CHALMERS
UNIVERSITY OF TECHNOLOGY

SSY281 - Multi Predictive Control

Analysis of linear state-space model of a DC-motor with Flywheel

---

Assignment - 1

ID-Number 43

Fikri Farhan Witjaksono fikrif@student.chalmers.se

January 30, 2020

1 PROBLEM STATEMENT

A pendulum with length l and point mass m, subject to gravity force and controlled by a motor at the pivot point, giving an external torque can be described by the following differential equation.

$$ml^2\ddot{\theta} = \tau + mglsin(\theta) \tag{1}$$

where θ is the angle relative to the vertical direction. By defining the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, and using the small angle approximation $sin(\theta) \approx \theta$, an approximate linear model in the state-space form can be rewritten as

$$\dot{x_1}(t) = \dot{x_2}(t), \tag{2}$$

$$\dot{x_2}(t) = g/l \cdot \dot{x_1}(t) + 1/ml^2 \cdot \tau(t) = \alpha x_1(t) + \beta u(t) \tag{3}$$

$$y(t) = x_1(t) \tag{4}$$

where $u(t) = \tau(t)$

By rearranging the equation (2), (3), (4) as state space equations, we could obtain the result as shown below.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u(t) \tag{5}$$

Since we assume that h = 0.1, $\alpha = 0.5$, $\beta = 1$, we could substitute them to form the state space equation as shown below

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{6}$$

2 QUESTIONS

2.1 Question a: Discrete time state space equation definition

In order to convert from continuous time system to discrete time system equation, we could use the equation below

$$x(k+1) = e^{Ach} \cdot x(k) + \int_0^h e^{A_c s} B_c \mathrm{d}s \cdot u(k) \tag{7}$$

where

$$A_d = e^{Ach} \tag{8}$$

$$B_d = \int_0^h e^{A_c s} B_c \mathrm{d}s \tag{9}$$

From the calculation in Matlab using the syntax **sys=ss(Ac,Bc,CC,Dc)** and **sysd = c2d(sys,h)** we obtain the following values shown below

$$A_d = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \end{bmatrix} \tag{10}$$

$$B_d = \begin{bmatrix} 0.0050 \\ 0.1001 \end{bmatrix} \tag{11}$$

$$C_d = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{12}$$

$$D_d = \begin{bmatrix} 0 \end{bmatrix} \tag{13}$$

2.2 Question b: Discrete time state space equation including the computational delay

If we include the computational delay to the discrete time state space equation as shown in equation (5) above, we could obtain the following equation in place of equation (5) as we have two different control signal due to computational delay inclusion.

$$x(k{+}1) = e^{Ach}{\cdot}x(kh){+}e^{A_c(h-\tau)}\int_0^h e^{A_c s}B_c\mathrm{d}s{\cdot}u(k{-}1){+}\int_0^{h-\tau} e^{A_c s}B_c\mathrm{d}s{\cdot}u(k) = Ax(k){+}B_1u(k{-}1){+}B_2u(k) \tag{14}$$

Hence, we could define the state space matrices

$$A = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \end{bmatrix} \tag{15}$$

$$B_1 = \begin{bmatrix} 0.0038 \\ 0.0501 \end{bmatrix} \tag{16}$$

$$B_2 = \begin{bmatrix} 0.0013 \\ 0.0500 \end{bmatrix} \tag{17}$$

Equation (12) could be put in a new state space model shown as below

$$\xi(k+1) = \begin{bmatrix} A & B_1 \\ 0 & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} B_2 \\ 1 \end{bmatrix} u(k) \tag{18}$$

where we have new A,B,C matrices defined as A_a, B_a, C_a as shown below

$$A_a = \begin{bmatrix} A & B_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.0025 & 0.1001 & 0.0038 \\ 0.0500 & 1.0025 & 0.0501 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$$B_a = \begin{bmatrix} B_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 0.0013 \\ 0.0500 \\ 1.0000 \end{bmatrix} \tag{20}$$

$$C_a = \begin{bmatrix} C_c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{21}$$

where we introduce the augmented state vector

$$\xi(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} \tag{22}$$

2.3 Question c: Controllability and Observability Check

A system is said to be controllable if all its states can reach any final state from any initial state within a finite time. The system is said to be controllable if its controllability matrix is full rank. The controllability matrix can be obtained as shown in equation 23 below.

