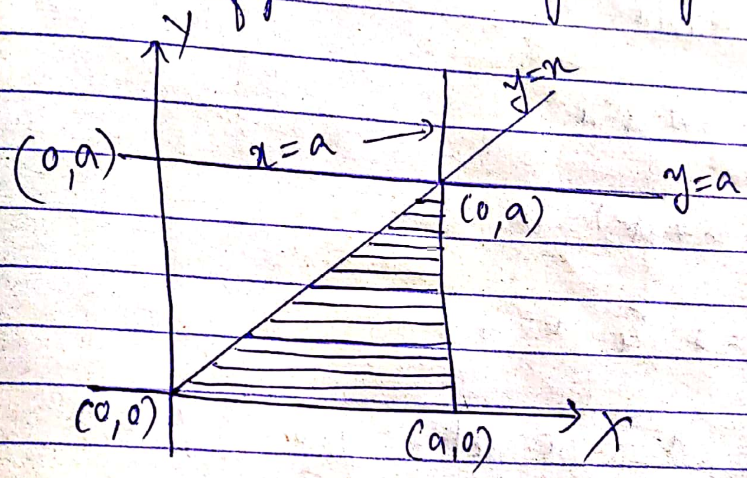
Double & Triple Integration

Evaluation of a Double Integral by changing the order of integration.

1) Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing the order of integration

$\Rightarrow$ The region bounded by the curves $x=y$, $x=a$ embedded between the lines $y=0$, $y=a$ is shown in figure



On changing the order x varies from 0 to a and y varies from 0 to x

$$I = \int_{x=0}^{a} \int_{y=0}^{x} x \cdot \frac{1}{x^2+y^2}\,dy\,dx$$

$$= \int_{x=0}^{a} x \cdot \frac{1}{x}\left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\tan^{-1}1 - \tan^{-1}0\right] dx$$

$$I = \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}\left[x\right]_0^a = \frac{\pi a}{4}$$

2) Change the order of integration & hence evaluate

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

⇒ We have,

$$I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{y=2\sqrt{ax}} dy\,dx$$

$$\frac{x^2}{4a} = 2\sqrt{ax}$$

$$x^4 = 64a^3x$$

$$x(x^3 - 64a^3) = 0$$

$$\Rightarrow \quad x = 0 \;\&\; x = 4a$$

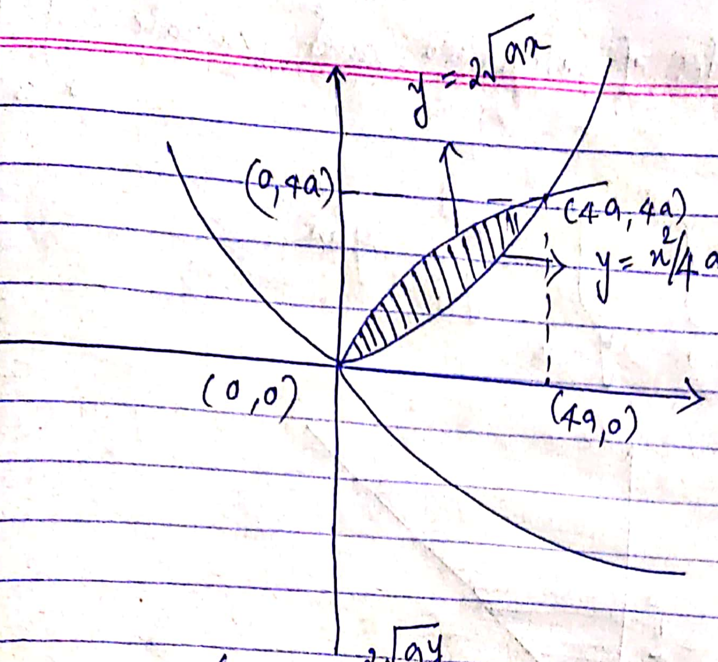
From $y = \frac{x^2}{4a}$ we get $y=0$ & $y = 4a$

