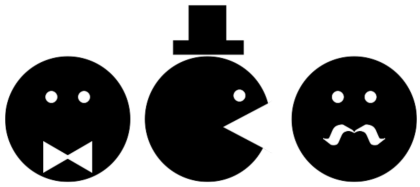
# Worksheet 11a: PGMs II*

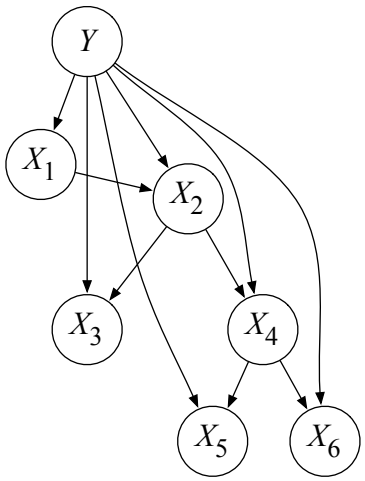
COMP90051 Statistical Machine Learning

Semester 2, 2020

**Exercise 1.** Mr. and Ms. Pacman have been searching for each other in the Pacman world (see http://ai.berkeley.edu/project_overview.html). Ms. Pacman has been pregnant with a baby, and this morning she has given birth to Pacbaby (congratulations, Pacmans!). To train Pacbaby to avoid encountering ghosts in the maze,[1] the Pacmans are trying to teach Pacbaby to distinguish Pacmen (pl.) from ghosts using discriminative visual features such as the presence of a bowtie, hat, mustache, etc.



Pacbaby has noticed that the features are not independent—nearly everyone who has a hat has a mustache, while those with bowties are always clean shaven. She decides to use a tree-augmented Naive Bayes model (TANB) to account for conditional dependencies. A TANB is an extension of a Naive Bayes model, where features are no longer assumed conditionally independent given the binary class $Y \in \{1, -1\}$ (Pacman or not-Pacman, respectively). Let $X_1, X_2, \ldots, X_6$ be the random variables corresponding to the features that Pacbaby observes. The TANB model arranges vertices in a tree-structured Bayes net with $Y$ at the root:




*Based on Berkeley CS188 section

[1] Ghosts are nice enough not to eat Pacbaby, but they will take all his money.

**(a)** Assume all features $X_1, \ldots, X_6$ are observed in the TANB model. What is the classification rule? Your answer should be in terms of the prior and conditional probabilities.

*Solution.* We need to find the value of the class $Y = y$ that maximises the posterior $p(y|X_1 = x_1, X_2 = x_2, \ldots, X_6 = x_6)$. Using Bayes' Theorem and the Bayes net factorisation of the joint distribution:

$$\begin{aligned} y^* &= \arg\max_y p(y|x_1, \ldots, x_6) \\ &= \arg\max_y p(y, x_1, \ldots, x_6) \\ &= \arg\max_y p(y)p(x_1|y)p(x_2|x_1, y)p(x_3|x_2, y)p(x_4|x_2, y)p(x_5|x_4, y)p(x_6|x_4, y) \end{aligned}$$

□

When we perform a marginalisation operation—i.e. removing a variable from a joint distribution, we perform a sum over the product of all factors that include that random variable. For example, marginalising over $X_4$ in the joint distribution above involves a factor containing four random variables.

$$\sum_{X_4} \underbrace{p(X_4|X_2, Y)p(X_5|X_4, Y)p(X_6|X_4, Y)}_{\phi(X_2, X_4, X_5, X_6)}$$

This induces a dependency between all the random variables in the factor except the variable being marginalised—all subsequent operations will have to treat $X_2, X_5, X_6$ together ($X_4$ is summed out). Assuming there is no special algebraic structure in the summand that can be exploited, the complexity is exponential in the number of different random variables in the summand. Thus the overall complexity of the variable elimination algorithm is dominated by the number of variables in the largest elimination factor, $\phi(\ldots)$. Determining the optimal (lowest-complexity) elimination ordering is intractable, but a useful heuristic is to find an ordering that minimises the size of the largest factor generated.

**(b)** Specify an elimination order that is efficient for the query $p(Y|X_5 = x_5)$ in the TANB model above. How many variables are in the biggest factor induced by variable elimination with your ordering? Which variables are they?

