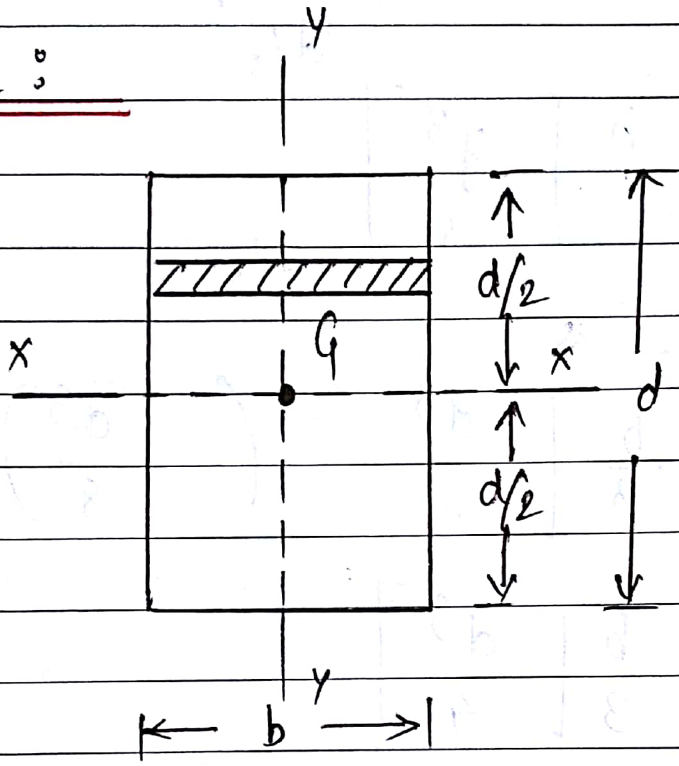
Unit IV

★ Moment of Inertia by Integration (From First Principles.

★ Rectangle :



Consider a rectangle of dimensions b×d as shown in the figure. Consider an elemental strip of length b and width dy at a distance 'y' from the centroidal X-axis.

$$dI_{xx} = y^2 \cdot dA$$

$$dA = b \cdot dy$$

$$dI_{xx} = y^2 \cdot b \cdot dy$$

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$$I_{xx} = \int dI_{xx} = b\int_{-d/2}^{d/2} y^2 \cdot dy$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= \frac{b}{3}\left[\frac{d^3}{8} - \left(-\frac{d^3}{8}\right)\right]$$

$$= \frac{b}{3}\left[\frac{d^3}{4}\right]$$

$$I_{xx} = \frac{bd^3}{12}.$$

Similarly, $I_{yy} = \dfrac{db^3}{12}$

The moment of inertia about the base AB can be obtained using parallel axes theorem. The distance AB from $x-x$ axis is $\dfrac{d}{2}$

$$I_{AB} = I_g + A \times d^2$$

where d = distance betn two parallel axes.

$$I_{AB} = \frac{bd^3}{12} + (bd)\left(\frac{d}{2}\right)^2$$

$$I_{AB} = \frac{bd^3}{3}$$

The polar moment of inertia can be obtained using perpendicular axes theorem.

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{bd^3}{12} + \frac{db^3}{12}$$

$$I_{zz} = \frac{bd}{12}\left[b^2 + d^2\right]$$

The radius of gyration about any axis can be obtained using $k = \sqrt{\frac{I}{A}}$, $A = b \times d$