Thus the point of intersections

$$(0,0) \;\&\; (4a, 4a)$$

On changing the order of integration we have y varying from 0 to $4a$

And x varying from $y^2/4a$ to $2\sqrt{ay}$



$$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx\, dy$$

$$= \int_{y=0}^{4a} \Big[x \Big]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy$$

$$= \left[2\sqrt{a}.\left(\frac{y^{3/2}}{3/2} \right) - \frac{1}{4a}\left(\frac{y^3}{3} \right) \right]_{y=0}^{4a}$$

$$= 2\sqrt{a}.\frac{2}{3}(4a)^{3/2} - \frac{1}{4a}\frac{(4a)^3}{3}$$

$$= \frac{4\sqrt{a}}{3}\, 4a.\sqrt{4a} - \frac{1}{4a}\frac{(4a)^3}{3}$$

$$= \frac{1}{3}\left[32a^2 - 16a^2 \right] = \frac{16}{3}a^2$$

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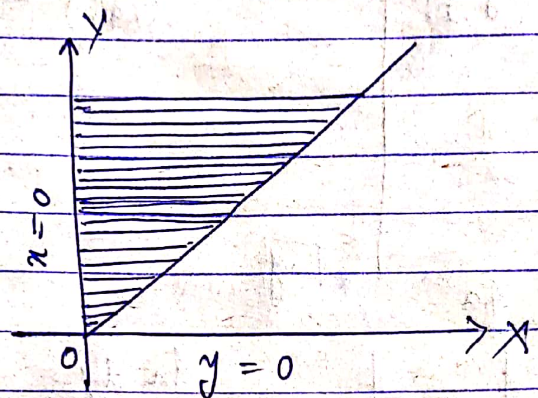
3) Change the order of integration and evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$

$\Rightarrow$

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx$$



On changing the order we must have $y = 0$ to ∞ and $x = 0$ to y

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} \, dx \, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [y] \, dy$$

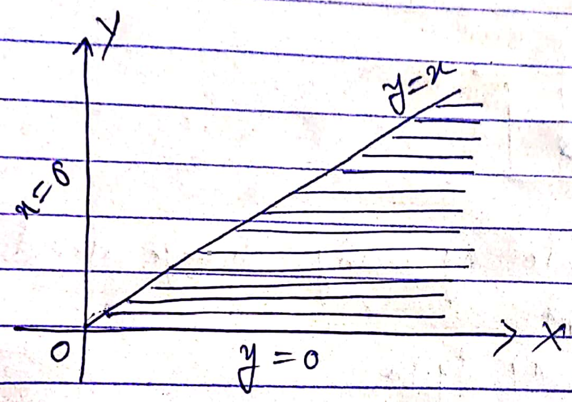
$$= -[e^{-y}]_0^\infty$$

$$\boxed{I = 1}$$

4) Evaluate $\int_0^\infty \int_0^x x\, e^{-x^2/y}\, dy\, dx$ by changing the order of integration.

$\Rightarrow$

$$I = \int_{x=0}^{\infty} \int_{y=0}^{x} x\, e^{-x^2/y}\, dy\, dx$$



The region is as shown in the figure
On changing the order of integration
$y = 0$ to ∞ ; $x = y$ to ∞

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x\, e^{-x^2/y}\, dx\, dy$$

Put $\frac{x^2}{y} = t$

$\therefore \frac{2x}{y} dx = dt$

$x\, dx = \frac{y\, dt}{2}$

When $x = y$, $t = y$
When $x = \infty$, $t = \infty$

$$\therefore I = \int_{y=0}^{\infty} \int_{t=y}^{\infty} e^{-t} \cdot \frac{y}{2}\, dt\, dy$$

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$$= \int_{y=0}^{\infty} \frac{y}{2} \left[-e^{-t}\right]_{t=y}^{\infty} dy$$

$$= \frac{1}{2} \int_{y=0}^{\infty} y e^{-y} \, dy$$

$$= \frac{1}{2} \left[y(-e^{-y}) - (1)(e^{-y}) \right]_0^{\infty}$$

$$\boxed{I = \frac{1}{2}}$$

(A) Evaluate, By using change of order

H.W

1) $\displaystyle\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

2) $\displaystyle\int_0^1 \int_{e^x}^{e} \frac{dy \, dx}{\log y}$

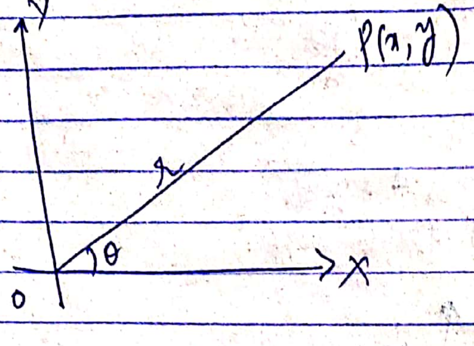
3) $\displaystyle\int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2 \, dx \, dy}{\sqrt{y^4 - a^2x^2}}$

