SSY281 - Multi Predictive Control

MPC Stability

Assignment - 5

ID-Number 43

Fikri Farhan Witjaksono fikrif@student.chalmers.se

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1 QUESTIONS

1.1 Question 1: Find a Lyapunov Function to prove system stability

In order to answer this question, we need to find the Lyapunov function in the quadratic form $V = x^T Sx$ with S positive definite, which fullfills the requirement that the function V has functions $\alpha_i \in K_\infty, i = 1, 2$ and a positive definite function α_3 such that for any $x \in \mathbb{R}^n$,

$$V(x) \geq \alpha_1(|x|) \tag{1}$$

$$V(x) \leq \alpha_2(|x|) \tag{2}$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|) \tag{3}$$

Hence, through the use of MATLAB function '**dlyap**' which gives solution to Discrete-time Matrix Lyapunov Equation,we obtain the S matrix below

$$S = \begin{bmatrix} 2.8526 & 0.0711 \\ 0.0711 & 1.0684 \end{bmatrix} \tag{4}$$

From equation above, we could conclude that S matrix is positive definite, hence it is a Lyapunov's function since the first, second and third condition is satisfied. The third condition could be observed as satisfied as shown below

$$V(x^+) - V(x) = x^T A^T SAx - x^T Sx = x^T (A^T SA - S))x \tag{5}$$

By using dlyap, we are setting such that $A^T SA - S = -Q$. Since the S that we got from equation (4) is positive definite, the third condition of the Lyapunov's function is satisfied as well, hence it is a Lyapunov's function.

1.2 Question 2: Prove that $V(u, x_0)$ is a Lyapunov function

Assume from the problem formulation we have

$$V(x(0), u(0 : \infty)) = \sum_{i=0}^{\infty}(x^T(i)Qx(i) + u^T(i)Ru(i)) \tag{6}$$

By splitting the sum-series into two different terms we have

$$V(x(0), u(0 : N-1)) = \sum_{i=0}^{N-1}(x^T(i)Qx(i) + u^T Ru(i)) + \sum_{N}^{\infty}(x^T(i)Qx(i) + u^T Ru(i)) \tag{7}$$

By minimizing the finite time horizon criterion above we have

$$V(x(0), u(0 : N-1)) = \sum_{i=0}^{N-1}(x^T(i)Qx(i) + u^T Ru(i)) + x^T(N)Px(N) \tag{8}$$

where P is the solution to the algebraic Riccati equation. From here, we could divide the cost function above into the stage cost (l(x,u)) and terminal cost ($V_f(x)$) as shown below

$$l(x, u) = x^T Qx + u^T Ru \tag{9}$$

$$V_f(x) = V_\infty^{uc} = x^T Px \tag{10}$$

If we look at the stage cost and its weighting matrix Q and R, we could see that the weighting matrix are positive semi-definite and positive definite respectively. Consequently the sum will be positive semi-definite as well, hence satisfying the equation (1) and (2) above. In order to satisfy the equation (3) above, we need to get the value of $V(u, x_1) - V(u, x_0)$ which is negative definite(e.g. P which is positive definite and it is in the form of $-x_0^T P x_0$).

$$V(x^+) - V(x) = V((A + BK)x_0) - V(x_0) \tag{11}$$

$$= -x_0^T Q x_0 - (Kx_0)^T R(Kx_0)$$

$$= -x_0^T (Q + K^T RK)x_0$$

Since the weighting matrix Q and R are positive semi-definite and positive definite respectively, then the matrix P which is $Q + K^T RK$ becomes positive definite. Consequently, cost function $V(u, x_1) - V(u, x_0)$ becomes negative semi-definite which satisfies the equation (3) above, hence it is a Lyapunov function. Calculated P is as shown below

$$P = \begin{bmatrix} 13.6586 & 7.9930 \\ 7.9930 & 39.4371 \end{bmatrix} \tag{12}$$

We could see that it is positive definite and hence preposition above is proven.