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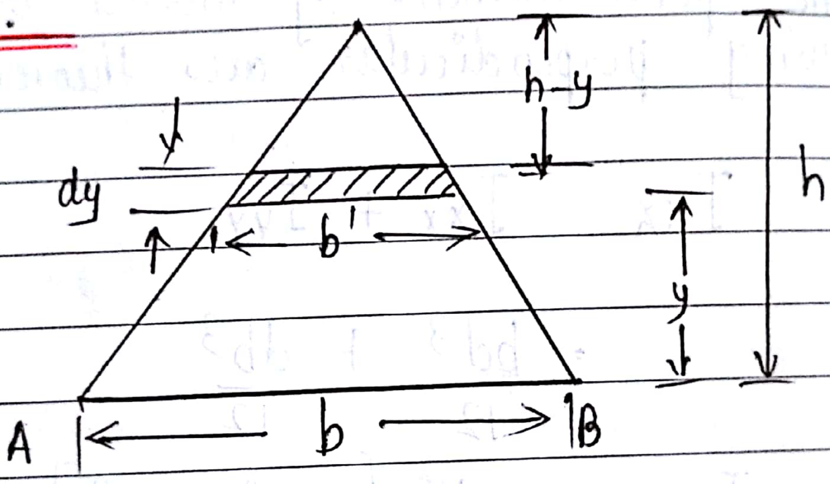
$$K_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{bd^3}{12 \times b \times d}} = \frac{d}{2\sqrt{3}}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{db^3}{12 \times b \times d}} = \frac{b}{2\sqrt{3}}$$

$$K_{zz} = \sqrt{\frac{I_{zz}}{A}} = \sqrt{\frac{\frac{bd}{12}\left(b^2+d^2\right)}{b \times d}} = \frac{\sqrt{b^2+d^2}}{2\sqrt{3}}$$

★ **Triangle :**



Consider a triangle of base 'b' and height 'h' as shown. Choose a horizontal strip of length 'b'' and width dy at a distance y from the base as differential element

By similar triangles

$$\frac{b'}{b} = \frac{h-y}{h}$$

$$b' = \left(\frac{h-y}{h}\right) \times b$$

Moment of inertia of the strip about base is,

$$dI_{AB} = y^2 \cdot dA$$

$$dA = b' \cdot dy$$

$$dA = \left[\frac{h-y}{h}\right] \cdot b \cdot dy$$

$$dI_{AB} = b\left[\frac{h-y}{h}\right] y^2 \cdot dy$$

$$I_{AB} = \frac{b}{h} \int_0^h (h-y) \cdot y^2 \cdot dy$$

$$= \frac{b}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$I_{AB} = \frac{bh^3}{12}$$

The centroid of triangle is at a height of $h/3$ from the base AB. Using parallel axes theorem.

$$I_{AB} = I_G + Ad^2$$

$$A = \frac{1}{2} bh \quad , \quad d = \frac{h}{3}$$

$$I_G = I_{AB} - Ad^2$$

$$= \frac{dh^3}{12} - \frac{1}{2} bh \left(\frac{h}{3}\right)^2$$

$$I_G = \frac{bh^3}{36}.$$

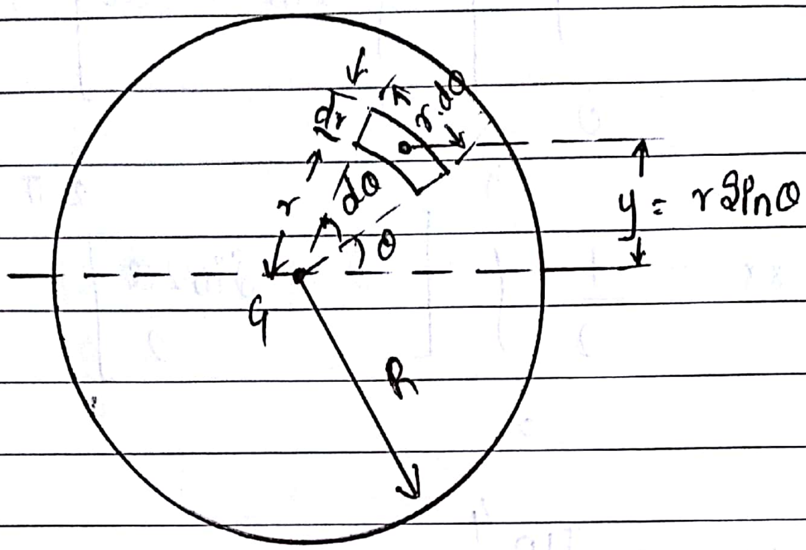
The radius of gyration

$$K_G = \sqrt{\frac{I_G}{A}} = \frac{h}{3\sqrt{2}}$$

★ Circle:

