CHALMERS
UNIVERSITY OF TECHNOLOGY

SSY281 - Multi Predictive Control

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# Linear Quadratic and Receding Horizon Control

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**Assignment - 2**

**ID-Number 43**

Fikri Farhan Witjaksono fikrif@student.chalmers.se

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# 1 PROBLEM STATEMENT

Consider the following statement

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$
$$y(k) = Cx(k)$$

and the following quadratic cost function

$$V_N(x(0), u(0:N-1)) = \sum_{k=0}^{N-1} (x(k)^T Q x(k) + u(k)^T R u(k)) + x(N)^T P_f x(N), \tag{2}$$

with Q,R,$P_f$ positive definite matrices. A finite-time LQ controller can be found as by solving the following Problem

$$\begin{aligned} \min_{u(0:N-1)} \quad & V_N(x(0), u(0:N-1)) \\ \text{s.t} \quad & x(k+1) = Ax(k) + Bu(k) \end{aligned} \tag{3}$$

where

$$u(0:N-1) = u(0), u(1), ..., u(N-1) \tag{4}$$

The following value have to be used in the rest of the assignment.

$$A = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \end{bmatrix}, B = \begin{bmatrix} 0.0050 \\ 0.1001 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{5}$$

$$Q = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, P_f = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, R = 0.5 \tag{6}$$

# 2 QUESTIONS

## 2.1 Question a: Find LQ controller by using Dynamic Programming method

This problem could be solved by applying the backwards dynamic programming (Backward DP) recursive method to the LQ problem. The method itself basically could be summarized with the general expressions as shown below.

1. **Find the Riccati Equation at N-1 after inputting P(N)**

$$P(k-1) = Q + A^T P(k) A - A^T P(k) B (R + B^T P(k) B)^{-1} B^T P(k) A, P(N) = P_f \tag{7}$$

2. **Solve the sequence of optimal control laws to find K(N)**

$$K(k) = -(R + B^T P(k+1) B)^{-1} B^T P(k+1) A \tag{8}$$

where

$$u^0(k; x) = K(k)x, k = 0, ...., N-1$$

3. **Repeat the step 2.1.1 in order to obtain P(N-2) by using obtained P(N-1)**

$$P(N-2) = Q + A^T P(N-1) A - A^T P(N-1) B (R + B^T P(N-1) B)^{-1} B^T P(N-1) A, P(N-1) = a \tag{9}$$

4. **Solve the sequence of optimal control laws to find K(N-1)**

$$K(N-1) = -(R + B^T P(N) B)^{-1} B^T P(N) A \tag{10}$$

5. **Repeat these steps above until we obtained desired $P_0$ and $K_0$**

## 2.2 Question b: Find shortest N that makes the system asympotically stable using DP method

In this question, we could find the shortest N by doing the step 2.1.5 until we get $K_0$ and $P_0$. Shortest N could be find by iteratively inputting N starting from 1 until we find that the eigenvalue $A + BK_0$ converges to 1. This is due to the fact that the requirement to get the asymptotically stable $u(k)$ for discrete time system is if we get the solution is that all the eigenvalues of the system matrix must have magnitudes less than 1. Mathematically, it could be shown as below

$$\sqrt{(Re(z))^2 + (Im(z))^2} \leq 1 \tag{11}$$

Hence, from calculation in MATLAB, we could find that $\mathbf{N = 3}$. Moreover, $K_0$ and $P_0$ which is the value at final iteration is as shown below

$$K_0 = \begin{bmatrix} -0.8013 & -0.9150 \end{bmatrix} \tag{12}$$

$$P_0 = \begin{bmatrix} 20.2347 & 3.1363 \\ 3.1363 & 4.4259 \end{bmatrix} \tag{13}$$

The eigenvalue of the system matrix is shown below

$$\lambda_i = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.9547 + 0.0311i \\ 0.9547 - 0.0311i \end{bmatrix} \tag{14}$$

## 2.3 Question c: Find the stationary solution $P_f$ of the Riccati equation using provided value and find shortest N that makes the system asymptotically stable and compare with question (a) and (b)

