SSY281 - Multi Predictive Control

MPC practice and Kalman Filter

Assignment - 3

ID-Number 43

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1 PROBLEM STATEMENT (FIRST PART)

Consider the following statement. The following value have to be used in the question a,b,c of the assignment.

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \\ 0.25 & 0 \\ 0 & 0.6 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \tag{1}$$

2 QUESTIONS

2.1 Question a: Calculation of the state and input set-points $(\mathbf{x}_s, u_s)$

The set-point tracking problem could be solved throught solving the equation shown below

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}, y_{sp} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.5 \\ 1.25 \\ -2.25 \\ 2.5 \\ -1.5 \end{bmatrix}$$

The corresponding state and input set-points is shown as the equation below

$$x_s = \begin{bmatrix} 2.5 \\ -1.5 \\ 1.25 \\ -2.25 \end{bmatrix}, u_s = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix} \tag{3}$$

Moreover, it is shown by inputting x_s and u_s found above in equation 3 to the system, it will result in output as below. Hence, *Regulation to y_s is achievable.*

$$y = y_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{4}$$

2.2 Question b: Same problem as above except that only the first control input is available for control

Assume that we have different B due to only first control input is available. The B is shown as below

$$B = \begin{bmatrix} 0.5 \\ 0 \\ 0.25 \\ 0 \end{bmatrix} \tag{5}$$

In this problem, we need to find the steady state that minimizes two-norm of the output error. Since we have more outputs than inputs due to the fact that the second control input is unavailable, We need to solve the optimization problem as shown below

$$\begin{aligned} \min_{x_s,u_s} \quad & \|Cx_s - y_{sp}\|_Q^2 \\ \text{s.t} \quad & Eu_s \leq e \\ & FCx_s \leq f \end{aligned} \tag{6}$$

This could be done through the use of the 'quadprog' function in MATLAB that will calculate the solution by defining the optimization problem as below matching the optimization problem as in equation 5.

$$\begin{aligned} \min_{x} \quad & \frac{1}{2}x^T Hx + f^T x \\ \text{s.t} \quad & Ax \leq b \\ & A_{eq}x = b_{eq} \\ & l_b \leq x \leq u_b \end{aligned} \tag{7}$$

Moreover, in the simulation the 'prob2struct' MATLAB function was also used to convert optimization problem to solver form. In order to do that, first, define the optimization variables as shown below. Optimization variable is a symbolic variable that could be used to describe the problem's objective and constraints

$$x = \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{8}$$

Then define a as the definition of the minimization problem as symbolic variable as below

$$a = Cx_s - y_{sp} = \begin{bmatrix} x_1 + x_2 - 1 \\ x_3 + x_4 + 1 \end{bmatrix} \tag{9}$$

as well as the objective function in the problem object as an expression in the named variables.

$$objective = a^T Ha \tag{10}$$

Next step would be to define constraints in the problem object as expressions in the named variables. This step could be shown as below.

$$0.5 \cdot x_1 - 0.5 \cdot x_5 = 0 \tag{11}$$

$$0.4 \cdot x_2 = 0 \tag{12}$$

$$0.5 \cdot x_3 - 0.25 \cdot x_5 = 0 \tag{13}$$

$$0.4 \cdot x_4 = 0 \tag{14}$$

$$x_1 + x_2 \geq 1 \tag{15}$$

$$x_3 + x_4 \leq -1 \tag{16}$$

The last step would be to solve the optimization problem using the 'quadprog' function and the result is as shown below

$$x = \begin{bmatrix} 0.9954 \\ 0 \\ -0.8730 \\ 0 \\ 0.5408 \end{bmatrix} \tag{17}$$

And since the set point that shows the state set-points (x_s) and input set-points (u_s) is corresponds to the first until fourth as well as fifth element of matrix x in equation (17) respectively,

$$x_s = \begin{bmatrix} 0.9954 \\ 0 \\ -0.8730 \\ 0 \end{bmatrix}, u_s = 0.5408 \tag{18}$$