1.3 Question 3

1.3.1 a. Find a shortest N that stabilizes the system

The shorthest N such that the RH controller stabilizes the system could be found by solving for the Dynamic Programming solution of the cost function $V_N(x)$.The closed loop system $x(k + 1) = (A + BK)x(k)$ is stable if the spectral radius of A+BK is strictly less than 1 (e.g. The spectral radius of the is $max_{i=1,...,n}|\lambda_i|$, where $\lambda_1, ..., \lambda_n$ are the system's eigenvalues). Then, by the method, the shortest horizon such that the system is stable is obtained as N=5 which gives the eigenvalues as below

$$\lambda_i = \begin{bmatrix} 0.9070 \\ 0.2058 \end{bmatrix} \tag{13}$$

1.3.2 b. Discuss the effect of Q on the stability when N=1

If we set the horizon length N=1 and consider the cost function including a scaling factor a which is an integer $a \in \mathbb{R}$ to weighting matrix Q,

$$V(x(0), u(0 : N - 1)) = x^T(1)P_f x(1) + \sum_{i=0}^{1-1} (a \cdot x^T(i)Qx(i) + u(i)^T Ru(i)) \tag{14}$$

$$= x^T(1)P_f x(1) + (a \cdot x^T(0)Qx(0) + u(0)^T Ru(0))$$

We could observe that the stability will be influenced by both term P_f and Q since both are considered as independent variables. if we keep increasing the scaling factor a gradually, it will result in the eigenvalues converging towards the origin. At a =5, we will have a marginally stable system as both eigenvalues are at the unit circle limit = 1. Then,if we keep increasing a after it is marginally stable condition until the factor of 10, both eigenvalues will move towards the origin. Meanwhile, increasing above a factor of 10 will makes both eigenvalues to be diverging against each other, with one of them converging to origin while the other one is converging towards the limit of the unit circle($\lambda_1 \to 1, \lambda_2 \to 0$). This happens at very large value of scaling factor ($a \to \infty$)

1.3.3 c. Find P_f for stable system

This question can be solved by finding the the solution to the LQ problem using the Riccati equation. This could be done by using the MATLAB function (**'idare'**) with the inputs A,B,Q,R. The riccati equation is as shown below

$$P(k-1) = Q + A^T P(k)A - A^T P(k)B(B^T P(k)B + R)^{-1} B^T P(k)A, P(N) = P_f \tag{15}$$

and the control policy as shown below

$$u^0(k;x) = K(k)x, k = 0, ..., N-1 \tag{16}$$

$$K(k) = -(R + B^T P(k+1)B)^{-1} B^T P(k+1)A$$

By the method above,we could obtain P_f which is the stationary solution for infinte horizon as

$$P_f = \begin{bmatrix} 13.6586 & 7.9930 \\ 7.9930 & 39.4371 \end{bmatrix} \tag{17}$$

1.3.4 d. Find another R which the system is stable

By trial and error, it was found that at R = 0.2 will gives us a marginally stable system (e.g. $\lambda_{1,2} = 1$) and as R is being decreased by the increment of 0.1 (e.g. R=0.19), we will get a stable system (e.g. $\lambda_{1,2} = \{0.9788, 0.9788\}$). And if we keep increasing until R=0, we will get a marginally stable system. This shows that it requires tuning of the weighting matrix R to get an stable system rather than a marginally stable system.

1.4 Question 4: Find a Lyapunov Function to prove system stability

Since we assume that N = 1, we could obtain the Receding Horizon controller directly as

$$K = -(B^T P_f B + R)^{-1} \cdot B^T P_f A \tag{18}$$

If we simplify the equation above by only considering the second term ($B^T P_f A$), we will see that the product between them is always zero as shown below

$$B^T P_f A = \underbrace{\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}}_{\text{1st term}} \underbrace{\begin{bmatrix} p_1 & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & p_n \end{bmatrix}}_{\text{2nd term}} \underbrace{\begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ 0 & \dots & 0 \end{bmatrix}}_{\text{3rd term}} \tag{19}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x & \dots & x & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ x & \dots & x & 0 \end{bmatrix} = 0$$

where x denotes the nonzero result of element-wise multiplication between the 2nd term and 3rd term of equation (19). Big zero in the 2nd term of equation (19) denotes that the upper triangular and lower triangular elements of the matrix is zero hence only have diagonal elements.

Consequently, the controller gain K will always be zero for all different values of Q and R. Hence, we could conclude that the RH controller won't be able to stabilize the system regardless of Q and R. This could be shown in the equation below

$$\begin{aligned} x^+ &= (A + BK)x \\ &= (A + 0)x \\ &= Ax \end{aligned} \tag{20}$$

Hence

$$eig(A + BK) = eig(A) \tag{21}$$

It is shown above that A is unstable for all values of Q and R since the controller does not stabilize the system due to $K = 0$.

