SSY285 - Linear Control System Design

Analysis of linear state-space model of a DC-motor with Flywheel

Assignment - 2

Group 11 - Mechanical

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1 PROBLEM STATEMENT

The DC motor with flywheel which was the focus of assignment-1, is considered for this assignment as well. Refer figure-1.

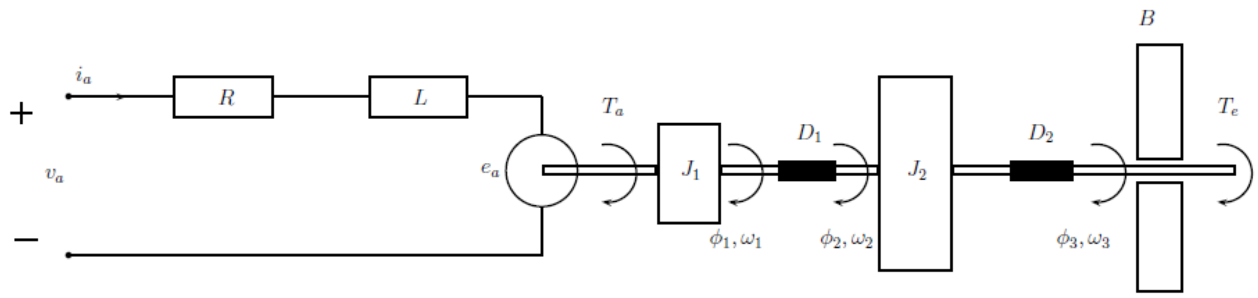

Figure 1: DC motor with flywheel

The starting point for this assignment is the linear state space models that were obtained in the previous assignment, from questions b and c. Matrices A and B are given as below:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & \frac{-D_2}{B} & 0 & 0 \\ \frac{-D_1}{J_1} & \frac{D_1}{J_1} & 0 & \frac{K_T K_e}{RJ_1} & 1 \\ \frac{D_1}{J_2} & \frac{D_1+D_2}{J_1} & \frac{D_2}{J_2} & 0 & 1 \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ \frac{K_T}{RJ_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

Matrices C and D for Case 1 and Case 2 are given as below-

Case 1:

$$y(t) = \begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix} \tag{3}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

Case 2:

$$y(t) = \begin{bmatrix} i_a \\ \omega_3 \end{bmatrix} \tag{6}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \frac{K_e}{R} & 0 \\ 0 & \frac{D_2}{B} & \frac{-D_2}{B} & 0 & 0 \end{bmatrix} \tag{7}$$

$$D = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix} \tag{8}$$

The values of the parameters that will be used for numerical calculations are given in table-1.

Table 1: Numerical values of parameters

Parameter	**Value**	Unit
R	1	Ω
K_e	10^{-1}	Vs/rad
K_t	10^{-1}	Nm/A
J_1	10^{-5}	kgm^2
J_2	$4 * 10^{-5}$	kgm^2
B	$2 * 10^{-3}$	Nms
D_1	20	Nm/rad
D_2	2	Nm/rad

2 QUESTIONS

2.1 Question a: Controllability and observability of the system

A system is said to be controllable if all its states can reach any final state from any initial state within a finite time. The system is said to be controllable if its controllability matrix is full rank. The controllability matrix can be obtained as shown in equation 9.

$$W_r = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} \tag{9}$$

Further, using the rank function in MATLAB the rank of the matrix is determined and if the rank is equal to the number of states in the system, then the system is said to be controllable. The oberservability of the system allows to determine the state of the system by measuring the inputs and outputs, provided the system is observable. To evaluate this property of the system, the observability matrix given by equation-10, is checked for full rank.

$$W_o = \begin{bmatrix} C \\ AC \\ A^2C \\ A^3C \\ A^4B \end{bmatrix} \tag{10}$$

Since there are two subsystems, two observable matrices are obtained.The system was determined as observable for the first subsystem and not observable for the second subsystem.The same matlab function ”rank” was used to check the rank of the observability matrix.

2.2 Question b: Stability and Detectability of the system

Since the system was determined to be controllable in question a, this directly implies that the system is stabilizable. Further, to determine if the system is detectable the condition given by equation 11 should be satisfied.

$$n = rank \begin{bmatrix} \lambda_i - A \\ C \end{bmatrix} \tag{11}$$

To calculate this matrix, the eigen values of matrix A with non negative real part are used to check if the matrix has a full rank for both the subsystems as discussed in question a. The eigen values for the matrix A are given in Table-2.

The only eigen value with non negative real part is 0. With this result, one can conclude that the Case 1 is detectable and Case 2 is not detectable.

