Hand in Problem 1 – Applied Signal Processing

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1st Task

(a). In order to find the minimum sample rate which guarantees that the continuous signal $x_s(t)$ can be perfectly reconstructed from the sample signal assuming the noise is present, we could use the Nyquist rate. The calculation is shown below :

$$\omega_0 = 13\pi \cdot 10^3 \; [\frac{rad}{s}]$$

$$\omega_s = 2 \cdot \omega_0 = 2 \cdot 13\pi \cdot 10^3 = 26\pi \cdot 10^3 \; [\frac{rad}{s}]$$

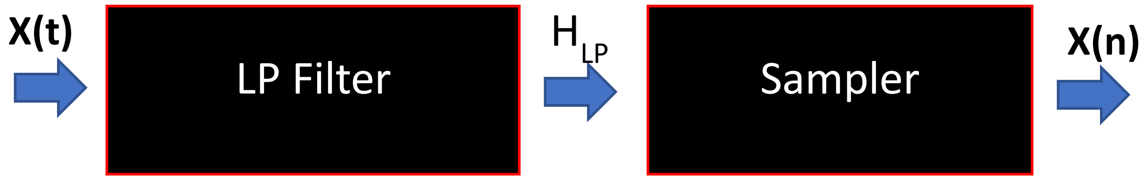

Figure 1. Scheme of the DSP

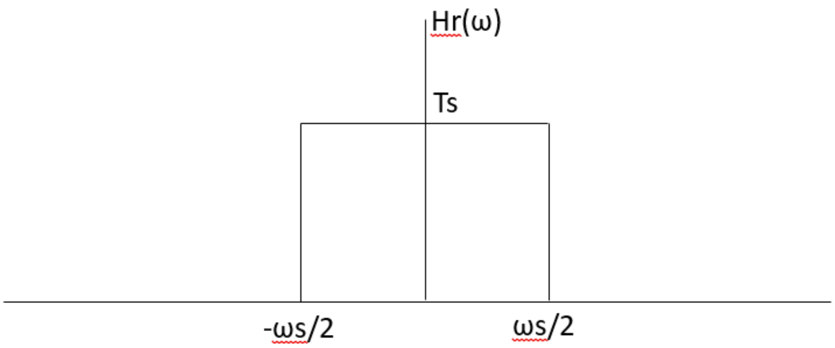

Figure 2. Frequency Response of Ideal Reconstruction Filter

By using the Filter as shown in Figure 2, we obtain the reconstructed signal frequency response from the original as shown in Figure 3 in the Continuous Time domain and Figure 4 in the Discrete time domain below.