1.5 Question 5: Find a Lyapunov Function to prove system stability

1.5.1 (a). Test R values which gives stable system for N =1

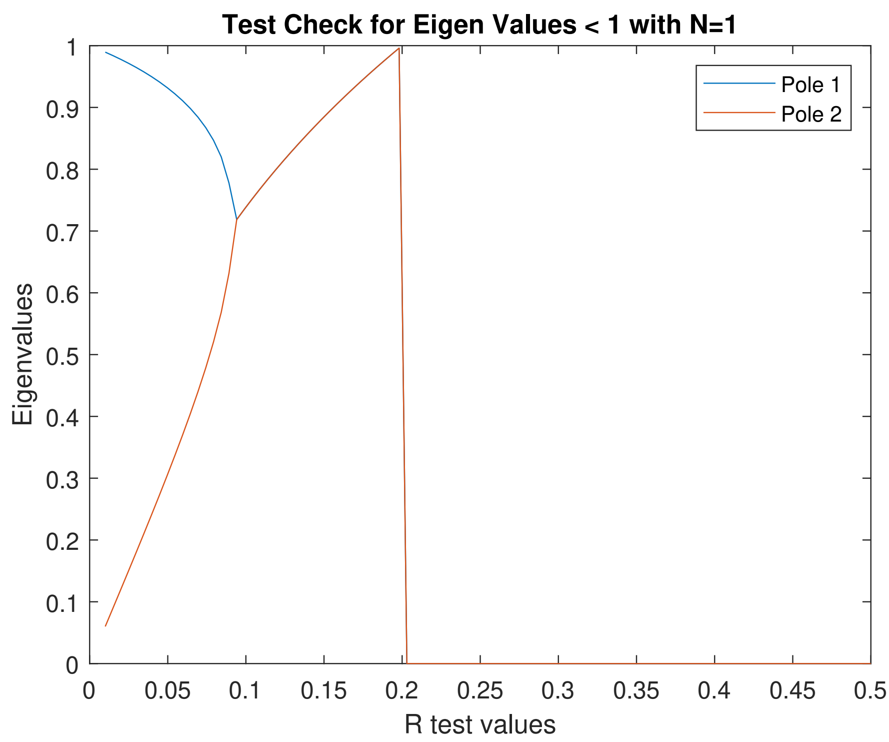

Figure 1: Test Check for Eigenvalues $(\lambda_{1,2}) < 1$ with $N = 1$

Since we assume that N = 1, we could obtain the Receding Horizon controller directly as below since solution of the Riccati equation P will give zero result due to short horizon.

$$K = -(B^T P_f B + R)^{-1} \cdot B^T P_f A \tag{22}$$

To obtain a stable system, the closed loop system eigenvalues needs to be less than 1. It could be shown above in figure 1 that the system will be stable as its eigenvalues is inside the unit circle is in the interval of $R \in [0, 0.2]$.