Consider a circle of radius R. Choose a

differential element of angular width '$d\theta$' at angle 'θ' and radial width dr at distance r from the centre as shown.



$$dA = (dr)(r\,d\theta)$$

$$dA = r.dr.d\theta$$

The distance of this area element from centroidal x-axis which is the diameter is

$$y = r\sin\theta$$

$$dI_{xx} = y^2.dA$$

$$= (r\sin\theta)^2.r\,dr.d\theta$$

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$$dI_{xx} = r^3 \sin^2\theta \cdot dr \cdot d\theta$$

$$I_{xx} = \int_0^R \left[\int_0^{2\pi} \sin^2\theta \cdot d\theta \right] r^3 \cdot dr$$

$$I_{xx} = \frac{1}{2} \int_0^R \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} \cdot r^3 \cdot dr$$

$$I_{xx} = \frac{\pi R^4}{4}$$

similarly,

$$I_{yy} = \frac{\pi R^4}{4}$$

$$R = \frac{D}{2}$$

$$I_{xx} = I_{yy} = \frac{\pi D^4}{64}$$

$$K_{xx} = \sqrt{\frac{I_{xx}}{A}}, \quad A = \pi R^2$$

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$$K_{xx} = K_{yy} = \frac{R}{2}$$

The polar moment of inertia is

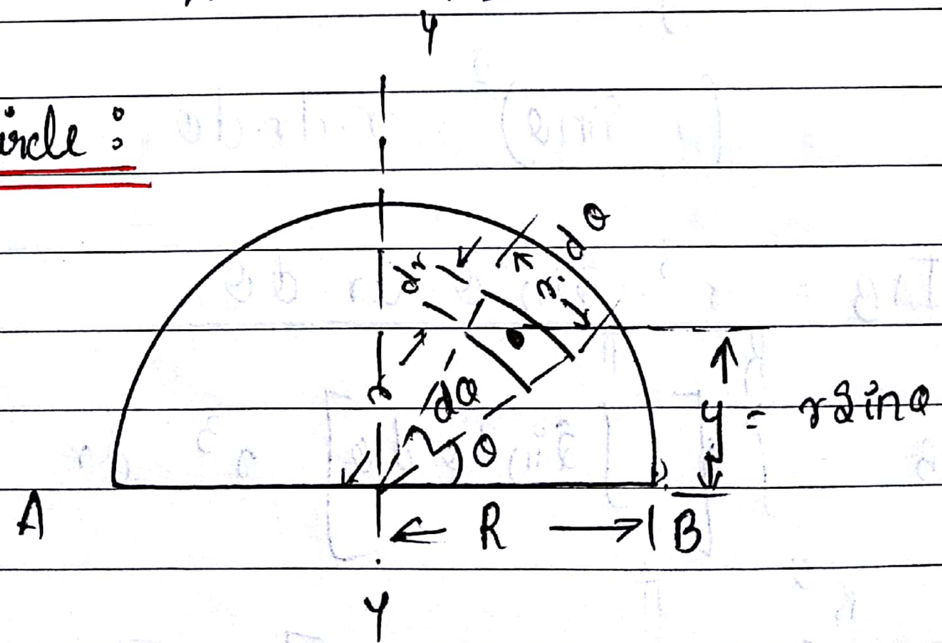
$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{\pi R^4}{4} + \frac{\pi R^4}{4}$$

$$I_{zz} = \frac{\pi D^4}{32} = \frac{\pi R^4}{2}$$

$$K_{zz} = \sqrt{\frac{I_{zz}}{A}} = \frac{R}{\sqrt{2}}$$

★ **Semi-circle :**



Consider a semi-circle of radius R as shown.

