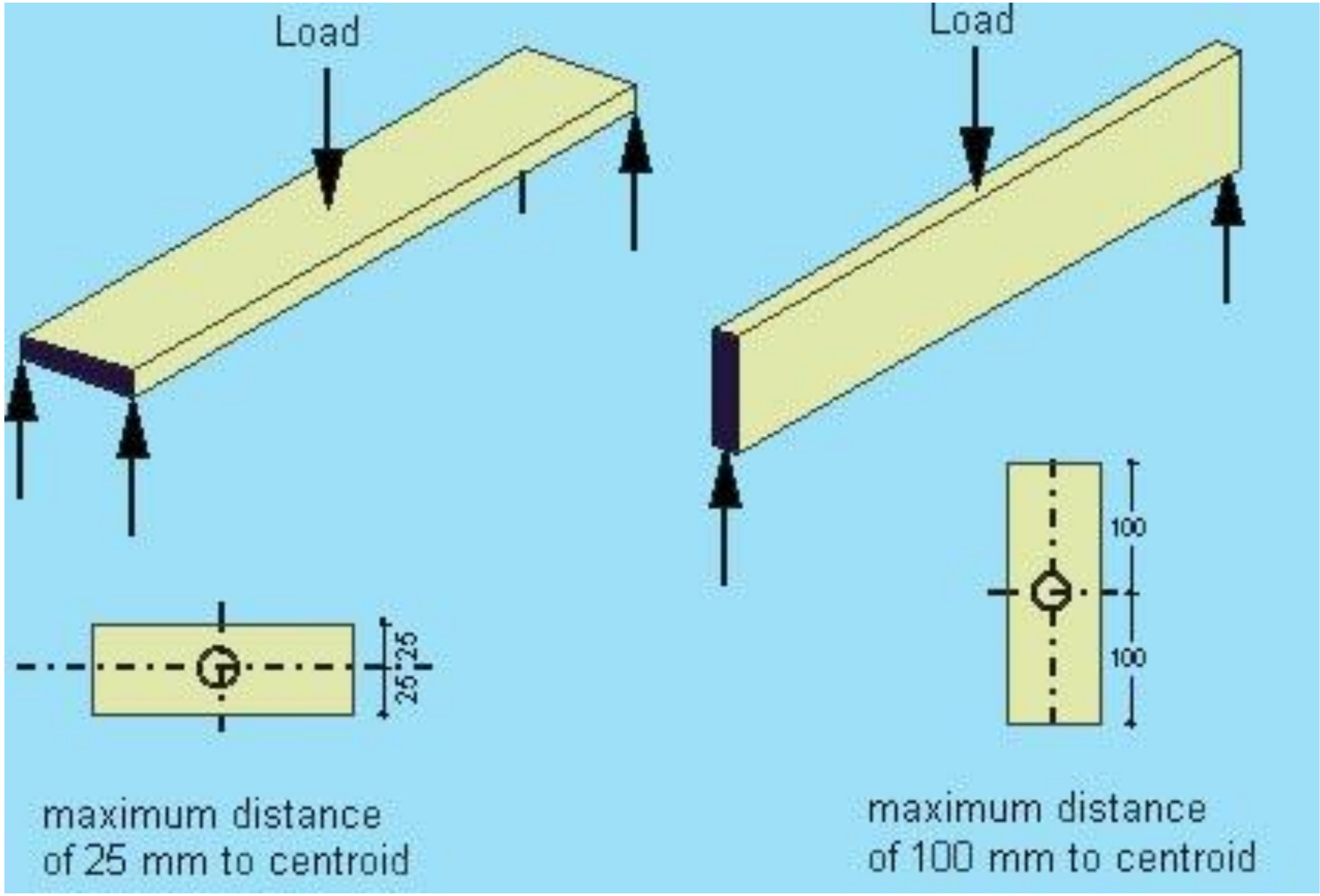
Moment of Inertia

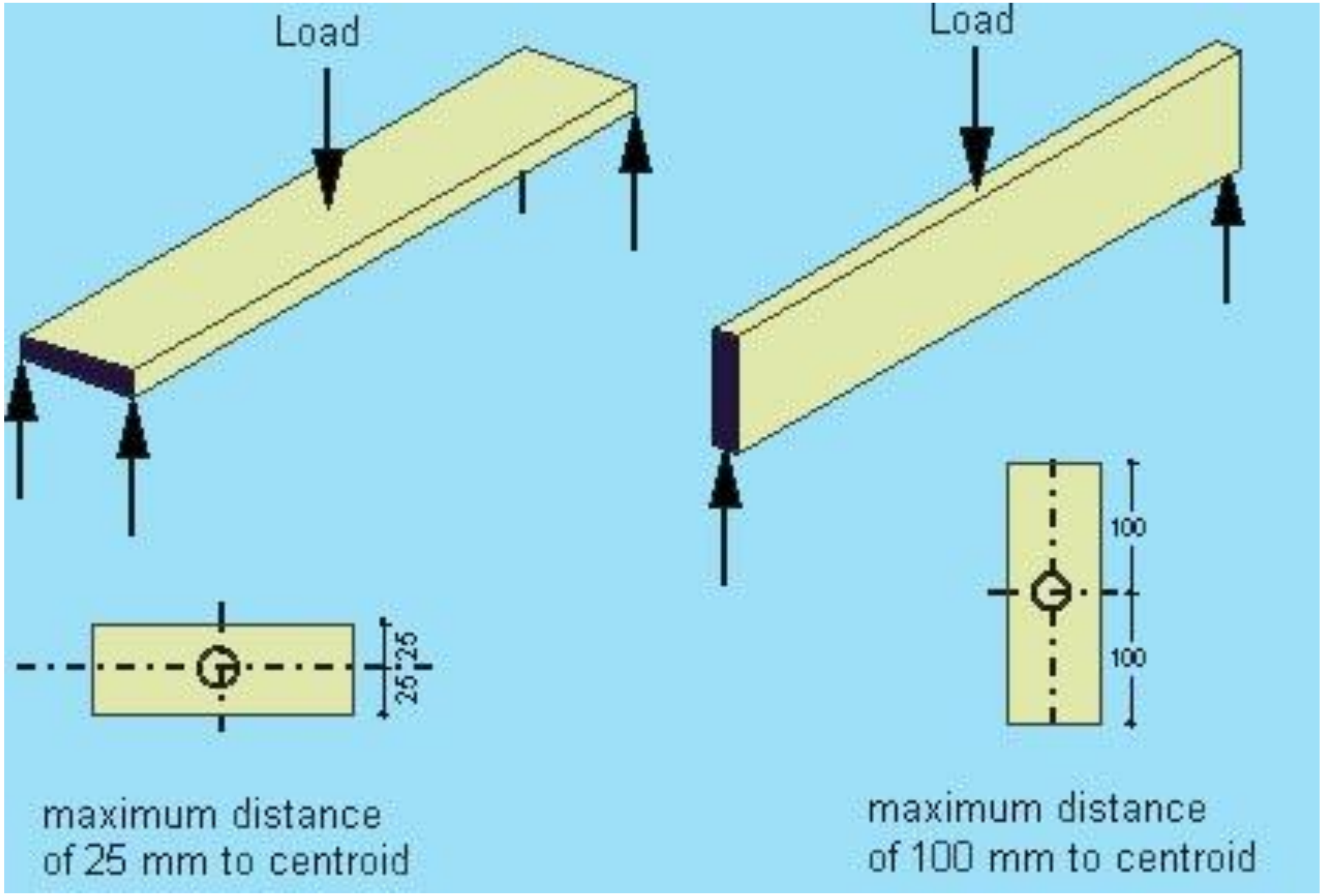
Area Moment of Inertia or Second Moment of Area

Introduction to the concept



- Two simply supported beams shown
- Which of the two bends more, the beam on the left or the one at the right?
- Observe the rectangular shapes shown below. These are called the cross-sections of the beams
- Also observe that the cross-sectional area (area of the rectangle) of both beams is the same. Only their orientations are changed

Introduction to the concept



- The correct answer: The beam on the left bends more
- So this tells us that by just changing the orientation of the beam cross-section the amount of bending changes
- But why does that happen?

