i.e. x varies from $\frac{4-y^2}{4}$ to $4-y^2$

and y varies from 0 to 2

$\therefore$ Required area $= 2\int_0^2 \int_{\frac{4-y^2}{4}}^{4-y^2} dx\,dy$

$$= 2\int_0^2 \left[x\right]_{\frac{4-y^2}{4}}^{4-y^2} dy$$

$$= 2\int_0^2 \left[4-y^2 - \frac{4-y^2}{4}\right] dy$$

$$= \frac{2}{4}\int_0^2 (12 - 3y^2)\, dy$$

$$= \frac{1}{2}\left[12y - y^3\right]_0^2 = \frac{1}{2}\left[24-8\right] = 8.$$

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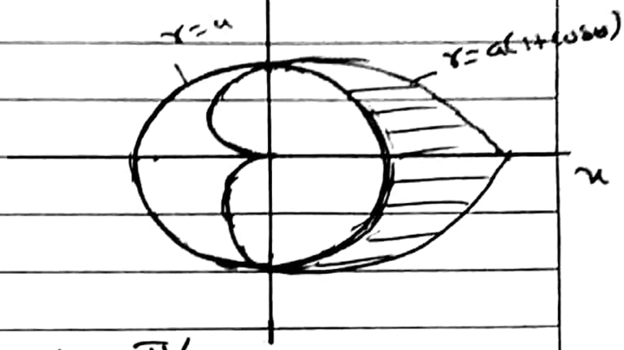
8) Find the area lying inside the cardiod $r = a(1+\cos\theta)$ and outside the $r = a$.



Sol^n By symmetry

Required Area = 2 times area above the line $\theta = 0$

For which θ varies from 0 to $\pi/2$ and r varies from a to $a(1+\cos\theta)$

$$\text{Required Area} = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}\left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= a^2\int_0^{\pi/2}(1+\cos^2\theta + 2\cos\theta - 1)\,d\theta$$

$$= a^2\int_0^{\pi/2}(2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2\left[(2\sin\theta)_0^{\pi/2} + \frac{(2-1)}{2}\times\frac{\pi}{2}\right]$$

$$= a^2\left[2 + \frac{\pi}{4}\right] = \frac{a^2}{4}(8+\pi)$$

9) Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral

Sol^n Since the sphere is symmetrical about all 8 octants.

$\therefore$ Required volume $V = 8$ times volume of 1^{st} quadrant

for which x varies from 0 to a

y " " 0 to $\sqrt{a^2-x^2}$

z " " 0 to $\sqrt{a^2-x^2-y^2}$

$$\therefore R.V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[z\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{(a^2-x^2-y^2)}\,dy\,dx$$

$$= 8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\,dy\right]dx$$

[formula P.T.O]

$$= 8\int_0^a\left[\frac{y\sqrt{(a^2-x^2)-y^2}}{2}+\frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a\left[0+(a^2-x^2)\frac{\pi}{2}-0\right]dx$$

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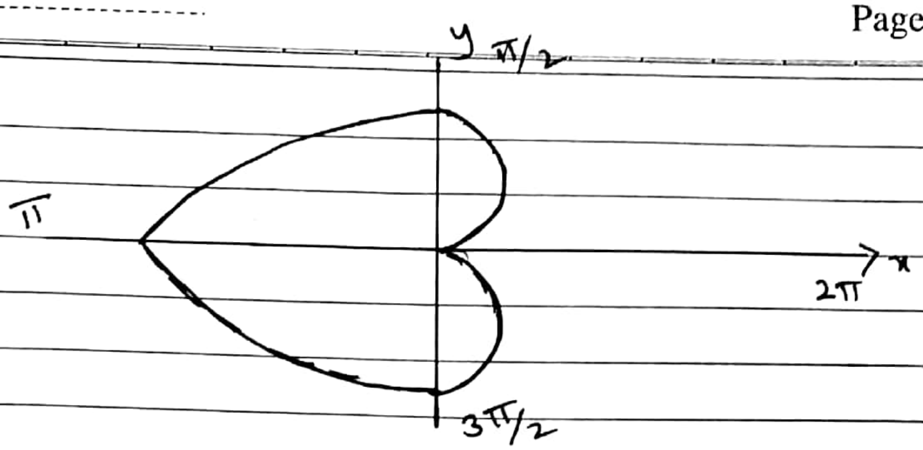
$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a = 2\pi \left[a^3 - \frac{a^3}{3} \right] = \frac{4\pi a^3}{3}$$

Formula

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$



θ	0	$\pi/4$	$2\pi/4$	$3\pi/4$	$4\pi/4$	$5\pi/4$	$6\pi/4$	$7\pi/4$	$8\pi/4$
r									

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	0	a	$2a$	a	0

$$\boxed{r = a(1 - \cos\theta)}$$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$2a$	a	0	a	$2a$

$$\boxed{r = a(1 + \cos\theta)}$$