$$W_r = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} \tag{23}$$

Furthermore, using the rank function in MATLAB the rank of the matrix is determined and if the rank is equal to the number of states in the system, then the system is said to be controllable. The observability of the system allows to determine the state of the system by measuring the inputs and outputs, provided the system is observable. To evaluate this property of the system, the observability matrix given by equation 24 below, is checked for full rank.

$$W_o = \begin{bmatrix} C \\ AC \\ A^2C \\ A^3C \\ A^4B \end{bmatrix} \tag{24}$$

In MATLAB, we could also use the commands ctrb, obsv and rank to check both the controllability, observability and rank of the both matrices respectively. The MATLAB showed that all 3 systems (system (2),(3),(4)) are all both controllable and observable.

2.4 Question d: Define a non zero matrix C that makes system (2) unobservable

Let we define the matrix C as 2x1 matrix with element C1 and C2 where both are non-zeros.

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{25}$$

and A_c as previously obtained

$$A_c = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \tag{26}$$

Thus, the observability W_o could be obtained as below

$$W_o = \begin{bmatrix} C \\ A_c \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ 0.5C_2 & C_1 \end{bmatrix} \tag{27}$$

If we reduce equation (27), we would find that the matrix W_o is not full rank, hence not observable. Furthermore, taking determinant of equation (27) will result in the equation as below

$$C_1^2 - 0.5C_2^2 = 0 \tag{28}$$

Hence, we get C_1 and C_2 as below

$$C_1 = \sqrt{0.5}C_2 \tag{29}$$

Furthermore, we can collect the two C elements in to a unified C matrix as below

$$C = \begin{bmatrix} \sqrt{0.5}C_2 & C_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \tag{30}$$

2.4.1 Conclude if it makes system (3) unobservable as well

Let we use C as shown in equation (30) above and the matrix A as below

$$A_d = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \end{bmatrix} \tag{31}$$

We obtained the observability matrix W_o as shown below

$$W_o = \begin{bmatrix} C \\ A_c \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.4142 \\ 1.0733 & 1.5178 \end{bmatrix} \tag{32}$$

Hence, the rank check of matrix W_o shows that it is not full rank, therefore become unobservable

2.4.2 Conclude if it makes system (4) unobservable as well

By using the same method as in section 2.4.1 above the delayed discrete time matrix A as below

$$A_a = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \end{bmatrix} \tag{33}$$

We obtained the observability matrix W_o as shown below

$$W_o = \begin{bmatrix} C \\ A_a \end{bmatrix} = \begin{bmatrix} 1.0025 & 0.1001 & 0.0038 \\ 0.0500 & 1.0025 & 0.0501 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$n = rank \begin{bmatrix} \lambda_i - A \\ C \end{bmatrix} \tag{35}$$

The rank check of equation (35) shows that it is not full rank. Hence, we could conclude that it is not observable.

It is shown from results in question 2.4 that the discrete time system and the discrete time system with computational delay will be unobservable if we the same nonzero matrix that makes continuous time system unobservable in each corresponding system (3) and (4). Moreover, we could conclude that the observability in continuous system will most likely influence the observability in discrete time (DT) system as well as DT system with computational delay.

2.5 Question e: Calculation of x_k of system (3)

Using the C matrix defined in equation (25) above, as well as the given $y(k), y(k+1)$ and $u(k)$, we could express the discrete time system in compact version in the case z = y as shown below

$$x^+ = Ax + Bu \tag{36}$$

$$y = Cx \tag{37}$$

$$y^+ = Cx^+ = C(Ax + Bu) = CAx + CBu \tag{38}$$

The state could be calculated using the state observer/estimator which could be constructed as below

$$\hat{x}^+ = A + Bu + L(y - C\hat{x}) \tag{39}$$

$$\tilde{x}^+ = (A - LC)\tilde{x} \tag{40}$$

Hence, if A-LC is stable ($\lambda \in$ LHP) as well as the original system is observable, we could assign purely stable eigenvalues such that estimation of x ($\hat{x}$) will converge to 0 as t converges to ∞. This implied that $\hat{x}$ converges to x.

