LINEAR CONTROL SYSTEM DESIGN: ASSIGNMENT – 1

DYNAMIC MODEL OF DC MOTOR WITH FLYWHEEL

CHALMERS

Mechanical - Group number – 11

Manikanta Venkatesh	*manven@student.chalmers.se*
Chintalapudi Aditya Reddy	*creddy@student.chalmers.se*
Fikri Farhan Witjaksono	*fikrif@student.chalmers.se*

PROBLEM STATEMENT:

Consider the system shown in figure 1, consisting of an electric DC- motor that drives a flywheel and is influenced by an external torque as well.

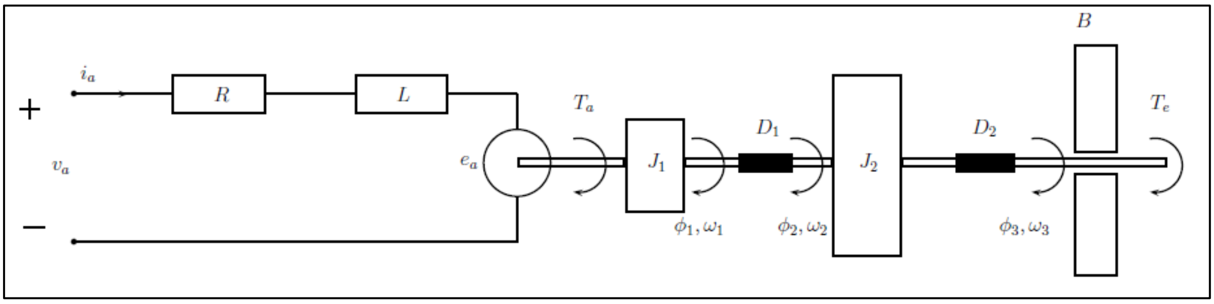

Figure 1: DC motor with flywheel

The system is characterised by different variables and parameters. Their symbols and meanings are given in Table 1.

Table 1: Symbols and meanings of different variables and parameters

Symbol	Meaning
v_a	External voltage applied to the rotor
i_a	Rotor current
e_a	Induced rotor voltage
T_a	Rotor produced torque
T_e	External torque applied to flywheel axis
ϕ_1, ϕ_2, ϕ_3	Angles
ω_1, ω_2, ω_3	Angular speeds
L	Rotor inductance
R	Rotor resistance
K_E	Coefficient related to induced voltage
K_T	Rotor torque constant
J_1	Rotor inertia
J_2	Flywheel inertia
D_1 and D_2	Flexibility of the axis (torsional springs) on each side of flywheel
B	Dynamic (linear) friction proportional to the angular speed
T	Total torque

TASKS:

Task a: Formulation of mathematical model of the system

Applying Kirchhoff's law to the electrical part of the system,

$$v_a = Ri_a + L\frac{di_a}{dt} + e_a \tag{1}$$

From the given relation $e_a = K_E\omega_1$, equation 1 can be written as –

$$v_a = Ri_a + L\frac{di_a}{dt} + K_E\omega_1 \tag{2}$$

Further, laws of mechanics such as angular form of Hooke's law can be applied for the mechanical part of the system which yields the following equations-

$$Torque, T = J\dot{\omega}$$

$$J_1\dot{\omega}_1 = T_a - D_1(\phi_1 - \phi_2) \tag{3}$$

Using the given relation $T_a = K_T i_a$, equation 4 can be written as –

$$J_1\dot{\omega}_1 = K_T i_a - D_1(\phi_1 - \phi_2) \tag{4}$$

Further, equations   can be formulated as

$$J_2\dot{\omega}_2 = D_1(\phi_1 - \phi_2) - D_2(\phi_2 - \phi_3) \tag{5}$$

$$B\omega_3 = D_2(\phi_2 - \phi_3) + T_e \tag{6}$$

Using Equation 2,3, 4,5 and 6, the state space model for the system can be formed as follows. All these linear differential equations are of the order 1.