From the result below, we could conclude that *the system could not be regulated to the set-point of y_s since it results in different output*

$$y_s \neq y = \begin{bmatrix} 0.7681 \\ -0.3013 \end{bmatrix} \tag{19}$$

2.3 Question c: Same problem as question (a) except that the C defined as below

Assume that we have different C due to only first row of control output is available. Moreover, in this problem we have to find the steady state that minimizes two-norm of the inpt while the output error is zero. The C matrix is shown as below

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \tag{20}$$

The method to solve this problem is similar to the one that we used in question (b). However, since the number of inputs are more than controlled outputs ($p_z < m$), the optimization problem to find the feasible steady-state targets becomes as shown below

$$\begin{aligned} \min_{x_s, u_s} \quad & \|u_s - u_{sp}\|^2_{R_s} + \left\| C_y x_s - y_{sp}{}^2_{Q_s} \right\| \\ \text{s.t} \quad & E u_s \leq e \\ & F C_z x_s \leq f \end{aligned} \tag{21}$$

$$x = \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{22}$$

Then define A and B as the first and second term of the minimization problem expressed as symbolic variables

$$A = u_s - u_{sp} = \begin{bmatrix} x_5 - 1 \\ x_6 - 1 \end{bmatrix} \tag{23}$$

$$B = C_y x_s - y_{sp} = x_1 + x_2 - 1 \tag{24}$$

as well as the objective function in the problem object as an expression in the named variables. Q is defined as the weighting on A whereas R is defined as the weighting on B.

$$objective = A^T Q A + B^T R B \tag{25}$$

Next step would be to define constraints in the problem object as expressions in the named variables. This step result could be shown as below.

$$0.5 \cdot x_1 - 0.5 \cdot x_5 = 0 \tag{26}$$

$$0.4 \cdot x_2 - 0.4 \cdot x_6 = 0 \tag{27}$$

$$0.5 \cdot x_3 - 0.25 \cdot x_5 = 0 \tag{28}$$

$$0.4 \cdot x_4 - 0.6 \cdot x_6 = 0 \tag{29}$$

$$x_1 = 0 \tag{30}$$

$$x_1 + x_2 \geq 1 \tag{31}$$

The last step would be to solve the optimization problem using the 'quadprog' function and the result is as shown below

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1.5 \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

And since the set point that shows the state set-points (x_s) and input set-points (u_s) is corresponds to the first until fourth as well as fifth until sixth of matrix x in equation (17) respectively,

$$x_s = \begin{bmatrix} 0 \\ 1.0000 \\ 0 \\ 1.5000 \end{bmatrix}, u_s = \begin{bmatrix} 0 \\ 1.0000 \end{bmatrix} \tag{33}$$

Moreover, it is shown by inputting x_s and u_s found above in equation (33) to the system, it will result in output as below. Hence, *Regulation to y_s is achievable.*

$$y = y_s = 1 \tag{34}$$

3 PROBLEM STATEMENT (SECOND PART)

Consider the following statement. The following value have to be used in the question d,e,f,g of the assignment.

$$A = \begin{bmatrix} 0.2681 & -0.00338 & -0.00728 \\ 9.703 & 0.3279 & -25.44 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -0.00537 & 0.1655 \\ 1.297 & 97.91 \\ 0 & -6.637 \end{bmatrix}, \tag{35}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_p = \begin{bmatrix} -0.1175 \\ 69.74 \\ 6.637 \end{bmatrix}, \tag{36}$$

Consider the three following disturbance models cases :

1. $n_d = 2$, $B_d = 0_{3x2}$,

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

,

2. $n_d = 3$, $B_d = 0_{3x3}$,

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

,

3. $n_d = 3$, $B_d = [0_{3x3} B_p]$,

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

,

3.1 Question d: Construct an augmented model with the given matrices above and Determine the detectability of each of the cases

In order to check the detectability, it is mandatory to find the non-negative real eigenvalue of matrix A of the augmented system first ($\mathbb{R}(\lambda) \geq 0$). The result is as shown below

$$\lambda_i = \begin{bmatrix} 0.2980 \\ 0.2980 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix} \tag{37}$$