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Choose a differential element of angular width '$d\theta$' at angle 'θ' and radial width 'dr' at distance 'r' from the centre. The area of the differential element will be.

$$dA = (r.d\theta).dr$$

The distance of this differential element from the diameter AB is

$$y = r\sin\theta$$

M.I. of differential element about AB is,

$$dI_{AB} = y^2.dA$$

$$= (r.\sin\theta)^2.r.dr.d\theta$$

$$dI_{AB} = r^3.\sin^2\theta.dr.d\theta$$

$$I_{AB} = \int_0^R \left[\int_0^\pi \sin^2\theta\, d\theta\right].r^3.dr$$

$$= \int_0^R \left[\int_0^\pi \left[\frac{1-\cos 2\theta}{2}\right] d\theta\right] r^3.dr$$

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$$= \frac{1}{2} \int_0^R \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} \cdot r^3 . dr$$

$$I_{AB} = \frac{\pi R^4}{8}$$

The centroid is at a distance of $\frac{4R}{3\pi}$ from AB.

$$I_{AB} = I_q + A.d^2$$

$$I_q = I_{AB} - Ad^2$$

$$A = \frac{\pi R^2}{2} \text{ and } d = \frac{4R}{3\pi}$$

$$I_q = \frac{\pi R^4}{8} - \left(\frac{\pi R^2}{2} \right) \left(\frac{4R}{3\pi} \right)^2$$

$$I_q = 0.11 R^4$$

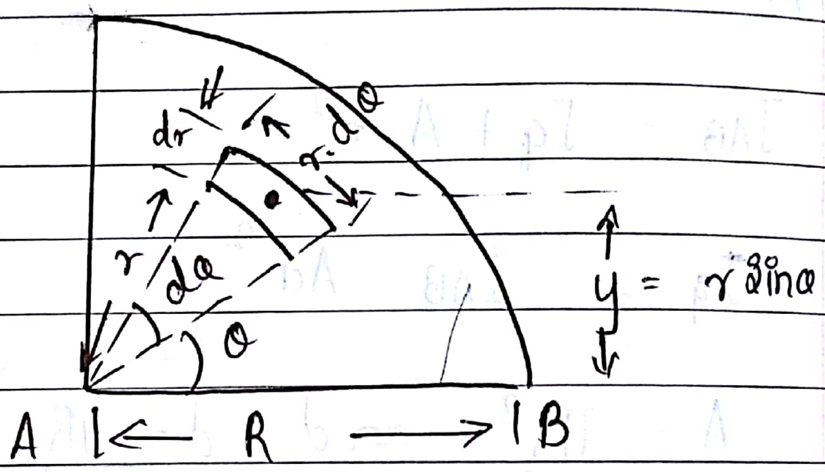
$$R = D/2$$

$$I_q = 0.00686 D^4$$

The moment of inertia about a centroidal axis perpendicular to diameter is.

$$I_q = \frac{\pi R^4}{8} = \frac{\pi D^4}{128}$$

* Quarter Circle:



Consider a quarter circle of radius R as shown in the figure. Choose a differential element of angular width 'dθ' at angle θ and radial width 'dr' at a distance r from the centre. The area of the differential element will be,

$$dA = (r.d\theta).dr$$
$$dA = r.dr.d\theta$$

The distance of this differential element from

AB is $y = r\sin\theta$

M.I. of differential element about AB is

$$dI_{AB} = y^2 . dA$$

$$= (r\sin\theta)^2 . (r.dr.d\theta)$$

$$dI_{AB} = r^3 . \sin^2\theta . dr.d\theta$$

$$I_{AB} = \int_0^R \left[\int_0^{\pi/2} \sin^2\theta . d\theta \right] . r^3 . dr$$

$$= \int_0^R \left[\int_0^{\pi/2} \left[\frac{1-\cos 2\theta}{2} \right] d\theta \right] . r^3 . dr$$