4) $\displaystyle\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$

(3) Evaluation by changing into polar co-ordinates

1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$

by changing to polar co-ordinates



In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$

$\because$ x, y varies from 0 to ∞
In the first quadrant θ varies from 0 to $\pi/2$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

$r^2 = t$. $r\,dr = \frac{dt}{2}$.

t also varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_{t=0}^{\infty} d\theta$$

$$= \frac{-1}{2} \int_{\theta=0}^{\pi/2} (0-1)\, d\theta$$

$$= \frac{1}{2} [\theta]_0^{\pi/2} = \pi/4 .$$

$I = \pi/4$

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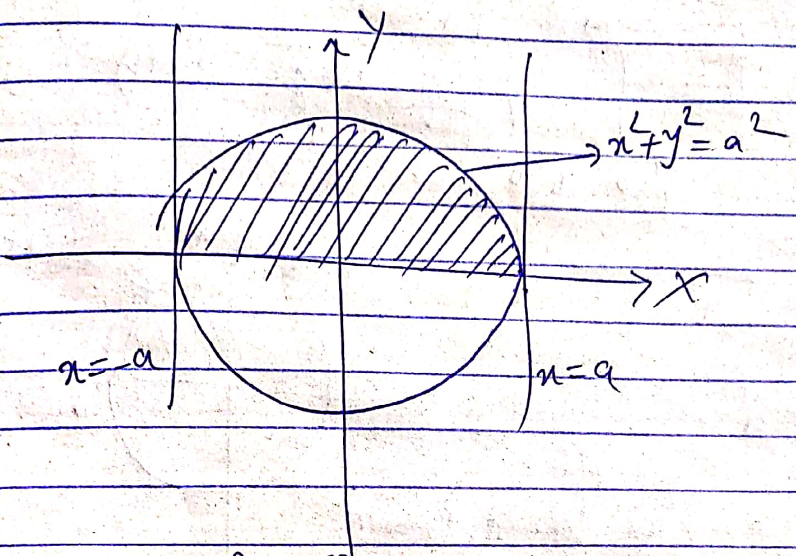
Change the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

into polars and hence evaluate the same

$\Rightarrow$



$\Rightarrow$ The region of integration is as shown in the fig

clearly θ varies from 0 to π

If $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

$a^2 = r^2 \Rightarrow r = a.$

$\therefore$ r varies from 0 to a

$dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} r \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{a^3}{3}\left[\theta\right]_0^{\pi} = \frac{a^3}{3}(\pi-0) = \frac{\pi a^3}{3}$$

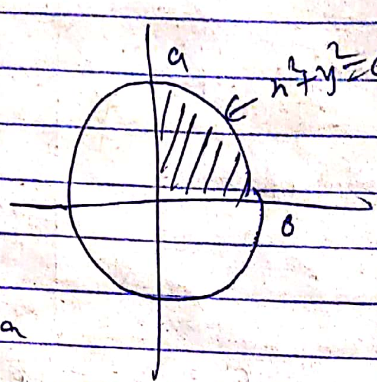
Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\,dy$ by changing into polars.

$\Rightarrow$

$$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\,dy$$

$x = \sqrt{a^2-y^2}$

$x^2 + a^2.$



$\because$ y varies from 0 to a

the region of integration is the first quadrant of the circle

In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$

ie $r^2 = a^2$

$r = a$

Also $x=0$, $y=0$ will give rise to $r=0$

& hence we can say that r varies from 0 to a

In the first quadrant θ varies from 0 to $\pi/2$

$dx\,dy = r\,dr\,d\theta$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r\sin\theta \cdot r \cdot r\, dr\,d\theta$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta\, dr\,d\theta$$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta \, dr \, d\theta$$

$$= \int_{r=0}^{a} r^3 \left[-\cos\theta\right]_0^{\pi/2} dr.$$

$$= \int_{r=0}^{a} -r^3 (0-1) \, dr$$

$$= \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

$$\boxed{I = \frac{a^4}{4}}$$

(4) $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2) \, dy \, dx$

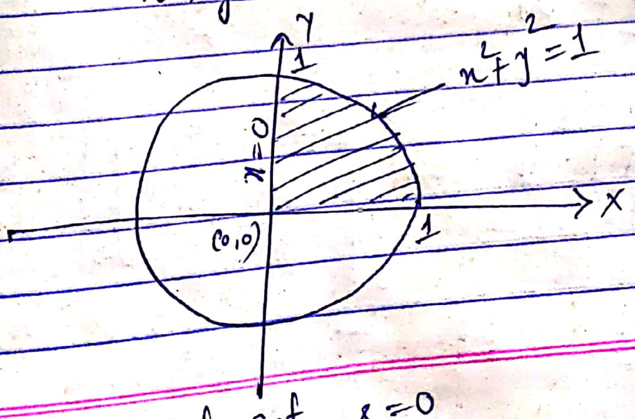
$\Rightarrow$ limits of y : $y=0$ to $y=1$