$$\text{Also, } \omega = \frac{d\phi}{dt} = \dot{\phi} \tag{7}$$

The state vector x(t) can be given as –

$$x(t) = \begin{bmatrix} i_a \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{8}$$

The system inputs are the input voltage and the external torque. The input vector u(t) can be defined as –

$$u(t) = \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{9}$$

Considering each state in the same order and calculating the derivative of the state using equations 2 to 6.

i) $x_1 = i_a$

$$\dot{x}_1 = \frac{di_a}{dt} = \frac{v_a}{L} - \frac{Ri_a}{L} - K_E\omega_1$$

$$\dot{x}_1 = \frac{di_a}{dt} = \frac{v_a}{L} - \frac{Rx_1}{L} - K_E x_5 \tag{10}$$

ii) $x_2 = \phi_1$

$$\dot{x}_2 = \dot{\phi}_1 = \omega_1 = x_5 \tag{11}$$

iii) $x_3 = \phi_2$

$$\dot{x}_3 = \dot{\phi}_2 = \omega_2 = x_6 \tag{12}$$

iv) $x_4 = \phi_3$

$$\dot{x}_4 = \dot{\phi}_3 = \frac{D_2}{B}(\phi_2 - \phi_3) + \frac{T_e}{B}$$

$$\dot{x}_4 = \dot{\phi}_3 = \frac{D_2}{B}(x_3 - x_4) + \frac{T_e}{B} \tag{13}$$

v) $x_5 = \omega_1$

$$\dot{x}_5 = \dot{\omega}_1 = \frac{K_T i_a}{J_1} - \frac{D_1}{J_1}(\phi_1 - \phi_2)$$

$$\dot{x}_5 = \dot{\omega}_1 = \frac{K_T x_1}{J_1} - \frac{D_1}{J_1}(x_2 - x_3) \tag{14}$$

vi) $x_6 = \omega_2$

$$\dot{x}_6 = \dot{\omega}_2 = \frac{D_1}{J_2}(\phi_1 - \phi_2) - \frac{D_2}{J_2}(\phi_2 - \phi_3)$$

$$\dot{x}_6 = \dot{\omega}_2 = \frac{D_1}{J_2}(x_2 - x_3) - \frac{D_2}{J_2}(x_3 - x_4) \tag{15}$$

Equations 8 to 15 are used to form the state space model of the system in the following form-

$$\dot{x}(t) = Ax(t) + B\,u(t) \tag{16}$$

$$y(t) = Cx(t) + D\,u(t) \tag{17}$$

Task b: Formulation of state space equation and calculation of matrices A and B for given conditions.

As per the given condition, Inductance L is assumed to be zero. This implies that, from equation 1, the value of current through the rotor, i_a will not be considered as a state for the given condition.

The state vector will now become-

$$x(t) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{18}$$

The system inputs will remain the same as the previous task -

$$u(t) = \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{19}$$

Proceeding further to form the state space model similar to the previous task -

i) $x_1 = \phi_1$

$$\dot{x}_1 = \dot{\phi}_1 = \omega_1 = x_4 \tag{20}$$

ii) $x_2 = \phi_2$

$$\dot{x}_2 = \dot{\phi}_2 = \omega_2 = x_5 \tag{21}$$

iii) $x_3 = \phi_3$

$$\dot{x}_3 = \dot{\phi}_3 = \frac{D_2}{B}(\phi_2 - \phi_3) + \frac{T_e}{B}$$

$$\dot{x}_3 = \dot{\phi}_3 = \frac{D_2}{B}(x_2 - x_3) + \frac{T_e}{B} \tag{22}$$

iv) $x_4 = \omega_1$

The expression for current through rotor i_a can be written as –

$$e_a = K_E\omega_1 = i_a R$$

$$i_a = \frac{K_E\omega_1}{R}$$

$$\dot{x}_4 = \dot{\omega}_1 = \frac{K_T K_e}{R J_1}\omega_1 - \frac{D_1}{J_1}(\phi_1 - \phi_2)$$

$$\dot{x}_4 = \dot{\omega}_1 = \frac{K_T K_e}{R J_1}x_4 - \frac{D_1}{J_1}(x_1 - x_2) \tag{23}$$

v) $x_5 = \omega_2$

$$\dot{x}_5 = \dot{\omega}_2 = \frac{D_1}{J_2}(\phi_1 - \phi_2) - \frac{D_2}{J_2}(\phi_2 - \phi_3)$$
$$\dot{x}_5 = \dot{\omega}_2 = \frac{D_1}{J_2}(x_1 - x_2) - \frac{D_2}{J_2}(x_2 - x_3) \qquad (24)$$