However, since we use the C matrix which results in unobservable system, therefore we could not estimate $x(k)$ (e.g. since W_o is not full rank).

2.6 Question f: Find Controller gain and plot the step response of the closed-loop system

First, we consider the system (3) is in closed-loop system with state-feedback control

$$u(k) = -Kx(k) + K_r r \tag{41}$$

and controller gain K. After that, we want to have the Discrete Time (DT) ploes to match the Continuous Time (CT) poles $\lambda_{1,2} = -4 \pm 6i$. We first convert the CT poles into DT poles as shown below.

$$p_{1d} = e^{\lambda_1 h} = 0.5532 + 0.3785i \tag{42}$$

$$p_{2d} = e^{\lambda_2 h} = 0.5532 - 0.3785i \tag{43}$$

Collecting the two poles will give us

$$p_d = \begin{bmatrix} 0.5532 + 0.3785i & 0.5532 - 0.3785i \end{bmatrix} \tag{44}$$

Then, we calculate the controller gain such that the DT system poles corresponds to CT poles using 'place' function. In the DT system with delay, we append zero as the 3rd element of the K matrix. The controller gain is calculated for both DT system without and with computational delay K_1 and K_2 is obtained respectively as below

$$K_1 = \begin{bmatrix} 34.7708 & 7.2399 \end{bmatrix} \tag{45}$$

$$K_2 = \begin{bmatrix} 34.7708 & 7.2399 & 0 \end{bmatrix} \tag{46}$$

The closed loop DT system A matrix for system (3) and (4) could be calculated as

$$A_{cl(3)} = A_d - B_d K_1 = \begin{bmatrix} 0.8286 & 0.0639 \\ -3.4299 & 0.2779 \end{bmatrix} \tag{47}$$

$$A_{cl(4)} = A_a - B_a K_2 = \begin{bmatrix} 0.9590 & 0.0910 & 0.0038 \\ -1.6889 & 0.6404 & 0.0501 \\ -34.7708 & -7.2399 & 0 \end{bmatrix} \tag{48}$$

The reference gain for system (3) and (4) by using the equation below which is expressed with 'dcgain' MATLAB function with slightly different in negativity of the gain in system (4).

$$K_{r(3)} = D_d + C_d(I_2 – A_d)^{-\mathbf{1}} B_d = 89.8284 \tag{49}$$

$$K_{r(4)} = -(D_a + C_a(I_2 – A_a)^{-\mathbf{1}} B_a) = 254.8570 \tag{50}$$

After obtaining the reference gain as well as the closed loop A matrix for system (3), the new state space equation was constructed for both system (3) and (4) as shown below.
For System (3) :

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0.8286 & 0.06387 \\ -3.43 & 0.2779 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.4493 \\ 8.99 \end{bmatrix} u(t) \tag{51}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{52}$$

For System (4) :

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0.959 & 0.09103 & 0.003752 \\ -1.689 & 0.6404 & 0.05007 \\ -34.77 & -7.24 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0.3186 \\ 12.75 \\ 254.9 \end{bmatrix} u(t) \tag{53}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{54}$$

For system (5), in order to recover the closed-loop behaviour of the system (3), 0 is appended to the previously obtained poles in equation (44)

$$p_{new} = \begin{bmatrix} 0.5532 + 0.3785i & 0.5532 - 0.3785i & 0 \end{bmatrix} \tag{55}$$

This new poles is then used to calculate the new feedback gain as below

$$K_3 = \begin{bmatrix} 34.9736 & 8.9834 & 0.4055 \end{bmatrix} \tag{56}$$

The process is roughly the similar from here as in the system (3) and (4) except for the fact that we use a new feedback gain in the calculation. The result of the calculation in MATLAB is shown below