limits of x : $x=0$ to $x=\sqrt{1-y^2}$

The region of integration bounded by the line $x=0$ and the circle $x^2+y^2=1$

Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the

$x^2+y^2 = r^2$ ie $r^2=1 \Rightarrow r=1$



$x=0$, $y=0$ will get $r=0$.

& hence we can say that r varies from 0 to 1
In the Ist quadrant θ varies from 0 to $\pi/2$

$$dxdy = r\,dr\,d\theta$$

$$I = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2 \cdot r\,dr\,d\theta$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^3\,dr\,d\theta$$

$$= \int_{r=0}^{1} r^3 \left[\int_{\theta=0}^{\pi/2} d\theta \right] dr$$

$$= \int_{r=0}^{1} r^3 \left[\theta \right]_0^{\pi/2} dr$$

$$= \int_{r=0}^{1} r^3 \left(\frac{\pi}{2} \right) dr = \left[\frac{r^4}{4} \cdot \frac{\pi}{2} \right]_0^1$$

$$= \frac{\pi}{8}$$

(7) $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

$\Rightarrow$ Limits of x : $x = \frac{y^2}{4a}$ to $x = y$

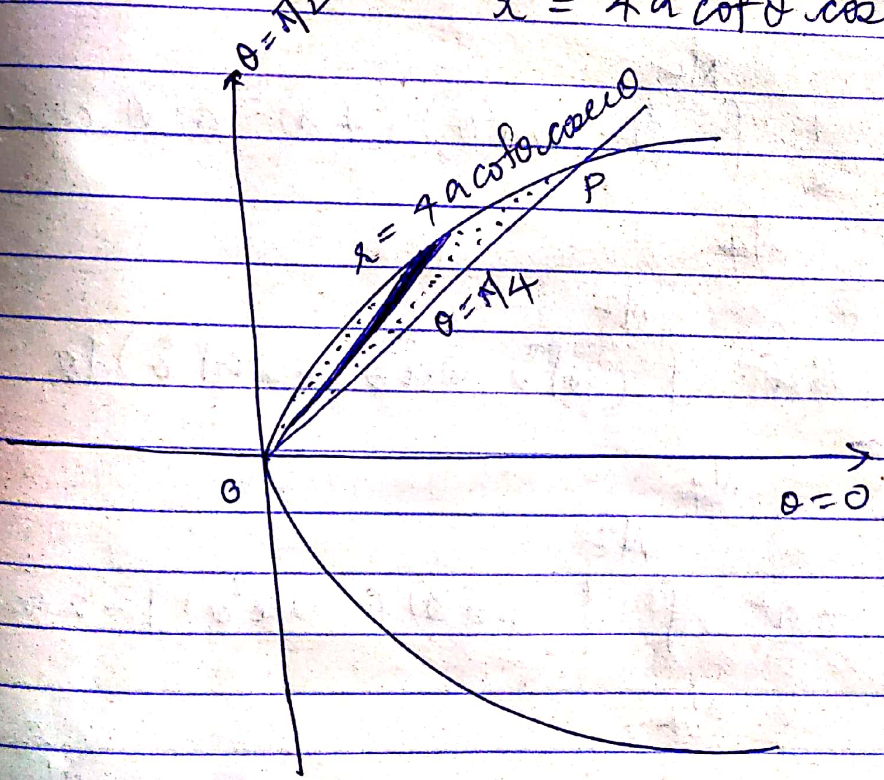
Limits of y : $y = 0$ to $y = 4a$

The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$

Putting $x = r\cos\theta$, $y = r\sin\theta$. polar form of the line $y = x$ is $r\sin\theta = r\cos\theta \Rightarrow \theta = \pi/4$

parabola $y^2 = 4ax$ is $r^2\sin^2\theta = 4ar\cos\theta$

$r = 4a\cot\theta\,\operatorname{cosec}\theta$



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Limits of r : $r=0$ to $r=4a\cot\theta\,\mathrm{cosec}\,\theta$