State space model in the form of equations 16 is given below -

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & -\frac{K_T K_e}{RJ_1} & 0 \\ \frac{D_1}{J_2} & -\frac{(D_1 + D_2)}{J_1} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ \frac{K_T}{RJ_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ T_e \end{bmatrix}$$

The values of matrices A and B are given below-

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & \frac{K_T K_e}{RJ_1} & 0 \\ \frac{D_1}{J_2} & -\frac{(D_1+D_2)}{J_1} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ \frac{K_T}{RJ_1} & 0 \\ 0 & 0 \end{bmatrix}$$

Task c: Calculation of matrices C and D for given conditions

The output equation is of the form, $y(t) = Cx(t) + Du(t)$

<u>1st case:</u> $y(t) = \begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix}$

The output equation for the first case can be written as -

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ T_e \end{bmatrix}$$

Matrices C and D for the first case would be –

$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>2nd case:</u> $y(t) = \begin{bmatrix} i_a \\ \omega_3 \end{bmatrix}$

The output equation for the second case can be written as -

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{K_e}{R} & 0 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix} \begin{bmatrix} v_a \\ T_e \end{bmatrix}$$

Matrices C and D for the second case would be –

$C = \begin{bmatrix} 0 & 0 & 0 & -\frac{K_e}{R} & 0 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}$

Task d: Calculation of eigen values of matrix A

<u>Given values:</u>

R = 1 [Ω]; $K_E = 10^{-1} [\frac{Vs}{rad}]$; $K_T = 10^{-1} [\frac{Nm}{A}]$; $J_1 = 10^{-5} [kgm^2]$

$J_2 = 4 \cdot 10^{-5} [kgm^2]$; B = $2 \cdot 10^{-3}$ [Nms]; $D_1 = 20 [\frac{Nm}{rad}]$; $D_2 = 2 [\frac{Nm}{rad}]$

Substituting these values into the matrix A, following is obtained-

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 \cdot 10^3 & -1 \cdot 10^3 & 0 & 0 \\ -2 \cdot 10^6 & 2 \cdot 10^6 & 0 & -1 \cdot 10^3 & 0 \\ 5 \cdot 10^5 & -5.5 \cdot 10^5 & 5 \cdot 10^4 & 0 & 0 \end{bmatrix}$$

Eigen values for the above matrix A are calculated using MATLAB and the results are given in Table 2.

Table 2: Eigen values of matrix A divided by 10^3

λ_1	$-0.3914 + 1.4791i$
λ_2	$-0.3914 - 1.4791i$
λ_3	0
λ_4	-0.2708
λ_5	-0.9464

Observations: From table 2, it can be inferred that none of the eigen values has a positive real part. Also, one of the eigen values is zero. From this we cannot conclude that the system is stable. Thus, we need to analyse the matrices G(s) for both the cases which can be generated using the in-built state space to transfer function (ss2tf) function in MATLAB.

Further, finding out the poles of the determined transfer function (table 3), 4 sets of poles are obtained as it is a 2 by 2 matrix and the denominators of each entry of the matrix are not the same.