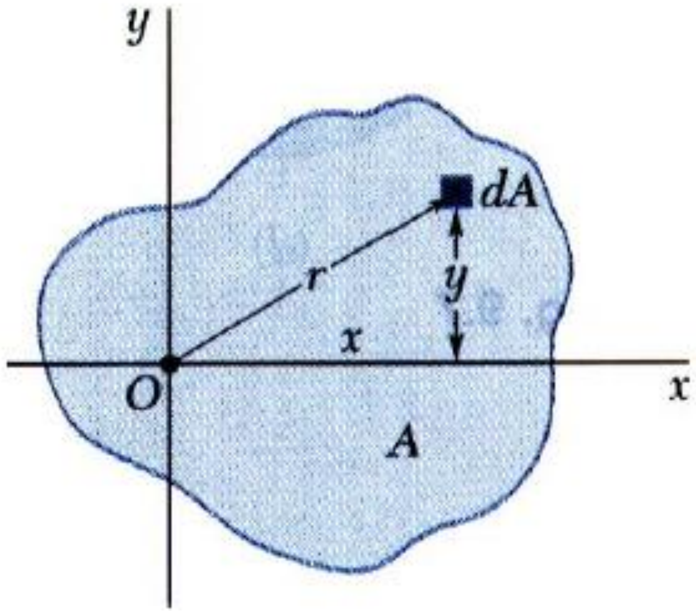
- The reason why this happens is change in one of the cross-sectional properties
- This property is called *Moment of Inertia* or *Second Moment of Area*
- We shall get back to this example again

Definition

- The second moment of area, or second area moment and also known as the area moment of inertia, is a geometrical property of an area which reflects how its points are distributed with regard to an arbitrary axis.
- The second moment of area is typically denoted with either an I (for an axis that lies in the plane) or with a J (for an axis perpendicular to the plane).
- Its unit of dimension when working with the International System of Units is meters to the fourth power, m^4 or mm^4

Mathematical Form of Area Moment of Inertia

- Consider an arbitrary shape of area *A.*
- Let it be required to determine the area moment of inertia (second moment of area) of this shape about the x- and y-axes
- To do this consider an infinitesimally small (it simply means very, very small) part of the shape. Let the area of the small part be *dA*
- Let the centroid of area *dA* be at a perpendicular distance of *x* from the y-axis and *y* from the x-axis as shown in figure



Mathematical Form of Area Moment of Inertia

- Taking first moment of the area *dA* about x-axis, we get

$$M = y \times dA$$

- Now taking second moment of the area *dA* about x-axis, we get

$$dI_x = y^2 \times dA$$

- By definition, second moment of area *dA* is the area moment of inertia of the area about x-axis dI_x
- Now, to get the area moment of inertia of the entire area *A* about x-axis, I_x we must integrate the above equation over the entire area *A*

$$I_x = \int dI_x = \int y^2 \times dA$$

- Similarly, the area moment of inertia of the entire area *A* about y-axis, I_y we must take the second moment of the area *dA* about y-axis and integrate to get the moment of inertia of the entire area *A*

$$I_y = \int dI_y = \int x^2 \times dA$$

Mathematical Form of Area Moment of Inertia

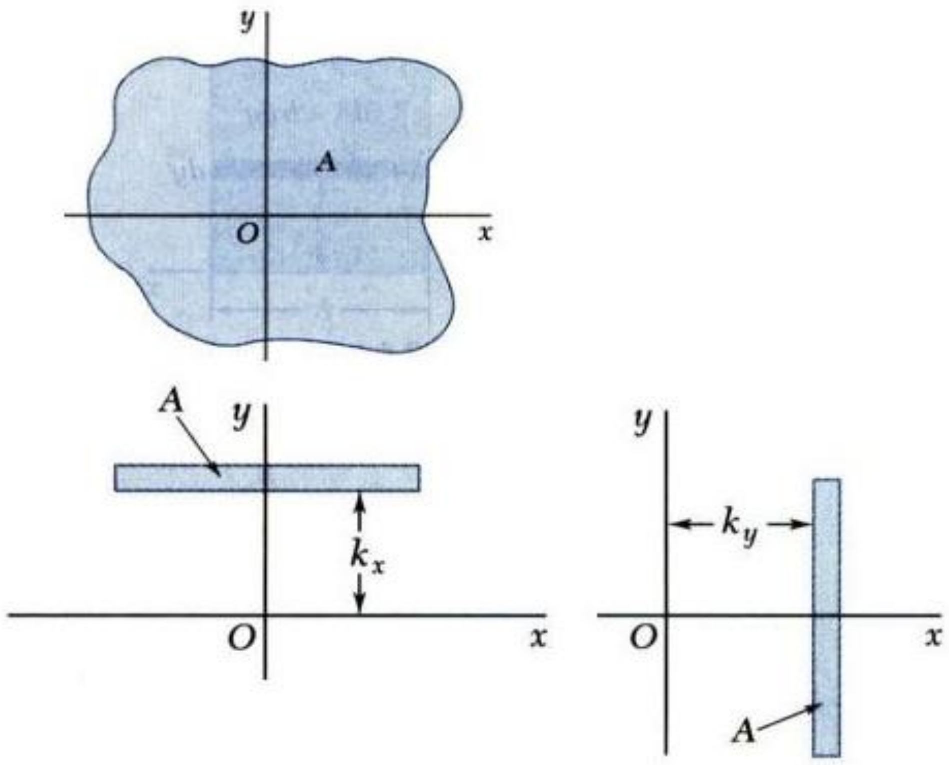
- Also, the moment of inertia of the area *A* about z-axis (axis perpendicular to the plane of area) is given by