Limits of θ : $\theta=\pi/4$ to $\theta=\pi/2$

$$I=\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}\,dx\,dy$$

$$=\int_{\pi/4}^{\pi/2}\int_0^{4a\cot\theta\,\mathrm{cosec}\,\theta}\frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}\,r\,dr\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}(1-2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\,\mathrm{cosec}\,\theta}d\theta$$

$$=\frac{1}{2}\int_{\pi/4}^{\pi/2}(1-2\sin^2\theta)(4a)^2\cot^2\theta\,\mathrm{cosec}^2\theta\,d\theta$$

$$=8a^2\int_{\pi/4}^{\pi/2}(\cot^2\theta\,\mathrm{cosec}^2\theta-2\cot^2\theta)\,d\theta$$

$$=8a^2\int_{\pi/4}^{\pi/2}\{-(\cot^2\theta)(-\mathrm{cosec}^2\theta)\}-2\,\mathrm{cosec}^2\theta+2]$$

$$=8a^2\left|-\frac{\cot^3\theta}{3}+2\cot\theta+2\theta\right|_{\pi/4}^{\pi/2}$$

$$=8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

Change of Variables of Integration.

1) Using the transformation $x+y=u$, $y=uv$ show that

$$\int_0^1 \int_0^{1-x} e^{y/x+y}\, dy\, dx = \frac{1}{2}(e-1)$$

$\Rightarrow$ $x+y=u$ $\qquad$ $y=uv$

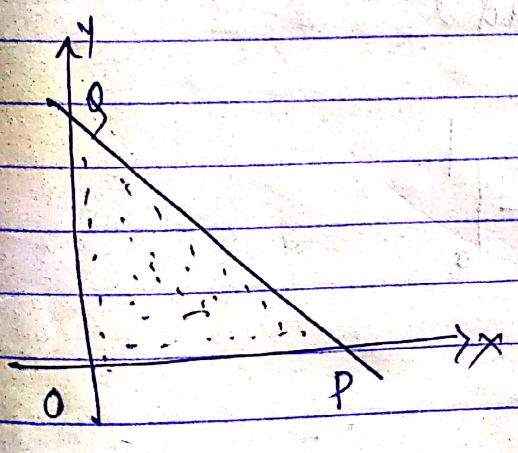
$x = u(1-v)$ $\qquad$ $y=uv$

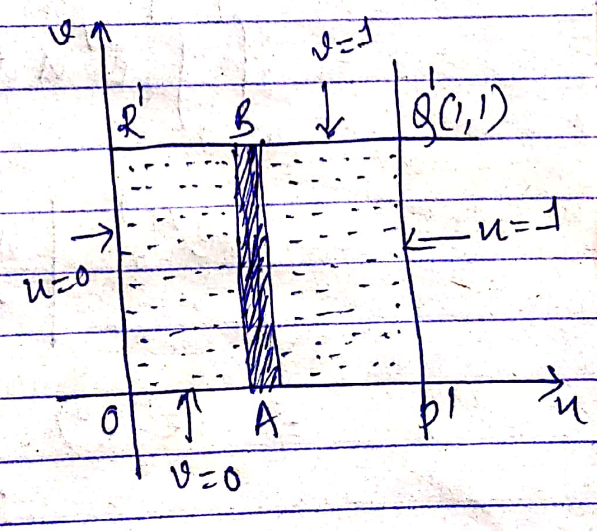
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

$$dx\,dy = |J|\, du\,dv = u\,du\,dv$$

limits of y : $y=0$ to $y=1-x$
limits of x : $x=0$ to $x=1$





The region in xy-plane is the triangle OPQ bounded by the lines $x=0$, $y=0$ & $x+y=1$

Under the transformation $x=u(1-v)$ & $y=uv$

① the line $x=0$ gets transformed to the line $u=0$ or $v=1$

② the line $y=0$ gets transformed to the line $u=0$ or $v=0$

③ the line $x+y=1$ gets transformed to the line $u=1$

The triangle OPQ in the xy-plane gets transformed to the square $OP'Q'R'$ in uv-plane bounded by the lines $u=0$, $v=0$, $u=1$ & $v=1$