Table 3: Values of poles of the transfer function G(s) divided by 10^3

Poles	G (1,1)	G (1,2)	G (2,1)	G (2,2)
p_1	- 0.3914 + 1.4791i	- 0.3914 + 1.4791i	- 0.3914 + 1.4791i	- 0.3914 + 1.4791i
p_2	- 0.3914 - 1.4791i	- 0.3914 - 1.4791i	- 0.3914 - 1.4791i	- 0.3914 - 1.4791i
p_3	- 0.9464	- 0.9464	- 0.9464	- 0.9464
p_4	- 0.2708	- 0.2708	- 0.2708	- 0.2708
p_5	0	0	-	-

The first four poles are the same for all the entries of the transfer function matrix G1(s). The 5th pole for the first row of the matrix is at origin which implies that the pole polynomial is given by denominator of these two entries which is the least common denominator. Also, the presence of zero in the poles concludes that the system is marginally stable and fully stable.

Further, when the subcase (2) of c is analysed. Since all the denominators of every entry in the matrix G(s) is the same, the least common denominator is easy to find and the poles are as shown in table 4. This system can be concluded stable for this case since all the poles have negative real value and thus are in the left half-plane.

Table 4: Poles of G(s) divided by 10^3 for subcase (2)

p_1	- 0.3914 + 1.4791i
p_2	- 0.3914 - 1.4791i
p_3	- 0.9464
p_4	- 0.2708

Task e: Simulation of state vector components and angular velocities over time

The system was built in a Simulink model and simulated with the external voltage v_a and external torque T_e as step inputs. The voltage was set at a final value of 10V after 1 second of simulation while the torque was given a final value of (-0.1 Nm) after 4 seconds of simulation. The variation of speeds and position (angles) are as shown below in figure 2. The plot looks like there are only two outputs. However, this is because of the reason that all the three angle values overlap and the speeds overlap after the step.

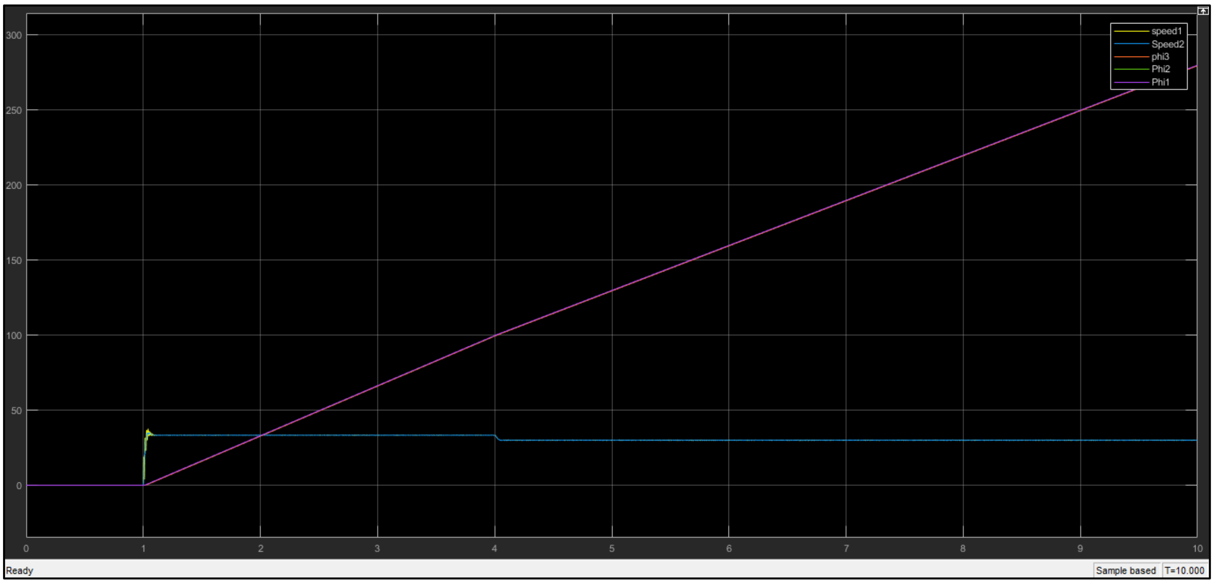

Figure 2: Development of angular speeds and angles with time.