1.5.2 (b). Test R values which gives stable system for N =2

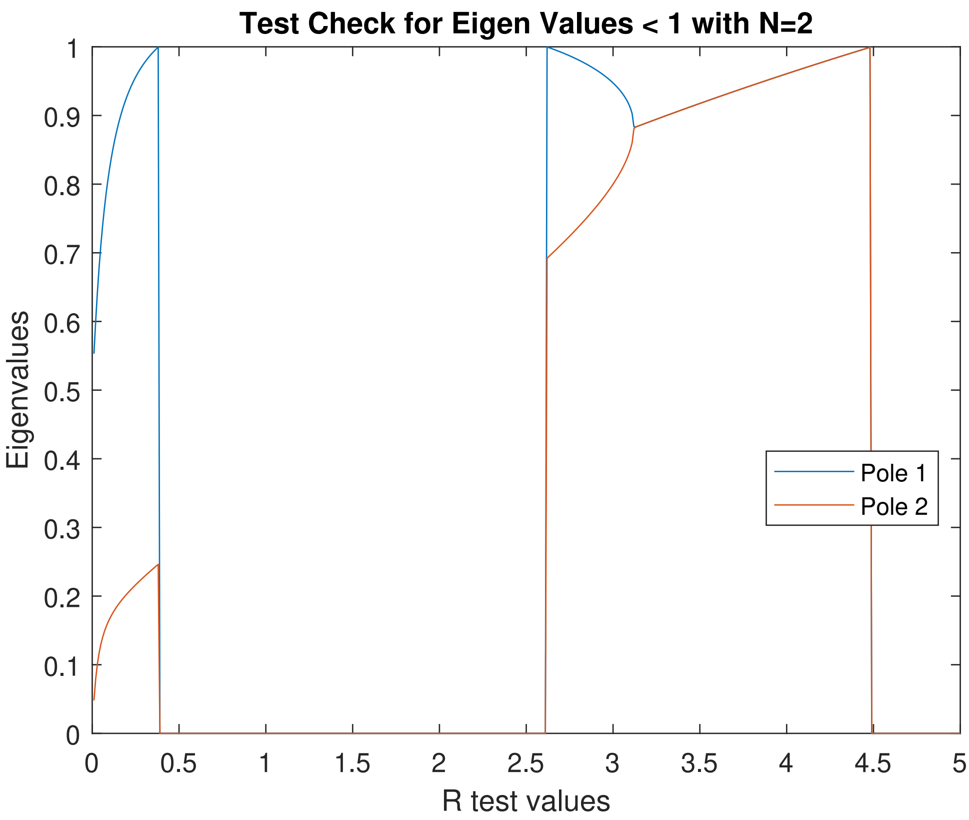

Figure 2: Test Check for Eigenvalues $(\lambda_{1,2}) < 1$ with $N = 1$

For N =2, we could use the Dynamic programming approach to find K and P values as the solution of the Riccati equation. From figure (2) above, it is shown that the system behaviour alternates between stable and unstable condition. The stable condition is within the range of $R \in [0, 0.38] \cup [2.61, 4.49]$ and the unstable condition is within the range of $R \in [0.39, 2.60] \cup [4.49, 5]$.

1.6 Question 6: Find a Lyapunov Function to prove system stability

To prove that the system (2) is stable when controlled with an infinite-time LQ controller,first we could use the 3rd condition and modify it as below due to the fact that we have infinite horizon

$$x_0^T(A + BK)^T P_f(A + BK)x_0 - x_0^T P_f x_0 = -x_0^T S x_0 \tag{23}$$

which could be simplified into the form below

$$(A + BK)P(A + BK) - P = -S \tag{24}$$

Next, by replacing K with equation (16), we would get the Left Hand Side (LHS) equation to be

$$(A^T P - A^T PB(B^T PB + R)^{-1}B^T P)(A - B(B^T PB + R)^{-1}B^T PA) - P \tag{25}$$

Then, by inserting P from equation (15) which is the solution to the Riccati equation and simplifying the terms, we would get

$$A^T PB(R + B^T PB)^{-1}(B^T PB(B^T PB + R)^{-1} - 1)B^T PA - Q \tag{26}$$

Simplifying further by decomposing the term $(B^TPB + R)^{-1}$ gives us

$$A^TPB(R + B^TPB)^{-1}(B^TPB - R - B^TPB)(R + B^TPB)^{-1}B^TPA - Q \tag{27}$$

Since we have $B^TPB - B^TPB$ inside one of the term, we can simplify it into only -R as below

$$\underbrace{A^TPB(R + B^TPB)^{-1}}_{\text{1st term}}(-R)\underbrace{(R + B^TPB)^{-1}B^TPA}_{\text{2nd term}} - Q \tag{28}$$

We know K is defined as in equation (22) above, then the 1st term is in fact could be replaced by $-K^T$ and the 2nd term with $-K$ which gives us

$$\underbrace{-K^T(-R) - K}_{\text{1st term}} - Q \tag{29}$$

Canceling the minus sign in the 1st term between $-K^T$ and $-K$ and combining with the Right Hand Side (RHS) equation gives

$$\underbrace{K^T(-R)K}_{\text{V(f(x))}} - \underbrace{Q}_{\text{V(x)}} = -S \tag{30}$$

which is in the form of the Lyapunov function 3rd condition. Thus, since we know that in the system (2) we have Q as positive semi-definite as well as R defined as positive definite, this will gives us the LHS equation ($V(x^+)$-$V(x)$) to be negative semi-definite. Consequently, we will have S as positive semi-definite. Therefore, V decays along solutions of the system, and since S is positive semi-definite the decay persists until the state x approaches the origin, for any $x \in \mathbb{R}^n$.