$$I_z = J = \int r^2 \times dA$$

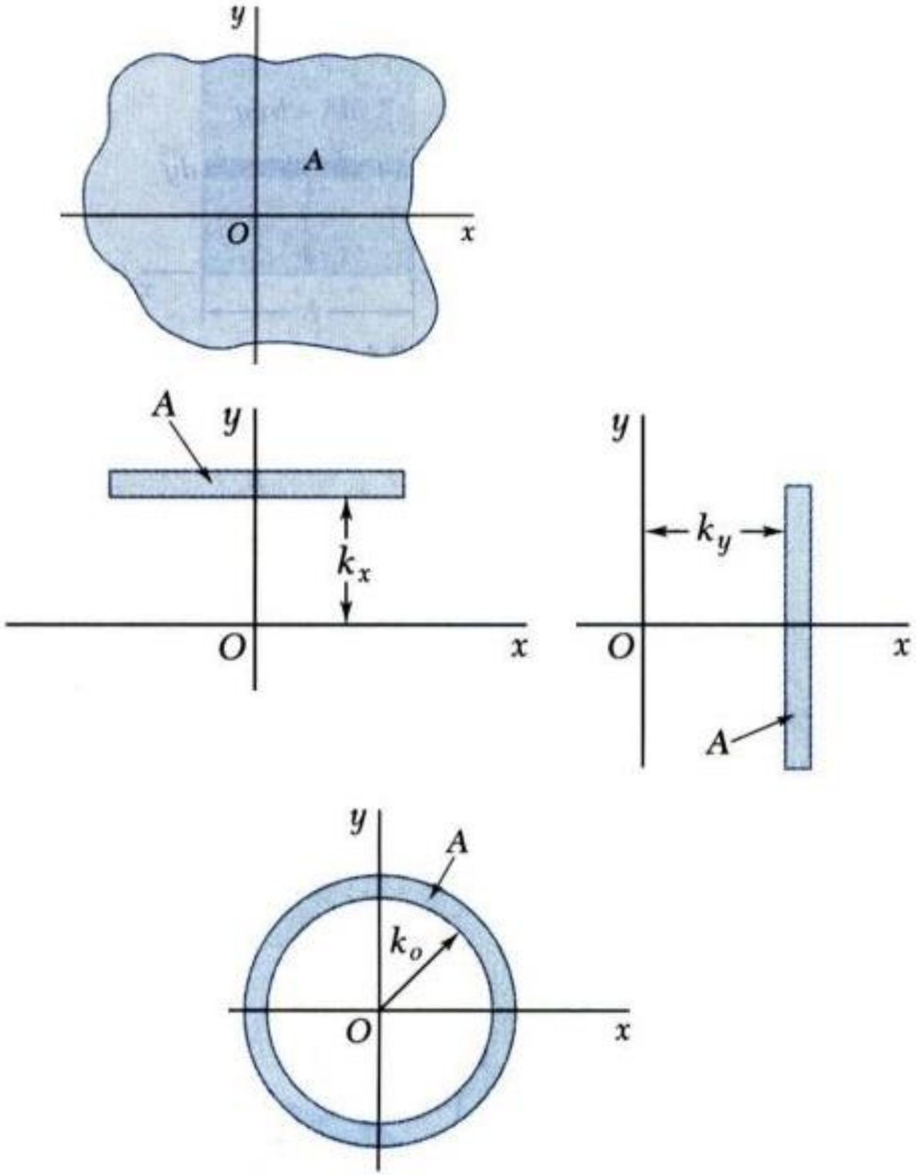
- Area moment of inertia about z-axis is also called *Polar Moment of Inertia (J)* of the area *A*