However, when the same plot is zoomed in, it can be seen that when the voltage is applied the DC motor starts to spin and rotates the flywheel and the angles grow linearly. As it can be seen from the figure 3, that when the motor starts to spin, one of the starts rotating first and the movement is quickly spread out to the other axis. Further, the speeds are synchronized after a very little time. But, until the synchronization of speeds, oscillations can be observed in the speed and this can be attributed to one of the eigen values being zero. Due, to the presence of one of the zero in the eigen value, it is marginally stable and not completely stable.

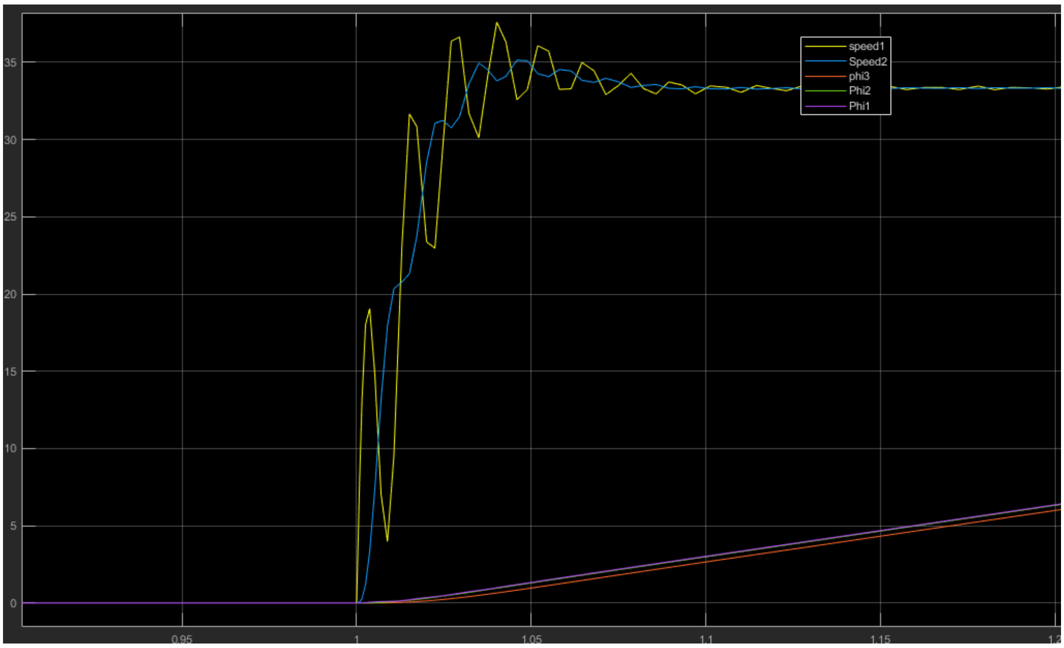

Figure 3: Angular speed transition after 1 second due to step voltage.

Further when the three speeds are analysed with respect to inputs (Voltage and external torque), one can see observe that in figure 4, speed 3 (ω_3) has a sudden drop due to the resting torque after 4 mins of simulation. This can be attributed to the directional relation of torque to angle 3 (ϕ_3). However, the other speeds also undergo a step drop due to the relation to ϕ_3 equation. Also, the speed 1 can be seen oscillating. This is because the eigen value which was 0, is associated with speed 1 through equation (ϕ_1).

