Model Predictive Control

Question 1. Find the steady state (x_s, u_s) for the following system to get $y_s = 1$

$$x(k+1) = \begin{bmatrix} 0.5 & 0.2 \\ -0.2 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

A steady state (x_s, u_s) of the system satisfies the equation

$$\begin{bmatrix} I-A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

The task to bring the system output y to a desired, constant set point y_{sp} is termed setpoint tracking which requires $Cx_s = y_{sp}$ and the condition for setpoint tracking becomes

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$

$$\begin{bmatrix} I - \begin{bmatrix} 0.5 & 0.2 \\ -0.2 & 0.5 \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.2 \\ -0.2 & 0.5 \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.5 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & -0.2 & 0 \\ 0.2 & 0.5 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{s1} \\ X_{s2} \\ U_s \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & -0.2 & 0 \\ 0.2 & 0.5 & -1 \\ 1 & 0 & 0 \end{bmatrix}}_{M}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{M^{-1} = \frac{1}{\det(M)} \times \mathrm{Adj}(M)}$$

$$\det(M) = 0.5(0+0) + 0.2(0+1) + 0(0-0.5)$$
$$= 0.2$$

$$M^T = \begin{bmatrix} 0.5 & 0.2 & 1 \\ -0.2 & 0.5 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0.5 & 0 \\ -1 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} -0.2 & 0 \\ 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} -0.2 & 0.5 \\ 0 & -1 \end{vmatrix} = 0.2$$

$$\begin{vmatrix} 0.2 & 1 \\ -1 & 0 \end{vmatrix} = 1 \qquad \begin{vmatrix} 0.5 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0.5 & 0.2 \\ 0 & -1 \end{vmatrix} = -0.5$$

$$\begin{vmatrix} 0.2 & 1 \\ 0.5 & 0 \end{vmatrix} = -0.5 \qquad \begin{vmatrix} 0.5 & 1 \\ -0.2 & 0 \end{vmatrix} = 0.2 \qquad \begin{vmatrix} 0.5 & 0.2 \\ -0.2 & 0.5 \end{vmatrix} = 0.25 + 0.04$$
$$= 0.29$$

$$\mathrm{Adj}(M) = \begin{bmatrix} 0 & 0 & 0.2 \\ 1 & 0 & -0.5 \\ -0.5 & 0.2 & 0.29 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2 \\ -1 & 0 & 0.5 \\ -0.5 & -0.2 & 0.29 \end{bmatrix}$$

$$M^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 0.2 \\ -1 & 0 & 0.5 \\ -0.5 & -0.2 & 0.29 \end{bmatrix}}_{\mathrm{Adj}(M)} \times \underbrace{0.2}_{\det(M)} = \begin{bmatrix} 0 & 0 & 0.04 \\ -0.2 & 0 & 0.1 \\ -0.1 & -0.04 & 0.058 \end{bmatrix}$$

$$\begin{bmatrix} X_{s1} \\ X_{s2} \\ U_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.04 \\ -0.2 & 0 & 0.1 \\ -0.1 & -0.04 & 0.058 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.1 \\ 0.058 \end{bmatrix}$$