limits of v: $v=0$ to $v=1$
limits of u: $u=0$ to $u=1$

$$I = \int_0^1 \int_0^{1-x} e^{y/x+y}\, dx\, dy$$

$$= \int_0^1 \int_0^1 e^v u\, du\, dv$$

$$= \left[e^v\right]_0^1 \left|\frac{u^2}{2}\right|_0^1$$

$$= (e^1 - e^0)\cdot\frac{1}{2}$$

$$= \frac{1}{2}(e-1)$$

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Triple Integration

Evaluate

① $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$

$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$

$= \int_0^3 \int_0^2 \left(x + y + \frac{1}{2} \right) dy\, dx$

$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_0^2 dx$

$= \int_0^3 (2x + 2 + 1)\, dx$

$= \int_0^3 (2x+3)\, dx = \left[\frac{2x^2}{2} + 3x \right]_0^3$

$= (3)^2 + 3(3) = 18$

Evaluate

2) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2)\, dx\, dy\, dz$

$= \int_{-c}^{c} \int_{-b}^{b} \left[\frac{x^3}{3} + y^2x + z^2x \right]_{-a}^{a} dy\, dz$

$= \int_{-c}^{c} \int_{-b}^{b} \left(\frac{a^3}{3} + y^2a + z^2a + \frac{a^3}{3} + y^2a + z^2a \right) dy\, dz$

$= 2 \int_{-c}^{c} \int_{-b}^{b} \left(\frac{a^3}{3} + y^2a + z^2a \right) dy\, dz$

$$= 2\int_{-c}^{c}\left(\frac{a^3}{3}y + \frac{y^3}{3}a + z^2ay\right)_{-b}^{b} dz$$

$$= 2\int_{-c}^{c}\left(\frac{a^3}{3}b + \frac{b^3}{3}a + z^2ab + \frac{a^3}{3}b + \frac{b^3a}{3} + z^2ab\right) dz$$

$$= 2\int_{-c}^{c}\left(\frac{2a^3b}{3} + \frac{2b^3a}{3} + 2z^2ab\right) dz$$

$$= 4\int_{-c}^{c}\left(\frac{a^3b}{3} + \frac{b^3a}{3} + z^2ab\right) dz$$

$$= 4\left[\frac{a^3b}{3}z + \frac{b^3a}{3}z + \frac{abz^3}{3}\right]_{-c}^{c}$$

$$= 4\left[\frac{a^3bc}{3} + \frac{b^3ac}{3} + \frac{abc^3}{3} + \frac{a^3bc}{3} + \frac{b^3ac}{3} + \frac{abc^3}{3}\right]$$

$$= 4\left[\frac{2a^3bc}{3} + \frac{2b^3ac}{3} + \frac{2abc^3}{3}\right]$$

$$= \frac{8}{3}\left[a^3bc + b^3ac + abc^3\right]$$

$$= \frac{8}{3}abc\left[a^2+b^2+c^2\right]$$

3) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$= \int_0^a \int_0^x \left[e^{x+y+z} \right]_0^{x+y} dy\, dx$

$= \int_0^a \int_0^x \left[e^{x+y+x+y} - e^{x+y+0} \right] dy\, dx$

$= \int_0^a \int_0^x \left[e^{2(x+y)} - e^{(x+y)} \right] dy\, dx$

$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y} \right]_0^x dx$

$= \int_0^a \left[\frac{e^{2(x+x)}}{2} - e^{x+x} - \frac{e^{2x}}{2} + e^{x} \right] dx$

$= \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right) dx$

$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right) dx$

$= \left[\frac{e^{4x}}{8} - \frac{3}{4} e^{2x} + e^{x} \right]_0^a$

$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{1}{8} - \frac{3}{4} + 1$

$= \frac{1}{8} \left[e^{4a} - 6e^{2a} + 8e^{a} + 1 \right]$

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*) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\,dy\,dz$

$$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{xyz^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{2} (1-x^2-y^2) \, dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{xy}{2} - \frac{x^3y}{2} - \frac{xy^3}{2} \right] dx\,dy$$

$$= \int_0^1 \left[\frac{xy^2}{4} - \frac{x^3y^2}{4} - \frac{xy^4}{8} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left[\frac{x(1-x^2)}{4} - \frac{x^3(1-x^2)}{4} - \frac{x(1-x^2)^2}{8} \right] dx$$

$$= \frac{1}{8} \int_0^1 \left[2x - 2x^3 - 2x^3 + 2x^5 - x(1+x^4-2x^2) \right] dx$$

$$= \frac{1}{8} \int_0^1 \left[2x - 4x^3 + 2x^5 - x - x^5 + 2x^3 \right] dx$$

$$= \frac{1}{8} \int_0^1 (x^5 - 2x^3 + x) \, dx$$

$$= \frac{1}{8} \left[\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{8} \left[\frac{1}{6} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} \right]$$

$$= \frac{1}{48}$$

Triple Integral in cylindrical co-ordinates:-

Find the Volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin \phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a \sin \phi$ and ϕ varies from 0 to π.