Table 2: Eigenvalues of A, divided by 10^3

λ_1	-0.3914 + 1.4791i
λ_2	-0.3914 - 1.4791i
λ_3	0
λ_4	-0.2708
λ_5	-0.9464

2.3 Question c: Calculations using MATLAB code and condition number check

Considering the numerical values from Table 1, the controllability and observability matrices for both cases are calculated using MATLAB functions 'ctrb' and 'obsv' respectively. The controllability and observability of the system in the subcases could be decided through investigating whether the controllability (K) and observability ($\mathcal{O}$) matrix is in fact full rank. For a full rank matrix K and $\mathcal{O}$, once conclude that the system is controllable and observable, respectively. On the other hand, if either rank of the matrix K or $\mathcal{O}$ is not full rank, the system is not controllable or not observable respectively.

The observability and controllability matrices for Case 1 are given by equations 11 and 12.

$$\mathcal{O}(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 500*10^3 & -550*10^3 & 50*10^3 & 0 & 0 \\ 500*10^3 & -550*10^3 & 50*10^3 & 0 & 1 \\ 0 & 50*10^6 & -50*10^6 & 500*10^3 & -550*10^3 \\ 0 & 50*10^6 & -50*10^6 & 500*10^3 & -550*10^3 \\ -1.275*10^{12} & 1.2525*10^{12} & 22.5*10^9 & -500*10^6 & 50*10^6 \\ -1.275*10^{12} & 1.2525*10^{12} & 22.5*10^9 & -500*10^6 & 50*10^6 \\ 1.025*10^{15} & -1.005*10^{15} & -20*10^{12} & -775*10^9 & 1.2525*10^{12} \end{bmatrix} \tag{12}$$

$$K(1) = \begin{bmatrix} 0 & 0 & 10^4 & 0 & 10^{-7} & 0 & -1*10^{10} & 0 & 3*10^{13} & 5*10^{13} \\ 0 & 0 & 0 & 0 & 0 & 2.5*10^7 & 5*10^9 & -2.5*10^{10} & -5*10^{12} & 1.125*10^{13} \\ 0 & 500 & 0 & -5*10^5 & 0 & 5*10^8 & 0 & -4.75*10^{11} & 5*10^{12} & 4.5*10^{14} \\ 10^4 & 0 & 10^{-7} & 0 & -1*10^{10} & 0 & 3*10^{13} & 5*10^{13} & -16 & -1*10^{17} \\ 0 & 0 & 0 & 2.5*10^7 & 5*10^8 & -2.5*10^{10} & -5*10^{12} & 1.125*10^{13} & -7.75*10^{15} & -1*10^{16} \end{bmatrix} \tag{13}$$

The rank of both these matrices are 4 which is less than the full rank = 5. Hence, one can conclude that the system for case 1 is both uncontrollable and unobservable.

The observability and controllability matrices for Case 2 are given by equations 13 and 14.

$$\mathcal{O}(2) = \begin{bmatrix} 0 & 0 & 0 & -0.1 & 0 \\ 0 & 1000 & -0.001 & 0 & 0 \\ 2*10^5 & -2*10^5 & 0 & 100 & 0 \\ 0 & -1*10^6 & 10^6 & 0 & 10^3 \\ -2*10^8 & 2*10^8 & 0 & 1*10^5 & -2*10^5 \\ 5*10^8 & 4.5*10^8 & -9.5*10^8 & 0 & -10^6 \\ -3*10^{11} & -3.1*10^{11} & -10^{10} & -3*10^8 & 2*10^8 \\ -5*10^{11} & -4*10^{11} & 9*10^{11} & 5*10^8 & 4.5*10^8 \\ 7*10^{14} & -7.2*10^{14} & 2*10^{13} & 1.831*10^{-4} & 3.1*10^1 1 \\ -7.75*10^{14} & 1.653*10^{15} & -8.775*10^{14} & -10^{12} & -4*10^{11} \end{bmatrix} \tag{14}$$

$$K(2) = \begin{bmatrix} 0 & 0 & 10^4 & 0 & 10^{-7} & 0 & -1*10^{10} & 0 & 3*10^{13} & 5*10^{13} \\ 0 & 0 & 0 & 0 & 0 & 2.5*10^7 & 5*10^9 & -2.5*10^{10} & -5*10^{12} & 1.125*10^{13} \\ 0 & 500 & 0 & -5*10^5 & 0 & 5*10^8 & 0 & -4.75*10^{11} & 5*10^{12} & 4.5*10^{14} \\ 10^4 & 0 & 10^{-7} & 0 & -1*10^{10} & 0 & 3*10^{13} & 5*10^{13} & -16 & -1*10^{17} \\ 0 & 0 & 0 & 2.5*10^7 & 5*10^8 & -2.5*10^{10} & -5*10^{12} & 1.125*10^{13} & -7.75*10^{15} & -1*10^{16} \end{bmatrix} \tag{15}$$

In addition, based on the calculation of the rank of the matrix $\mathcal{O}$ and K, it was discovered that for case 2, the system is both uncontrollable and unobservable due to rank of both matrices being 4 (not full rank).