*Solution.* Here the query variable (the variable of interest) is $Y$ while the evidence (observed variables) is $X_5 = x_5$. We saw last week that to find the posterior:

- We use Bayes' Theorem to express the posterior as proportional to the joint over the query and evidence.
- We express the joint over the query and evidence as the full joint, marginalised over the remaining variables.
- We use the Bayes net structure to factorise the full joint as the product of simpler conditional probability tables.

$$\begin{aligned} p(Y|x_5) \propto p(Y, x_5) &= \sum_{X_1, X_2, X_3, X_4, X_6} p(Y, X_1, X_2, X_3, X_4, x_5, X_6) \\ &= \sum_{X_1, X_2, X_3, X_4, X_6} p(Y)p(X_1|Y)p(X_2|X_1, Y)p(X_3|X_2, Y)p(X_4|X_2, Y)p(x_5|X_4, Y)P(X_6|X_4, Y) \end{aligned}$$

Note that marginalising $X_3$ and $X_6$ in the factored distribution above sums to one. Hence we only have to consider the sums over $X_1, X_2, X_4$. Also note that $X_1$ and $X_4$ are connected to

one undetermined variable ($X_2$), but $X_2$ is connected to two undetermined variables ($X_1, X_4$). This suggests we should use the variable elimination algorithm to eliminate either of $X_1, X_4$ first before eliminating $X_2$, to minimise the number of variables in the largest factor, for example, walking up the tree:

$$\begin{aligned}
p(Y|x_5) &= \sum_{X_1,X_2,X_4} p(Y)p(X_1|Y)p(X_2|X_1,Y)p(X_4|X_2,Y)p(x_5|X_4,Y) \\
&= p(Y)\sum_{X_1} p(X_1|Y)\sum_{X_2} p(X_2|X_1,Y)\sum_{X_4} \underbrace{p(X_4|X_2,Y)p(x_5|X_4,Y)}_{\phi(X_2,X_4,Y)} \\
&= p(Y)\sum_{X_1} p(X_1|Y)\sum_{X_2} \underbrace{p(X_2|X_1,Y)m(X_2)}_{\phi(X_1,X_2,Y)} \\
&= p(Y)\sum_{X_1} p(X_1|Y)m(X_1) \\
&= p(Y)m_1
\end{aligned}$$

The largest factor size is 3. Convince yourself the reverse order $(X_1, X_2, X_4)$ is similar. What happens if we attempt to eliminate $X_2$ before $X_1$ or $X_4$?

$$p(Y|x_5) = \sum_{X_1,X_4} p(Y)p(X_1|Y)p(x_5|X_4,Y)\sum_{X_2} \underbrace{p(X_2|X_1,Y)p(X_4|X_2,Y)}_{\phi(X_1,X_2,X_4,Y)}$$

Notice we get a factor of size 4. Remember that the time complexity of inference is exponential in the size of the largest factor, so we shouldn't eliminate $X_2$ first. Any ordering that eliminates one of $X_1$ or $X_4$ before $X_2$ is acceptable, and the size of the largest factor should be 3. By the variable elimination algorithm, the last variable to be eliminated should be $Y$, which is needed to normalise the final result. □

**(c)** Specify an elimination order that is efficient for the query $p(X_3|X_5 = x_5)$ in the TANB model above. How many variables are in the biggest factor induced by variable elimination with your ordering? Which variables are they?

*Solution.* Repeating the arguments above with query $X_3$ and evidence $X_5 = x_5$:

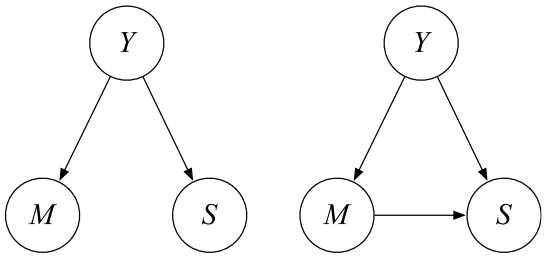
$$\begin{aligned}
p(X_3|x_5) \propto p(X_3, x_5) &= \sum_{Y,X_1,X_2,X_4,X_6} p(Y, X_1, X_2, X_3, X_4, x_5, X_6) \\
&= \sum_{Y,X_1,X_2,X_4} p(Y)p(X_1|Y)p(X_2|X_1,Y)p(X_3|X_2,Y)p(X_4|X_2,Y)p(x_5|X_4,Y)
\end{aligned}$$