Radius of Gyration

- **Radius of gyration** of a shape about an axis is defined as the distance to a point which would have a moment of inertia the same as the shape's actual distribution of area, if the total area of the shape were concentrated at that point



Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$

Important Theorems related to Area MI

There are two important theorems related to MI

1. Perpendicular axis theorem: Useful to determine the polar moment of inertia of a given shape when the MI's about x- and y-axes are known
2. Parallel axis theorem: Useful to determine the MI of any given composite area when the MI's of all the component shapes are known

   (Composite area: area made up of basic geometric shapes)

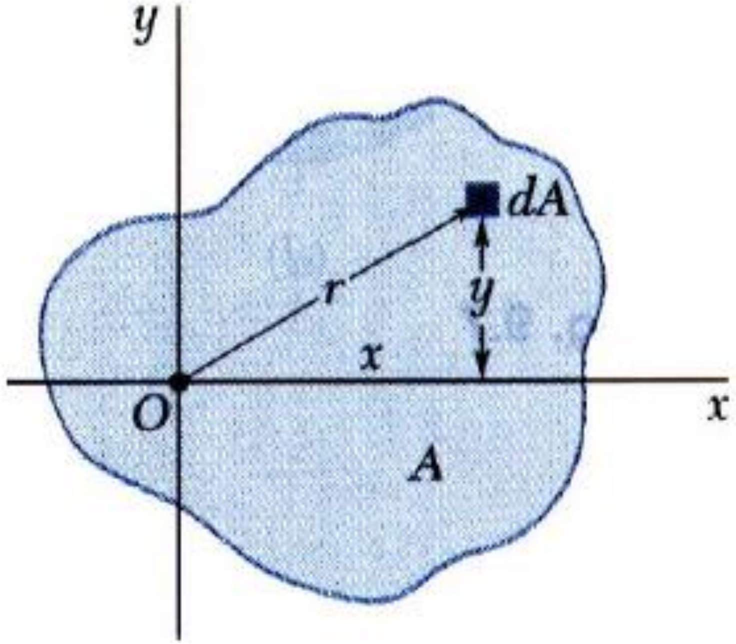
Perpendicular Axis Theorem - Proof

- The **perpendicular axis theorem** states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about the two axes at right angles to each other, in its own plane intersecting each other at the point where the perpendicular axis passes through it.
- Define perpendicular axes x, y, and z (which meet at origin O) so that the body lies in the xy plane, and the z axis is perpendicular to the plane of the body. Let I_x, I_y and I_z be moments of inertia about axis x, y, z axes respectively, the perpendicular axis theorem states that

$$I_z = I_x + I_y$$

(the z-axis is perpendicular to the screen and cannot be seen)

Perpendicular Axis Theorem - Proof



- We know that

$$I_z = J = \int r^2 \times dA$$

- From the figure it is seen that

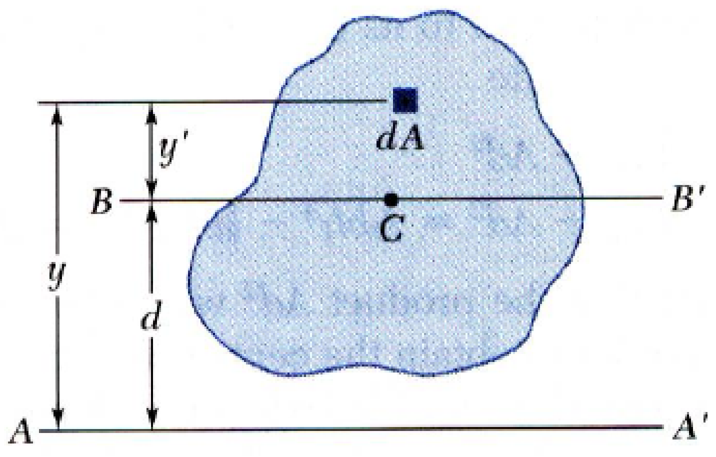
$$r^2 = x^2 + y^2$$ (Pythogoras' Theorem)

- Substituting in the equation for I_z we get

$$I_z = J = \int (x^2 + y^2) \times dA = \int x^2 \times dA + \int y^2 \times dA$$

$$\therefore I_z = J = I_x + I_y$$

Perpendicular Axis Theorem - Proof



- Consider moment of inertia I of an area A with respect to the axis AA'

$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2 \qquad \textit{parallel axis theorem}$$

Other form of writing this equation is

$I_{AA'} = I_G + Ad^2$