The condition number is the measure of sensitivity of the matrix U for each Controllability and Observability matrix is shown in equation (16) and (17) respectively,

$$U_K = K^{-1}x \tag{16}$$

$$U_{\mathcal{O}} = \mathcal{O}^{-1}x \tag{17}$$

which is denoted by κ_K (Ratio between the maximum over minimum single element of eigenvalue of Controllability matrix KK^T) and $\kappa_{\mathcal{O}}$ (Ratio between the maximum over minimum single element of eigenvalue of Observability matrix $\mathcal{O}\mathcal{O}^T$)

$$\kappa_K = \frac{\sigma_{max}(K)}{\sigma_{min}(K)} = \frac{\lambda_{max}(KK^T)}{\lambda_{min}(KK^T)} \tag{18}$$

$$\kappa_{\mathcal{O}} = \frac{\sigma_{max}(\mathcal{O})}{\sigma_{min}(\mathcal{O})} = \frac{\lambda_{max}(\mathcal{O}\mathcal{O}^T)}{\lambda_{min}(\mathcal{O}\mathcal{O}^T)} \tag{19}$$

The results from MATLAB calculations indicate very large value of condition number (κ) for both controllabillity and observability in both cases 1 and 2

$$\kappa_{K1} = 1.02*10^{12} \tag{20}$$

$$\kappa_{K2} = 1.02*10^{12} \tag{21}$$

$$\kappa_{\mathcal{O}(1)} = 3.12*10^{15} \tag{22}$$

$$\kappa_{\mathcal{O}(2)} = 1.2*10^{16} \tag{23}$$

If the controllability and observability of the system in question (c) is compared to its counterpart question (b), it can be observed that the rank is different (4). This might lead to the conclusion that the system is not controllable or observable. However, this contradicts to the results that were obtained in question a and b. To explain this ambiguity one can turn to condition number which is calculated using MATLAB function "cond". This number represents the relationship between the highest and the lowest number of the matrix. The condition number for all the three matrices are very high (in the order of 10^{15}) which implies that the matrix is ill-conditioned and it is easier to obtain numerical errors. Due to this reason MATLAB does not compute the matrices and the ranks given are incorrect.

2.4 Question d: Calculation of discrete time system matrix

The discrete time system matrix A_d was calculated by using the Zero Order Hold (ZOH) principle from equation (24).

$$A_d = e^{A_c T_s} \tag{24}$$

The result of discrete time matrix (A_d) is given by equation (25).

$$A_d = \begin{bmatrix} 0.401 & 0.5964 & 0.002618 & 0.0004499 & 0.0002341 \\ 0.2076 & 0.7749 & 0.01753 & 5.851 * 10^{-5} & 0.0009191 \\ 0.05851 & 0.5686 & 0.3729 & 1.309 * 10^{-5} & 0.0003505 \\ -782.8 & 773.7 & 9.085 & -0.04892 & 0.5964 \\ 342.5 & -371 & 28.43 & 0.1491 & 0.7749 \end{bmatrix} \tag{25}$$

2.5 Question e: Calculation of discrete time input matrix with ZOH principle

Similar to the previous question, the discrete time matrix (B_d) is determined using the ZOH principle as given by equation (26).

$$B_d = \int_0^{T_s} e^{A_c t} B_c \mathrm{d}t \tag{26}$$

The result from both methods i.e., discrete time matrix (B_d) is given below.

$$B_d = \begin{bmatrix} 0.003154 & 0.0002857 \\ 0.0001595 & 0.003211 \\ 0.125 & 0.3168 \\ 4.499 & 1.309 \\ 0.5851 & 8.763 \end{bmatrix} \tag{27}$$

2.6 Question f: Check for the minimal order of the discrete time state-space model

For a system to be minimal order, it has to be both observable and controllable. Thus, as a test for minimal order, the controllability and the observability is determined for both subsystems but in this case, with the new discrete time matrices (A_d) and (B_d).

It was found that both the subsystems are controllable but however the subsystem 2 was not found to be observable. Thus, it can be concluded that the sub system 1 is of minimal order and the sub system 2 is not of minimal order.

Further, for a discrete system, if the eigen values are to be stable, the absolute value needs to be lower than 1. The eigenvalues of A_d is given below-

$$\lambda_{Ad} = \begin{bmatrix} 0.0619 + 0.6733i \\ 0.0619 - 0.6733i \\ 1 \\ 0.7627 \\ 0.3881 \end{bmatrix} \tag{28}$$

It can be observed from the above table that all the eigen values are within the stability region expect one of the eigen value which equals to 1. This implies that both the sub systems are marginally stable.

3 REFERENCES

[1] SSY285 - Linear Control System Design Assignment-2 handout
[2] SSY285 – Linear Control System Design – Lecture slides
[3] Control Theory - Multivariable and Nonlinear Methods by Torkel Glad and Lennart Ljung