In this question, we could find the stationary solution $P_f$ of the Riccati equation using the values 'dare' MATLAB function whose usage is to find unique stabilizing solution X of the discrete-time algebraic Riccati equation. It also calculated the closed-loop eigenvalue L. The result of the 'dare' function (X) is then considered as terminal cost gain. After that, by defining it as the terminal cost gain as well as inputting N starting from 1 until the shortest N possible which gives asymptotically stable system is found (e.g. the requirement as shwon in eq.(11) is fullfilled)), it is found that *no matter what value of N is inputted will not have an effect on the system since $P_f$ is defined as terminal cost gain, coming from a stationary solution of the Riccati equation.* Therefore, asymptotically stable system is automatically achieved and the eigenvalues of the system matrix could be shown as follows

$$\lambda_i = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.8543 + 0.0791i \\ 0.8543 - 0.0791i \end{bmatrix} \tag{15}$$

Compared to the results in question (a) and (b), *it could be shown that the result in this part is independent of the length of N whereas in question (b), as the length of N is increased, the chance that eigenvalues of the system matrices of satisfying equation (11) above becomes higher and the magnitudes were converged to 1.*

## 2.4 Question d: Solve optimization problem (3) with batch approach solution

$$\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \tag{16}$$

$$x = \Omega x(0) + \Gamma u \tag{17}$$

From equation (15) and (16), we could find that the Omega and Gamma could be obtained by he matrix below

$$\Omega = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \tag{18}$$

$$\Gamma = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} \tag{19}$$

The calculated result of equation (17) and (18) is as shown below

$$\Omega = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.0500 & 1.0025 \\ 1.0402 & 0.4054 \\ 0.2025 & 1.0402 \\ 1.0402 & 0.4054 \\ 0.2025 & 1.0402 \\ 1.0402 & 0.4054 \\ 0.2025 & 1.0402 \end{bmatrix} \tag{20}$$

$$\Gamma = \begin{bmatrix} 0.0050 & 0 & 0 & 0 \\ 0.1001 & 0 & 0 & 0 \\ 0.0150 & 0.0050 & 0 & 0 \\ 0.1006 & 0.1001 & 0 & 0 \\ 0.0251 & 0.0150 & 0.0050 & 0 \\ 0.1016 & 0.1006 & 0.1001 & 0 \\ 0.0354 & 0.0251 & 0.0150 & 0.0050 \\ 0.1031 & 0.1016 & 0.1006 & 0.1001 \end{bmatrix}$$

$\bar{Q}$ and $\bar{R}$ are the weightings used to solve the LQ in the Batch solution method as shown below.

$$\bar{Q} = \begin{bmatrix} Q & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & 0 & Q & 0 \\ 0 & 0 & 0 & R \end{bmatrix} \tag{21}$$

$$\bar{R} = \begin{bmatrix} R & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & R \end{bmatrix}$$

The weightings could be expressed as below from the MATLAB function.

$$\bar{Q} = \begin{bmatrix} 5 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 5 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 5 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{R} = \begin{bmatrix} 0.5 & 0 & \dots & 0 \\ 0 & 0.5 & 0 & 0 \\ \vdots & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

The Batch solution $(V_N(x(0), u)$ is quadratic in u, we can solve for the optimal control vector by the equation as shown below:

$$u^0 = -(\Gamma^T \bar{Q} \Gamma + \bar{R})^{-1} \Gamma^T \bar{Q} \Omega x(0) \tag{22}$$

Hence, for equation (20), $K_0$ could be obtained as below which is used in the matlab code

$$K_0 = -(\Gamma^T \bar{Q} \Gamma + \bar{R})^{-1} \Gamma^T \bar{Q} \Omega \tag{23}$$

The solution for equation (22) is shown as follows

$$K_0 = \begin{bmatrix} -0.8352 & -0.9730 \\ -0.4849 & -0.6704 \\ -0.2175 & -0.4012 \\ -0.0577 & -0.1802 \end{bmatrix} \tag{24}$$

The optimal cost to go is shown below

$$V_N^0(x(0), u) = x^T(0)(Q + \Omega^T \bar{Q} \Omega - \Omega^T \bar{Q} \Gamma (\Gamma^T \bar{Q} \Gamma + \bar{R})^{-1} \Gamma^T \bar{Q} \Omega) x(0) \tag{25}$$

The $P_0$ could be obtained by the equation below from equation (22)

$$P_0 = (Q + \Omega^T \bar{Q} \Omega - \Omega^T \bar{Q} \Gamma (\Gamma^T \bar{Q} \Gamma + \bar{R})^{-1} \Gamma^T \bar{Q} \Omega) \tag{26}$$

The solution for equation (25) is shown as follows

$$P_0 = \begin{bmatrix} 25.7995 & 6.7731 \\ 6.7731 & 6.8159 \end{bmatrix} \tag{27}$$