Where marginalising over $X_6$ sums to 1 and does not effect inference. Notice that $Y$ and $X_2$ are connected to 3 vertices in the TANB, which means that eliminating either early on will result in a factor of size 4 (try it for yourself). Hence we should eliminate one of $X_1$ or $X_4$ before eliminating $X_2$ or $Y$—any ordering of the remaining three random variables will be fine. For example:

$$\begin{aligned}
p(X_3|x_5) &\propto \sum_Y p(Y)\sum_{X_1} p(X_1|Y)\sum_{X_2} p(X_2|X_1,Y)p(X_3|X_2,Y)\sum_{X_4} \underbrace{p(X_4|X_2,Y)p(x_5|X_4,Y)}_{\phi(X_2,X_4,Y)} \\
&= \sum_Y p(Y)\sum_{X_1} p(X_1|Y)\sum_{X_2} \underbrace{p(X_2|X_1,Y)p(X_3|X_2,Y)m(X_2,Y)}_{\phi(X_1,X_2,Y)} \\
&= \sum_Y p(Y)\sum_{X_1} \underbrace{p(X_1|Y)m(X_1,Y)}_{\phi(X_1,Y)} \\
&= \sum_Y p(Y)m(Y)
\end{aligned}$$

The largest factor $\phi$ generated is of size 3 for $(X_2, X_4, Y), (X_1, X_2, Y)$. □

**Exercise 2.** Consider the Bayes nets below over the nodes $Y$ (Pacbaby sees Pacman or not), $M$ (Pacbaby sees a moustache), and $S$ (Pacbaby sees sunglasses).




(NB) (TANB)

Empirically:

- Pacbaby observes $Y = 1$ or $Y = -1$ (Pacman or not) 50% of the time.
- Given $Y = 1$, Pacbaby observes $M = 1$ (moustache) 50% of the time and $S = 1$ (sunglasses) 50% of the time.
- When Pacbaby observes $Y = -1$, the frequency of observations are identical (equal probabilities of $M = 1, -1$, $S = 1, -1$).
- When Pacbaby observes $Y = 1$, anyone with a moustache wears sunglasses and anyone without a moustache does not wear sunglasses.
- If $Y = -1$ the presence/absence of a moustache has no influence on sunglasses.

**(a)** Based on the above information, fill in Pacbaby's conditional probability tables.

*Solution.*

For NB (left model)

| $y$ | $\mathbb{P}(Y = y)$ |
|---|---|
| 1 | .5 |
| −1 | .5 |

| | $\mathbb{P}(M = m \mid Y = y)$ | |
|---|---|---|
| | $y = 1$ | $y = -1$ |
| $m = 1$ | .5 | .5 |
| $m = -1$ | .5 | .5 |

| | $\mathbb{P}(S = s \mid Y = y)$ | |
|---|---|---|
| | $y = 1$ | $y = -1$ |
| $s = 1$ | .5 | .5 |
| $s = -1$ | .5 | .5 |

For TANB (right model)

| $y$ | $\mathbb{P}(Y = y)$ |
|---|---|
| 1 | .5 |
| −1 | .5 |

| | $\mathbb{P}(M = m \mid Y = y)$ | |
|---|---|---|
| | $y = 1$ | $y = -1$ |
| $m = 1$ | .5 | .5 |
| $m = -1$ | .5 | .5 |

| | $\mathbb{P}(S = s \mid Y = y, M = m)$ | | | |
|---|---|---|---|---|
| | $y = 1$ | | $y = -1$ | |
| | $m = 1$ | $m = -1$ | $m = 1$ | $m = -1$ |
| $s = 1$ | 1 | 0 | .5 | .5 |
| $s = -1$ | 0 | 1 | .5 | .5 |

□

**(b)** Pacbaby sees someone with a moustache and wearing a pair of sunglasses. What prediction does the NB model make? What probability does it assign to its prediction? What prediction does Pacbaby's TANB model make? What probability does it assign to its prediction?

*Solution.* For the NB model, Pacbaby assigns probability 1/2 to each label and is undecided.

$$\begin{aligned} y^*_{NB} &= \arg\max_y p(Y=y|M=1,S=1) \\ &= \arg\max_y p(Y=y)p(M=1|Y=y)p(S=1|Y=y) \\ &= \arg\max_y \begin{cases} (1/2)^3, & \text{for } Y=1 \\ (1/2)^3, & \text{for } Y=-1 \end{cases} \end{aligned}$$

For the TANB model, features are no longer assumed conditionally independent given observation of $y$:

$$\begin{aligned} y^*_{TANB} &= \arg\max_y p(Y=y|M=1,S=1) \\ &= \arg\max_y p(Y=y)p(M=1|Y=y)p(S=1|M=1,Y=y) \\ &= \arg\max_y \begin{cases} (1/2)^2, & \text{for } Y=1 \\ (1/2)^3, & \text{for } Y=-1 \end{cases} \end{aligned}$$

Normalising, Pacbaby assigns a probability of 2/3 that she saw a Pacman. □