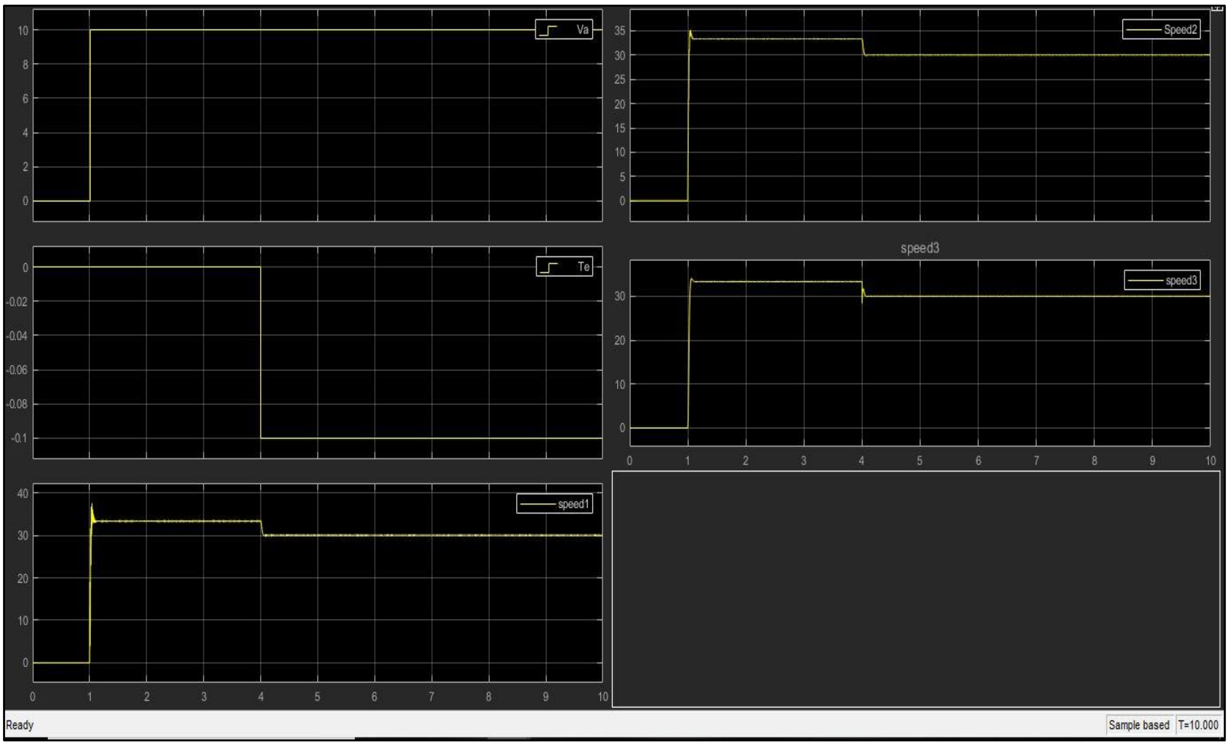

Figure 4: Variation of angular speeds in response to the input voltage and external torque.

Task f: Calculation of poles and transmission zeroes

Assuming that the external torque is zero, the transfer function is calculated for the second case in Task c. This condition will give the following updated matrices for B and D. Matrices A and C, however, will remain the same.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & \frac{K_T K_e}{RJ_1} & 0 \\ \frac{D_1}{J_2} & -\frac{(D_1+D_2)}{J_1} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_T}{RJ_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -\frac{K_e}{R} & 0 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{bmatrix}$$

Using MATLAB, these matrices were put into a state space model and then the corresponding transfer function is calculated. Further, poles and zeroes of the transfer function is calculated using MATLAB functions.

For further calculations, the second column of the matrix B and D were removed since it corresponds to zero input (No external torque). In other words, matrices B and D will become a column vector. The corresponding transfer function will be in the order of 2x1 matrix and the polynomials are given in figure 5. The values of poles and zeroes for the 1st term of the transfer function is given in Table 5.

```
s =

  From input to output...
       s^5 + 1000 s^4 + 2.55e06 s^3 + 2.5e09 s^2 + 1e11 s + 0.1773
   1:  -----------------------------------------------------------
       s^5 + 2000 s^4 + 3.55e06 s^3 + 3.05e09 s^2 + 6e11 s - 0.1069

                                 5e12 s
   2:  -----------------------------------------------------------
       s^5 + 2000 s^4 + 3.55e06 s^3 + 3.05e09 s^2 + 6e11 s - 0.1069

Continuous-time transfer function.
```

Figure 5: Values of the polynomials of the resulting transfer function.