## 2.5 Question e: Find shortest N that makes the system asymptotically stable using Batch approach

For this problem, it is required to find the eigenvalues of system matrix using a certain value of N starting from 1 until the equation (11) above is satisfied. Using the MATLAB function, we could find the solution as below,

$$N_{shortest} = 4 \tag{28}$$

$$K_0 = \begin{bmatrix} -0.8352 & -0.9730 \\ -0.4849 & -0.6704 \\ -0.2175 & -0.4012 \\ -0.0577 & -0.1802 \end{bmatrix} \tag{29}$$

The eigenvalues of the system matrix could be found as below

$$\lambda_i = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.8964 \\ 0.6471 \end{bmatrix} \tag{30}$$

### 2.6 Question f : Design 4 RHC controllers as well as simulate, tune and plot them in the same figure using the Batch solution Approach

Using the cost function defined in equation (2) and using the value of Q and $P_f$ in equation (6), simulate the obtained closed-loop systems for 20s starting from

$$x(0)^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{31}$$

Here, in this part of question, new function that gave solution using the batch method was defined as **BS43part2.m**. After that, plot the systems inputs and outputs for the four controller with value defined in the 4 different combination of numerical values below

1. R = 0.5 and N = 5
2. R = 0.5 and N = 15
3. R = 0.05 and N = 5
4. R = 0.05 and N = 15

The result of the plot is shown as follows

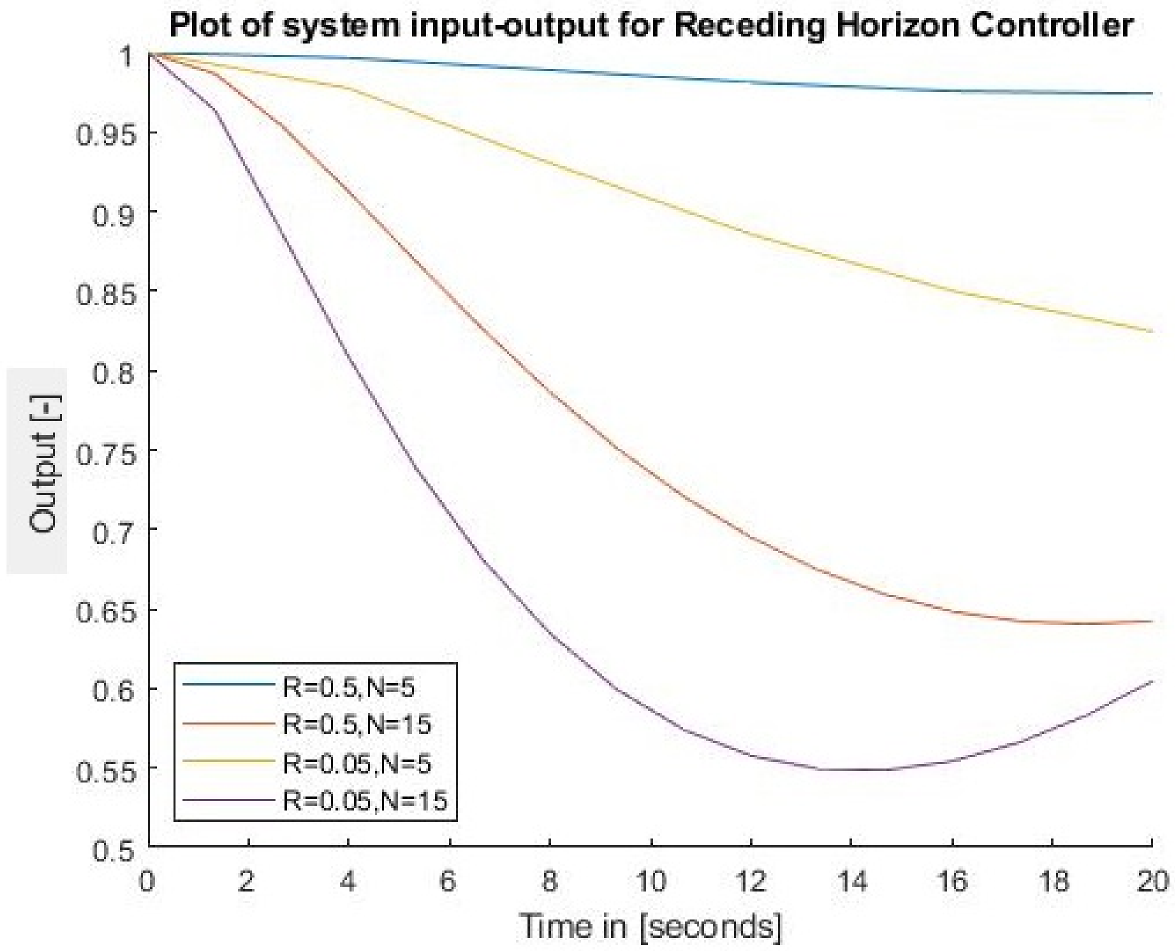


Figure 1: Plot of 4 RHC Controllers Input and Outputs

By observing Figure 1 above, we could see that by choosing the first combination, we would obtain the highest output (e.g. most optimal) as well as less oscillation (e.g.stable system). However, we could observe that only small control actions which might be gibing us slow response to disturbances. On the other hand, by scaling the first combination by dividing R by 10 (e.g. in second combination), we would get lower output than the first one due to the fact that we put less weighting on the input control move.
As the length of iteration N is prolonged, we would generally get lower output (e.g. less optimal) as could be observed from combination 2 and 4. However, theoretically, in order to guarantee that the LQ based RHC controller gives a stable closed-loop system (e.g.satisfying eq.(11)), it is better to use extended horizon length. This could observed from the calculated eigenvalues of the each scheme as shown below which shows that the eigenvalues are relatively more inside the unit circle if we extend the horizon.

1. **Scheme 1**
$$\lambda_i = \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}0.9399 + 0.0558i\\ 0.9399 - 0.0558i\end{bmatrix} \tag{32}$$