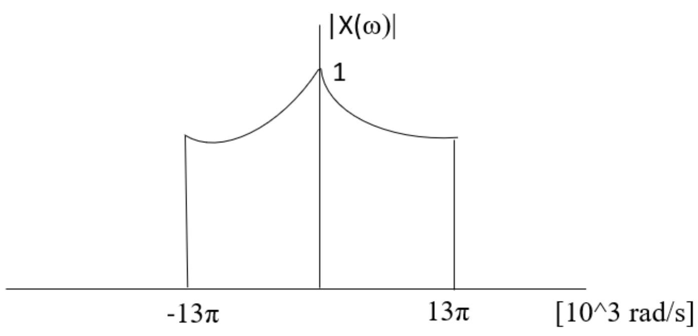

Figure 3. Continuous Time FT Domain Frequency Response

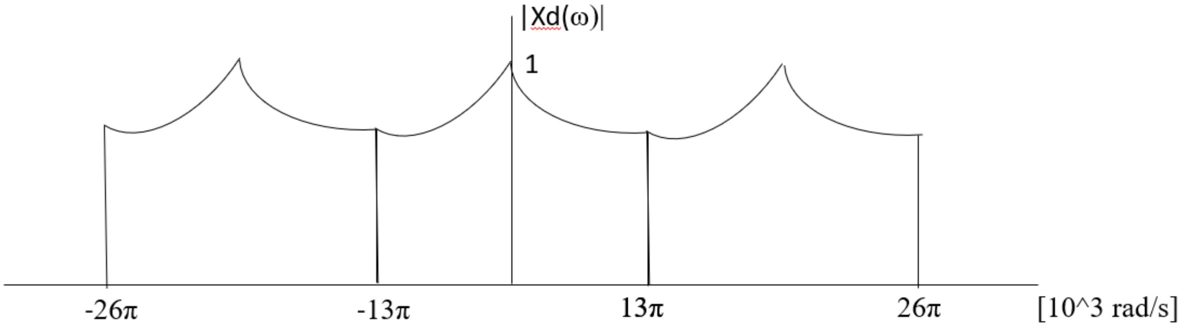

Figure 4. Discrete Time FT Domain Frequency Response

(b) In this task, we want to have the magnitude of sampled noise signal at $|\omega| < 13\pi \times 10^3$ to be 20 times lower than the magnitude for the sampled and filtered signal for $\omega = 0$.

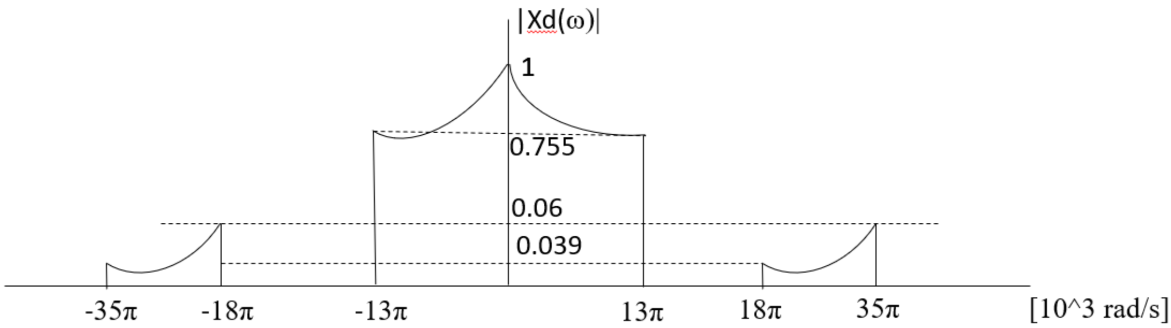

Figure 5. Continuous Time FT Domain of Filtered Signal + Noise

Figure 5 above shows the initial condition of the signal and noise.

In order to solve the problem, we could start by using the filter frequency function as shown below.

$$H(\omega) = \frac{1}{1 + j\omega'/\omega_0}$$

Calculating for $|H(\omega)| = |\frac{1}{2}|$ due to the fact that the maximum magnitude of the frequency response of the Noise is 10 times smaller before filtering and it becomes 20 times smaller after filtering.

$$\frac{1}{1 + j\omega'/\omega_0} = |\frac{1}{2}|$$

By flipping both Left Hand Side (LHS) and Right Hand Side (RHS) Equation we get,

$$1 + \frac{j\omega\prime}{\omega_0} = |2|$$

Quadratic operation of LHS to remove the absolute terms in RHS gives

$$1 + \frac{\omega'^2}{\omega_0^2} = 4 \rightarrow \omega' = \omega_0\sqrt{3} = 2.598\pi \cdot 10^4 \, [\frac{rad}{s}]$$

Since the frequency signal of interest that is of importance to be covered is $1.3\pi \cdot 10^4 [\frac{rad}{s}]$

The minimum sampling frequency that satisfy our requirements would be

$$\omega' + \omega_0 = 2.598\pi \cdot 10^4 + 1.3\pi \cdot 10^4 = 3.898\pi \cdot 10^4 \left[\frac{rad}{s}\right]$$

Task 2

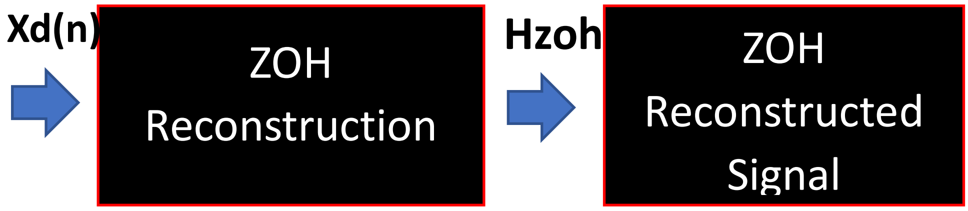

Figure 6. Reconstruction Process Scheme

Let the discrete time sinusoidal signal is as shown below

$$x_d(n) = \sin(2\pi n f_0 f_s) \qquad (1)$$

where $f_0 = 4\ kHz\ and\ f_s = sample\ rate = 15\ kHz$

Initially, before reconstructing the signal, we have to find Discrete Fourier Transform plot after sampling equation which is periodic as shown below

$$X_d(\omega) = X_d(\omega + k\omega_s) \qquad \text{for k = 1,2,3,...n} \qquad (2)$$

The reconstructed signal could be found by filtering the the discrete time sinusoidal signal with H_{ZOH}.