Hence we have to do the detectability test using the Hautus Lemma. If it satisfies the Hautus lemma, the rank of the test matrix should be equal to dimension of the row and column of matrix A and B respectively (n=3).

$$n_{detect} = rank(\begin{bmatrix} \lambda_i I - A_e & C_e \end{bmatrix}) = 3 \tag{38}$$

Hence, from the test above, The result below is obtained:

1. Case 1: Detectable
2. Case 2: Not Detectable
3. Case 3: Detectable

3.2 Question e: Design a Kalman Filter for each detectable augmented model to estimate its states

First, to design the Kalman Filter, we need to express the discrete time detectable augmented system (Case 1 and 3) into the form as below

$$x[n+1] = Ax[n] + Bu[n] + Gw[n] \tag{39}$$

$$y[n] = Cx[n] + Du[n] + Hw[n] + v[n]$$

And then, we need to determine the **G,D,H,w,v** matrix in order to the standard input to the **'kalman'** function in MATLAB. Then, we need to define the covariance data as

$$E(w[n]w[n]^T) = Q_w, \tag{40}$$

$$E(v[n]v[n]^T) = Q_v,$$

$$E(w[n]v[n]^T) = N_n,$$

The covariance matrices are defined as below. The scaling is done by changing the scale of Q_w and Q_v by increasing/decreasing with a factor of 10 depending how much we trust the measurement vs the system model that we have. *If we increase the factor for Q_w, that means we trust more the measurement than the model and vice versa with Q_v.*

$$Q_w = scale_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

$$Q_v = scale_2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Moreover, the matrices G, D and H are defined as below

$$G_{case1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, G_{case3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, H_{case1} = H_{case3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

$$D_{case1} = D_{case3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Kalman filter estimator has the following equation

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L(k)[y(k) - C\hat{x}(k|k-1)] \tag{43}$$

and the Kalman Gain has the gain equation as below

$$L(k) = AP(k)C^T[CP(k)C^T + R]^{-1} \tag{44}$$

And regarding the equation above, the resulting matrix could be divided into two parts, the gain for state estimation (L_x) and the gain for disturbance estimation (L_d) according to equation below

$$\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}^+ = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}) \tag{45}$$

3.3 Question f: Find matrix M_{ss} for each detectable augmented system

In this problem, we are only going to control the first and 3rd input by determining the matrix H and the new output as $C_s = HC$ where H is

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

The new augmented system from the previously checked detectable system could be constructed as below due to the fact that the input is becoming more than the controlled outputs ($p_z < m$),

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d} \\ z_{sp} - HC_d\hat{d} \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ 162.2662 & 0.6718 & 13.9573 & -112.2438 & -18.2728 \\ 0 & 0 & 0 & 0 & 1.0000 \\ 84.0857 & -0.4229 & -4.1415 & -65.6454 & 10.1456 \\ 0 & 0 & 0.1507 & 0 & 0 \end{bmatrix}$$

Hence M_{ss} could be defined as below according to different B_d and C_d for each corresponding detectable augmented cases (e.g. cases 1 and 3).

$$M_{ss} = \begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_d \\ z_{sp} - HC_d \end{bmatrix} \tag{48}$$

3.4 Question g: Determine whether the controller can remove offset using RHC

First, we calculate for the steady state target for both cases 1 and 3 respectively by using the obtained value of M_{ss} that was calculated using equation (48) and input them to equation (46) to include the distrubance.

Next, we use the modified Batch solution approach by including the control horizon length M into the equation as shown below.

$$\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \tag{49}$$

$$x = \Omega x(0) + \Gamma u \tag{50}$$

From equation (15) and (16), we could find that the Omega and Gamma could be obtained by he matrix below

$$\Omega = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \tag{51}$$

$$\Gamma = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} \tag{52}$$

4 REFERENCES

[1] SSY281 - Multi Predictive Control Assignment-3 handout
[2] SSY281 - Multi Predictive Control Lecture Notes
[3] Model Predictive Control: Theory, Computation, and Design by James B. Rawlings,David Q. Mayne and Moritz M. Diehl
[4] https://en.wikipedia.org/wiki/Hautuslemma