2. **Scheme 2**
$$\lambda_i = \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}0.8584 + 0.0809i\\ 0.8584 - 0.0809i\end{bmatrix} \tag{33}$$
3. **Scheme 3**
$$\lambda_i = \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}0.8581\\ 0.6817\end{bmatrix} \tag{34}$$
4. **Scheme 4**
$$\lambda_i = \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}0.7607\\ 0.6968\end{bmatrix} \tag{35}$$

## 2.7 Question g : Calculate the optimal solution when the cost function and its constraint are given and include the system dynamics

In solving this problem, consider that the system equation (1) with cost function (2) is solved using the following constraint below

$$\begin{aligned} F_1x + G_1u &= h_1 \\ F_2x + G_2u &\leq h_2 \end{aligned} \tag{36}$$

By using quadprog function which is defined as below , and by considering the optimization variable as below

$$z = \begin{bmatrix}x\\ u\end{bmatrix} = \begin{bmatrix}x(1)\\ \vdots\\ x(N)\\ u(0)\\ \vdots\\ u(N-1)\end{bmatrix} \tag{37}$$

where quadprog function could be defined as below

$$\begin{aligned} \min_{x} \quad & \frac{1}{2}z^T H z + f^T x \\ \text{s.t} \quad & Ax \leq b, \end{aligned} \tag{38}$$

$$A_{eq}x = b_{eq}$$

Then, we could find A,b,$A_{eq}$,$b_{eq}$ as we compare between equation (36) and equation (38) as defined below

$$A = \begin{bmatrix} F_2 & G_2 \end{bmatrix}, b = h_2 \tag{39}$$

$$A_{eq} = \begin{bmatrix} F_1 & G_1 \end{bmatrix}, b_{eq} = h_1 \tag{40}$$

Note that the optimization constraints (F1,G1,G2,G2,b,$b_{eq}$) should be defined numerically before we can solve the problem using quadprog function. By comparing the cost function equation (2) and the cost function in the quadprog function above, it is easy to observe that the matrix H is as shown below

$$H = \begin{bmatrix} Q & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \dots & P_f & 0 & \dots & 0 \\ \vdots & 0 & 0 & R & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 \end{bmatrix} \tag{41}$$

After z is obtained from equation (38), the final cost could be rewritten as below

$$V_N(z) = \frac{1}{2}z^T Hz = \frac{1}{2}\begin{bmatrix} x(1)^T & \dots & x(N)^T & u(0)^T & \dots & u(N-1)^T \end{bmatrix} \begin{bmatrix} Q & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \dots & P_f & 0 & \dots & 0 \\ \vdots & 0 & 0 & R & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(1) \\ \vdots \\ x(N) \\ u(0) \\ \vdots \\ u(N-1) \end{bmatrix} \tag{42}$$

# 3 REFERENCES

[1] SSY281 - Multi Predictive Control Assignment-2 handout
[2] SSY281 - Multi Predictive Control Lecture Notes
[3] Model Predictive Control: Theory, Computation, and Design by James B. Rawlings,David Q. Mayne and Moritz M. Diehl