The magnitude of the ZOH filter could be found from the impulse response of the ZOH filter which is

shown as below

$$H_{ZOH}(\omega) = FT[h_{ZOH}(t)] = \Delta t e^{\frac{-j\omega\Delta t}{2}} \frac{\sin\left(\frac{\omega\Delta t}{2}\right)}{\frac{\omega\Delta t}{2}} = \Delta t e^{\frac{-j\omega\Delta t}{2}} \frac{\sin\left(\pi\frac{\omega}{\omega_s}\right)}{\pi\frac{\omega}{\omega_s}} \qquad (3)$$

Simplification of equation (3) will results in the magnitude of the frequency response equation assuming that the frequency function is zero for all $\omega = k\omega_s$, k = $\pm 1, \pm 2$, ...Then It is shown below

$$|H_{ZOH}| = \Delta t sinc\left(\frac{\omega}{\omega_0}\right) \qquad (4)$$

where Δt is the sampling period = 1/fs, Δt is chosen since $H_{ZOH}(0) = \Delta t$ and sinc function denotes as

$$sinc\left(\frac{\omega}{\omega_0}\right) = \frac{\sin\left(\pi\frac{\omega}{\omega_0}\right)}{\pi\frac{\omega}{\omega_0}} \qquad (5)$$

The reconstructed signal would be obtained from the equation below

$$X(\omega) = H_{ZOH}(\omega)X_d(\omega) \qquad (6)$$

Moreover, the magnitude of the reconstructed signal components could be calculated as below

$$|X(\omega)| = |H_{ZOH}| \cdot X_d(\omega) \qquad (7)$$

Equation (6) and (7) represents filtered discrete time sinusoidal signal.

The obtained Discrete Fourier Transform Plot would look like as shown in Figure 7 below due to symmetry.