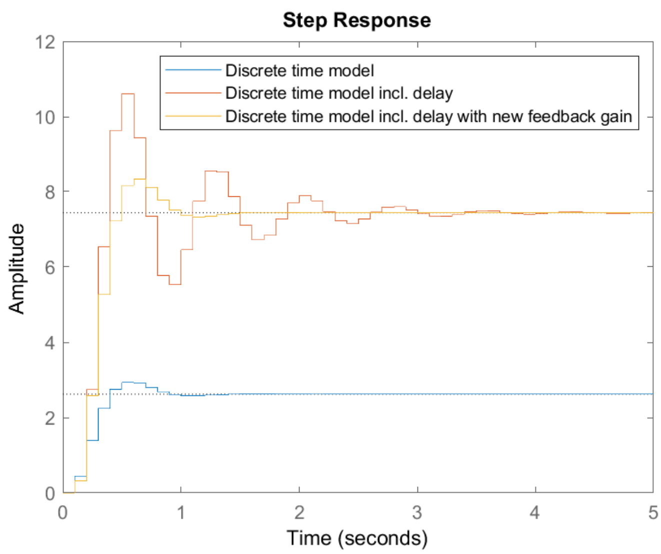
$$A_{cl(5)} = A_a - B_a K_3 = \begin{bmatrix} 0.9588 & 0.0889 & 0.0032 \\ -1.6990 & 0.5532 & 0.0298 \\ -34.9736 & -8.9834 & -0.4055 \end{bmatrix} \tag{57}$$

Hence, the state space equation can be obtained as shown below For System (5) :

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0.9588 & 0.08885 & 0.003245 \\ -1.699 & 0.5532 & 0.02979 \\ -34.97 & -8.983 & -0.4055 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0.3186 \\ 12.75 \\ 254.9 \end{bmatrix} u(t) \tag{58}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_2(t) \end{bmatrix} \tag{59}$$

After that, the step response is plotted using 'Step" function in MATLAB. The result is shown below



If we notice the figure above, it is shown that the maximum amplitude of the step response for the system (3) has the lowest amplitude. The amplitude of system (4) is considerably higher than the system (3) as well as it has a bit more oscillating behaviour compare to system (3). This is due to much higher reference gain that the system (4) has as well as the introduction of additional zero to the feedback gain causes the oscillating behaviour. Furthermore, for system (5), the maximum amplitude decreases a little bit as we introduce additional zero to the poles matrix which causes lower feedback gain compare to system (4). However, as the new system (5) recover the closed loop behaviour of system (3), the oscillation significantly reduced implying that it is more stable than system (4).

2.7 Question g: Find the steady-state and output and plot the output of the system

Here, we introduced the constant set point tracking $y_{sp} = \frac{\pi}{6}$ and then we tried to find the steady-state (x_s) and output (y_s). The system needs to satisfy the following equation.

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{s_p} \end{bmatrix} \tag{60}$$

The result of calculation of equation (60) above is shown below by inversing the first term matrix and move it to the Right Hand Plane (RHP).

$$x_s = \begin{bmatrix} 0.5236 \\ 0 \\ -0.2618 \end{bmatrix} \tag{61}$$

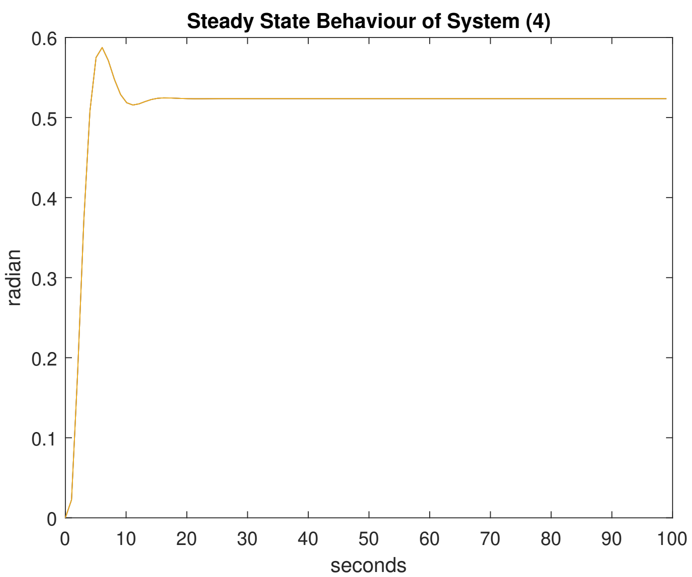
$$u_s = -0.2618 \tag{62}$$

After that, in order to plot the output, we used the for loop which use the equation below which take account the difference between the steady-state input and output to the previous System (4) without set-point tracking.

$$x_{k+1} = (A_a - B_a K_3)(x_k - x_s) + x_s \tag{63}$$

$$y_k = C_a(x_k - x_s) + y_s \tag{64}$$

Then, we could plot the result as shown below in Figure 2



If we observe Figure (2) above, we could see that the output is being directed towards the set-point tracking angle of $\frac{\pi}{6}$, and because of this, initially the output response will show higher than the point that was being set before being stabilized after 10 seconds. The set point tracking proved to be useful in this matter.