Table 5: Values of poles and zeroes for the 1st term of the transfer function G(s) with no external torque, divided by 10^3

Poles		Zeroes	
p_1	- 0.3914 + 1.4791i	z_1	- 0.0015 + 1.5834i
p_2	- 0.3914 - 1.4791i	z_2	- 0.0015 - 1.5834i
p_3	- 0.9464	z_3	- 0.9553
p_4	- 0.2708	z_4	- 0.0418
p_5	0	z_5	0

The values of poles for the second term in the transfer function is same as the first term. And the values of zeroes are 0. In other words, same values of poles for both the polynomial terms and no zeroes for the second polynomial.

<u>Observations:</u>

In control theory, a SIMO (Single Input Multi- Output) system is said to be of minimum phase if all the poles and zeroes of its transfer function are in the left half plane of its s-plane representation. In case of question f, it can be seen that the real part of the poles and zeroes are in the left half plane, rendering the system to be of minimal phase.

REFERENCES

1. SSY285 - Linear Control System Design Assignment-1 handout
2. SSY285 – Linear Control System Design – Lecture slides
3. Control Theory - Multivariable and Nonlinear Methods by Torkel Glad and Lennart Ljung

APPENDIX – MATLAB CODE

```
clc

clear all
close all
```

Question d - Calculation of Eigen values of A

```
clc

clear all
close all

R = 1;
```

```
Ke = 0.1;

Kt = 0.1;

J1 = 1e-5;

J2 = 4e-5;

Bf = 2e-3;

D1 = 20;

D2 = 2;



A = [ 0 0 0 1 0 ;

     0 0 0 0  1;

     0 (D2/Bf) (-D2/Bf) 0 0 ;

     (-D1/J1) (D1/J1) 0 (-(Kt*Ke)/(R*J1)) 0;

     (D1/J2) ((-D1-D2)/J2) (D2/J2) 0 0];



Eigenvalue = eig(A)



%  Calculation of poles and zeroes for 1st case
B1= [0 0;
    0 0;
    0 (1/Bf);
   (Kt/(R*J1)) 0 ;
     0 0];

C1 = [0 1 0 0 0;
     0 0 0 0 1];

D1 = zeros(2);

G1 = ss(A,B1,C1,D1);
s1 = tf(G1);

for n=1:1:2
    for i=1:1:2
[zz1,pp1,kk1] = tf2zp(s1.numerator{n,i},s1.denominator{n,i})
```

```
    end
end

% Calculation of poles and zeroes for 2nd case
B2= [0 0;
    0 0;
    0 (1/Bf);
   (Kt/(R*J1)) 0 ;
     0 0];

C2 = [ 0 0 0 ((-Ke)/R) 0;
     0 (D2/Bf) (-D2/Bf) 0 0];

D2 = [(1/R) 0;
    0 (1/Bf)];

G2 = ss(A,B2,C2,D2);
s2 = tf(G2);

for n=1:1:2
    for i=1:1:2
[zz2,pp2,kk2] = tf2zp(s2.numerator{n,i},s2.denominator{n,i})
    end
end
```

Question f - Calculation of poles and transmission zeroes with no external torque

```
R = 1;
Ke = 0.1;
Kt = 0.1;
J1 = 1e-5;
J2 = 4e-5;
Bf = 2e-3;
D1 = 20;
D2 = 2;

A = [ 0 0 0 1 0 ;
     0 0 0 0  1;
     0 (D2/Bf) (-D2/Bf) 0 0 ;
     (-D1/J1) (D1/J1) 0 (-(Kt*Ke)/(R*J1)) 0;
     (D1/J2) ((-D1-D2)/J2) (D2/J2) 0 0];

B3= [0;
    0;
    0 ;
   (Kt/(R*J1)) ;
     0];

C3 = [ 0 0 0 ((-Ke)/R) 0;
     0 (D2/Bf) (-D2/Bf) 0 0];

D3 = [(1/R); 0];
```

```
G3 = ss(A,B3,C3,D3);
 s3 = tf(G3);

for n=1:1:2
    [zz3,pp3,kk3] = tf2zp(s3.numerator{n,1},s3.denominator{n,1})
end
```

Published with MATLAB® R2019b