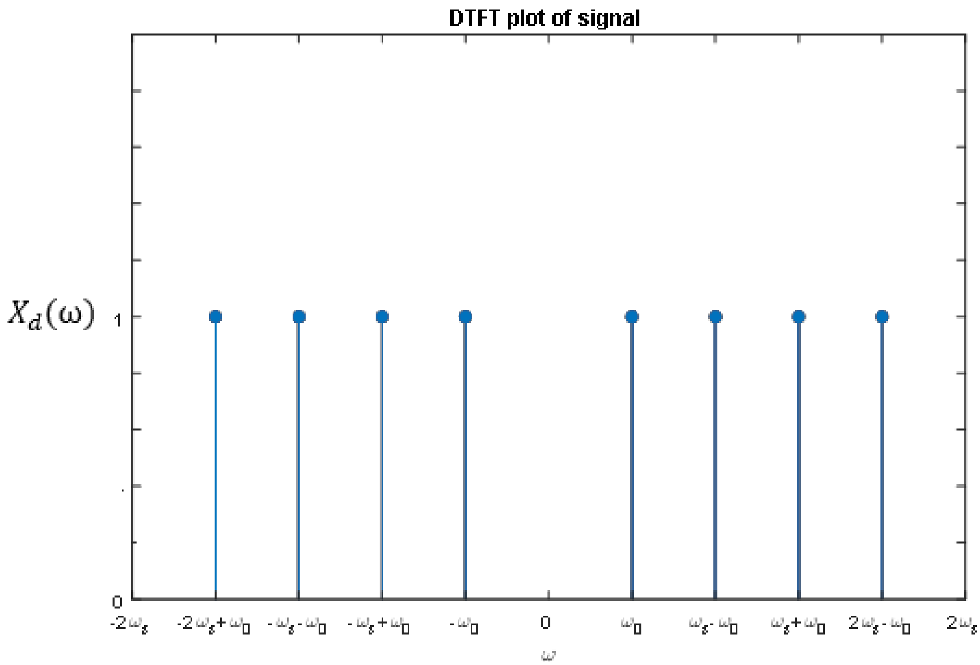

Figure 7. DTFT Plot of the discrete signal $x_d(n)$

<u>Sampling Frequency</u>

$$\omega_s = 15 \cdot 2\pi = 9.42 \cdot 10^4 \left[\frac{rad}{s}\right]$$

<u>Fundamental Components Frequency</u>

$$\omega_0 = 4 \cdot 2\pi = 2.51 \cdot 10^4 \left[\frac{rad}{s}\right]$$

$$-\omega_0 = 4 \cdot 2\pi = 2.51 \cdot 10^4 \left[\frac{rad}{s}\right]$$

<u>1st Harmonic Frequencies</u>

$$\omega_s - \omega_0 = (9.42 - 2.51) \cdot 10^4 = 6.93 \cdot 10^4 \left[\frac{rad}{s}\right]$$

$$-\omega_s + \omega_0 = (-9.42 + 2.51) \cdot 10^4 = -6.93 \cdot 10^4 \left[\frac{rad}{s}\right]$$

2nd Harmonic Frequencies

$$\omega_s + \omega_0 = (9.42 + 2.51) \cdot 10^4 = 11.93 \cdot 10^4 \left[\frac{rad}{s}\right]$$

$$-\omega_s - \omega_0 = (-9.42 - 2.51) \cdot 10^4 = -11.93 \cdot 10^4 \left[\frac{rad}{s}\right]$$

3rd Harmonic Frequencies

$$2\omega_s - \omega_0 = (18.84 - 2.51) \cdot 10^4 = 16.33 \cdot 10^4 \left[\frac{rad}{s}\right]$$

$$-2\omega_s + \omega_0 = (-18.84 + 2.51) \cdot 10^4 = -16.33 \cdot 10^4 \left[\frac{rad}{s}\right]$$

The calculated magnitude of the fundamental component based on the equation (6) is shown below

Table 1. Magnitude of Frequency Response Components

Components	Magnitude of Components
Fundamental	0.89
1st Harmonic	0.32
2nd Harmonic	0.19
3rd Harmonic	0.14

In addition the ZOH reconstruction plot $X(\omega)$ is shown below in Figure 8. The frequency component plots will look like a stairstep function from the Fundamental and 3rd Harmonic. As we go to the higher harmonics, we will have higher pitch (e.g. 1st harmonic has higher pitch than original component and the 2nd harmonics has higher pitch than 1st harmonics).

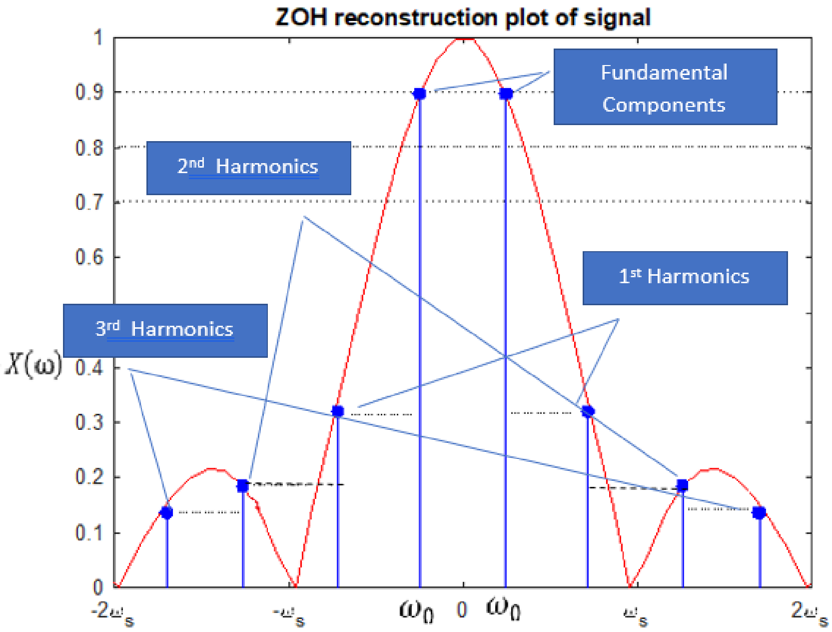

Figure 8. ZOH reconstruction plot of $X(\omega)$