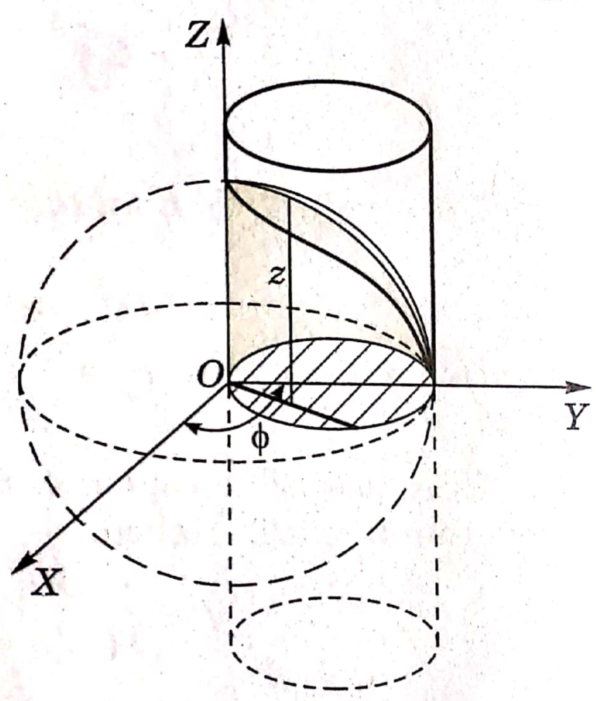


Fig. 7.29

Hence the required volume $= 2\int_0^{\pi}\int_0^{a \sin \phi}\int_0^{\sqrt{(a^2-\rho^2)}} \rho\, dz\, d\rho\, d\phi$

$$= 2\int_0^{\pi}\int_0^{a \sin \phi} \rho\sqrt{(a^2 - \rho^2)}\, d\rho\, d\phi = 2\int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a \sin \phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi} (1 - \cos^3 \phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Evaluate the following by changing to spherical polar Co-ordinates.

Example 3: Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the region bounded by the sphere $x^2+y^2+z^2=a^2$.

Solution:

1. Putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ equation of the sphere $x^2+y^2+z^2=a^2$ reduces to $r=a$.
2. For the complete sphere, limits of $r: r=0$ to $r=a$

limits of θ: $\theta=0$ to $\theta=\pi$

limits of ϕ : $\phi=0$ to $\phi=2\pi$

$$I=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}}=\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\int_0^{a}\frac{r^2+a^2-a^2}{\sqrt{a^2-r^2}}dr$$

$$=|\phi|_0^{2\pi}|-\cos\theta|_0^{\pi}\int_0^{a}\left(\frac{a^2}{\sqrt{a^2-r^2}}-\sqrt{a^2-r^2}\right)dr$$

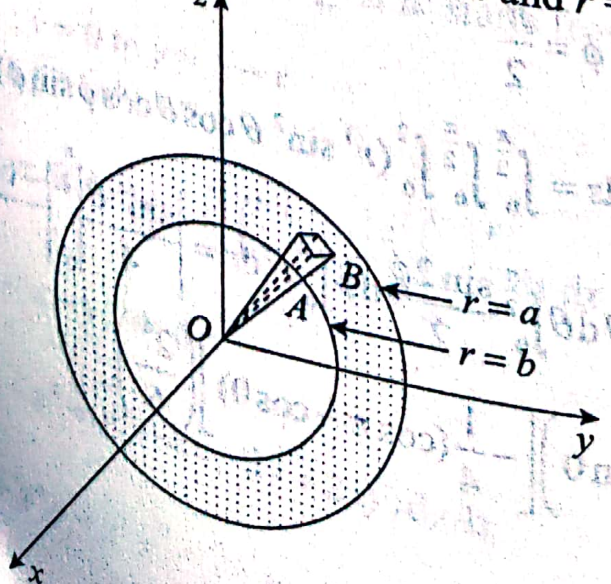
$$=(2\pi)(-\cos\pi+\cos 0)\left|a^2\sin^{-1}\frac{r}{a}-\frac{r}{2}\sqrt{a^2-r^2}-\frac{a^2}{2}\sin^{-1}\frac{r}{a}\right|_0^a$$

$$=4\pi\left(\frac{a^2}{2}\sin^{-1}1\right)=4\pi\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}=\pi^2a^2$$

Example 4: Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ over the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a>b>0$.