$$= \frac{1}{2} \int_0^R \frac{\pi}{2} . r^3 . dr$$

$$I_{AB} = \frac{\pi}{16} R^4$$

$$I_{AB} = I_G + Ad^2$$

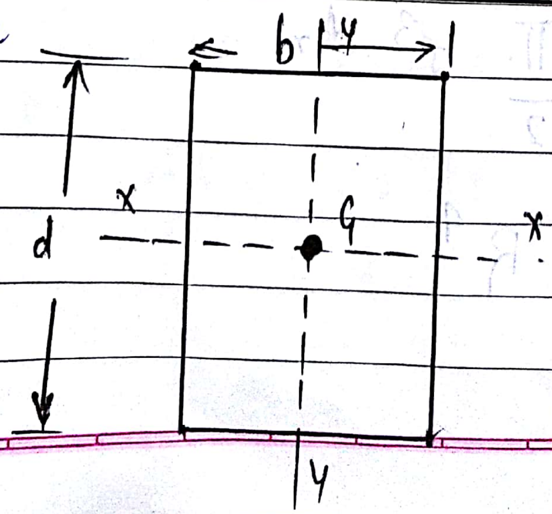
$$I_G = I_{AB} - Ad^2$$

$A = \frac{\pi R^2}{4}$, $d = \frac{4R}{3\pi}$ as the centroid is at a distance of $4R/3\pi$ from AB

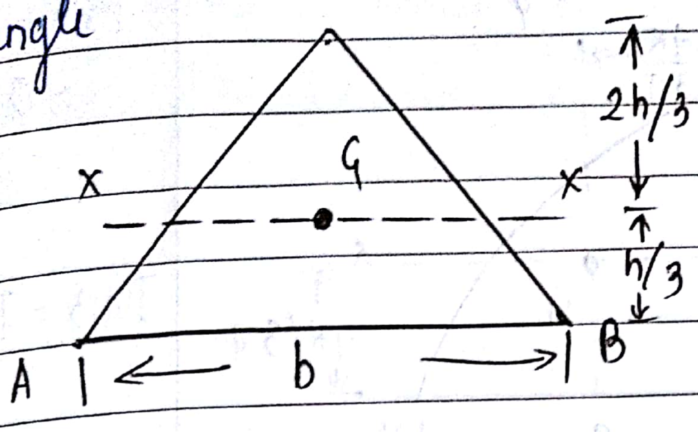
$$I_G = \frac{\pi R^4}{16} - \left(\frac{\pi R^2}{4}\right)\left(\frac{4R}{3\pi}\right)^2$$

$$I_G = 0.055R^4$$

★ Moment of Inertia of different Geometrical Figures.

Sr. No.	Basic Area Description	Figure	M.I.
1.	Rectangle		$I_{xx} = bd^3/12$ $I_{yy} = db^3/12$ $I_{AB} = bd^3/3$ $I_{zz} = \frac{bd}{12}(b^2+d^2)$

