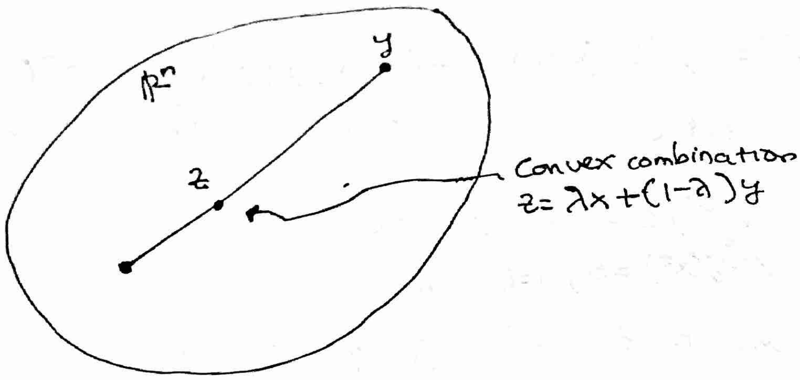
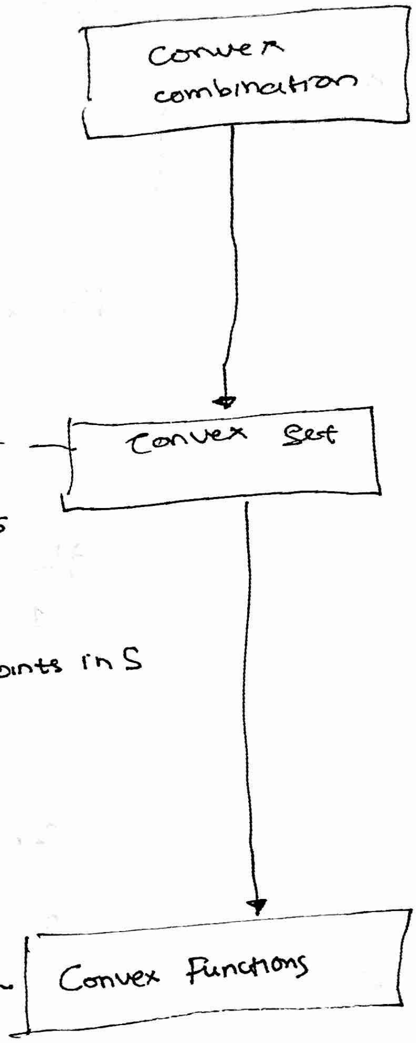
Micro Assignment 6

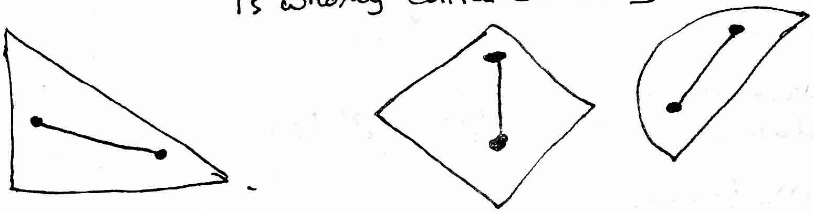
Question 1. What is the difference between a convex set and a convex function?

Given two points $x, y \in \mathbb{R}^n$, a <u>convex combination</u> of them is any point of the form $z = \lambda x + (1-\lambda) y, \lambda \in [0,1]$





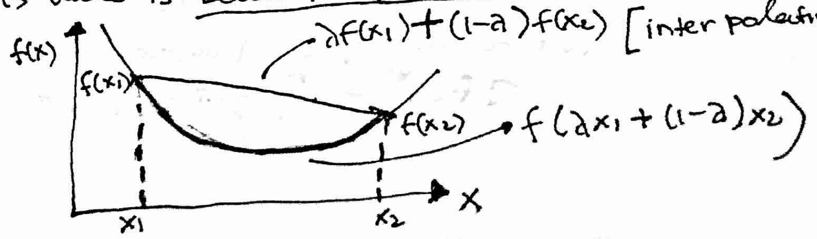
- A set of $S \subseteq \mathbb{R}^n$ is a convex set if it contains <u>all convex combinations</u> of any two points within it
- Graphically: A set of points S is a convex set "if the line segment joining any two points in S is wholly contained in S"



- Let S be a <u>convex set</u>. The function $f(x): S \to \mathbb{R}$ is a <u>convex function</u> if for any two points x_1, x_2 in S

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \ \lambda \in [0,1]$$

- f(x) is convex if its value is <u>below the interpolation</u> formed between any two points



Question 2. Consider the following optimization problem

$$\min_{x} \overbrace{(x_1^2 + x_2^2)}^{f(x)}$$

$$\text{s.t } x_1 \le 0$$

$$x_1 + x_2 = 1$$

Write the KKT conditions and solve the optimization problem. Which constraints are active? Does the strong duality hold?

Answer:

For a problem in the following form,

$$\min \ f(x) \quad \cdots (1)$$

$$\text{s.t } g_i(x) - b_i \ge 0 \quad \text{for } i = 1, \dots, k \quad \cdots (2)$$

$$g_i(x) - b_i = 0 \quad \text{for } i = k+1, \dots, m \quad \cdots (3)$$

The necessary KKT conditions are:

1. Feasibility $\longrightarrow$ $g_i(x) - b_i$ is feasible $\xrightarrow[\text{to}]{\text{applied}}$ (2), (3)
2. No direction which improves objective and is feasible $\longrightarrow$ $\nabla f(\underbrace{x^*}_{\text{optimal}}) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$ $\xrightarrow[\text{to}]{\text{applied}}$ (1), (3)
3. Complementary slackness $\longrightarrow$ $\lambda_i^*(g_i(x^*) - b_i) = 0$ $\xrightarrow[\text{to}]{\text{applied}}$ (2)
4. Positive Lagrange multipliers $\longrightarrow$ $\lambda_i^* \ge 0$ $\xrightarrow[\text{to}]{\text{applied}}$ (2)

Assume both constraints are binding

KKT #1 $\quad g_1 = 0, \ g_2 = 1$ $\quad$ We can check the sign of the Lagrange multipliers to see if that's a good assumption.

KKT #2 $\quad$ Write out the Lagrange multiplier equations

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

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$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0 \longrightarrow 2x_1 - \lambda_1(1) - \lambda_2(1) = 0$$

$$-\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0 \longrightarrow 2x_2 - \lambda_1(0) - \lambda_2(1) = 0$$

$$2 - \lambda_2 = 0$$

$$\boxed{\lambda_2 = 2}$$

$$g_1 = 0 \longrightarrow x_1 = 0$$

$$g_2 = 1 \longrightarrow x_1 + x_2 = 1$$

Solution: $x_1 = 0$, $x_2 = 1$, $\lambda_1 = -2$, $\lambda_2 = 2$

This solution violates the KKT #4
the Lagrange multiplier has to be positive,

Hence, since $\lambda_1 < 0$, the constraint λ_1 is inactive
since $\lambda_2 \geq 0$, the constraint λ_2 is active.

As the strong duality condition requires that the following statements are satisfied:

(1) x^* is a global optimum

(2) there are μ^*, λ_2^* such that the KKT conditions hold.

$$\mu^* \to \lambda_1^*$$

Hence, since the condition (2) is not satisfied,

"STRONG DUALITY DOES NOT HOLD"