2.8 Question h: Find the augmented system with the disturbance and check the stabilizability and detectability

Here, The DT system (4) is given a constant disturbance d(k). We could augment the state vector with disturbance and get the new system as shown below

$$\xi(k+1) = A_e \xi k + B_e u(k) \tag{65}$$

$$y(k) = C_e \xi(k) \tag{66}$$

The equation (65) and (66) above could be represented by the equation below to allow for calculation

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} A_a & B_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B_a \\ 0 \end{bmatrix} u(k) \tag{67}$$

$$y = \begin{bmatrix} C_a & C_d \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \tag{68}$$

The result of calculation of equation (67) and (68) above gives us the following

$$Ae = \begin{bmatrix} 1.0025 & 0.1001 & 0.0038 & 0 \\ 0.0500 & 1.0025 & 0.0501 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \tag{69}$$

$$Be = \begin{bmatrix} 0.0013 \\ 0.0500 \\ 1.0000 \\ 0 \end{bmatrix} \tag{70}$$

The new augmented system is then check for controllability before being checked for stabilizability. After checking for controllability, it is found out that the controllability matrix as shown in equation (23) is full rank. However, it is found out that the system has non-negative eigenvalues as shown below

$$\lambda_i = \begin{bmatrix} 1.0025 \\ 2.0038 \\ -0.0012 \\ 0 \end{bmatrix} \tag{71}$$

Hence we have to do the stabilizablity and detectability test using the Hautus Lemma as shown below

$$n_{stab} = rank(\begin{bmatrix} \lambda_i - A_e \\ C_e \end{bmatrix}) \tag{72}$$

$$n_{detect} = rank(\begin{bmatrix} \lambda_i - A_e & B_e \end{bmatrix}) \tag{73}$$

where λ_i is defined as the non-negative elements of equation (71). After that, we should determine if the rank in equation (72) and (73) is full rank (e.g. equal to number of row of the square matrix A_e. By this test, we could determine that all non-zero eigenvalues are stabilizable and detectable.

2.9 Question i: Design a linear controller to place the four poles with the three poles same as question (f) and the last one at 1

In this question, same procedure was done as in question (f), however the last additional pole was changed and the system use the one as defined in question (h). The result of the new feedback gain K is shown below

$$K_4 = \begin{bmatrix} 0.1071 & 1.9629 & 0.8002 & 1.9552 \end{bmatrix} \tag{74}$$

If the last additional pole is moved to zero, the pole placement design command "place" would not work since it cannot place poles with multiplicity greater than rank(B). On the other hand, if the last additional pole is greater than 1, the system step response become unstable and the amplitude goes to a very large number. These two conditions are undesirable conditions.

Therefore, the only feasible value to create linear controller is if the last pole is placed at 1.

2.10 Question j: Design a linear observer for the augmented system (6) with poles placed at (0.1,0.2,0.3,0.4)

The requirements in order to be able to design a linear observer, the observability matrix has to be full rank. The result of the linear observer design will have the gain as shown below

$$L = \begin{bmatrix} 2.0050 \\ 23.5796 \\ 0.0252 \\ 23.0055 \end{bmatrix} \tag{75}$$

The eigenvalue of the system matrix of the observer is shown below

$$\lambda_i(A_e - LC_e) = \begin{bmatrix} 0.4000 \\ 0.3000 \\ 0.2000 \\ 0.1000 \end{bmatrix} \tag{76}$$

The equation above shows that the estimation of x ($\hat{x}$) converges to the real state x as the error dynamics $e(k+1)$ converges to zero.

3 REFERENCES

[1] SSY281 - Multi Predictive Control Assignment-1 handout
[2] SSY281 - Multi Predictive Control Lecture Notes
[3] Control Theory - Multivariable and Nonlinear Methods by Torkel Glad and Lennart Ljung