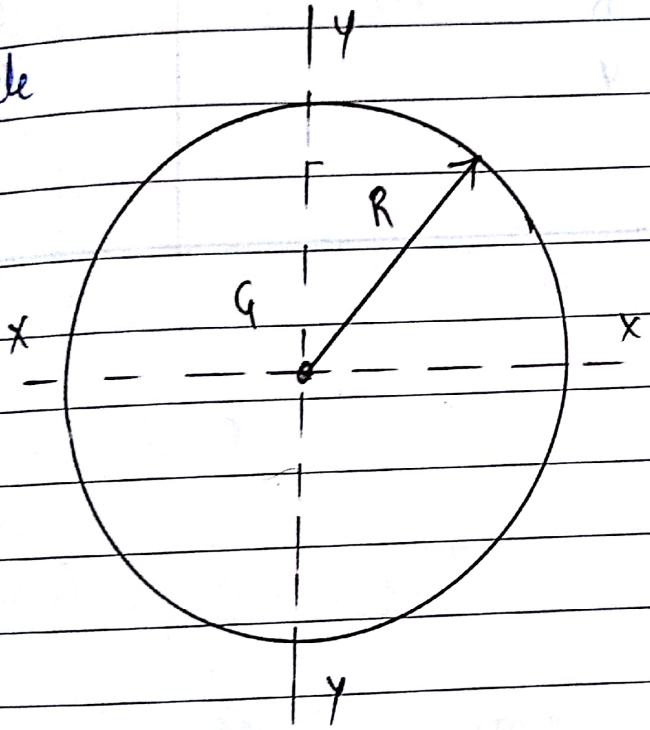
2. Triangle



$$I_{xx} = \frac{bh^3}{36}$$

$$I_{AB} = \frac{bh^3}{12}$$

3. Circle

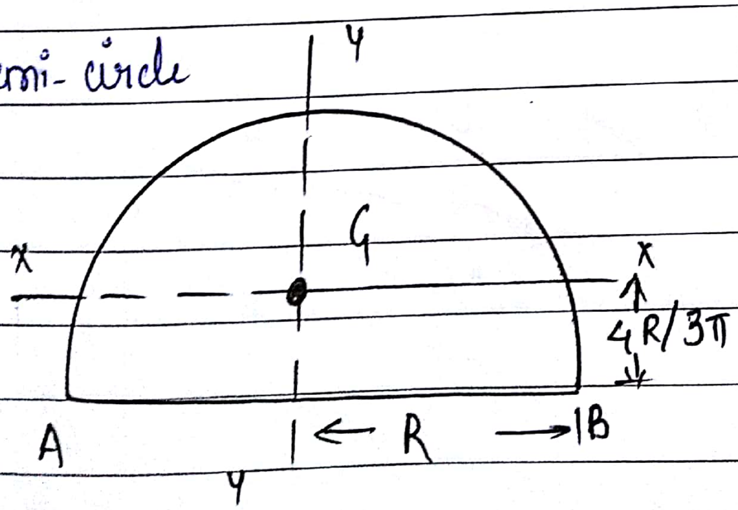


$$I_{xx} = I_{yy} = \frac{\pi R^4}{4}$$

$$= \frac{\pi D^4}{64}$$

$$I_{zz} = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

4. Semi-circle

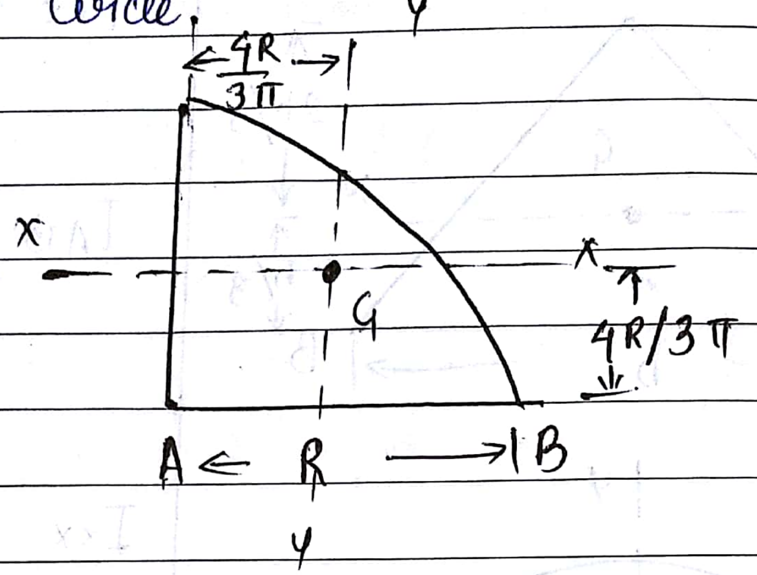


$$I_{xx} = 0.11R^4$$

$$I_{yy} = \frac{\pi R^4}{8} = \frac{\pi D^4}{128}$$

$$I_{AB} = \frac{\pi R^4}{8} = \frac{\pi D^4}{128}$$

5. Quarter Circle.



$I_{xx} = I_{yy}$
$= 0.055R^4$

$I_{AB} = \frac{\pi R^4}{16}$
$= \frac{\pi D^4}{256}$

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