Solution:

1. Putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, equation of the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ reduces to $r=a$ and $r=b$ respectively.



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2. Draw an elementary radius vector OAB from the origin in the region. This radius vector enters in the region from the sphere $r = b$ and leaves the region at the sphere $r = a$.
Limits of r : $r = b$ to $r = a$.
3. For complete sphere, limits of θ: $\theta = 0$ to $\theta = \pi$

limits of ϕ: $\phi = 0$ to $\phi = 2\pi$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_b^{a} \frac{r^2 \sin\theta}{r}\, dr\, d\theta\, d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_b^{a} r\, dr$$

$$= |\phi|_0^{2\pi} \left|-\cos\theta\right|_0^{\pi} \left|\frac{r^2}{2}\right|_b^{a} = 2\pi(-\cos\pi + \cos 0)\frac{(a^2 - b^2)}{2} = 2\pi(a^2 - b^2)$$

Example 5: Evaluate $\iiint z^2 dx\, dy\, dz$ over the region common to the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 2x$.

Solution:

1. Putting $x = r\cos\theta, y = r\sin\theta, z = z$, equation of the
   (i) sphere $x^2 + y^2 + z^2 = 4$ reduces to

$$r^2 + z^2 = 4$$

$$z^2 = 4 - r^2$$

   (ii) cylinder $x^2 + y^2 = 2x$ reduces to
   $r^2 = 2r\cos\theta, r = 2\cos\theta$

2. Draw an elementary volume parallel to z-axis in the region. This elementary volume starts from the part of the sphere $z^2 = 4 - r^2$, below xy-plane and terminates on the part of the sphere $z^2 = 4 - r^2$, above xy-plane.

   Limits of r : $z = -\sqrt{4 - r^2}$ to $z = \sqrt{4 - r^2}$

3. Projection of the region in $r\theta$ plane is the circle $r = 2\cos\theta$.
4. Draw an elementary radius vector OA in the region ($r = 2\cos\theta$) which starts from the origin and terminates on the circle $r = 2\cos\theta$

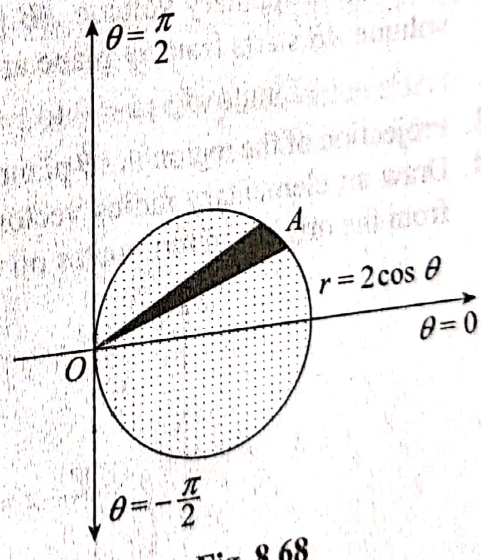

Fig. 8.68

Limits of r : $r = 0$ to $r = 2\cos\theta$

Limits of θ : $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

$$I = \iiint z^2 dx\, dy\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2 \cdot r\, dz\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \left|\frac{z^3}{3}\right|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dr\, d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} 2(4 - r^2)^{\frac{3}{2}} r\, dr\, d\theta$$

$$=\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\cos\theta}\left[-(4-r^2)^{\frac{3}{2}}(-2r)\,\mathrm{d}r\right]\mathrm{d}\theta$$

$$=-\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\frac{2(4-r^2)^{\frac{5}{2}}}{5}\right|_{0}^{2\cos\theta}\mathrm{d}\theta \qquad \left[\because \int[f(r)]^n f'(r)\mathrm{d}r=\right.$$

$$=-\frac{2}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2^5\sin^5\theta-2^5\right)\mathrm{d}\theta$$

$$=-\frac{2}{15}\left[0-2^5|\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \qquad \left[\begin{array}{l}\because \int_{-a}^{a} f(\theta)\mathrm{d}\theta=0, \text{ if } f(-\theta \\ \sin^5(-\theta)=-\sin^5\theta\end{array}\right.$$

$$=\frac{2^6\pi}{15}=\frac{64\pi}{15}$$